Classification of two-dimensional fermionic and bosonic topological orders

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A classification of 2D fermionic and bosonic topological order

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The string-net approach by Levin and Wen, and the local unitary transformation approach by Chen, Gu, and Wen, provide ways to classify topological orders with gappable edge in 2D bosonic systems. The two approaches reveal that the mathematical framework for 2+1D bosonic topological order with gappable edge is closely related to unitary fusion category theory. In this paper, we generalize these systematic descriptions of topological orders to 2D fermion systems. We find a classification of 2+1D fermionic topological orders with gappable edge in terms of the following set of data $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta}_{jkn,\chi\delta}, d_i)$, that satisfy a set of non-linear algebraic equations. The exactly soluble Hamiltonians can be constructed from the above data on any lattices to realize the corresponding topological orders. When $F^{ij}_k = 0$, our result recovers the previous classification of 2+1D bosonic topological orders with gappable edge.

Contents

I. INTRODUCTION

Understanding phases of matter is one of the central problems in condensed matter physics. Landau symmetry breaking theory,[1,2] as a systematic theory of phases and phase transitions, becomes a cornerstone of condensed matter theory. However, at zero temperature, the symmetry breaking states described by local order parameters are basically direct product states. It is hard to believe that various direct product states can describe all possible quantum phases of matter.

Based on adiabatic evolution, one can show that gapped quantum phases at zero temperature correspond to the equivalence classes of local unitary (LU) transformations generated by finite-time evolutions of local hermitian operators $\tilde{H}(\tau)$:[3–7]

$$|\Phi\rangle \sim |\Phi'\rangle \quad \text{iff } |\Phi'\rangle = \mathcal{T}\mathrm{e}^{\mathrm{i}\int d\tau \tilde{H}(\tau)}|\Phi\rangle. \tag{1}$$

It turns out that there are many gapped quantum states that cannot be transformed into direct product states through LU transformations. Those states are said to have a long range entanglement. Thus, the equivalence classes of LU transformations, and hence the quantum phases of matter, are much richer than direct product states and much richer than what the symmetry breaking theory can describe. Different patterns of long range entanglement correspond to different quantum phases that are beyond the symmetry-breaking/order-parameter description[8] and direct-product-state description. The patterns of long range entanglement really correspond to the topological orders[9,10] that describe the new kind of orders in quantum spin liquids and quantum Hall states.[11–19]

In absence of translation symmetry, the above LU transformation can be expressed as a quantum circuit, which corresponds to a discretized LU transformation. The discretized LU transformation is more convenient to use. The gapped quantum phases can be more effectively studied and even be classified through the discretized LU transformations.[6,7,20–22]

After discovering more and more kinds of topological orders, it becomes important to gain a deeper understanding of topological order under a certain mathematical framework. We know that symmetry breaking orders can be understood systematically under the mathematical framework of group theory. Can topological orders be also understood under some mathematical framework? From the systematic construction of topologically ordered states based on string-nets[20] and the systematic description of non-Abelian satistics[23], it appears that tensor category theory may provide the underlying mathematical framework for topological orders.[24]

However, the string-net and the LU transformation approaches[6,7,20] only provide a systematic understanding for topological orders in qubit systems (*ie* quantum spin systems or local boson systems). Fermion systems can also have non trivial topological orders. In this paper, we will introduce a systematic theory for topological orders in interacting fermion systems (with interacting boson systems as special cases). Our approach is based on the LU transformations generated by local hermitian operators that contain even number of fermion operators. It allows us to classify and construct a large class of topological orders in fermion systems. The mathematical framework developed here may be related to the theory of enriched categories,[25] which can be viewed as a generalization of the standard tensor category theory[26,27]

To gain a systematic understanding of topological order in fermion systems, we first need a way to label those fermionic topological orders. In this paper, we show that a large class of fermionic topological orders (which include bosonic topological orders as special cases) can be labeled by a set of tensors: $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta}_{jkn,\chi\delta}, d_i)$. Certainly, not every set of tensors corresponds to a valid fermionic topological order. We show that only the tensors that satisfy a set of non-linear equations correspond to valid fermionic topological orders. The set of non-linear equations obtained here is a generalization of the non-linear equations (such as the pentagon identity) in a tensor category theory. So our approach is a generalization of tensor category theory and the string-net approach for bosonic topological orders. We would like to point out that the framework developed here not only leads to a classification of fermionic topological orders, it also leads to a more general classification of bosonic topological orders than the string-net and the related approaches.[6,20]

From a set of the data $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta}_{jkn,\chi\delta}, d_i)$, we can obtain the parent Hamiltonian as a sum of projectors. We believe that the Hamiltonian is unfrustrated. Its zero-energy ground state realizes the fermionic/bosonic topological order described by the data.

In section II, we give a careful discussion on what is a local fermion system. In section III, we introduce fermionic local unitary transformations. We then use fermionic local unitary transformations to define quantum phases for local fermion systems and fermionic topological orders. In section IV, we use fermionic local unitary transformations to define a wave function renormalization flow for local fermion systems. In section V, we discuss the fixed points of the wave function renormalization flow, and use those fixed points to classify a large class of fermionic (and bosonic) topological orders. In section VI, we comment on its relation to categorical framework. In section VII, we give a few simple examples. In appendix A, we discuss the definition of branching structure for a trivalent graph. In appendix B, we discuss the fermionic structure of the support space. In appendix C, we derive the ideal Hamiltonian from the data that characterize the fermionic/bosonic topological orders.

II. LOCAL FERMION SYSTEMS

Local boson systems (*ie* local qubit systems) and local fermion systems have some fundamental differences. To reveal those differences, in this section, we are going to define local fermion systems carefully. To contrast local fermion systems with local boson systems, let us first review the definition of local boson systems.

A. Local bosonic operators and bosonic states in a local boson model

A local boson quantum model is defined through its Hilbert space V and its local boson Hamiltonian H. The Hilbert space V of a local boson quantum model has a structure $V = \otimes V_{\boldsymbol{i}}$ where $V_{\boldsymbol{i}}$ is the local Hilbert space on the site $\boldsymbol{i}$. A local bosonic operator is defined as an operator that act within the local Hilbert space $V_{\boldsymbol{i}}$, or as a finite product of local bosonic operators acting on nearby sites. A local boson Hamiltonian H is a sum of local bosonic operators. The ground state of a local boson Hamiltonian H is called a bosonic state.

B. Local fermionic operators and fermionic states in a local fermion model

Now, let us try to define local fermion systems. A *local fermion quantum model* is also defined through its Hilbert space V and its local fermion Hamiltonian H_f. First let us introduce *fermion operator* $c^\alpha_{\boldsymbol{i}}$ at site $\boldsymbol{i}$ as operators that satisfy the anticommutation relation

$$c_{\boldsymbol{i}}^{\alpha} c_{\boldsymbol{j}}^{\beta} = -c_{\boldsymbol{j}}^{\beta} c_{\boldsymbol{i}}^{\alpha}, \qquad c_{\boldsymbol{i}}^{\alpha} (c_{\boldsymbol{j}}^{\beta})^{\dagger} = -(c_{\boldsymbol{j}}^{\beta})^{\dagger} c_{\boldsymbol{i}}^{\alpha}, \tag{2}$$

for all $\boldsymbol{i} \neq \boldsymbol{j}$ and all values of the α, β indices. We also say $c_{\boldsymbol{i}}^{\alpha}$ acts on the site $\boldsymbol{i}$. The fermion Hilbert space V is the space generated by the fermion operators and their hermitian conjugate:

$$V = \{[c_{\boldsymbol{j}}^{\beta} (c_{\boldsymbol{i}}^{\alpha})^{\dagger} ...]|0\rangle\}. \tag{3}$$

Due to the anticommutation relation (2), V has a form $V = \otimes V_{\boldsymbol{i}}$ where $V_{\boldsymbol{i}}$ is the local Hilbert space on the site $\boldsymbol{i}$. We see that the total Hilbert space of a fermion system has the same structure as a local boson system.

Using the Hilbert space $V = \otimes V_{\boldsymbol{i}}$, an explicit representation of the fermion operator $c_{\boldsymbol{i}}^{\alpha}$ can be obtained. First, each local Hilbert space can be splitted as $V_{\boldsymbol{i}} = V_{\boldsymbol{i}}^0 \oplus V_{\boldsymbol{i}}^1$. We also choose an ordering of the site label $\boldsymbol{i}$. Then $c_{\boldsymbol{i}}^{\alpha}$ has the following matrix representation

$$c_{\boldsymbol{i}}^{\alpha} = C_{\boldsymbol{i}}^{\alpha} \prod_{\boldsymbol{j}<\boldsymbol{i}} \Sigma_{\boldsymbol{j}}^3,$$

$$C_{\boldsymbol{i}}^{\alpha} = \begin{pmatrix} 0 & A_{\boldsymbol{i}}^{\alpha} \\ B_{\boldsymbol{i}}^{\alpha} & 0 \end{pmatrix}, \qquad \Sigma_{\boldsymbol{i}}^3 = \begin{pmatrix} I_{\boldsymbol{i}}^0 & 0 \\ 0 & -I_{\boldsymbol{i}}^1 \end{pmatrix}, \tag{4}$$

where $I_{\boldsymbol{i}}^0$ is the identity matrix acting in the space $V_{\boldsymbol{i}}^0$ and $I_{\boldsymbol{i}}^1$ is the identity matrix acting in the space $V_{\boldsymbol{i}}^1$. The matrix $C_{\boldsymbol{i}}^{\alpha}$ maps a state in $V_{\boldsymbol{i}}^0$ to a state in $V_{\boldsymbol{i}}^1$, and vice versa. We note that

$$C_{\boldsymbol{i}}^{\alpha} \Sigma_{\boldsymbol{i}}^3 = -\Sigma_{\boldsymbol{i}}^3 C_{\boldsymbol{i}}^{\alpha}. \tag{5}$$

We see that a fermion operator is not a local bosonic operator. The product of an odd number of fermion operators and any number of local bosonic operators on nearby sites is called a *local fermionic operator*.

Let us write the eigenvalue of $\Sigma_{\boldsymbol{i}}^3$ as $(-)^{s_{\boldsymbol{i}}}$. The states in $V_{\boldsymbol{i}}^0$ have $s_{\boldsymbol{i}} = 0$ and are called bosonic states. The states in $V_{\boldsymbol{i}}^1$ have $s_{\boldsymbol{i}} = 1$ and are called fermionic states. We can view $s_{\boldsymbol{i}}$ as the fermion number on site $\boldsymbol{i}$.

A *local fermion Hamiltonian* H_f is a sum of terms: $H = \sum_P O_P$, where $\sum_P$ sums over a set of regions. Each term O_P is a product of an even number of local fermionic operators and any number of local bosonic operators on a finite region P. Such kind of terms is called *pseudo-local bosonic operator* acting on the region. In other words, a local fermion Hamiltonian is a sum of pseudo-local bosonic operators. The ground state of a local fermion Hamiltonian H_f is called a *fermionic state*.

Note that, beyond 1D, a pseudo-local bosonic operator is in general not a local bosonic operator. So a local fermion Hamiltonian H_f (beyond 1D) in general is not a local boson Hamiltonian defined in the last subsection. In this sense, a local boson system and a local fermion system are fundamentally different despite they have the same Hilbert space. When viewed as a boson system, a local fermion Hamiltonian corresponds to a non-local boson Hamiltonian (beyond 1D). Thus classifying the quantum phases of local fermion systems corresponds to classifying the quantum phases of a particular kind of non-local boson systems.

III. FERMIONIC LOCAL UNITARY TRANSFORMATION AND TOPOLOGICAL PHASES OF FERMION SYSTEMS

Similar to the local boson systems, the finite-time evolution generated by a local fermion Hamiltonian defines an equivalence relation between gapped fermionic states:

$$|\psi(1)\rangle \sim |\psi(0)\rangle \text{ iff } |\psi(1)\rangle = \mathcal{T}[\mathrm{e}^{\,\mathrm{i} \int_0^1 dg\, \tilde{H}_f(g)}]|\psi(0)\rangle \tag{6}$$

where $\mathcal{T}$ is the path-ordering operator and $\tilde{H}(g) = \sum_{\boldsymbol{i}} O_{\boldsymbol{i}}(g)$ is a local fermion Hamiltonian (*ie* $O_{\boldsymbol{i}}(g)$ is a pseudo-local bosonic operator which is a product of even local fermionic operators). We will call $\mathcal{T}[\mathrm{e}^{\,\mathrm{i} \int_0^1 dg\, \tilde{H}_f(g)}]$ a fermion local unitary (fLU) evolution. We believe that the equivalence classes of such an equivalence relation are the universality classes of the gapped quantum phases of fermion systems.

The finite-time fLU evolution introduced here is closely related to *fermion quantum circuits with finite depth*. To define fermion quantum circuits, let us introduce piecewise fermion local unitary operators. A piecewise fermion local unitary operator has a form $U_{pwl} = \prod_i \mathrm{e}^{\,\mathrm{i} H_f(\boldsymbol{i})}$, where $\{H_f(\boldsymbol{i})\}$ is a Hermitian operator which is a pseudo-local bosonic operator that acts on a region labeled by $\boldsymbol{i}$. Note that regions labeled by different $\boldsymbol{i}$'s are not overlapping. $U_{\boldsymbol{i}} = \mathrm{e}^{\,\mathrm{i} H_f(\boldsymbol{i})}$ is called a fermion unitary operator. The size of each region is less than some finite number l. The unitary operator U_{pwl} defined in this way is called a fermion piece-wise local unitary operator with range l. A fermion quantum circuit with depth M is given by the product of M fermion piece-wise local unitary operators: $U_{circ}^M = U_{pwl}^{(1)} U_{pwl}^{(2)} \cdots U_{pwl}^{(M)}$. We believe that finite time fLU evolution can be simulated with a constant depth fermion quantum circuit and vice versa. Therefore, the equivalence relation eqn. (6) can be equivalently stated in terms of constant depth fermion quantum circuits:

$$|\psi(1)\rangle \sim |\psi(0)\rangle \text{ iff } |\psi(1)\rangle = U_{circ}^M |\psi(0)\rangle \tag{7}$$

where M is a constant independent of system size. Because of their equivalence, we will use the term "fermion Local Unitary Transformation" to refer to both fermion local unitary evolution and constant depth fermion quantum circuit in general.

Just like boson systems, the equivalence classes of fermionic local unitary transformations correspond to the universality classes that define phases of matter. Since here we do not include any symmetry, the equivalence classes actually correspond to topologically ordered phases. Such topologically ordered phases will be called fermionic topologically ordered phases.

IV. FERMIONIC LOCAL UNITARY TRANSFORMATION AND WAVE FUNCTION RENORMALIZATION

After defining the fermionic topological orders as the equivalence classes of many-body wave functions under

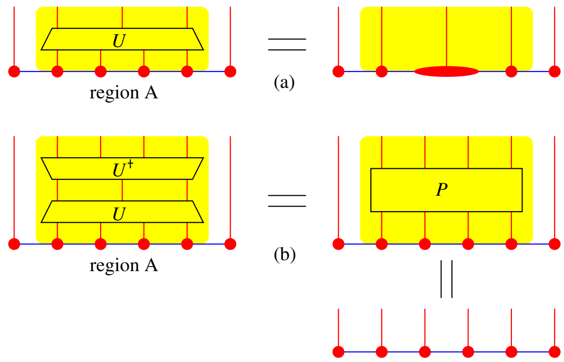


FIG. 1: (Color online) (a) A fermion local unitary (gfLU) transformation U_g acts in region A of a fermionic state $|\psi\rangle$ are formed by bosonic and fermionic operators in the region A. U_g always contain even numbers of fermionic operators. (b) $U_g^\dagger U_g = P$ is a projector, whose action does not change the state $|\psi\rangle$.

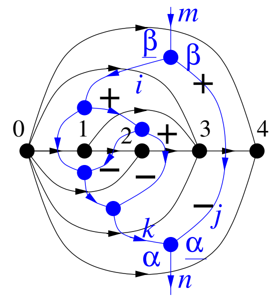


FIG. 2: (Color online) The lattice is a graph (described by the blue lines) with branching structure. The black lines describe the dual lattice. The state on the edges are labeled by $i, j, k, m, n = 0, \cdots, N$. The state on the edges are labeled by α, β. The vertices are labeled by $\underline{\alpha}$, $\underline{\beta}$. The $\underline{\alpha}$ vertex has two incoming edges and α has a range $\alpha = 1, \cdots, N_n^{kj}$. The $\underline{\beta}$ vertex has two incoming edges and β has a range $\beta = 1, \cdots, N_{ij}^m$.

fLU transformations, we like to use the fLU transformations, or more precisely the generalized fermion local unitary (gfLU) transformation, to define a wave function renormalization procedure. The wave function renormalization can remove the non-universal short-range entanglement and make generic complicated wave functions to flow to some very simple fixed-point wave functions. The simple fixed-point wave functions can help us to classify fermionic topological orders.

Let us first define the gfLU transformation U_g more carefully. Consider a state $|\psi\rangle$. Let ρ_A be the entanglement density matrix of $|\psi\rangle$ in region A. ρ_A may act in a subspace of the Hilbert space in region A. The subspace is called the support space $\tilde V_A$ of region A (see Fig. 1(a)). Let $|\tilde\phi_{\tilde i}\rangle$ be a basis of this support space $\tilde V_A$, and $|\phi_i\rangle$ be a basis of the full Hilbert space V_A of region A. The gfLU transformation U_g is the projection from the full Hilbert space V_A to the support space $\tilde V_A$. So up to some unitary transformations, U_g is a hermitian projection operator:

$$
\begin{aligned}
&U_g = U_1 P_g U_2, \qquad P_g^2 = P_g, \qquad P_g^\dagger = P_g,\\
&U_1^\dagger U_1 = 1, \qquad U_2^\dagger U_2 = 1.
\end{aligned} \tag{8}
$$

The matrix elements of U_g are given by $\langle \tilde\phi_{\tilde a}|\phi_i\rangle$. We will call such a gfLU transformation a primitive gfLU transformation. A generic gfLU transformation is a product of several primitive gfLU transformations which may contain several hermitian projectors and unitary transformations, for example, $U_g = U_1 P_g U_2 P_g' U_3$.

To understand the fermionic structure of U_g, we note that the support space $\tilde V_A$ has a structure $\tilde V_A = \tilde V_A^0 \oplus \tilde V_A^1$ (see appendix B 1), where $\tilde V_A^0$ has even numbers of fermions and $\tilde V_A^1$ has odd numbers of fermions. This means that U_g contains only even numbers of fermionic operators (*ie* U_g is a pseudo-local bosonic operator).

We also regard the inverse of U_g, $U_g^\dagger$, as a gfLU transformation. An fLU transformation is viewed as a special case of gfLU transformations where the degrees of freedom are not changed. Clearly $U_g^\dagger U_g = P$ and $U_g U_g^\dagger = P'$ are two hermitian projectors. The action of P does not change the state $|\psi\rangle$ (see Fig. 1(b)). Thus despite the degrees of freedom can be reduced under the gfLU transformations, no quantum information of the state $|\psi\rangle$ is lost under the gfLU transformations.

We note that the gfLU transformations can map one wave function to another wave function with fewer degrees of freedom. Thus it can be viewed as a wave function renormalization group flow. If the wave function renormalization leads to fixed-point wave functions, then those fixed-point wave functions can be much simpler, which can provide an efficient or even one-to-one labeling scheme of fermionic topological orders.

V. WAVE FUNCTION RENORMALIZATION AND A CLASSIFICATION OF FERMIONIC TOPOLOGICAL ORDERS

As an application of the above fermionic wave function renormalization, in this section, we will study the structure of fixed-point wave functions under the wave function renormalization. This will lead to a classification of fermionic topological orders.

A. Quantum state on a graph

Since the wave function renormalization may change the lattice structure, we will consider quantum state defined on a generic trivalent graph G. The graph has a branching structure as described by appendix A: Each edge has an orientation and each vertex has two incoming or one incoming edges. Each edge has $N+1$ states, labeled by $i = 0, ..., N$. Each vertex also has physical states. The number of the states depends on the states on the connected edges and they are labeled by $\alpha = 1, ..., N_k^{ij}$ or $\beta = 1, ..., N_{ij}^k$ for vertices with two incoming and one outgoing edges. (see Fig. 2).

Despite the similar look between α index and $\underline{\alpha}$ index, the two indices are very different. $\underline{\alpha}$ index labels the vertices while α index labels the state on a vertex. In this paper, we very often use α to label states on vertex $\underline{\alpha}$.

The states on the edge are always bosonic. However, the states on the vertices may be fermionic. We introduce, for example, F_k^{ij} to indecate the number of fermionic states on the vertex: $\alpha = 1, ..., B_k^{ij}$ label the bosonic(fermion parity even) states and $\alpha = 1 + B_k^{ij}, ..., F_k^{ij} + B_k^{ij}$ label the fermionic(fermion parity odd) states. Here

$$B_k^{ij} = N_k^{ij} - F_k^{ij} \tag{9}$$

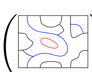
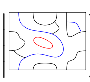
is the number of bosonic states on the vertex. Similarly, we can introduce N_{ij}^k and F_{ij}^k, to indecate the number of states and fermionic states on vertices with one incoming edges and two outgoing edges. In this paper, we will assume that

$$N_k^{ij} = N_{ij}^k, \quad B_k^{ij} = B_{ij}^k, \quad F_k^{ij} = F_{ij}^k, \tag{10}$$

as required by unitarity.

We introduce $s_k^{ij}(\alpha)$ to indicate whether a vertex state labeled by α is bosonic or fermionic: $s_k^{ij}(\alpha) = 0$ if the α-state is bosonic and $s_k^{ij}(\alpha) = 1$ if the α-state is fermionic. Here the vertex connects to three edges i, j, and k (see Fig. 2). Each graph with a given $\alpha, \beta, ...i, j, ...$ labeling (see Fig. 2) corresponds to a state and all such labeled graphs form an orthonormal basis. Our fixed-point state is a superposition of those basis states

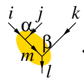
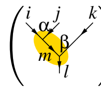
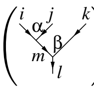
$$|\psi_{\text{fix}}\rangle = \sum_{\text{all conf.}} \psi_{\text{fix}} \Big(\Big) \Big| \Big\rangle . \tag{11}$$

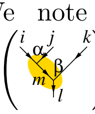



In the string-net approach, we made a very strong assumption that the above graphic states on two graphs are the same if the two graphs have the same topology. However, since different vertices and edges are really distinct, a generic graph state does not have such an topological invariance. To consider more general states, in this paper, we would like to weaken such a topological requirement. We will consider vertex-labeled graphs (v-graphs) where each vertex is assigned an index $\underline{\alpha}$. Two v-graphs are said to be topologically the same if one graph can be continuously deformed into the other in such a way that vertex labeling of the two graphs matches. In this paper, we will consider the graph states that depend only on the topology of the v-graphs. Those states are more general than the graph states that depend only on the topology of the graphs without vertex labeling. Such a generalization is important in obtaining interesting fermionic fixed-point states on graphs.

B. The structure of a fixed-point wave function

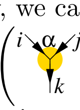
Before describing the wave function renormalization, we examine the structure of entanglement of a fixed point wave function ψ_{fix} on a v-graph. First let us introduce the concept of support space with a fixed boundary state.

We examine the wave function on a patch, for example, . The fixed-point wave function ψ_{fix}  (only the relevant part of the graph is drawn) can be viewed as a function of α, β, m: $\phi_{ijkl,\Gamma}(\alpha, \beta, m) = \psi_{\text{fix}}$  if we fix i, j, k, l and the indices on the other part of the graph. (Here the indices on the other part of the graph are summarized by Γ.) As we vary the indices Γ on the other part of the graph (still keep i, j, k, and l fixed), the wave function of α, β, m, $\phi_{ijkl,\Gamma}(\alpha, \beta, m)$, may change. All those $\phi_{ijkl,\Gamma}(\alpha, \beta, m)$ form a linear space of dimension D_l^{ijk}. D_l^{ijk} is an important concept that will appear later. We note that the two vertices α and β and the edge m form a region surrounded by the edges i, j, k, l. So we will call the dimension-D_l^{ijk} space the support space V_l^{ijk} and D_l^{ijk} the support dimension for the state ψ_{fix} on the region surrounded by a fixed boundary state i, j, k, l.

We note that in the fixed-point wave function ψ_{fix} , the number of choices of α, β, m is $N_l^{ijk} = \sum_{m=0}^{N} N_m^{ij} N_l^{mk}$. Thus the support dimension D_l^{ijk} satisfies $D_l^{ijk} \leq N_l^{ijk}$. Here we will make an important assumption – the saturation assumption: *The fixed-point wave function saturates the inequality:*

$$D_l^{ijk} = N_l^{ijk} \equiv \sum_{m=0}^{N} N_m^{ij} N_l^{mk}. \tag{12}$$

In general, we will make the similar saturation assumption for any tree graphs. We will see that the entanglement structure described by such a saturation assumption is invariant under the wave function renormalization.

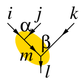
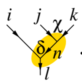
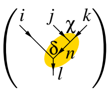
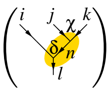
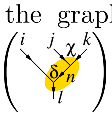
Similarly, we can define D_k^{ij} as the support dimension of the Φ_{fix}  on a region bounded by links i, j, k. Since the region contains only a single vertex $\underline{\alpha}$, we have $D_k^{ij} \leq N_k^{ij}$. The saturation assumption requires that

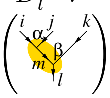
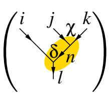
$$D_k^{ij} = N_k^{ij}. \tag{13}$$

In fact, this is how N_k^{ij} is defined.

We note that under the saturation assumption, the structure of the support dimensions for tree graphs is encoded in the N_k^{ij} tensor. Here N_k^{ij} plays a similar role as the pattern of zeros in a classification of fractional quantum Hall wave functions.[28]

C. The first type of wave function renormalization: F-move

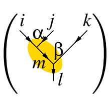
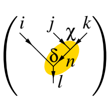
Our wave function renormalization scheme contains two types of renormalization. The first type of renormalization does not change the degrees of freedom and corresponds to a local unitary transformation. It corresponds to locally deforming the v-graph  to . (The parts that are not drawn are the same.) The fixed-point wave function on the new v-graph is given by $\psi_{\text{fix}}\Big($  $\Big)$. Again, such a wave function can be viewed as a function of χ, δ, n: $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n) = \psi_{\text{fix}}\Big($  $\Big)$ if we fix i,j,k,l and the indices on the other part of the graph. The support dimension of the state $\psi_{\text{fix}}\Big($  $\Big)$ on the region surrounded by i,j,k,l is $\tilde D^{ijk}_l$. Again $\tilde D^{ijk}_l \leq \tilde N^{ijk}_l$, where $\tilde N^{ijk}_l \equiv \sum_{n=0}^N N^{jk}_n N^{in}_l$ is the number of choices of χ, δ, n. The saturation assumption implies that $\tilde N^{ijk}_l = \tilde D^{ijk}_l$.

The two fixed-point wave functions $\psi_{\text{fix}}\Big($  $\Big)$ and $\psi_{\text{fix}}\Big($  $\Big)$ are related via a local unitary transformation. Thus

$$D^{ijk}_l = \tilde D^{ijk}_l, \tag{14}$$

which implies

$$\sum_{m=0}^N N^{ij}_m N^{mk}_l = \sum_{n=0}^N N^{jk}_n N^{in}_l. \tag{15}$$

We note that the support space of $\psi_{\text{fix}}\Big($  $\Big)$ and $\psi_{\text{fix}}\Big($  $\Big)$ should have the same number of fermionic states. Thus eqn. (15) can be splitted as

$$\sum_{m=0}^N B^{ij}_m B^{mk}_l + F^{ij}_m F^{mk}_l = \sum_{n=0}^N B^{jk}_n B^{in}_l + F^{jk}_n F^{in}_l, \tag{16}$$

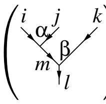
$$\sum_{m=0}^N B^{ij}_m F^{mk}_l + F^{ij}_m B^{mk}_l = \sum_{n=0}^N B^{jk}_n F^{in}_l + F^{jk}_n B^{in}_l. \tag{17}$$

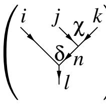
We express the above unitary transformation in terms of the tensor $F^{ijm,\alpha\beta}_{kln,\chi\delta}$, where $i,j,k,... = 0,...,N$, and $\alpha = 1,...,N^{ij}_k$, *etc* :

$$\phi_{ijkl,\Gamma}(\alpha,\beta,m) \simeq \sum_{n=0}^N \sum_{\chi=1}^{N^{jk}_n} \sum_{\delta=1}^{N^{in}_l} F^{ijm,\alpha\beta}_{kln,\chi\delta} \tilde\phi_{ijkl,\Gamma}(\chi,\delta,n) \tag{18}$$

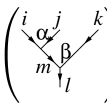
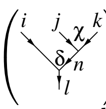
or graphically as

$$\psi_{\text{fix}}\Bigg(\quad\Bigg) \simeq \sum_{n\chi\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta}\, \psi_{\text{fix}}\Bigg(\quad\Bigg). \tag{19}$$

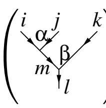


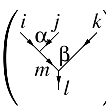


where the vertices carrying the states labeled by $(\alpha,\beta,\chi,\delta)$ are labeled by $(\underline{\alpha}, \underline{\beta}, \underline{\chi}, \underline{\delta})$ (see Fig. 2). Here $\simeq$ means equal up to a constant phase factor. (Note that the total phase of the wave function is unphysical.) We will call such a wave function renormalization step an F-move.

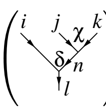
There is a subtlety in eqn. (19). Since some values of α, β, ... indices correspond to fermionic states, the sign of wave function depends on how those fermionic states are ordered. In (19), the wave functions $\psi_{\text{fix}}\Big($  $\Big)$ and $\psi_{\text{fix}}\Big($  $\Big)$ are obtained by assuming the fermionic states are order in a particular way:

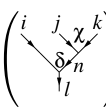
$$\Big|\psi_{\text{fix}}\Big(\quad\Big)\Big\rangle = \sum \psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}\Big(\quad\Big) |\alpha\beta\eta_1\eta_2...\rangle_{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$$





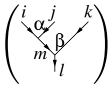
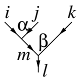
$$\Big|\psi_{\text{fix}}\Big(\quad\Big)\Big\rangle = \sum \psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}\Big(\quad\Big) |\chi\delta\eta_1\eta_2...\rangle_{\underline{\chi}\,\underline{\delta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots} \tag{20}$$

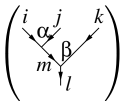


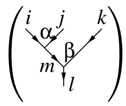


where $\sum$ sums over the all indices on the vertices and edges. η_i are indices on other vertices. Here $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ is a graph state where the $\underline{\alpha}$-vertex is in the $|\alpha\rangle$-state, the $\underline{\beta}$-vertex is in the $|\beta\rangle$-state, *etc* . We note that $|\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}$ is also a graph state where the $\underline{\alpha}$-vertex is in the $|\alpha\rangle$-state, the $\underline{\beta}$-vertex is in the $|\beta\rangle$-state. But if the $|\alpha\rangle$-state and the $|\beta\rangle$-state are fermionic (*ie* $s^{ij}_m(\alpha) = s^{mk}_l(\beta) = 1$), $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ and $|\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}$ will differ by a sign, since in $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ the fermion on $\underline{\beta}$-vertex is created before the fermion on $\underline{\alpha}$-vertex is created, while in $|\beta\alpha...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ the fermion on $\underline{\alpha}$-vertex is created before the fermion on $\underline{\beta}$-vertex is created. In general we have

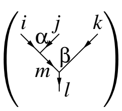
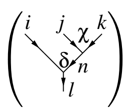
$$|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...} = (-)^{s^{ij}_m(\alpha) s^{mk}_l(\beta)} |\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}. \tag{21}$$

We see that subscript $\underline{\alpha}\,\underline{\beta}...$ in $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ is important to properly describe such an order dependent sign. Similarly, we must add the superscript in the wave function as well, as in $\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$ , since the amplitude of the wave function depends on both the labeled graph  and the ordering of the vertices $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2....$ Such a wave function has the following sign dependence:

$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$ 

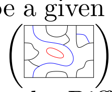
$= (-)^{s_m^{ij}(\alpha)s_l^{mk}(\beta)}\psi_{\text{fix}}^{\underline{\beta}\,\underline{\alpha}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$  . (22)

Thus eqn. (19) should be more properly written as

$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\cdots}$  $\simeq \sum_{n\chi\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta}\,\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}\cdots}$  , (23)

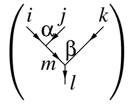
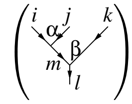
where the superscripts $\underline{\alpha}\,\underline{\beta}...$ and $\underline{\chi}\,\underline{\delta}...$ describing the order of fermionic states are added in the wave function.

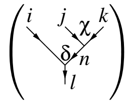
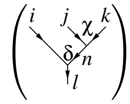
Since the sign of the wave function depends on the ordering of fermionic states, the F-tensor may also depend on the ordering. In this paper, we choose a particular ordering of fermionic states to define the F-tensor as described by $\underline{\alpha}\,\underline{\beta}...$ and $\underline{\chi}\,\underline{\delta}...$ in eqn. (23). In such a canonical ordering, we create a fermion on the $\underline{\beta}$-vertex before we create a fermion on the $\underline{\alpha}$-vertex. Similarly, we create a fermion on the $\underline{\delta}$-vertex before we create a fermion on the $\underline{\chi}$-vertex.

We have seen that, to describe a given fermionic state, the fermionic wave function ψ_{fix}  depends on the order of the fermions on the graph. Different choices of fermion orders lead to different fermionic wave functions even for the same fermionic state. To avoid such order dependence of the fermionic wave function (even for the same fermionic state), in the following, we would like to introduce one Majorana number $\theta_{\underline{\alpha}}$ on each vertex $\underline{\alpha}$ to rewrite a wave function that does not depend on the ordering of fermionic states on vertices. The Majorana numbers satisfy

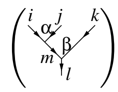
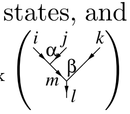
$$\theta^2_{\underline{\alpha}} = 1, \quad \theta_{\underline{\alpha}}\theta_{\underline{\beta}} = -\theta_{\underline{\beta}}\theta_{\underline{\alpha}} \text{ for any } \alpha \neq \beta,$$
$$\theta^\dagger_{\underline{\alpha}} = \theta_{\underline{\alpha}}, \quad (\theta_{\underline{\alpha}}...\theta_{\underline{\beta}})^\dagger = \theta_{\underline{\beta}}...\theta_{\underline{\alpha}}. \qquad (24)$$

We introduce the following wave function with Majorana numbers:

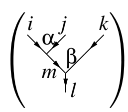
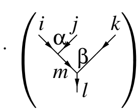
Ψ_{fix}  $= [\theta_{\underline{\alpha}}^{s_m^{ij}(\alpha)}\theta_{\underline{\beta}}^{s_l^{mk}(\beta)}...]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\cdots}$ 

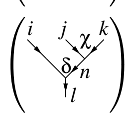
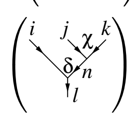
Ψ_{fix}  $= [\theta_{\underline{\delta}}^{s_l^{in}(\delta)}\theta_{\underline{\chi}}^{s_n^{jk}(\chi)}...]\psi_{\text{fix}}^{\underline{\delta}\,\underline{\chi}\cdots}$ 

(25)

where the order of the Majorana numbers $(\theta_{\underline{\alpha}}\theta_{\underline{\beta}}...)$ is tied to the order $\underline{\alpha}\,\underline{\beta}...$ in the superscript that describes the order of the fermionic states. We see that, by construction, the sign of Ψ_{fix}  does not depend on the order of the fermionic states, and this is why the Majorana wave function Ψ_{fix}  does not carry the superscript $(\underline{\alpha}\,\underline{\beta}...)$.

We would like to mention that $(\theta_{\underline{\alpha}}, \theta_{\underline{\beta}})$ and $(\theta_{\underline{\chi}}, \theta_{\underline{\delta}})$ are treated as different Majorana numbers even when, for example, $\underline{\alpha}$ and $\underline{\chi}$ take the same value. This is because $\underline{\alpha}$ and $\underline{\chi}$ label different vertices regardless if $\underline{\alpha}$ and $\underline{\chi}$ have the same value or not. So a more accurate notation should be

Ψ_{fix}  $= [\theta_{\underline{\alpha}}^{s_m^{ij}(\alpha)}\theta_{\underline{\beta}}^{s_l^{mk}(\beta)}...]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\cdot\cdot}$ 

Ψ_{fix}  $= [\tilde\theta_{\underline{\chi}}^{s_n^{jk}(\chi)}\tilde\theta_{\underline{\delta}}^{s_l^{in}(\delta)}...]\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}\cdot\cdot}$  ,

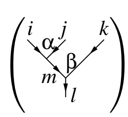
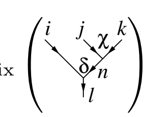
(26)

where $\theta_{\underline{\alpha}}$ and $\tilde\theta_{\underline{\chi}}$ are different even when $\underline{\alpha} = \underline{\chi}$. But in this paper, we will drop the ˜ and hope that it will not cause any confusions.

Let us introduce the F-tensor with Majorana numbers:

$$\mathcal{F}^{ijm,\alpha\beta}_{kln,\chi\delta} = \theta_{\underline{\alpha}}^{s_m^{ij}(\alpha)}\theta_{\underline{\beta}}^{s_l^{mk}(\beta)}\theta_{\underline{\delta}}^{s_l^{in}(\delta)}\theta_{\underline{\chi}}^{s_n^{jk}(\chi)}F^{ijm,\alpha\beta}_{kln,\chi\delta} \qquad (27)$$

We can rewrite (23) as

Ψ_{fix}  $\simeq \sum_{n\chi\delta}\mathcal{F}^{ijm,\alpha\beta}_{kln,\chi\delta}\Psi_{\text{fix}}$  . (28)

Such an expression is valid for any ordering of the fermion states.

From the graphic representation (23), We note that

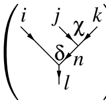
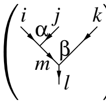
$$F^{ijm,\alpha\beta}_{kln,\chi\delta} = 0 \text{ when} \qquad (29)$$
$$N^{ij}_m < 1 \text{ or } N^{mk}_l < 1 \text{ or } N^{jk}_n < 1 \text{ or } N^{in}_l < 1,$$
$$\text{or } s^{ij}_m(\alpha) + s^{mk}_l(\beta) + s^{jk}_n(\chi) + s^{in}_l(\delta) = 1 \text{ mod } 2.$$

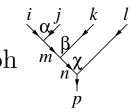
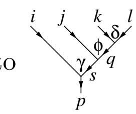
When $N^{ij}_m < 1$ or $N^{mk}_l < 1$, the left-hand-side of eqn. (23) is always zero. Thus $F^{ijm,\alpha\beta}_{kln,\chi\delta} = 0$ when $N^{ij}_m < 1$ or $N^{mk}_l < 1$. When $N^{jk}_n < 1$ or $N^{in}_l < 1$, wave function on the right-hand-side of eqn. (23) is always zero. So we can choose $F^{ijm,\alpha\beta}_{kln,\chi\delta} = 0$ when $N^{jk}_n < 1$ or $N^{in}_l < 1$. Also, $F^{ijm,\alpha\beta}_{kln,\chi\delta}$ represents a pseudo-local bosonic operator which contains even number of fermionic operators. Therefore $F^{ijm,\alpha\beta}_{kln,\chi\delta}$ is non-zero only when $s^{ij}_m(\alpha) + s^{mk}_l(\beta) + s^{jk}_n(\chi) + s^{in}_l(\delta) = 0 \bmod 2$.

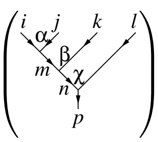
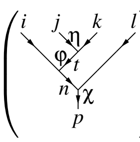
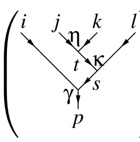
For fixed i, j, k, and l, the matrix $F^{ij,\frac{\alpha\beta}{\chi\delta}}_{kl,}$ with matrix elements $(F^{ij}_{kl})^{m,\alpha\beta}_{n,\chi\delta} = F^{ijm,\alpha\beta}_{kln,\chi\delta}$ is a matrix of dimension N^{ijk}_l (see (15)). Here we require the mapping $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n) \to \phi_{ijkl,\Gamma}(\alpha,\beta,m)$ generated by the matrix F^{ij}_{kl} to be unitary. Since, as we change Γ, $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n)$ and $\phi_{ijkl,\Gamma}(\alpha,\beta,m)$ span two N^{ijk}_l dimensional spaces. Thus we require F^{ij}_{kl} to be an $N^{ijk}_l \times N^{ijk}_l$ unitary matrix

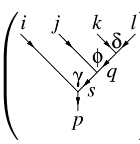
$$\sum_{n\chi\delta} F^{ijm',\alpha'\beta'}_{kln,\chi\delta} (F^{ijm,\alpha\beta}_{kln,\chi\delta})^* = \delta_{m,m'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}. \tag{30}$$

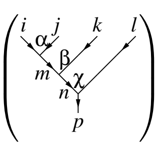
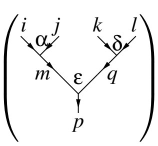
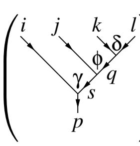
In this way, the F-move represents an fLU transformation. It is easy to see that the unitarity condition implies:

$\Psi_{\text{fix}}\Big($  $\Big) \simeq \sum_{m\alpha\beta} \left(\mathcal{F}^{ijm,\alpha\beta}_{kln,\chi\delta}\right)^\dagger \Psi_{\text{fix}}\Big($  $\Big)$. (31)

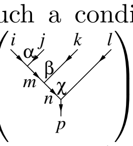
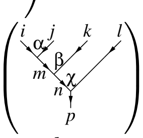
The F-move (28) can be viewed as a relationship between wave functions on different v-graphs that are only differ by a local transformation. Since we can locally transform one v-graph to another v-graph through different paths, the F-move (28) must satisfy certain self consistent conditions. For example the v-graph  can be transformed to  through two different paths; one contains two steps of local transformations and another contains three steps of local transformations as described by eqn. (28). The two paths lead to the following relations between the wave functions:

$\Psi_{\text{fix}}\Big($  $\Big) \simeq \sum_{t\eta\varphi} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \Psi_{\text{fix}}\Big($  $\Big) \simeq \sum_{t\eta\varphi;s\kappa\gamma} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \Psi_{\text{fix}}\Big($  $\Big)$

$\simeq \sum_{t\eta\kappa;\varphi;s\kappa\gamma;q\delta\phi} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \mathcal{F}^{jkt,\eta\kappa}_{lsq,\delta\phi} \Psi_{\text{fix}}\Big($  $\Big)$. (32)

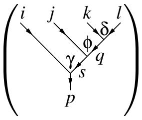
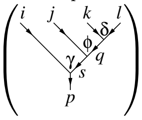
$\Psi_{\text{fix}}\Big($  $\Big) \simeq \sum_{q\delta\epsilon} \mathcal{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \Psi_{\text{fix}}\Big($  $\Big) \simeq \sum_{q\delta\epsilon;s\phi\gamma} \mathcal{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \mathcal{F}^{ijm,\alpha\epsilon}_{qps,\phi\gamma} \Psi_{\text{fix}}\Big($  $\Big)$, (33)

The consistence of the above two relations leads a condition on the F-tensor.

To obtain such a condition, let us fix i, j, k, l, p, and view $\psi_{\text{fix}}\Big($  $\Big)$ as a function of $\alpha, \beta, \chi, m, n$: $\phi(\alpha,\beta,\chi,m,n) = \psi_{\text{fix}}\Big($  $\Big)$. As we vary indices on the other part of graph, we obtain different wave functions $\phi(\alpha,\beta,\chi,m,n)$ which form a dimension D^{ijkl}_p space. In other words, D^{ijkl}_p is the support dimension of the state ψ_{fix} on the region α,β,χ,m,n with boundary state i,j,k,l,p (see the discussion in section V B). Since the number of choices of α,β,χ,m,n is $N^{ijkl}_p = \sum_{m,n} N^{ij}_m N^{mk}_n N^{nl}_p$, we have $D^{ijkl}_p \le N^{ijkl}_p$. Here we require a similar saturation condition as in (12):

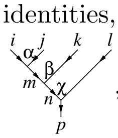
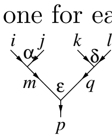
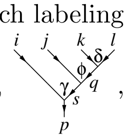
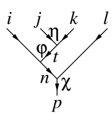
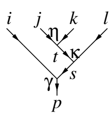
$$N^{ijkl}_p = D^{ijkl}_p \tag{34}$$

Similarly, the number of choices of δ,ϕ,γ,q,s in

ψ_{fix} $\Bigg($  $\Bigg)$ is also N_p^{ijkl}. Here we again assume $\tilde{D}_p^{ijkl} = N_p^{ijkl}$, where $\tilde{D}_p^{ijkl}$ is the support dimension of ψ_{fix} $\Bigg($  $\Bigg)$ on the region bounded by i, j, k, l, p.

So the two relations (33) and (32) can be viewed as two relations between a pair of vectors in the two D_p^{ijkl} dimensional vector spaces. As we vary indices on the other part of graph (still keep i, j, k, l, p fixed), each vector in the pair can span the full D_p^{ijkl} dimensional vector space. So the validity of the two relations (33) and (32) implies that

$$\sum_t \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \mathcal{F}^{jkt,\eta\kappa}_{lsq,\delta\phi}$$
$$\simeq \sum_{\epsilon=1}^{N_p^{mq}} \mathcal{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \mathcal{F}^{ijm,\alpha\epsilon}_{qps,\phi\gamma}. \tag{35}$$

which is the fermionic generalization of the famous pentagon identity. The above expression actually contains many different pentagon identities, one for each labeling scheme of the vertices in , , , , and . We obtain

$$\sum_t \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \mathcal{F}^{jkt,\eta\kappa}_{lsq,\delta\phi}$$
$$\simeq \sum_{\epsilon=1}^{N_p^{mq}} \mathcal{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \mathcal{F}^{ijm,\alpha\epsilon}_{qps,\phi\gamma}. \tag{36}$$

We can use the transformation

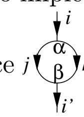
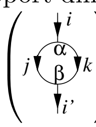
$$F^{ijm,\alpha\beta}_{kln,\chi\delta} \to \mathrm{e}^{\mathrm{i}\,\theta} F^{ijm,\alpha\beta}_{kln,\chi\delta} \tag{37}$$

to change $\simeq$ to $=$ in the above equation and remove the Majorana numbers to rewrite the above as

$$\sum_t \sum_{\eta=1}^{N_t^{jk}} \sum_{\varphi=1}^{N_n^{it}} \sum_{\kappa=1}^{N_s^{tl}} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi}$$
$$= (-)^{s_m^{ij}(\alpha)s_q^{kl}(\delta)} \sum_{\epsilon=1}^{N_p^{mq}} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}. \tag{38}$$

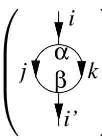
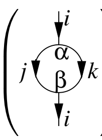
The above fermionic pentagon identity (38) is a set of nonlinear equations satisfied by the rank-10 tensor $F^{ijm,\alpha\beta}_{kln,\chi\delta}$. The above consistency relations (38) are equivalent to the requirement that the local unitary transformations described by eqn. (28) on different paths all commute with each other up to a total phase factor.

D. The second type of wave function renormalization: O-move

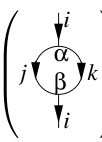
The second type of wave function renormalization does change the degrees of freedom and corresponds to a generalized local unitary transformation. One way to implement the second type renormalization is to reduce  to $\downarrow i$, so that we still have a trivalent graph. This requires that the support dimension $D_{ii'}$ of the fixed-point wave function ψ_{fix} $\Bigg($  $\Bigg)$ is given by

$$D_{ii'} = \delta_{ii'}. \tag{39}$$

This implies that

$\psi_{\text{fix}} \Bigg($  $\Bigg) = \delta_{ii'} \psi_{\text{fix}} \Bigg($  $\Bigg).$ (40)

The second type renormalization can now be written as (since $D_{ii} = 1$)

$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\cdots} \Bigg($  $\Bigg) \simeq O_i^{jk,\alpha\beta} \psi_{\text{fix}}^{\underline{\eta}\cdots} \Big(\downarrow i \Big).$ (41)

where the ordering of the vertices is described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}...$. We will call such a wave function renormalization step an O-move. Here $O_i^{jk,\alpha\beta}$ satisfies

$$\sum_{k,j} \sum_{\alpha=1}^{N_i^{jk}} \sum_{\beta=1}^{N_i^{jk}} O_i^{jk,\alpha\beta} (O_i^{jk,\alpha\beta})^* = 1 \tag{42}$$

and

$$O_i^{jk,\alpha\beta} = 0, \text{ if } N_i^{jk} < 1 \text{ or } s_i^{jk}(\alpha) + s_i^{jk}(\beta) = 1 \bmod 2. \tag{43}$$

The condition (42) ensures that the two wave functions on the two sides of eqn. (41) have the same normalization. We note that the number of choices for the four indices (j, k, α, β) in $O_i^{jk,\alpha\beta}$ must be equal or greater than 1:

$$D_i = \sum_{jk} (N_i^{jk})^2 \geq 1. \tag{44}$$

In fact, we should have a stronger condition: the number of choices for the four indices (j, k, α, β) that correspond to bosonic states must be equal or greater than 1

$$D_i = \sum_{jk} (B_i^{jk})^2 + (F_i^{jk})^2 \geq 1. \tag{45}$$

The wave functions in eqn. (41) is defined with respect to the ordering of the fermionic states described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots$. Let us introduce

$$\Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\end{array}\right) = [\theta_{\underline{\alpha}}^{s_i^{jk}(\alpha)}\theta_{\underline{\beta}}^{s_i^{jk}(\beta)}\theta_{\underline{\eta}}^{s(\eta)}\ldots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\end{array}\right),$$

$$\Psi_{\text{fix}}\left(\downarrow i\right) = [\theta_{\underline{\eta}}^{s(\eta)}\ldots]\psi_{\text{fix}}^{\underline{\eta}\ldots}\left(\downarrow i\right) \tag{46}$$

and

$$\mathcal{O}_i^{jk,\alpha\beta} = \theta_{\underline{\alpha}}^{s_i^{jk}(\alpha)}\theta_{\underline{\beta}}^{s_i^{jk}(\beta)} O_i^{jk,\alpha\beta}. \tag{47}$$

We can rewrite (41) as

$$\Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\end{array}\right) \simeq \mathcal{O}_i^{jk,\alpha\beta}\Psi_{\text{fix}}\left(\downarrow i\right) \tag{48}$$

which is valid for any ordering of the fermionic states.

E. The third type of wave function renormalization: Y-move

The third type of wave function renormalization also changes the degrees of freedom. The support space of $\downarrow i\ \ \downarrow j$ is one dimensional, while the support space of $\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}$ is $\sum_k (N_k^{ij})^2$ dimensional. So the wave function $\psi_{\text{fix}}\left(\downarrow i\ \ \downarrow j\right)$ is a particular vector in the support space of $\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}$. Thus, the third type of wave function renormalization takes the following form

$$\sum_{k,\alpha\beta} Y^{ij}_{k,\alpha\beta}\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots}\left(\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}\right) \simeq \psi_{\text{fix}}^{\underline{\eta}\ldots}\left(\downarrow i\ \ \downarrow j\right) \tag{49}$$

where the ordering of the vertices is described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots$. We will call such a wave function renormalization step a Y-move. We can choose

$$Y^{ij}_{k,\alpha\beta} = 0,\ \text{if}\ N_k^{ij} < 1\ \text{or}\ s_k^{ij}(\alpha) + s_k^{ij}(\beta) = 1 \bmod 2. \tag{50}$$

The wave functions in eqn. (49) is defined with respect to the ordering of the fermionic states described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots$. Let us introduce

$$\Psi_{\text{fix}}\left(\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}\right) = [\theta_{\underline{\alpha}}^{s_k^{ij}(\alpha)}\theta_{\underline{\beta}}^{s_k^{ij}(\beta)}\theta_{\underline{\eta}}^{s(\eta)}\ldots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots}\left(\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}\right),$$

$$\Psi_{\text{fix}}\left(\downarrow i\ \ \downarrow j\right) = [\theta_{\underline{\eta}}^{s(\eta)}\ldots]\psi_{\text{fix}}^{\underline{\eta}\ldots}\left(\downarrow i\ \ \downarrow j\right) \tag{51}$$

and

$$\mathcal{Y}^{ij}_{k,\alpha\beta} = \theta_{\underline{\beta}}^{s_k^{ij}(\beta)}\theta_{\underline{\alpha}}^{s_k^{ij}(\alpha)} Y^{ij}_{k,\alpha\beta}. \tag{52}$$

We can rewrite (49) as

$$\sum_{k,\alpha\beta}\mathcal{Y}^{ij}_{k,\alpha\beta}\Psi_{\text{fix}}\left(\begin{array}{c} i\searrow\alpha\swarrow j\\ \downarrow k\\ i\swarrow\beta\searrow j\end{array}\right) \simeq \Psi_{\text{fix}}\left(\downarrow i\ \ \downarrow j\right) \tag{53}$$

which is valid for any ordering of the fermionic states.

F. A relation between $\mathcal{O}_i^{jk,\alpha\beta}$ and $\mathcal{Y}_k^{ij,\alpha\beta}$

We find that the following wave function has two ways of reduction:

$$\begin{aligned}\sum_{\beta\gamma}\mathcal{Y}^{jk}_{i,\beta\gamma}\Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\\ j\ \begin{smallmatrix}\gamma\\ \lambda\end{smallmatrix}\ k\\ \downarrow i\end{array}\right) &\simeq \Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \lambda\end{smallmatrix}\ k\\ \downarrow i\end{array}\right)\\ &\simeq \mathcal{O}_i^{jk,\alpha\lambda}\Psi_{\text{fix}}\left(\downarrow i\right),\end{aligned} \tag{54}$$

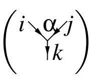
$$\begin{aligned}&\sum_{\beta\gamma}\mathcal{Y}^{jk}_{i,\beta\gamma}\Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\\ j\ \begin{smallmatrix}\gamma\\ \lambda\end{smallmatrix}\ k\\ \downarrow i\end{array}\right)\\ &\simeq \sum_{\beta\gamma}\mathcal{Y}^{jk}_{i,\beta\gamma}\mathcal{O}_i^{jk,\gamma\lambda}\Psi_{\text{fix}}\left(\begin{array}{c}\downarrow i\\ j\ \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\ k\\ \downarrow i\end{array}\right)\\ &\simeq \sum_{\beta\gamma}\mathcal{Y}^{jk}_{i,\beta\gamma}\mathcal{O}_i^{jk,\gamma\lambda}\mathcal{O}_i^{jk,\alpha\beta}\Psi_{\text{fix}}\left(\downarrow i\right)\end{aligned} \tag{55}$$

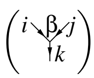
The two reductions should agree, which leads to the condition

$$\mathcal{O}_i^{jk,\alpha\lambda} \simeq \sum_{\beta\gamma}\mathcal{Y}^{jk}_{i,\beta\gamma}\mathcal{O}_i^{jk,\gamma\lambda}\mathcal{O}_i^{jk,\alpha\beta} \tag{56}$$

G. A "gauge" freedom

We note that the following transformation changes the wave function, but does change fixed-point property and the phase described by the wave function:

$$\Psi_{\text{fix}}$$


$$\to \sum_\beta f^{ij,\alpha}_{k,\beta} \Psi_{\text{fix}}$$


$$, \tag{57}$$

where f^{ij}_k is a unitary matrix

$$\sum_\beta f^{ij,\alpha}_{k,\beta}(f^{ij,\alpha'}_{k,\beta})^* = \delta_{\alpha\lambda'}. \tag{58}$$

Similarly, we have unitary transformation $f^{k,\alpha}_{ij,\beta}$ for vertices with two incoming edges and one outgoing edge. Such transformations corrrespond to a choice of basis and should be regarded as an equivalent relation.

The above transformation induce the following transformation on $(F^{ijm,\alpha\beta}_{kln,\gamma\lambda}, O^{jk,\alpha\beta}_i, Y^{ij}_{k,\alpha\beta})$:

$$\begin{aligned} O^{jk,\alpha\beta}_i &\to f^{i,\alpha}_{jk,\alpha'} f^{jk,\beta}_{i,\beta'} O^{jk,\alpha'\beta'}_i, \\ Y^{ij}_{k,\alpha\beta} &\to (f^{ij,\alpha'}_{k,\alpha})^*(f^{k,\beta'}_{ij,\beta})^* Y^{ij}_{k,\alpha'\beta'}, \\ F^{ijm,\alpha\beta}_{kln,\chi\delta} &\to f^{ij,\alpha}_{m,\alpha'} f^{mk,\beta}_{l,\beta'} (f^{jk,\chi'}_{n,\chi})^* (f^{in,\delta'}_{l,\delta})^* F^{ijm,\alpha'\beta'}_{kln,\chi'\delta'}. \end{aligned} \tag{59}$$

We can use the above "gauge" degree of freedom to choose

$$O^{jk,\alpha\beta}_i = O^{jk,\alpha}_i \delta_{\alpha\beta}, \quad O^{jk,\alpha}_i \geq 0. \tag{60}$$

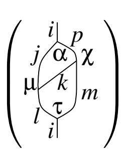
$O^{jk,\alpha}_i$ is chosen to be a real number.

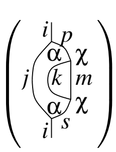
Then eqn. (56) implies that $Y^{ij}_{k,\alpha} \simeq 1/O^{ij,\alpha}_k$, and we can choose the phase of $Y^{ij}_{k,\alpha}$ to make

$$Y^{ij}_{k,\alpha} = 1/O^{ij,\alpha}_k. \tag{61}$$

H. Dual F-move and a relation between $O^{jk,\alpha}_i$ and $F^{ijm,\alpha\beta}_{kln,\delta\chi}$

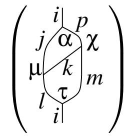
We also find another wave function that can have two ways of reduction as well:

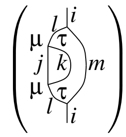
$$\Psi_{\text{fix}}$$


$$\simeq \sum_s \mathcal{F}^{jkl,\mu\tau}_{mis,\chi\alpha} \Psi_{\text{fix}}$$


$$\simeq \mathcal{F}^{jkl,\mu\tau}_{mip,\chi\alpha} O^{km,\chi}_p O^{jp,\alpha}_i \Psi_{\text{fix}} \left(\downarrow i \right)$$

$$\simeq \mathcal{F}^{jkl,\mu\tau}_{mip,\chi\alpha} O^{km,\chi}_p O^{jp,\alpha}_i (-)^{s^{km}_s(\chi)+s^{jp}_i(\alpha)} \Psi_{\text{fix}} \left(\downarrow i \right).$$

$$\Psi_{\text{fix}}$$


$$\simeq \tilde{\mathcal{F}}^{jkl,\mu\tau}_{mip,\chi\alpha} \Psi_{\text{fix}}$$


$$\begin{aligned} &\simeq \tilde{\mathcal{F}}^{jkl,\mu\tau}_{mip,\chi\alpha} \mathcal{O}^{jk,\mu}_l \mathcal{O}^{lm,\tau}_i \Psi_{\text{fix}} \left(\downarrow i \right) \\ &\simeq \tilde{\mathcal{F}}^{jkl,\mu\tau}_{mip,\chi\alpha} O^{jk,\mu}_l O^{lm,\tau}_i \Psi_{\text{fix}} \left(\downarrow i \right). \end{aligned} \tag{62}$$

All the edges have a canonical orientation from up to down, and $\tilde{\mathcal{F}}^{ijm,\alpha\beta}_{kln,\chi\delta}$, the dual so-called F-move, can be expressed as:

$$\tilde{\mathcal{F}}^{ijm,\mu\tau}_{kln,\chi\alpha} = \theta^{s^{jp}_i(\alpha)}_{\underline{\alpha}} \theta^{s^{km}_p(\chi)}_{\underline{\chi}} \theta^{s^{jk}_l(\mu)}_{\underline{\mu}} \theta^{s^{lm}_i(\tau)}_{\underline{\tau}} \tilde{F}^{ijm,\mu\tau}_{kln,\chi\alpha} \tag{63}$$

This allows us to obtain another condition

$$\tilde{F}^{jkl,\mu\tau}_{mip,\chi\alpha} = F^{jkl,\mu\tau}_{mip,\chi\alpha} O^{km,\chi}_p O^{jp,\alpha}_i (O^{lm,\tau}_i)^{-1} (O^{jk,\mu}_l)^{-1} \tag{64}$$

We require $\tilde{F}^{jkl,\mu\tau}_{mip,\chi\alpha}$ to be unitary, which leads to

$$\begin{aligned} &\sum_{l\mu\tau} (F^{jkl,\mu\tau}_{mip',\chi'\alpha'})^* \frac{O^{km,\chi'}_{p'} O^{jp',\alpha'}_i}{O^{lm,\tau}_i O^{jk,\mu}_l} F^{jkl,\mu\tau}_{mip,\chi\alpha} \frac{O^{km,\chi}_p O^{jp,\alpha}_i}{O^{lm,\tau}_i O^{jk,\mu}_l} \\ &= \sum_{l\mu\tau} \left(F^{jkl,\mu\tau}_{mip',\chi'\alpha'} \right)^* F^{jkl,\mu\tau}_{mip,\chi\alpha} \frac{O^{km,\chi'}_{p'} O^{jp',\alpha'}_i O^{km,\chi}_p O^{jp,\alpha}_i}{(O^{lm,\tau}_i O^{jk,\mu}_l)^2} \\ &= \delta_{pp'} \delta_{\chi\chi'} \delta_{\alpha\alpha'}, \end{aligned} \tag{65}$$

or

$$\sum_{l\mu\tau} \frac{(F^{jkl,\mu\tau}_{mip',\chi'\alpha'})^* F^{jkl,\mu\tau}_{mip,\chi\alpha}}{(O^{lm,\tau}_i O^{jk,\mu}_l)^2} = \frac{\delta_{pp'}\delta_{\chi\chi'}\delta_{\alpha\alpha'}}{(O^{km,\chi}_p O^{jp,\alpha}_i)^2}, \tag{66}$$

The above condition can be satisfied by the following ansatz

$$O^{ij,\alpha}_k = \sqrt{\frac{d_i d_j}{D d_k}} \delta^{ij}_k, \quad D = \sum_l d_l^2, \quad d_i > 0, \tag{67}$$

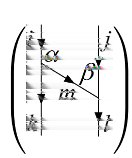
where $\delta^{jk}_i = 1$ for $N^{jk}_i > 0$ and $\delta^{jk}_i = 0$ for $N^{jk}_i = 0$. From eqn. (42), we find that d_i satisfy

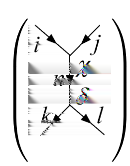
$$\sum_{ij} d_i d_j N^{ij}_k = d_k D, \quad D = \sum_l d_l^2. \tag{68}$$

The solution of such an equation gives us the quantum dimension d_i.

I. H-move and an additional constraint between $O^{jk,\alpha}_i$ and $F^{ijm,\alpha\beta}_{kln,\delta\chi}$

Let us consider a new type of move – H-move.

$$\Psi_{\text{fix}}$$


$$\simeq \sum_{n\chi\delta} \mathcal{H}^{kim,\alpha\beta}_{jln,\chi\delta} \Psi_{\text{fix}}$$


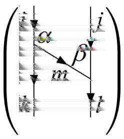
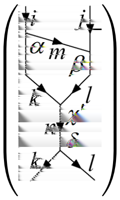
$$. \tag{69}$$

Again, we use the convention that all vertices have a canonical ordering from up to down. Similar to the F-move, $\mathcal{H}^{kim,\alpha\beta}_{jln,\chi\delta}$ can be expressed as:

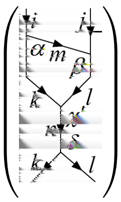
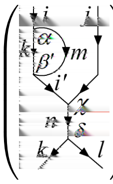
$$\mathcal{H}^{kim,\alpha\beta}_{jln,\chi\delta} = \theta^{s^{ij}_m(\alpha)}_{\underline{\alpha}} \theta^{s^{mk}_l(\beta)}_{\underline{\beta}} \theta^{s^{in}_l(\delta)}_{\underline{\delta}} \theta^{s^{jk}_n(\chi)}_{\underline{\chi}} H^{kim,\alpha\beta}_{jln,\chi\delta} \tag{70}$$

In the following, we will show how to compute the coefficients $H^{kim,\alpha\beta}_{jln,\chi\delta}$ from $F^{ijm,\alpha\beta}_{kln,\chi\delta}$ and d_i.

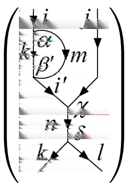
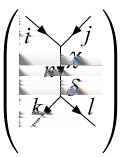
First, by applying the Y-move, we have:

Ψ_{fix}  $\simeq \sum_{n,\chi'\delta} \mathcal{Y}^{kl}_{n,\chi'\delta} \Psi_{\text{fix}}$  . (71)

Next, by applying an inverse F-move, we obtain:

Ψ_{fix}  $\simeq \sum_{i',\beta'\chi} \left(\mathcal{F}^{kmi',\beta'\chi}_{jnl,\beta\chi'}\right)^\dagger \Psi_{\text{fix}}$  . (72)

Finally, by applying the O-move, we end up with:

Ψ_{fix}  $\simeq \mathcal{O}^{km,\alpha\beta'}_i \delta_{ii'} \Psi_{\text{fix}}$  . (73)

All together, we find:

$$\mathcal{H}^{kim,\alpha\beta}_{jln,\chi\delta} = \sum_{\chi'\beta'} \mathcal{Y}^{kl}_{n,\chi'\delta} \left(\mathcal{F}^{kmi,\beta'\chi}_{jnl,\beta\chi'}\right)^\dagger \mathcal{O}^{km,\alpha\beta'}_i \tag{74}$$

Under the proper gauge choice Eq.(60), we can further express the coefficients $H^{ijm,\alpha\beta}_{kln,\chi\delta}$ as:

$$\begin{aligned} H^{kim,\alpha\beta}_{jln,\chi\delta} &= Y^{kl}_{n,\delta} (F^{kmi,\alpha\chi}_{jnl,\beta\delta})^* O^{km,\alpha}_i \\ &= (F^{kmi,\alpha\chi}_{jnl,\beta\delta})^* (O^{kl,\delta}_n)^{-1} O^{km,\alpha}_i \end{aligned} \tag{75}$$

The unitarity condition for H-move requires that:

$$\sum_{n\chi\delta} \frac{F^{km'i,\alpha\chi}_{jnl,\beta'\delta'} (F^{kmi,\alpha\chi}_{jnl,\beta\delta})^*}{(O^{kl,\delta}_n)^2} = \frac{\delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}}{(O^{km,\alpha}_i)^2}, \tag{76}$$

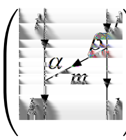
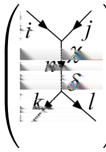
With the special ansatz Eq.(67), we can further simplify the above expressions as:

$$H^{kim,\alpha\beta}_{jln,\chi\delta} = \sqrt{\frac{d_m d_n}{d_i d_l}} \left(F^{kmi,\alpha\chi}_{jnl,\beta\delta}\right)^* \tag{77}$$

and

$$\sum_{n\chi\delta} d_n F^{km'i,\alpha\chi}_{jnl,\beta'\delta'} (F^{kmi,\alpha\chi}_{jnl,\beta\delta})^* = \frac{d_i d_l}{d_m} \delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}, \tag{78}$$

Similarly, we can also construct the dual-H move:

Ψ_{fix}  $\simeq \sum_{n\chi\delta} \tilde{\mathcal{H}}^{kim,\alpha\beta}_{jln,\chi\delta} \Psi_{\text{fix}}$  . (79)

and we can express $\tilde{\mathcal{H}}^{kim,\alpha\beta}_{jln,\chi\delta}$ as:

$$\tilde{\mathcal{H}}^{kim,\alpha\beta}_{jln,\chi\delta} = \theta^{s^{mk}_l(\beta)}_{\underline{\beta}} \theta^{s^{ij}_m(\alpha)}_{\underline{\alpha}} \theta^{s^{in}_l(\delta)}_{\underline{\delta}} \theta^{s^{jk}_n(\chi)}_{\underline{\chi}} \tilde{H}^{kim,\alpha\beta}_{jln,\chi\delta}, \tag{80}$$

where the coefficients $\tilde{\mathcal{H}}^{kim,\alpha\beta}_{jln,\chi\delta}$ can be expressed as:

$$\begin{aligned} \tilde{H}^{kim,\alpha\beta}_{jln,\chi\delta} &= Y^{kl}_{n,\delta} F^{imk,\alpha\delta}_{lnj,\beta\chi} O^{ml,\beta}_j \\ &= F^{imk,\alpha\chi}_{lnj,\beta\delta} (O^{kl,\delta}_n)^{-1} O^{ml,\beta}_j \end{aligned} \tag{81}$$

Again, with the special ansatz Eq.(67), we have:

$$\tilde{H}^{kim,\alpha\beta}_{jln,\chi\delta} = \sqrt{\frac{d_m d_n}{d_j d_k}} F^{imk,\alpha\delta}_{lnj,\beta\chi} \tag{82}$$

It is easy to see that the unitarity condition for dual H-move is automatically satisfied if H-move is unitary.

J. Summary of fixed-point gfLU transformations

To summarize, the conditions (15, 17, 45, 30, 29, 38, 68 form a set of non-linear equations whose variables are N^{ij}_k, F^{ij}_k, $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$, d_i, Let us collect those conditions and list them below

- $\sum_{m=0}^{N} N^{ij}_m N^{mk}_l = \sum_{n=0}^{N} N^{jk}_n N^{in}_l$
- $\sum_{m=0}^{N} B^{ij}_m F^{mk}_l + F^{ij}_m B^{mk}_l = \sum_{n=0}^{N} B^{jk}_n F^{in}_l + F^{jk}_n B^{in}_l$
- $\sum_{jk} (B^{jk}_i)^2 + (F^{jk}_i)^2 \geq 1,$
- $N^{jk}_i = B^{jk}_i + F^{jk}_i.$ (83)

- $\sum_{n\chi\delta} F^{ijm',\alpha'\beta'}_{kln,\chi\delta}(F^{ijm,\alpha\beta}_{kln,\chi\delta})^* = \delta_{m,m'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$,
- $F^{ijm,\alpha\beta}_{kln,\chi\delta} = 0$ when
  $N^{ij}_m < 1$ or $N^{mk}_l < 1$ or $N^{jk}_n < 1$ or $N^{in}_l < 1$,
  or $s^{ij}_m(\alpha) + s^{mk}_l(\beta) + s^{jk}_n(\chi) + s^{in}_l(\delta) = 1 \bmod 2$.
- $\sum_t \sum_{\eta=1}^{N^{jk}_t} \sum_{\varphi=1}^{N^{it}_n} \sum_{\kappa=1}^{N^{tl}_s} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi}$

$$= (-)^{s^{ij}_m(\alpha)s^{kl}_q(\delta)} \sum_{\epsilon=1}^{N^{mq}_p} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}. \quad (84)$$

- $\sum_{i,j} d_i d_j N^{ij}_k = d_k D, \quad D = \sum_l d_l^2. \quad (85)$

- $\sum_{n\chi\delta} d_n F^{km'i,\alpha\chi}_{jnl,\beta'\delta'}(F^{kmi,\alpha\chi}_{jnl,\beta\delta})^* = \frac{d_i d_l}{d_m}\delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}, \quad (86)$

Finding N^{ij}_k, F^{ij}_k, $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$, and d_i that satisfy such a set of non-linear equations corresponds to finding a fixed-point gfLU transformation that has a non-trivial fixed-point wave function. So the solutions (N^{ij}_k,F^{ij}_k,$F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$,$d_i$) give us a characterization of fermionic topological orders (and bosonic topological orders as a special case where $F^{ij}_k = 0$).

We would like to stress that, although the solutions (N^{ij}_k,F^{ij}_k,$F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$,$d_i$) decscribe 2+1D fermionic topological orders with gappable edge, the correspondence is not one to one. Given a set of solutions, the transformation in eqn. (59) on $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$ will generate another set of solutions (since the equitions for d_i and $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$ decouple). The two sets of solutions describe the same topologically ordered phase. Also eqn. (59) does not include all the redundancy: two solutions that are not related by "gauge" transformation eqn. (59) may still describe the same fermionic topological orders. We need to compute the modula transformation T and S from the data (N^{ij}_k,F^{ij}_k,$F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$,$d_i$) to determine the 2+1D topological order.[9,29,30]

VI. CATEGORICAL FRAMEWORK

To provide a conceptual understanding of our generalization of string-net model, we discuss briefly the categorical picture which underlies earlier algebraic manipulations. Such a mathematical framework will provide more examples for our fermionic string-net Hamiltonians in Appendix B 2.

A string-net or Levin-Wen Hamiltonian can be easily constructed using $6j$-symbols from a unitary fusion category $\mathcal{C}$. The elementary excitations of the model form a unitary modular tensor category (UMTC) $\mathcal{E}$, which turns out to be the quantum double $Z(\mathcal{C})$ of the input category $\mathcal{C}$. A priori, the output modular category $\mathcal{E}$ is not necessarily related to the input category $\mathcal{C}$. Therefore, it is conceivable that similar Hamiltonians can be constructed from some other algebraic data where the elementary excitations still form a UMTC, which is not necessarily a quantum double. This is explored in Ref. 6. In the preceding sections, we generalize the string-net model by including fermionic degrees of freedom.

The mathematical framework for such a generalization is the theory of enriched categories.[25] An enriched category is actually not a category, just like a quantum group is not a group. We will consider only special enriched categories, which we call *projective super fusion categories.* The ordinary unitary fusion categories are enriched categories over the category of Hilbert spaces, while projective super fusion categories are enriched categories over the category of super Hilbert spaces up to *projective even* unitary transformations.

To the physically inclined readers, the use of category theory in condensed matter physics seems to be unjustifiably abstract. We would argue that the abstractness of category theory is actually its virtue. Topological properties of quantum systems are independent of the microscopic details and are non-local. A framework to encode such properties is necessarily blind to microscopic specifics. Therefore, philosophically category theory could be extremely relevant, as we believe.

A. Projective super tensor category

We use super vector spaces to accommodate fermionic states, and generalize the composition of linear transformations to one only up to overall phases—a possibility allowed by quantum mechanics. The projective tensor category of vector spaces is the category of vector spaces and linear transformations composed up to overall phases, and the category of super vector spaces is the tensor category of $\mathbb{Z}_2$-graded vector spaces and all *even* linear transformations.

In the categorical language, a fusion category is a rigid finite linear category with a simple unit. Equivalently, it can be defined using $6j$-symbols: an equivalence class of solutions of pentagons satisfying certain normalizations[27]. Fermionic $6j$-symbols $\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}$ in eqn. (28) with certain normalizations define a projective super fusion category if they satisfy fermionic pentagon equations eqn. (35). However, the setup used in this paper may only generate a subclass of projective super fusion category.

B. Super tensor category from super quantum groups

The trivial example of a super tensor category is the category of $\mathbb{Z}_2$-graded vector spaces and all linear transformations. More interesting examples of super tensor categories can be constructed from the representation theory of super quantum groups.

Super quantum groups are deformations of Lie superalgebras.[31,32] Though a mathematical theory analogous to quantum group exists, the details have not been worked out enough for our application here. In literature, the categorical formulation focuses on the invariant spaces of even entwiners, while for our purpose, we need to consider all entwiners. In particular, we are not aware of work on Majorana valued Clebsch-Gordon coefficients, therefore, we will leave the details to future publications.

VII. SIMPLE SOLUTIONS OF THE FIXED-POINT CONDITIONS

In this section, let us discuss some simple solutions of the fixed-point conditions (83, 84, 85, 86) for the fixed-point gfLU transformations $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta}_{kln,\gamma\lambda}, d_i)$.

A. Solutions from group cohomology

Many bosonic solutions can be constructed from a finite group G. Here we treat the edge index $i, j, k, \cdots$ as elements in the group: $i, j, k \in G$ with group mulitplication $i \cdot j \in G$. We choose the fusion coefficient as

$$N^{ij}_k = \begin{cases} 1, & \text{if } i \cdot j = k \\ 0, & \text{if } i \cdot j \neq k \end{cases}$$
$$F^{ij}_k = 0. \tag{87}$$

Since $N^{ij}_k = 0, 1$, we can drop the indices α, β on vertices. $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$ that satisfies eqn. (84) is given by

$$F^{ijm}_{kln} = \omega_3(i, j, k) N^{ij}_m N^{mk}_l N^{jk}_n N^{in}_l, \tag{88}$$

where $\omega_3(i, j, k)$ is the 3-cocycle in group cohomology class $\mathcal{H}^3[G, U(1)]$.[33,34] In this case, the self-consistent condition Eq.(84) for F-tensor becomes the cocycle equation for $\omega_3(i, j, k)$:

$$\omega_3(i, j, k)\omega_3(i, j \cdot k, l)\omega_3(j, k, l) = \omega_3(i \cdot j, k, l)\omega_3(i, j, k \cdot l) \tag{89}$$

d_i that satisfies eqn. (85) is given by

$$d_i = 1. \tag{90}$$

Such a solution describes a "twisted" gauge theory in 2+1D.[35–37] If we choose a trivial 3-cocycle $\omega_3(i, j, k) = 1$, the solution will describe a standard gauge theory with gauge group G in 2+1D.

B. Solutions from group supercohomology

Similarly, many fermionic solutions can be constructed from a finite group G. Again we treat the edge index $i, j, k, \cdots$ as elements in the group: $i, j, k \in G$ with group mulitplication $i \cdot j \in G$. We choose the same fusion coefficient N^{ij}_k as for bosonic solutions, but with nonzero F^{ij}_k:

$$N^{ij}_k = \begin{cases} 1, & \text{if } i \cdot j = k \\ 0, & \text{if } i \cdot j \neq k \end{cases}$$
$$F^{ij}_k = n_2(i, j) N^{ij}_k \neq 0. \tag{91}$$

where $n_2(i, j) \in \mathbb{Z}_2$ valued on $0, 1$ is 2-cocycle in the obstruction free subgroup of group cohomology class $B\mathcal{H}^2[G, \mathbb{Z}_2]$. By obstruction free, we mean that for any $n_2(i, j)$ satisfying the 2-cocycle condition:

$$n_2(i, j) + n_2(i \cdot j, k) = n_2(i, j \cdot k) + n_2(j, k), \tag{92}$$

the following ± 1 valued function:

$$(-)^{n_2(i,j)n_2(k,l)}, \tag{93}$$

must be a coboundary in $\mathcal{B}^4[G, U(1)]$ when we view it as a 4-cocycle with $U(1)$ coefficient. Each element in $B\mathcal{H}^2[G, \mathbb{Z}_2]$ become a valid Z_2-graded structure for fermion systems.

On the other hand, since $N^{ij}_k = 0, 1$, we can again drop the indices α, β on vertices. $F^{ijm,\alpha\beta}_{kln,\gamma\lambda}$ that satisfies eqn. (84) is given by

$$F^{ijm}_{kln} = \tilde{\omega}_3(i, j, k) N^{ij}_m N^{mk}_l N^{jk}_n N^{in}_l, \tag{94}$$

where $\omega_3(i, j, k)$ is the 3-supercocycle in group supercohomology class $\mathcal{H}^3_f[G, U(1)]$[33,34], which satisfies:

$$\begin{aligned} &\tilde{\omega}_3(i, j, k)\tilde{\omega}_3(i, j \cdot k, l)\tilde{\omega}_3(j, k, l) \\ &= (-)^{n_2(i,j)n_2(k,l)}\tilde{\omega}_3(i \cdot j, k, l)\tilde{\omega}_3(i, j, k \cdot l) \end{aligned} \tag{95}$$

Again d_i that satisfies eqn. (85) is given by

$$d_i = 1. \tag{96}$$

Such a solution describe a fermionic gauge theory in 2+1D, e.g. the recently proposed fermionic toric code.[38]

VIII. SUMMARY

Using string-net condensations and LU transformations (or in other words, unitary fusion category theory), we have obtained a classification of 2+1D topological orders with gappable edge in bosonic systems.[6,20] An interacting fermionic system is a non-local bosonic system. So classifying topological orders in fermion systems appears to be a very difficult problem.

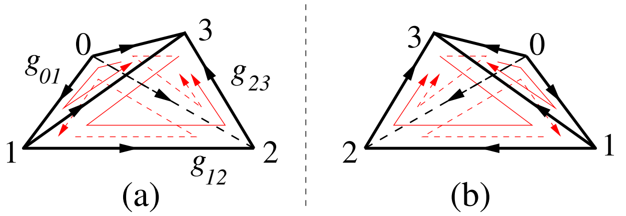


FIG. 3: (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

In this paper, we introduced fLU and gfLU transformations, which allow us to develop a general theory for a large class of fermionic topological orders. We propose that 2+1D topological orders with gappable edge in fermionic systems can be classified by the data $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta}_{jkn,\chi\delta}, d_i)$ that satisfy a set of non-linear algebraic equations (83), (84), (85), and (86). Such a result generalizes the string-net result[6,20] to fermionic cases. We hope our approach to be a starting point for establishing a mathematical framework for topological orders in interacting fermion systems.

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Appendix A: Branching structure of 2D graph

To define a lattice model a space M, we first triangulate of the space M to obtain a complex M_{tri}. We will call a cell in the complex as a simplex. In order to define a generic lattice theory on the complex M_{tri}, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.[33,34,39] A branching structure is a choice of orientation of each edge in the n-dimensional complex so that there is no oriented loop on any triangle (see Fig. 3).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, *etc* . So the simplex in Fig. 3a has the following vertex ordering: $0, 1, 2, 3$.

The branching structure also gives the simplex (and its sub simplexes) an orientation denoted by $s_{ij\cdots k} = \pm$. Fig. 3 illustrates two 3-simplices with opposite orientations $s_{0123} = +$ and $s_{0123} = -$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

In this paper, we will only consider 2D space. The graph that we use to define our lattice model is the dual graph of the 2D complex M_{tri}. The branching structure of M_{tri} leads a branching structure of our graph: each vertex of the graph cannot have three incoming edges or three outgoing edges.

Appendix B: The parent Hamiltonian for fixed point wavefunctions

1. The fermionic structure of support space

To understand the fermionic structure of the support space $\tilde V_A$, let us first study the structure of ρ_A. Let $|\phi_i\rangle$ be a basis of the Hilbert space of the region A and $|\bar\phi_{\bar i}\rangle$ be a basis of the Hilbert space of the region outside of A. $|\psi\rangle$ can be expanded by $|\phi_i\rangle \otimes |\bar\phi_{\bar i}\rangle$:

$$|\psi\rangle = \sum_{i,\bar i} C_{i,\bar i} |\phi_i\rangle \otimes |\bar\phi_{\bar i}\rangle. \tag{B1}$$

Then the matrix elements of ρ_A is given by

$$(\rho_A)_{ij} = \sum_{\bar i} (C_{i\bar i})^* C_{j\bar i}. \tag{B2}$$

For a fermion system, the Hilbert space on a site, V_i has a structure: $V_i = V_i^0 \oplus V_i^1$, where states in V_i^0 have even numbers of fermions and states in V_i^1 have odd numbers of fermions. The Hilbert space on the region A, V_A, has a similar structure $V_A = V_A^0 \oplus V_A^1$, where states in V_A^0 have even number of fermions and states in V_A^1 have odd numbers of fermions. Let $|\phi_{i,\alpha}\rangle$ be a basis of V_A^α. Similarly, the Hilbert space on the region outside of A, $V_{\bar A}$, also has a structure $V_{\bar A} = V_{\bar A}^0 \oplus V_{\bar A}^1$. Let $|\bar\phi_{\bar i,\alpha}\rangle$ be a basis of $V_{\bar A}^\alpha$. In this case, $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \sum_{i\alpha;\bar i\beta} C_{i,\alpha;\bar i,\beta} |\phi_{i,\alpha}\rangle \otimes |\bar\phi_{\bar i,\beta}\rangle. \tag{B3}$$

the matrix elements of ρ_A can now be expressed as

$$(\rho_A)_{i,\alpha;j,\beta} = \sum_{\bar i,\gamma} (C_{i,\alpha;\bar i,\gamma})^* C_{j,\beta;\bar i,\gamma}. \tag{B4}$$

Since the fermion number mod 2 is conserved, we may assume that $|\psi\rangle$ contains even numbers of fermions. This means $C_{i,\alpha;\bar i,\gamma} = 0$ when $\alpha + \gamma = 1$ mod 2. Hence, we find that

$$(\rho_A)_{i,\alpha;j,\beta} = 0, \text{ when } \alpha + \beta = 1 \text{ mod } 2. \tag{B5}$$

Such a density matrix tells us that the support space $\tilde V_A$ has a structure $\tilde V_A = \tilde V_A^0 \oplus \tilde V_A^1$, where $\tilde V_A^0$ has even numbers of fermions and $\tilde V_A^1$ has odd numbers of fermions. This means that U_g contains only even numbers of fermionic operators (*ie* U_g is a pseudo-local bosonic operator).

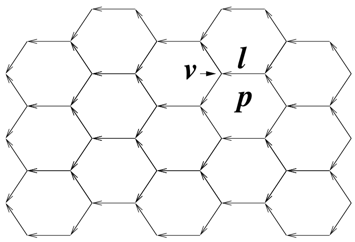


FIG. 4: A honeycomb lattice. The vertices are labeled by $\boldsymbol{v}$, hexagons by $\boldsymbol{p}$, and links by $\boldsymbol{l}$.

2. Compute the parent Hamiltonian

In the section V, we have constructed the fixed-point wave functions from the solutions $(N^{ij}_k, F^{ij}_k, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, d_i)$ of the self consistent conditions. In this section, we will show that those fixed-point wave functions on a honeycomb lattice (see Fig. 4) are exact gapped ground state of a local Hamiltonian

$$\hat{H} = \sum_{\boldsymbol{v}} (1 - \hat{Q}_{\boldsymbol{v}}) + \sum_{\boldsymbol{p}} (1 - \hat{B}_{\boldsymbol{p}}) \tag{B6}$$

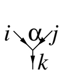
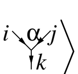
where $\sum_{\boldsymbol{v}}$ sums over all vertices and $\sum_{\boldsymbol{p}}$ sums over all hexagons.

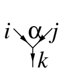
The Hamiltonian $\hat{H}$ should act on the Hilbert space V_G formed by all the graph states. It turns out that it is more convenient to write down the Hamiltonian if we expand the Hilbert space by adding an auxiliary qubit to each vertex:

$$V^{ex}_G = V_G \otimes (\otimes_{\boldsymbol{v}} V_{qubit}) \tag{B7}$$

where $\otimes_{\boldsymbol{v}}$ goes over all vertices and V_{qubit} is the two dimensional Hilbert space of an auxiliary qubit $|I\rangle$, $I = 0, 1$. So in the expanded Hilbert space V^{ex}_G, the states on a vertex $\boldsymbol{v}$ are labeled by $|\alpha\rangle \otimes |I\rangle$, $I = 0, 1$. V_G is embedded into V^{ex}_G in the following way: each vertex state $|\alpha\rangle$ in V_G correspond to the following vertex state $|\alpha\rangle \otimes |s_{ijk}(\alpha)\rangle$ in V^{ex}_G, where we have assume that the states on the three links connecting to the vertex are $|i\rangle$, $|j\rangle$, and $|k\rangle$. So the new auxiliary qubit $|I\rangle$ on a vertex is completely determined by (i, j, k, α) and does not represent an independent degree of freedom. It just tracks if the vertex state is bosonic or fermionic. The $|0\rangle$-state correspond to bosonic vertex states and the $|1\rangle$-state correspond to fermionic vertex states.

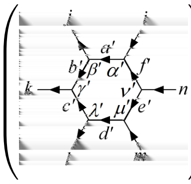
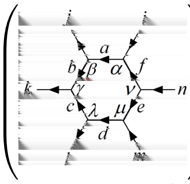
In the expanded Hilbert space, $\hat{Q}_{\boldsymbol{v}}$ in $\hat{H}$ acts on the states on the 3 links that connect to the vertex $\boldsymbol{v}$ and on the states $|\alpha\rangle \otimes |I\rangle$ on the vertex $\boldsymbol{v}$:

$\hat{Q}_{\boldsymbol{v}}$ |  ⟩ $\otimes |I\rangle =$ |  ⟩ $\otimes |I\rangle$ if $N^{ij}_k > 0$, $I = s^{ij}_k(\alpha)$,

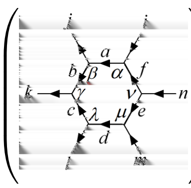
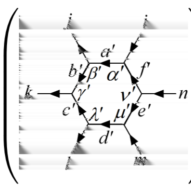
$\hat{Q}_{\boldsymbol{v}}$ |  ⟩ $\otimes |I\rangle = 0$ otherwise. (B8)

Clearly, $\hat{Q}_{\boldsymbol{v}}$ is a projector $\hat{Q}^2_{\boldsymbol{v}} = \hat{Q}_{\boldsymbol{v}}$. The $\hat{B}_{\boldsymbol{p}}$ operator in $\hat{H}$ acts on the states on the 6 links and the 6 vertices of the hexagon $\boldsymbol{p}$ and on the 6 links that connect to the hexagon. However, $\hat{B}_{\boldsymbol{p}}$ operator will not alter the states on the 6 links that connect to the hexagon. Let us define the Majorana number valued matrix element $\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$ as:

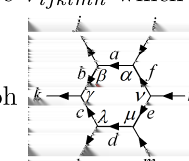
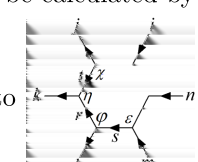
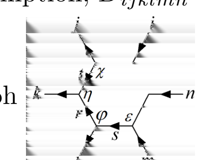
$\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$

$= \Big\langle \Psi_{\text{fix}}$ () $\Big|\, \hat{B}_{\boldsymbol{p}} \,\Big| \Psi_{\text{fix}}$ () $\Big\rangle$ (B9)

In order to compute $\mathcal{B}$, we consider the following local deformation:

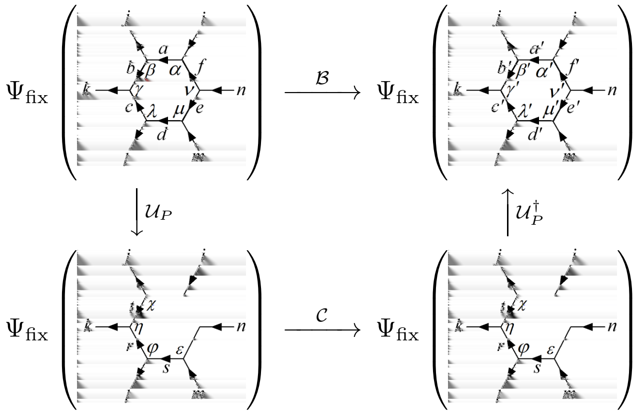
Ψ_{fix} () $\to \Psi_{\text{fix}}$ () (B10)

We note that the self consistent conditions satisfied by the F-tensor and O-tensor ensure that all those different ways to transform between the two above states lead to the same $\mathcal{B}$ matrix.

To understand how $\mathcal{B}$ acts on a state that is not in the support space, let us consider the dimension D_{ijklmn} of the support space V_{ijklmn} which can be calculated by deforming the graph  into  through a gfLU transformation $\mathcal{U}$. Under the saturation assumption, D_{ijklmn} is equal to the distinct labels in the graph  (with i, j, k, l, m, n fixed):

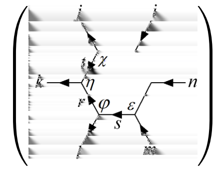
$$D_{ijklmn} = \sum_{trs} N^{jt}_i N^{tr}_k N^{rl}_s N^{nm}_s. \tag{B11}$$

In particular, we can have the following two paths that connect the two states (see Fig. 5):

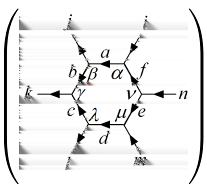


where $\mathcal{U}$ is a gfLU transformation. We find that

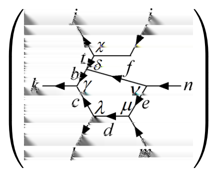
$$\mathcal{B} = \mathcal{U}_P^\dagger \mathcal{C} \mathcal{U}_P \tag{B12}$$

where $\mathcal{C}$, acting on Ψ_{fix} , is a dimension $D_{ijklmn} \times D_{ijklmn}$ identity matrix. In the following let us compute the explicit form of $\mathcal{U}_P$.

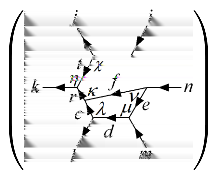
As seen in Fig. 5, let us first apply an inverse H-move, an F-move, a dual H-move, an inverse F-moves and finally one O-move, thus we obtain:

Ψ_{fix} 

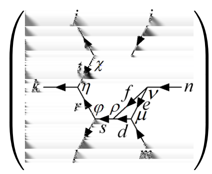
$$= \sum_{t\chi\delta} (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \Psi_{\text{fix}} \tag{B13}$$



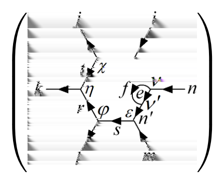
$$= \sum_{t\chi\delta}\sum_{r\kappa\eta} (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \mathcal{F}^{tfb,\delta\gamma}_{ckr,\kappa\eta} \Psi_{\text{fix}}$$



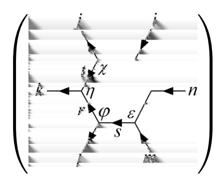
$$= \sum_{t\chi\delta}\sum_{r\kappa\eta}\sum_{s\rho\varphi} (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \mathcal{F}^{tfb,\delta\gamma}_{ckr,\kappa\eta} \tilde{\mathcal{H}}^{rfc,\kappa\lambda}_{dls,\rho\varphi} \Psi_{\text{fix}}$$



$$= \sum_{t\chi\delta}\sum_{r\kappa\eta}\sum_{s\rho\varphi}\sum_{n'\nu'\epsilon} (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \mathcal{F}^{tfb,\delta\gamma}_{ckr,\kappa\eta} \tilde{\mathcal{H}}^{rfc,\kappa\lambda}_{dls,\rho\varphi} (\mathcal{F}^{fen',\nu'\epsilon}_{msd,\mu\rho})^\dagger \Psi_{\text{fix}}$$



$$= \sum_{t\chi\delta}\sum_{r\kappa\eta}\sum_{s\rho\varphi}\sum_{\nu'\epsilon} (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \mathcal{F}^{tfb,\delta\gamma}_{ckr,\kappa\eta} \tilde{\mathcal{H}}^{rfc,\kappa\lambda}_{dls,\rho\varphi} (\mathcal{F}^{fen,\nu'\epsilon}_{msd,\mu\rho})^\dagger \mathcal{O}^{fe,\nu\nu'}_{n} \Psi_{\text{fix}}$$



Therefore, we finally derive:

$$(\mathcal{U}_p)^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{trs,\chi\eta\varphi\epsilon}(i,j,k,l,m,n) = (\mathcal{H}^{jit,\chi\delta}_{fba,\alpha\beta})^\dagger \mathcal{F}^{tfb,\delta\gamma}_{ckr,\kappa\eta} \tilde{\mathcal{H}}^{rfc,\kappa\lambda}_{dls,\rho\varphi} (\mathcal{F}^{fen,\nu'\epsilon}_{msd,\mu\rho})^\dagger \mathcal{O}^{fe,\nu\nu'}_{n} \tag{B14}$$

Also $\mathcal{U}_P$, containing only one O-move (see Fig. 5), has a form $\mathcal{U}_P = \mathcal{U}_1 \mathcal{P} \mathcal{U}_2$ where $\mathcal{U}_{1,2}$ are unitary matrices and $\mathcal{P}$ is a projection matrix. So the rank of $\mathcal{B}$ is equal or less than D_{ijklmn}. Since it is the identity in the D_{ijklmn}-

dimensional space V_{ijklmn}, the matrix $\mathcal{B}$ is a hermitian projection matrix onto the space V_{ijklmn}:

$$\sum_{a''\alpha''}\sum_{b''\beta''}\sum_{c''\gamma''}\sum_{d''\lambda''}\sum_{e''\mu''}\sum_{f''\nu''}\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a''\alpha'',b''\beta'',c''\gamma'',d''\lambda'',e''\mu'',f''\nu''}(i,j,k,l,m,n)\mathcal{B}^{a''\alpha'',b''\beta'',c''\gamma'',d''\lambda'',e''\mu'',f''\nu''}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$$
$$= \mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n) \tag{B15}$$

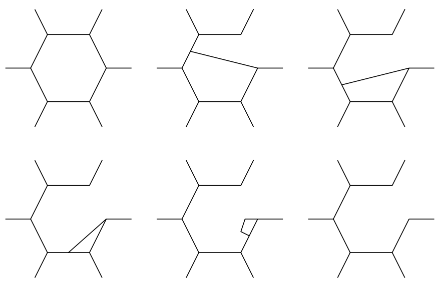


FIG. 5: $\mathcal{U}_P$ is generated by an inverse H-move, an F-move, a dual H-move, an inverse F-moves and finally one O-move, which turns a hexagon graph into a tree graph. $(\mathcal{U}_P)^\dagger$ turns a tree graph into a hexagon graph.

In the above calculation of the $\mathcal{B}$, we first insert a bubble on the a-link. We may also calculate $\mathcal{B}$ by first inserting a bubble on other lines. All those different calculations will lead to the same $\mathcal{B}$ matrix, as discussed above.

We note that $\hat{B}_{\boldsymbol{p}_1}\hat{B}_{\boldsymbol{p}_2}$ and $\hat{B}_{\boldsymbol{p}_2}\hat{B}_{\boldsymbol{p}_1}$ are generated by different combinations of F-moves and O-moves. Since the two combinations transform between the same pair of states, they give rise to the same relation between the two states. Therefore $\hat{B}_{\boldsymbol{p}_1}$ and $\hat{B}_{\boldsymbol{p}_2}$ commute

$$\hat{B}_{\boldsymbol{p}_1}\hat{B}_{\boldsymbol{p}_2} = \hat{B}_{\boldsymbol{p}_2}\hat{B}_{\boldsymbol{p}_1}. \tag{B16}$$

We see that the corresponding Hamiltonian $\hat{H}$ is a sum of commuting projectors and is exactly soluble.

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