Interacting topological insulator and emergent grand unified theory

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Interacting Topological Insulator and Emergent Grand Unified Theory

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Motivated by the Pati-Salam Grand Unified Theory [1] we study $(4+1)d$ topological insulators with $\mathrm{SU}(4) \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ symmetry, whose $(3+1)d$ boundary has 16 flavors of left-chiral fermions, which form representations $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$. The key result we obtain is that, without any interaction, this topological insulator has a $\mathbb{Z}$ classification, namely any quadratic fermion mass operator at the $(3+1)d$ boundary is prohibited by the symmetries listed above; while under interaction this system becomes trivial, namely its $(3+1)d$ boundary can be gapped out by a properly designed short range interaction without generating nonzero vacuum expectation value of any fermion bilinear mass, or in other words, its $(3+1)d$ boundary can be driven into a "strongly coupled symmetric gapped (SCSG) phase". Based on this observation, we propose that after coupling the system to a dynamical $\mathrm{SU}(4) \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ lattice gauge field, the Pati-Salam GUT can be fully regularized as the boundary states of a $(4+1)d$ topological insulator with a *thin* fourth spatial dimension, the thin fourth dimension makes the entire system generically a $(3+1)d$ system. The mirror sector on the opposite boundary will *not* interfere with the desired GUT, because the mirror sector is driven to the SCSG phase by a carefully designed interaction and is hence decoupled from the GUT.

1. INTRODUCTION

In the Standard Model of particle physics and the Grand Unified Theories (GUT), the gauge coupling is asymmetric between left and right handed fermions. This chiral gauge coupling makes it difficult to regularize the field theory as a full quantum theory on a lattice. The main obstacle of this lattice regularization is the fermi doubling theorem [2, 3], which states that both left and right handed fermions will arise at low energy for any lattice fermion model. Then when the lattice fermion is coupled to a gauge field, it will induce the same coupling between left and right fermions, which is inconsistent with the Standard Model or the GUT. In order to get around the fermi doubling theorem, one method is to realize the GUT on the $3d$ boundary of a $4d$ topological insulator (TI), or in other words at the domain wall of the mass of $4d$ Dirac fermion [4–6] [59]. Then there is a mirror sector of fermions with opposite chirality localized on the other opposite boundary, which is spatially separated from the GUT. Fermions at each boundary can naturally have a chiral coupling to the bulk gauge fields. However, this method requires subtle adjustment of the scale of the fourth dimension: if the fourth dimension is too large, the gauge boson in the bulk will be gapless and interfere with the low energy physics of the boundary; on the other hand if the fourth dimension is too small, then the GUT suffers from interference with its mirror sector on the other boundary [7].

In a GUT, effectively in every generation there are 16 left handed fermions, thus its mirror sector must have 16 right handed fermions with the same gauge coupling. It would be ideal if we can gap out the mirror sector without affecting the fermions in the GUT, *i.e.* decouple the mirror sector from low energy physics completely. Then we can regularize the GUT on the $3d$ boundary of a $4d$ TI with a very thin fourth dimension (which makes the bulk generically a $3d$ system), see Fig. 1. However, if the mirror sector is gapped out in the standard way, namely they are gapped out by condensing a boson field that couples to the mass operators of the mirror fermions, then the same boson field would couple to the fermions in the GUT and gap them out as well. Thus we seek the possibility to gap out the mirror sector while having zero fermion bilinear expectation value, $\langle \psi_a^\intercal \mathrm{i}\sigma^y \psi_b \rangle = 0$ in the mirror sector [60], for arbitrary flavor indices $a, b = 1, ..., 16$. We label this fully gapped phase of the mirror sector as "strongly coupled symmetric gapped phase" (SCSG phase).[61] (This phase was also called the "strongly coupled symmetric phase" or the "paramagnetic strong-coupling (PMS) phase" in literature, see appendix A for a review of the recent progress on the SCSG phase in the condensed matter community.)

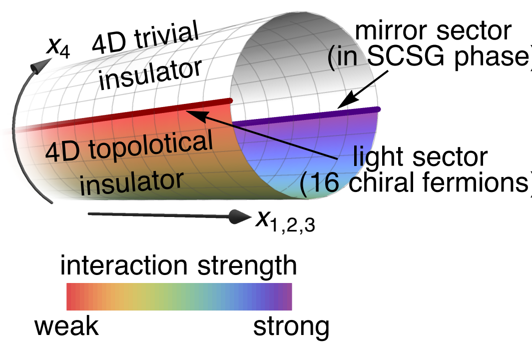

FIG. 1: (Color online.) Regularizing the GUT on a lattice with three extended dimensions $x_{1,2,3}$ and a compactified dimension x_4. The light sector (GUT) and the mirror sector are separated in the x_4 dimension, as two $3d$ boundaries of a $4d$ TI. The mirror sector is decoupled from GUT due to interaction, whose strength varies with x_4.

A SCSG phase with fully gapped but nondegenerate spectrum in the mirror sector is only possible when the system satisfies the following two necessary criteria:

(1) Based on the anomaly matching condition [8, 9], the system should *not* have any symmetry that would be anomalous in the mirror sector once the system is coupled to the gauge fields.

For instance the global charge U(1) symmetry of chiral fermions $\psi_{a,L} \to e^{\mathrm{i}\theta}\psi_{a,L}$, which was a key assumption for the no-go theorem proved in Ref. [2, 3], should not exist in the lattice model.

(2) The $4d$ bulk state is a nontrivial TI (hence must have gapless chiral fermions on its $3d$ boundary) without interaction; but under interaction it becomes a trivial state, which means that its boundary can be driven into the desired SCSG phase by interaction [62].

Notice that there must be a minimum nonzero critical interaction strength for this SCSG phase to exist, because a weak short-range four fermion interaction is irrelevant for $(3+1)d$ Dirac or chiral fermions. Thus we assume that the interaction on the mirror sector is stronger than the GUT, thus the interaction only gaps out the mirror sector. Alternatively, we can take a uniform interaction in the entire system, but make the kinetic energy stronger on the GUT but weaker on the mirror sector.

The second criterion mentioned above implies that the $4d$ bulk TI must be trivialized by interaction. This effect of interaction on TI was studied immensely in condensed matter community in the last few years [10–19], and now it is understood that in one, two, and three spatial dimensions, there are examples of topological insulators which are nontrivial in the noninteracting limit, but can be trivialized by certain *well-designed* interaction, namely their boundaries can be driven into the SCSG phase by interaction. Thus to obtain the desired lattice regularization for GUT, we need to demonstrate the following two results:

First of all, there is a $4d$ TI which in the noninteracting limit has massless chiral fermions on its $3d$ boundary, and the symmetry of the TI is precisely the same as the gauge symmetry of the GUT, so we can couple the system to the correct gauge fields;

Second, and most importantly, under interaction the $4d$ TI must become a trivial phase, thus its boundary can be driven into the SCSG phase.

There are two equivalent ways to prove a TI is trivialized by interaction: *(1)* one can directly show that the boundary of the TI is driven into the SCSG phase by certain interaction; *(2)* alternatively, we can also prove that the topological-to-trivial quantum phase transition in the bulk is "erased" by interaction, namely under interaction the "trivial insulator" and TI in the noninteracting limit can be connected adiabatically to each other without closing the bulk gap [63]. In section **2**, we will apply the *first* approach to a toy model, which is similar to the GUT in the sense that its $3d$ boundary has 16 gapless chiral fermions; In section **3**, we will use the *second* approach to show that the Pati-Salam GUT [1] emerges as the boundary of a $4d$ TI, and the mirror sector is decoupled in the IR because it can be driven into the SCSG phase by interaction.

The first lesson we learned from the studies of interacting TI is that, the SCSG phase does *not* exist for arbitrary flavors of fermions. It is now well-understood that in $0d$ and $1d$, SCSG phase only exists for $8n$ flavors of Majorana fermions with integer n; in $2d$, SCSG phase only exists for $16n$ flavors of Majorana fermions (a review of these previous results is given in appendix A). The interaction that realizes the SCSG phase must be a flavor mixing interaction term, whose explicit form was given in $0d$ and $1d$ [10, 11, 20]. Thus one has to carefully select the short range interaction terms to realize the SCSG phase.

We note here that the all-important SCSG phase of the mirror sector was also sought for in the past[7, 21–24]. This phase was first proposed in high energy physics community in Ref. [25] and it was called the Eichten-Preskill mechnism. But the existence of the SCSG phase was never firmly established. Recently new proposal of constructing SCSG phase based on classification of symmetry protected topological states (a generalization of topological insulator) was made in Ref. [26, 27], which is similar to the logic we presented in this section. Besides, SCSG phases for anomaly-free $(1+1)d$ systems were also discussed in Ref. [28]. In Ref. [26, 27], the general diagnosis for classification of fermionic SPT states was based on the computation of super-cocycles of symmetry group. In our current work, we will use a very different way of understanding classification of interacting TIs, which is more intuitive and more convenient to analyze compared with super-cocycle calculation, especially for the Lie groups involved in GUT. Meanwhile, our method not only demonstrates the existence of the SCSG phase, but also gives us guidance for constructing the specific interaction that realizes the SCSG phase. A more detailed review of Ref. [26, 27] and comparison with our work will be given in appendix A.4.

2. A TOY MODEL

Let us first start with a toy model, whose bulk theory is a $4d$ TI with a U(1) and $\mathbb{Z}_2$ symmetry, and we can use the same bulk band structure introduced for the $4d$ quantum Hall state in Ref. [27, 29]:

$$\begin{aligned} H_{\mathrm{TI}} &= \sum_{a=1}^{2}\sum_{\vec{k}} \psi^\dagger_{\vec{k},a}\Big(\sum_{i=1}^{4}\Gamma^i\sin(k_i)\Big)\psi_{\vec{k},a} \\ &+ \psi^\dagger_{\vec{k},a}\Gamma^5\Big(\sum_{i=1}^{4}\cos(k_i) - 4 + m\Big)\psi_{\vec{k},a}, \end{aligned} \tag{1}$$

where $\Gamma^{1,2,3} = \sigma^3 \otimes \sigma^{1,2,3}$, $\Gamma^4 = \sigma^1 \otimes \sigma^0$, $\Gamma^5 = \sigma^2 \otimes \sigma^0$ (with $\sigma^{1,2,3}$ being the Pauli matrices and σ^0 being the 2×2 identity matrix). $m > 0$ and $m < 0$ correspond to the topological and trivial insulators respectively. Close to the critical point $m = 0$, when expanded around $\vec{k} = 0$, Eq. (1) becomes the standard $4d$ Dirac fermion Hamiltonian: $H_{\mathrm{TI}} = \sum_{a=1,2} \sum_{\vec{k}} \psi^\dagger_{\vec{k},a} (\vec{k} \cdot \vec{\Gamma} + m\Gamma^5) \psi_{\vec{k},a}$.

The $3d$ boundary of this theory (which is the domain wall of mass m: Fig. 1) has precisely two flavors of chiral fermions (domain wall fermions):

$$H_{3d} = \int \mathrm{d}^3\boldsymbol{x} \ \sum_{a=1}^{2} \psi^\dagger_a (\mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{\partial}) \psi_a, \tag{2}$$

with $\sigma^{x,y,z}$ being the Pauli matrices in the spin space. The $\mathrm{U}(1)$ and $\mathbb{Z}_2$ symmetries act on the boundary chiral fermions as

$$\mathrm{U}(1) : \psi_a \to [e^{\mathrm{i}\tau^y \theta}]_{ab} \psi_b, \quad \mathbb{Z}_2 : \psi_a \to (\tau^y)_{ab} \psi_b, \tag{3}$$

where $\tau^{x,y,z}$ denote the Pauli matrices in the flavor space.

As long as we preserve the $\mathrm{U}(1) \times \mathbb{Z}_2$ symmetry, the $3d$ boundary can never be gapped without interaction for *arbitrary* copies of this system, because the only fermion bilinear mass terms that can gap out the boundary are the Cooper pair operators: $\psi_a^\intercal \mathrm{i}\sigma^y \psi_b + H.c.$ which inevitably break at least one of the symmetries. Thus this $4d$ TI has a $\mathbb{Z}$ classification with the $\mathrm{U}(1) \times \mathbb{Z}_2$ symmetry, at the noninteracting level (see appendix D for a proof of the classification).

In the following we will argue that short range interactions can reduce the classification of this $4d$ TI to $\mathbb{Z}_8$: local four-fermion interactions can gap out 8 copies of Eq. (2) and drive it into a SCSG phase. Notice that here the $\mathrm{U}(1)$ symmetry is not anomalous (it is analogous to the $B - L$ $\mathrm{U}(1)$ symmetry of the Standard Model), thus a SCSG phase in this case does not violate the anomaly matching condition.

Directly studying strong four-fermion interactions is difficult, so we will follow the same logic as in Ref. [17, 30–34]: we will first manually break a subgroup of the $\mathrm{U}(1) \times \mathbb{Z}_2$ symmetry by condensing an order parameter that transforms nontrivially under these symmetries. Then we will condense/proliferate the defects of the condensate to restore the broken symmetry. After condensing the defects, the order parameter becomes disordered and can be safely integrated out. This generates an effective interaction at low energy.

Let us first spontaneously break the $\mathrm{U}(1)$ symmetry by condensing an $\mathrm{O}(2)$ "superfluid" order parameter with unit length $\boldsymbol{n} = (n_1, n_2) \in \mathbb{R}$ at the $3d$ boundary, which couples to the fermions as:

$$H_{\mathrm{O}(2)} = \boldsymbol{n} \cdot (\mathrm{Re}[\psi^\intercal \tau^x \sigma^y \psi], \, \mathrm{Re}[\psi^\intercal \tau^z \sigma^y \psi]). \tag{4}$$

This superfluid order parameter gaps out the chiral fermions and breaks the $\mathrm{U}(1)$ symmetry, but preserves the $\mathbb{Z}_2$ symmetry in Eq. (3). The broken $\mathrm{U}(1)$ symmetry can be restored by condensing the vortex lines of the $\mathrm{O}(2)$ order parameter $\boldsymbol{n}$ in Eq. (4).

The dynamics of vortex lines can be systematically described in the dual formalism. In an ordinary $3d$ superfluid phase with spontaneous $\mathrm{U}(1)$ symmetry breaking, the $\mathrm{U}(1)$ Goldstone mode is dual to a rank-2 antisymmetric tensor field $B_{\mu\nu}$ defined as

$$J_\mu = \epsilon_{\mu\nu\rho\tau} \partial_\nu B_{\rho\tau}. \tag{5}$$

J_μ is the superfluid current. $B_{\mu\nu}$ is coupled to the vortex loops, which now can be described by a vector gauge field a_μ, and the fact that the vortex loop can never end corresponds to the Gauss law of the gauge field $\boldsymbol{\nabla} \cdot \boldsymbol{e} = 0$. The dynamics of vortex loops can be described by the following schematic dual action in the $4d$ Euclidean space lattice (which corresponds to the $(3{+}1)d$ boundary spacetime):

$$\begin{aligned} \mathcal{S} \ &= \ \sum_{\vec{x}} -t \cos(\nabla_\mu a_\nu - \nabla_\nu a_\mu - B_{\mu\nu}) \\ &- \ \frac{1}{e^2} (\epsilon_{\mu\nu\rho} \nabla_\mu B_{\nu\rho})^2 + \cdots \end{aligned} \tag{6}$$

The details of this standard duality formalism can be found in Ref. [35]. Depending on t/e^2, this model has two different phases: with $t/e^2 \ll 1$ the vortex loops are "small", and the system has one gapless gauge boson $B_{\mu\nu}$, which is the dual of the Goldstone mode in the superfluid phase; while with $t/e^2 \gg 1$, the vortex loops condense, and $B_{\mu\nu}$ and a_μ will both be gapped due to their coupling, this corresponds to the quantum disordered phase of superfluid.

The above dual action only applies when the vortex loop is "trivial", namely there is no extra low energy degree of freedom besides the vortex loops and superfluid Goldstone mode. This requires the fermion ground state with a vortex loop background be gapped and nondegenerate. For example, if the fermion ground state within a vortex loop is two fold degenerate, then the vortex loop will carry an extra flavor index $i = 1, 2$. In this case after the vortex loops condense, the system is *not* fully gapped, instead, it would enter a phase with a gapless photon excitation [36]; or in other words, after the superfluid order parameter is disordered, this system becomes a "$\mathrm{U}(1)$ spin liquid" phase. This is because $a_{1,\mu}$ and $a_{2,\mu}$ are both coupled to $B_{\mu\nu}$, thus when both $a_{1,\mu}$ and $a_{2,\mu}$ proliferate, $a_{+,\mu} = a_{1,\mu} + a_{2,\mu}$ will be rendered gapped by $B_{\mu\nu}$, while $a_{-,\mu} = a_{1,\mu} - a_{2,\mu}$ remains gapless since it is not coupled to any dual Goldstone mode. More details about quantum phases after proliferation of degenerate vortex loops can be found in Ref. [36].

Thus, *the desired SCSG phase is only possible when the defects in the condensate have a trivial spectrum.* we have to be careful with the core of the vortex line, since it is the singularity of the $\mathrm{O}(2)$ order parameter, and the fermions may become gapless along the vortex line. Now we have reduced our original $3d$ problem to a $1d$ problem inside a vortex line, which we can analyze much more reliably. In our current case, the vortex line of this O(2) order parameter traps $1d$ nonchiral Majorana fermion modes that are localized at the vortex line.[37] Upon solving the Dirac equation in the vortex background, we find that these modes are described by the Hamiltonian:

$$H_{1d} = \frac{1}{2}\int \mathrm{d}x\,(\chi_L \mathrm{i}\partial_x \chi_L - \chi_R \mathrm{i}\partial_x \chi_R). \tag{7}$$

and their transformation properties under the residual $\mathbb{Z}_2$ symmetries are:

$$\mathbb{Z}_2 : \chi_L \to \chi_L, \quad \chi_R \to -\chi_R. \tag{8}$$

With this symmetry, it is straightforward to verify that for arbitrary numbers of the $1d$ system Eq. (7), any fermion bilinear mass term is forbidden. For example, $\bar{\chi}\chi = 2i\chi_L\chi_R$ is forbidden by the $\mathbb{Z}_2$ symmetry Eq. (8), thus without interaction, this $1d$ system cannot be gapped without degeneracy, for arbitrary copies of this system Eq. (7), then this implies that without turning on certain interaction at the vortex core, a SCSG phase can *not* be obtained by condensing the vortex loops.

However, Ref. [10–12] showed that although all the fermion bilinear mass terms are forbidden in Eq. (7), when there are $8n$ copies of Eq. (7), a particular four fermion interaction term which preserves the $\mathbb{Z}_2$ symmetry still gaps out the $1d$ fermions with $\langle\bar{\chi}_a\chi_b\rangle = 0$ for arbitrary flavor index a, b. The specific form of this interaction was given in Ref. [10, 11, 20] and reviewed in appendix B, and it can also be concisely written as

$$H_{\text{int}} = -\frac{g}{2}\int \mathrm{d}x \sum_{a=1}^{7} (\chi_L^{\intercal}\gamma^a\chi_R + \text{H.c.})^2, \tag{9}$$

where the Majorana field $\chi_{L,R}$ has been extended to eight-component. The coupling matrices γ^a are the Gamma matrices of the SO(7) group in its 8-dimensional spinor representation, which, under a specific choice of basis, may be written as $\boldsymbol{\gamma} = (\sigma^{002}, \sigma^{323}, \sigma^{021}, \sigma^{203}, \sigma^{231}, \sigma^{123}, \sigma^{211})$ (hereinafter $\sigma^{ijk\cdots} \equiv \sigma^i \otimes \sigma^j \otimes \sigma^k \cdots$ denotes the tensor product of Pauli matrices). As proven in Ref. [10, 11, 20], such interaction can drive eight copies of the 1d system Eq. (7) into a SCSG phase at strong coupling. Then the O(2) vortex loops can condense to restore the U(1) symmetry and gap out the chiral fermions on the $3d$ boundary.

To explicitly implement our picture of condensing vortex loops, we need to control the dynamics of the vortex loops. In order to do this, we propose to add the following interacting Hamiltonian on the $4d$ lattice model:

$$H_{\text{total}} = H_{\text{int-4d}} + H_{\mathrm{O(2)}} + H[\boldsymbol{n}]. \tag{10}$$

H_{int} is a four-fermion interaction term which generates the Eq. (9) in every vortex loop, which will gap out the vortex loop without degeneracy. Its explicit form in the $4d$ bulk and at the $3d$ boundary reads (see appendix B for derivation):

$$\begin{aligned} H_{\text{int-4d}} &= -\frac{g}{2}\int \mathrm{d}^4\boldsymbol{x} \sum_{a=1}^{7} \mathrm{Re}[\psi^\intercal \tau^y \Gamma^2 \gamma^a \psi]^2, \\ H_{\text{int-3d}} &= -\frac{g}{2}\int \mathrm{d}^3\boldsymbol{x} \sum_{a=1}^{7} \mathrm{Re}[\psi^\intercal \tau^y \sigma^y \gamma^a \psi]^2. \end{aligned} \tag{11}$$

$H_{\mathrm{O(2)}}$ is the coupling between the O(2) vector $\boldsymbol{n}$ to the fermions on the lattice model, which generates coupling Eq. (4) on the $3d$ boundary. $H[\boldsymbol{n}]$ controls the dynamics of the O(2) vector $\boldsymbol{n}$, including the dynamics of the vortex loops. We parametrize $\boldsymbol{n}$ as $\boldsymbol{n} = (\cos(\hat{\phi}), \sin(\hat{\phi}))$, where $\hat{\phi} \in [0, 2\pi)$, and label the canonical momentum of $\hat{\phi}$ as $\hat{N}$, with $\hat{N} \in$ Integers. We propose the following Hamiltonian $H[\boldsymbol{n}]$:

$$\begin{aligned} H[\boldsymbol{n}] &= \sum_{\boldsymbol{x},\mu\neq\nu} -J\cos\left(\nabla_\mu\hat{\phi}\right) + V[\hat{N}(\boldsymbol{x})] \\ &+ K\cos\left(\nabla_\mu\nabla_\nu\hat{\phi}\right), \end{aligned} \tag{12}$$

where the sum is taken over all spatial positions $\boldsymbol{x}$ and directions μ, ν. The lattice derivatives are defined as $\nabla_\mu\hat{\phi}(\boldsymbol{x}) = \hat{\phi}(\boldsymbol{x}+\mu) - \hat{\phi}(\boldsymbol{x})$, $\nabla_\mu\nabla_\nu\hat{\phi}(\boldsymbol{x}) = \hat{\phi}(\boldsymbol{x}+\mu+\nu) - \hat{\phi}(\boldsymbol{x}+\nu) - \hat{\phi}(\boldsymbol{x}+\mu) + \hat{\phi}(\boldsymbol{x})$. $V[\hat{N}]$ is a local short range repulsive interaction of $\hat{N}$, whose explicit form has many choices, but the simplest possibility is $V[\hat{N}(\boldsymbol{x})] = v\big(\hat{N}(\boldsymbol{x})\big)^2$. When J dominates all the other terms, $\boldsymbol{n}$ is ordered, the O(2) symmetry is spontaneously broken, and the fermions acquire an ordinary fermion gap. If we start with a weak superfluid phase (a superfluid phase with a small stiffness), the K term will compete with the superfluid order by lowering the core energy of vortices, and we expect it to drive the system into a vortex condensate, with an appropriate choice of $V[\hat{N}]$. An analogue of $H[\boldsymbol{n}]$ in $2d$ was studied by quantum Monte Carlo in Ref. [38, 39]. It was shown in a spin-1/2 quantum XY model that when the ring exchange term K dominates J, it indeed drives a order-disorder quantum phase transition. Thus the K term can be effectively viewed as $\sim -K\rho_v^2$, where ρ_v is the local density of vortices.

Our toy model demonstrated that eight copies of the $4d$ TI Eq. (1) can be trivialized under a local fermion interaction with SO(7) × SO(2) symmetry. Since the mirror sector is driven into the SCSG phase, we can obtain 16 chiral fermions on the other $3d$ boundary with lattice regularization. In fact, the symmetry group can be further enlarged to SO(7) × SO(3). In that case, we introduce an O(3) order parameter with unit length $\boldsymbol{n} = (n_1, n_2, n_3) \in \mathbb{R}$ which couples to the boundary fermions as $H_{\mathrm{O(3)}} = \boldsymbol{n}\cdot(\mathrm{Re}[\psi^\intercal\tau^x\sigma^y\psi], \mathrm{Re}[\psi^\intercal\tau^z\sigma^y\psi], \mathrm{Im}[\psi^\intercal\sigma^y\psi])$. Following the similar defect condensation argument, we can first gap out the chiral fermions on the $3d$ boundary by ordering the O(3) order parameter at the price of breaking the SO(3) symmetry, and then we attempt to restore the symmetry by condensing the monopole defects of $\boldsymbol{n}$. Each monopole will trap eight Majorana zero modes χ (the calculation is identical to that in Ref. [40]), which can not be gapped out by any fermion bilinear terms because they are all forbidden by the SO(7) symmetry. Now the same $3d$ interaction in Eq. (11) will induce the following $0d$ interaction among the eight Majorana zero modes at the monopole core:

$$H_{\text{int}} = -\frac{g}{2}\sum_{a=1}^{7}(\chi^{\intercal}\gamma^{a}\chi)^{2}, \tag{13}$$

with the same set of γ^a matrices defined below Eq. (9). As shown in Ref. [10, 11, 20] and reviewed in appendix B, this $0d$ interaction can gap out the Majorana zero modes and stabilize a unique SO(7) singlet ground state in the monopole core. It can also be verified that the monopole defect is a boson, so it can condense to restore the SO(3) symmetry. Thus the chiral fermions in the mirror sector can also be driven into the SCSG phase with the larger symmetry SO(7) × SO(3) as well.

One can see that the symmetry group and the design of the interaction may vary from case to case, but the common features that we wish to emphasize are: *(1)* the interaction terms we turn on explicitly breaks the anomalous U(1) symmetry of the boundary chiral fermions, thus a SCSG phase is possible; *(2)* the counting of 16 chiral fermions is crucial, if the fermion flavor number is insufficient, the SCSG phase will not be realized and the mirror sector can not be decoupled by interaction.

The key of the analysis in this section is to show that a properly designed $4d$ bulk interaction can induce the *correct* $1d$ ($0d$) four fermion interaction Eq. (9) (Eq. (13)) inside the vortex loop (monopole core), which is known to be capable of driving the vortex loop (monopole core) into a SCSG phase [10, 11]. In the next section, we will also use the dimensional reduction argument, and we show that the Standard Model can be successfully regularized as part of the Pati-Salam GUT on the boundary of a $4d$ TI, and the mirror sector can be driven into the SCSG phase and hence decoupled in the IR.

3. PATI-SALAM GUT

Motivated by the Pati-Salam GUT whose gauge group is $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$, we may directly start from a $4d$ TI with $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ as its symmetry group. The lattice model of the $4d$ TI is of the same form as Eq. (1), expect that now $\psi_{\vec{k},a}$ (for each $a=1,2$ respectively) is extended to an eight-flavor Dirac fermion field. The Hamiltonian respects the $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry in the way that $\psi_{\vec{k},1}$ and $\psi_{\vec{k},2}$ form the representations $(\mathbf{4},\mathbf{2},\mathbf{1})$ and $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ respectively. Its 3d boundary theory still takes the same form as Eq. (2), but the boundary fermions ψ_a $(a=1,2)$ now transform under $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ like $(\mathbf{4},\mathbf{2},\mathbf{1})$ and $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ representations, which can be written out explicitly as

$$\begin{aligned}
\mathrm{SU}(4):&\ \psi_1\to e^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\rho}}\psi_1, \quad \psi_2\to e^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\rho}^*}\psi_2;\\
\mathrm{SU}(2)_1:&\ \psi_1\to e^{\mathrm{i}\boldsymbol{\theta}_1\cdot\boldsymbol{\mu}}\psi_1,\ \ \psi_2\to\psi_2;\\
\mathrm{SU}(2)_2:&\ \psi_1\to\psi_1, \qquad\quad \psi_2\to e^{\mathrm{i}\boldsymbol{\theta}_2\cdot\boldsymbol{\mu}}\psi_2.
\end{aligned} \tag{14}$$

$\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ denote the generators of SU(4) and SU(2) groups respectively.

As long as the $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry is preserved, the $3d$ boundary must remain gapless at the free-fermion level. Because all the fermion bilinear mass terms at the $3d$ boundary take the form of the spin-singlet Cooper pairing: $\psi_a^{\intercal}\mathrm{i}\sigma^y M\psi_b + H.c.$ (where $a,b=1,2$ and M is an arbitrary matrix in the color-flavor space), but such terms are forbidden by the SU(4) symmetry if $a=b$, and are forbidden by the $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry if $a\neq b$, therefore no fermion bilinear mass term can be added without breaking the $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry. Thus the $4d$ insulating phases with $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry is $\mathbb{Z}$ classified (see appendix D for a proof of the classification).

Another way of making the same statement is to say that, at the free-fermion level, the $4d$ bulk TI can not be smoothly tuned (while preserving the symmetry) into a trivial insulator without going through a gap-closing phase transition. Tuning the TI to trivial corresponds to driving the mass m of a bulk $4d$ Dirac fermion from positive to negative, and close to the quantum critical point $m=0$ and expanded at $\vec{k}=0$, the bulk theory reads:

$$H_{\text{TI}} = \int \mathrm{d}^4\boldsymbol{x}\sum_{a=1,2}\psi_a^{\dagger}(\mathrm{i}\vec{\Gamma}\cdot\vec{\partial}+m\Gamma^5)\psi_a, \tag{15}$$

where ψ_a for each a is an eight-flavor Dirac fermion which also carries SU(4) and SU(2) indices. While changing m, the fermion bulk gap will close at $m=0$. Without interaction, the gap-closing transition can not be circumvented, because there is no other symmetry-allowed mass terms to be added that can gap out the point $m=0$. For example, the Majorana mass terms $\psi_a^{\intercal}\mathrm{i}\Gamma^2 M\psi_b + H.c.$ could gap out the bulk criticality at $m=0$, however, as we have shown before, such terms are all forbidden by the $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry. So without interaction the $4d$ $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ TI and the trivial insulator are in different phases, separated by a phase transition that can not be avoid at the non-interacting level without breaking the symmetry.

However, as seen before (and also recently studied in literatures[10–19]), the classification of topological insulators can be reduced by interaction. Here, as we will show in the following, the classification of the $4d$ $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ TI is reduced from $\mathbb{Z}$ to trivial, meaning that under interaction the $4d$ TI and the trivial insulator are actually in the same phase, and the bulk phase transition between them can be avoid by strong-enough and properly-designed interactions, as shown in the phase diagram Fig. 2(a). In other words, the gapless bulk fermion at the $m=0$ critical point can be gapped out by interaction.

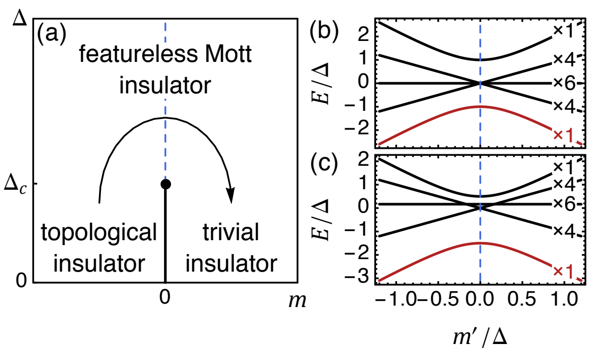

FIG. 2: (a) Schematic phase diagram of the $4d$ TI with $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry under interaction. There exist a critical interaction strength Δ_c, above which the topological-to-trivial transition can be circumvented. (b,c) The O(4) monopole core levels along a path connecting the $4d$ TI to the trivial insulator, parameterized by the reduced mass m'. The effective Hamiltonian in the monopole core reads $H=H_{\text{free}}+H_{\text{int}}$, where H_{int} is taken from (b) Eq. (18) or (c) Eq. (19). The 16-dimensional Hilbert space split according to SU(4) representations as $\mathbf{16}=\mathbf{1}+\mathbf{1}'+\mathbf{4}+\bar{\mathbf{4}}+\mathbf{6}$ with the unique ground state $|\mathbf{1}\rangle+|\mathbf{1}'\rangle$ (marked out in red). The dashed line marks out the $m'=0$ critical point, where degeneracy is avoided by interaction.

To show this, we will again implement the argument of defect proliferation/condensation, i.e. one may choose to break part of the symmetry by condensing certain fermion-bilinear order parameter, and then restore the symmetry by condensing topological defects of that order parameter field. Due to the fact $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2\simeq\mathrm{SO}(4)$, one can introduce the symmetry-breaking O(4) vector order parameter field $\vec{n}=(n_0,\boldsymbol{n})=(n_0,n_1,n_2,n_3)\in\mathbb{R}$, which couples to the bulk fermions as

$$H_{\mathrm{O}(4)}=\int d^4\boldsymbol{x}\, n_0\psi_1^\intercal \mathrm{i}\Gamma^2\psi_2+\boldsymbol{n}\cdot\psi_1^\intercal\Gamma^2\boldsymbol{\mu}\psi_2+\text{H.c.}. \quad (16)$$

The $\mathrm{SU}(2)_1$ and the $\mathrm{SU}(2)_2$ rotations act respectively as the left and the right isoclinic rotations on the O(4) vector $\vec{n}$. We first condense $\vec{n}$ to gap out the bulk fermions for all range of m (including $m=0$) at the price of breaking the $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ symmetry. In the $4d$ bulk, a O(4) vector order parameter has the hedgehog monopole topological defects, due to the fact $\pi_3[S^3]=\mathbb{Z}$. The broken symmetry is expected to be restored by condensing the O(4) vector monopole defects in the $4d$ bulk. Since our goal is to show that the critical point $m=0$ can be gapped out by interaction, we only need to demonstrate that under interaction the fermion spectrum inside the monopole will always be gapped and nondegenerate in the entire phase diagram.

By directly solving the Schrödinger equation (see appendix C for details), one can show that the monopole will trap four complex fermion localized modes f_i ($i=1,2,3,4$) forming a fundamental representation of the SU(4) symmetry. Since the SU(4) symmetry is not broken by the O(4) vector $\vec{n}$, the effective Hamiltonian of f_i must be SU(4) invariant. Without interaction, the only fermion bilinear Hamiltonian reads

$$H_{\text{free}}\sim m'\sum_{i=1}^{4}(f_i^\dagger f_i-1/2), \quad (17)$$

here the coefficient m' is proportional to the mass m of the bulk Dirac fermion. Thus by tuning m from negative to positive, the monopole core will close its spectrum gap at $m\sim m'=0$, and the monopole will be 16-fold degenerate at $m'=0$. Thus without four-fermion interaction inside the monopole, condensing the monopole will still lead to a bulk quantum phase transition at $m=0$ [64].

However, the monopole core spectrum can be completely changed by the following SU(4) invariant local four-fermion interaction

$$H_{\text{int}}=-\Delta(f_1f_2f_3f_4+\text{H.c.}), \quad (18)$$

At $m'=0$, H_{int} will lift the degeneracy among these fermion zero modes, and single out the unique ground state $(|0000\rangle+|1111\rangle)/\sqrt{2}$. $|0\rangle$ and $|1\rangle$ stand for the fermion occupation number eigenstates of the zero mode. A slightly different interaction (see appendix B for derivation) will play qualitatively the same role as Eq. (18):

$$\begin{aligned}H_{\text{int}}&=-\frac{g}{2}(f^\intercal\boldsymbol{\lambda}f+\text{H.c.})^2\\&=-24g(f_1f_2f_3f_4+\text{H.c.})-8g\sum_{i<j}\rho_i\rho_j,\\ \boldsymbol{\lambda}&=(\sigma^{12},\sigma^{20},\sigma^{32},-\mathrm{i}\sigma^{21},-\mathrm{i}\sigma^{02},-\mathrm{i}\sigma^{23}),\end{aligned} \quad (19)$$

where λ^a ($a=1,\cdots,6$) are six 4×4 matrices acting in the SU(4) color sector (forming the representation $\mathbf{6}$ of SU(4)), and $\rho_i=f_i^\dagger f_i-\frac{1}{2}$ ($i=1,2,3,4$) denote the fermion density operators. The first term is the interaction in Eq. (18) by identifying $\Delta=24g$, and the second term is a density-density interaction which does not qualitatively change the spectrum of monopole, as seen by comparing Fig. 2(b,c).

With the protection of the gap Δ, the ground state of the O(4) monopole evolves smoothly in all range of m without any level crossing with the excited states, as shown in Fig. 2(b,c). The interaction not only renders the monopole to a nondegenerate SU(4) singlet, it also makes the monopole a boson, this is because deep in the trivial phase of Eq. (1) and Eq. (15), *i.e.* when m is negative and it is the dominant energy scale of the system, the ground state of a monopole in the bulk must be a featureless boson. And we have proved that with interaction $H_{\rm int}$ the ground state of the monopole never has any level-crossing with excited states, thus the ground state of the monopole must remain as a boson for the entire range of m. Thus the monopole can be safely condensed to restore the broken $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ symmetry without causing ground state degeneracy or breaking other symmetries. After the monopole condensation, we end up with a symmetric gapped phase in the bulk for the entire range of m, meaning that the bulk phase transition between the $4d$ $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ TI and the trivial insulator can be removed by the interaction with sufficient strength, see Fig. 2(a). Note that the $3d$ boundary of the $4d$ TI is simply the spatial interface between the $4d$ TI and the trivial insulator (vacuum). Since the $4d$ TI can now smoothly evolve into the trivial insulator without gap-closing phase transition, the $3d$ interface between these two states (which can be viewed as an evolution in space) must also be driven into a SCSG phase by the same kind of interaction.

Again, to explicitly implement our picture of "condensing topological defects", we need to control the dynamics of the topological defects. In order to do this, we propose to add the following interacting Hamiltonian on the $4d$ lattice model:

$$H_{\rm total} = H_{\text{int-4d}} + H_{\mathrm{O}(4)} + H[\vec{n}]. \tag{20}$$

$H_{\text{int-4d}}$ is a $4d$ bulk interaction that will induce the correct four-fermion term at the monopole core, which gaps out the monopole for all range of m in the phase diagram (see appendix B for derivation):

$$H_{\rm int} = -\frac{g}{2}\int \mathrm{d}^4\boldsymbol{x}\ (\psi_1^\intercal\Gamma^2\mu^2\boldsymbol{\lambda}\psi_1 + \psi_2^\dagger\Gamma^2\mu^2\boldsymbol{\lambda}\psi_2^{\dagger\intercal} + \mathrm{H.c.})^2. \tag{21}$$

This interaction is manifestly $\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ invariant. $H_{\mathrm{O}(4)}$ is given by Eq. (16). $H[\vec{n}]$ is the Hamiltonian for the O(4) unit vector order parameter $\vec{n}$ that should control the dynamics of $\vec{n}$ and its topological defects:

$$H[\vec{n}] = \sum_{\boldsymbol{x},\mu} -J\left(\nabla_\mu\vec{n}\right)^2 + V[L^{ab}(\boldsymbol{x})] - K\rho_m(\boldsymbol{x})^2. \tag{22}$$

V is a local interaction between SO(4) angular momentum operator $L^{ab}(x)$ (which is conjugate to operator $\vec{n}(\boldsymbol{x})$), its simplest form could be $\sum_{a<b} v\big(L^{ab}(\boldsymbol{x})\big)^2$. When J dominates all the other terms, vector $\vec{n}$ would be ordered, and the fermions will acquire an ordinary mass gap. $\rho_m(\boldsymbol{x})$ is the local monopole density of the SO(4) vector $\vec{n}$. To define a monopole on a lattice, one can just follow the strategy of Ref. [41], which defined SO(3) monopole on a $3d$ cubic lattice and numerically studied its effects on phase transitions. Thus we can start with a weak order of $\vec{n}$ (when J and V terms are comparable with each other), and gradually increasing K. Then we expect that across a finite critical point, the K term will drive the system into a monopole condensate in the $4d$ bulk. And based on our argument presented before, the same interaction can drive the mirror sector on the $3d$ boundary into the desired SCSG phase.

Normally condensing a conserved bosonic point particle will lead to a gapless Goldstone mode. But in our case, inside an ordered phase of $\vec{n}$, monopoles have long range interaction, and the condensate of bosons with long range interaction can still have a gapped spectrum. This is precisely the Higgs mechanism. For example, condensing the vortices of a $(2+1)d$ superfluid will not lead to any gapless Goldstone mode, because in the standard dual formalism the vortex field is a complex boson which are coupled to a dual U(1) gauge field.

Our analysis above suggests that if we want to regularize the Pati-Salam GUT on a $3d$ lattice (a $4d$ lattice with a thin fourth dimension and a decoupled mirror sector), then a four-fermion interaction $H_{\text{int-4d}}$ is necessary. This four-fermion interaction creates/annihilates a four-fermion SU(4) singlet, thus breaks the baryon number (B) and lepton number conservation (L), but it still conserves $B-L$. For instance this four fermion term contains the standard dimension-6 operators that would lead to proton-decay: $qqql/\Lambda^2$. But here the UV cut-off Λ should be the lattice scale, which is higher than any other scale of the system. Thus the proton decay effect is expected to be much smaller than that predicted in the SU(5) GUT, which is suppressed by factor $1/\Lambda_{GUT}^2$.

4. SUMMARY

In this work we apply the latest progress in condensed matter physics towards understanding strongly interacting topological insulator to the long standing problem in high energy physics: How to regularize the SM or GUT on a lattice. In our approach, because the bulk topological insulator is trivialized by interaction, the mirror sector is in the SCSG phase and hence decoupled from the GUT in the infrared limit. Our current work heavily relies on the analysis of classification of topological insulators under interaction, and our argument of topological defects condensation leads to explicit construction of an interacting lattice Hamiltonian, whose low energy physics is described by the Pati-Salam GUT.

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[59] Throughout the paper, $3d$ and $4d$ represent the spatial dimensions of the boundary and bulk respectively, while $(3+1)d$ and $(4+1)d$ represent the space-time dimensions.
[60] In condensed matter physics, $\psi^\intercal i\sigma^y\psi$ is a Cooper pair operator; while in high energy physics, it is the Majorana mass of chiral fermion.
[61] In principle, it is also possible to drive the mirror sector into a fully symmetric topological order which has a gapped spectrum, but degenerate ground states. This case was immensely studied in condensed matter physics. But in our current work we focus on the case when the mirror sector is nondegenerately gapped by interaction, *i.e.* there is no topological order in the SCSG phase.
[62] Based on the first criterion, the global U(1) symmetry in the bulk must be explicitly broken, thus the term "insulator" is not entirely accurate. We use the term topological insulator because in our construction this anomalous U(1) symmetry is only broken by the four fermion interaction term, while it is preserved at the noninteracting level.
[63] It is believed (although not proved) that these two approaches are equivalent, namely if two $d-$dimensional states can be adiabatically connected by tuning a parameter (for instance the Dirac mass m) without closing the bulk gap, then it implies that the $(d-1)-$dimensional interface between these two states can be gapped and nondegenerate. To visualize this statement, one can just make a smooth and wide interface, over which the tuning parameter changes smoothly in space from one state to another, then the gap never closes at this smooth interface.
[64] If we project the $4d$ Dirac fermion Eq. (15) to the

monopole core, then $m = 0$ in the $4d$ bulk precisely coincides with $m' = 0$ in the monopole core; the parameter m in the lattice model will not be exactly proportional to m' in Eq. (17), but since H_{free} is the only noninteracting term inside a monopole core, tuning m from negative to positive in the lattice model will definitely cross the point $m' = 0$.

SUPPLEMENTAL MATERIAL

A. Review of SCSG Phase and Interacting TI in Lower Dimensions

The interacting fermionic topological insulator/superconductor (TI/TSC) has recently attracted much research attention in condensed matter physics. In the non-interacting limit, the TI/TSC has a fully gapped and non-degenerated bulk state with gapless fermionic boundary modes protected by symmetry. The gapless boundary fermions are also known as the domain wall fermions[4, 21, 22, 42] in high energy physics. It was first pointed out by Fidkowski and Kitaev[10, 11] that the classification of the TI/TSC can be reduced by the fermion interaction, namely certain non-trivial TI/TSC phases can actually be smoothly connected to the trivial phase under interaction, and correspondingly, their gapless boundaries can be driven into the strongly coupled symmetric gapped (SCSG) phase by the same interaction.

A1. SCSG Phase in $0d$: Boundary of $1d$ Systems

The simplest example is to consider the $0d$ boundary of a $1d$ TSC, which hosts Majorana fermion zero modes (the $0d$ analog of the domain wall fermions). The Majorana modes are denoted by the operators χ_a $(a = 1, \cdots, N)$ satisfying $\{\chi_a, \chi_b\} = 2\delta_{ab}$. Let us define a time-reversal symmetry $\mathbb{Z}_2^T$ ($\mathcal{T}^2 = +1$), which acts trivially on the Majorana modes as $\mathbb{Z}_2^T : \chi_a \to \mathcal{K}\chi_a$, where $\mathcal{K}$ is the complex conjugation operator (flipping the imaginary unit as $\mathcal{K}^{-1}\mathrm{i}\mathcal{K} = -\mathrm{i}$). Any fermion bilinear operator $\mathrm{i}\chi_a\chi_b$ will break the time-reversal symmetry, because χ_a transforms trivially but i gets a minus sign. So if we require the time-reversal symmetry, all the fermion bilinear terms will be ruled out from the $0d$ boundary Hamiltonian, and the $0d$ boundary fermions can not be gapped out in the free fermion limit no matter how many modes N there are. However the four-fermion interaction term will not break the time-reversal symmetry, since no factor i will be involved. For $N = 4$, the only interaction term that one can write down is $H = -J\chi_1\chi_2\chi_3\chi_4$. Pairing up the Majorana fermions to regular (complex) fermions $c_\uparrow = (\chi_1 + \mathrm{i}\chi_2)/2$, $c_\downarrow = (\chi_3 + \mathrm{i}\chi_4)/2$, and define the fermion number operator $n_\sigma = c_\sigma^\dagger c_\sigma$, then the interaction Hamiltonian can be written as $H = J(2n_\uparrow - 1)(2n_\downarrow - 1)$, which can be interpreted as a Hubbard interaction leading to a two-fold degenerated ground state (as a spin-1/2 doublet), for either sign of J. So if we have $N = 8$ Majorana zero mode, under the interaction, $\chi_{1,2,3,4}$ form a doublet and $\chi_{5,6,7,8}$ form another doublet, and the two doublets can be coupled together into a singlet (such as via the Heisenberg coupling), and the ground state degeneracy is completely removed by the interaction, which also implies that the expectation value of any fermion bilinear operator must vanish, because otherwise the ground state would be degenerated. The explicit form of the interaction is given by Fidkowski and Kitaev[10, 11]

$$\begin{aligned} H_{\mathrm{FK}} \sim &+ \chi_1\chi_2\chi_3\chi_4 + \chi_1\chi_2\chi_5\chi_6 + \chi_1\chi_2\chi_7\chi_8 + \chi_1\chi_3\chi_5\chi_7 - \chi_1\chi_3\chi_6\chi_8 - \chi_1\chi_4\chi_5\chi_8 - \chi_1\chi_4\chi_6\chi_7 \\ &- \chi_2\chi_3\chi_5\chi_8 - \chi_2\chi_3\chi_6\chi_7 - \chi_2\chi_4\chi_5\chi_7 + \chi_2\chi_4\chi_6\chi_8 + \chi_3\chi_4\chi_5\chi_6 + \chi_3\chi_4\chi_7\chi_8 + \chi_5\chi_6\chi_7\chi_8. \end{aligned} \tag{23}$$

This interaction looks rather involved and has a very high SO(7) symmetry, nevertheless it is not the only choice. There exist many other interactions (to be reviewed in Appendix B) that can also gap out eight Majorana fermions in $0d$. The point is that in this $0d$ fermion system, only when we have eight flavors of Majorana fermions, we can get a fully gapped spectrum and a non-degenerate ground state (i.e. a SCSG state). If the flavor number is insufficient ($N < 8$), the Majorana zero modes can not be completely gapped out.

The above $0d$ system is actually first realized as the boundary of the $1d$ Majorana fermion chain[43] with the $\mathcal{T}^2 = +1$ time-reversal symmetry (in the BDI[44, 45] symmetry class). The model Hamiltonian is defined on a $1d$ lattice of the length L, as $H = -\sum_{i=0}^{L-1} \mathrm{i}u_i\chi_i\chi_{i+1}$, where χ_i $(i = 1, \cdots, L)$ denotes the Majorana fermion operator on the site i, and the bond strength $u_i = 1 + (-1)^i\delta$ alternates along the chain, similar to the pattern of polyacetylene. If $\delta > 0$, both the bulk and the boundaries are fully gapped, and the system is in its trivial phase; if $\delta < 0$, the bulk is still fully gapped, but each boundary will host a dangling Majorana zero mode, and the system is in its non-trivial phase (known as a $1d$ TSC of BDI class). The time-reversal symmetry acts as $\mathbb{Z}_2^T : \chi_i \to \mathcal{K}(-1)^i\chi_i$, which will reduce on the boundary to precisely the same the time-reversal symmetry that we defined in the previous $0d$ example. Protected by the time-reversal symmetry, the boundary Majorana zero modes can not be gapped out at the free fermion level, and the $1d$ TSC is therefore $\mathbb{Z}$ classified in the absence of interaction. However, if we stack eight copies of the $1d$ TSC's together, the boundary Majorana zero modes (now there are eight zero modes) can be gapped out by the Fidkowski-Kitaev interaction Eq. (23) without breaking the time-reversal symmetry, and the classification of the $1d$ TSC is reduced from $\mathbb{Z}$ to $\mathbb{Z}_8$ under interaction. This phenomena is known as the interaction reduced classification of TI/TSC in condensed matter physics.

The interaction reduced classification indicates that eight copies of the $1d$ TSC is actually in the same phase as the $1d$ trivial insulator, such that one can (with the help of the interaction) smoothly tune eight copies of the $1d$ TSC to trivial without going through a phase transition while respecting the symmetry. First let us point out that without the interaction the bulk phase transition can not be avoid. To see this, let us assume $|\delta| \ll 1$, then the effective Hamiltonian of the $1d$ TSC at low-energy becomes a $1d$ Dirac fermion $H = \frac{1}{2}\int \mathrm{d}x\, \chi^\intercal(\mathrm{i}\partial_x\sigma^1 + m\sigma^2)\chi$ with the time-reversal symmetry $\mathbb{Z}_2^T : \chi \to \mathcal{K}\sigma^3\chi$. The Dirac mass m is proportional to the parameter δ in the lattice model, so that $m > 0$ ($m < 0$) corresponds to the TSC (trivial) phase. Tuning from $m > 0$ to $m < 0$, the bulk gap must close at $m = 0$, which triggers a phase transition. No matter how many copies of the $1d$ TSC we made, the time-reversal symmetry will always rule out the additional fermion bilinear mass terms (which all take the form of $\chi_a^\intercal \mathrm{i}\sigma^3 A_{ab}\chi_b$ with $A^\intercal = -A$) that anticommute with $m\sigma^2$, so the bulk phase transition is inevitable at the free fermion level. The interaction reduced classification means that the bulk criticality at $m = 0$ actually can be removed by the properly designed interaction.

For $1d$ TSC, this conclusions has been *rigorously* proven[10] at the field theory level using the bosonization approach. The precise conclusion of [10] is that, for 8 copies of the $1d$ TSC whose low energy field theory reads $H = \frac{1}{2}\int \mathrm{d}x \sum_{a=1}^{8} \chi_a^\intercal(\mathrm{i}\partial_x\sigma^1 + m\sigma^2)\chi_a$, there is a SO(7) invariant interaction (χ_a forms a spinor rep of SO(7)), which for the entire range of m, renders the spectrum gapped without ground state degeneracy, even for the point where $\langle\chi^\intercal\sigma^2\chi\rangle = 0$ (in the field theory this corresponds to the critical point $m = 0$).

Instead of reproducing the proof in Ref. [10], here we will present a more intuitive argument, which is analogous to the argument we gave in the main text for the Pati-Salam GUT. The advantage of this argument is that it can be easily generalized to any higher spatial dimensions. We first consider two copies of the $1d$ TSC coupled to a $\mathbb{Z}_2^T$ symmetry-breaking Ising field n, as described by the effective Hamiltonian $H = \frac{1}{2}\int \mathrm{d}x\, \chi^\intercal(\mathrm{i}\partial_x\sigma^{10} + m\sigma^{20} + n(x)\sigma^{32})\chi$ [59]. The strategy is that we first order the n field to locally gap out the critical fermions in the bulk at the expense of breaking the $\mathbb{Z}_2^T$ symmetry, then we disorder the Ising field n by condensing its kink defects to restore the symmetry. A fully gapped and non-degenerate bulk state can be obtained only if the kink is also fully gapped and non-degenerate. For two copies of the $1d$ TSC, it is found that the kink of $n(x)$ will trap two Majorana localized modes, which again defines a complex fermion localized mode c, and the tuning parameter m is coupled to the density of the complex fermion: $m(c^\dagger c - 1/2)$. Thus at $m = 0$, the kink has two degenerate states with opposite fermion parity. Thus the point $m = 0$ cannot be driven into a gapped and nondegenerate state by condensing the kinks. Further analysis shows that only when we have eight copies of the $1d$ TSC, the kink, which is a $0d$ object hosting eight Majorana localized modes, can be trivially gapped out by the interaction for the entire range of m (following the previous discussion of the $0d$ example). Then when and only when there are $8n$ $1d$ TSC, can we adiabatically connect $m > 0$ and $m < 0$ through condensing the kinks without closing the gap. And in this case the kink is a boson that can condense to restore the symmetry. Thus we arrive at the same conclusion that the $1d$ TSC is $\mathbb{Z}_8$ classified under interaction.

A2. SCSG phase in $1d$: boundary of $2d$ systems

From the above discussion, we can see there are two equivalent arguments to demonstrate that a TI/TSC is trivialized by interaction: (1) the boundary argument by showing that the boundary of TI/TSC can be driven to the SCSG phase by interaction, (2) the bulk argument by showing that the bulk topological-to-trivial phase transition can be removed by interaction. The study of the interaction reduced classification of TI/TSC is soon extended to higher spatial dimensions, such as $2d$ $p \pm \mathrm{i}p$ TSC [12–15] (D class), $3d$ ^{3}He-B TSC [16, 17] (DIII class), $4d$ TSC[18] (A class), and higher dimensions TI/TSC in general[19]. All these examples can be understood following either the boundary or the bulk arguments concluded above.

For example, it was shown that the $2d$ $p \pm \mathrm{i}p$ TSC with a $\mathbb{Z}_2$ symmetry is $\mathbb{Z}_8$ classified under interaction. Close to the topological-to-trivial phase transition, the bulk effective Hamiltonian in the free fermion limit is $H = \frac{1}{2}\int \mathrm{d}^2\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_1\sigma^{10} + \mathrm{i}\partial_2\sigma^{30} + m\sigma^{23})\chi$ with the $\mathbb{Z}_2 : \chi \to \sigma^{03}\chi$ symmetry, where $m > 0$ ($m < 0$) corresponds to the topological (trivial) phase. Following the boundary argument, the $1d$ boundary of a $2d$ $p \pm \mathrm{i}p$ TSC consists of a pair of counter-propagating Majorana edge modes, described by $H = \frac{1}{2}\int \mathrm{d}x\chi^\intercal(\mathrm{i}\partial_x\sigma^1)\chi$, which, at the field-theory level, is the same as the bulk theory of a single copy of the $1d$ (BDI class) TSC at its $m = 0$ critical point discussed in the last subsection. As eight copies of the $1d$ TSC can be trivialized by interaction (due to its $\mathbb{Z}_8$ classification) without generating any fermion bilinear order, it therefore suggests that eight copies of the $2d$ $p \pm \mathrm{i}p$ TSC can also be trivialized by interaction, as its boundary Majorana modes can be driven to the SCSG phase by the same kind of interaction. This boundary argument can be made precice[12] by the bosonization formalism as Ref. [10].

For the bulk argument, we can consider two copies of the $2d$ $p\pm\mathrm{i}p$ TSC close to the $m = 0$ critical point while coupling to an O(2) real boson field $\boldsymbol{n} = (n_1, n_2)$, as described by $H = \frac{1}{2}\int \mathrm{d}^2\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_1\sigma^{100} + \mathrm{i}\partial_2\sigma^{300} + m\sigma^{230} + n_1\sigma^{211} + n_2\sigma^{213})\chi$.

Following the "defect proliferation argument", we can first order the O(2) field $\boldsymbol{n}$ to locally gap out the bulk fermion criticality at the expense of breaking the $\mathbb{Z}_2$ symmetry, then we restore the symmetry by proliferating vortices of $\boldsymbol{n}$. It is found that the O(2) vortex will trap two Majorana localized modes. Then again only for eight copies of the $2d$ $p \pm ip$ TSC, the O(2) vortex will trap eight Majorana localized modes, which can be gapped out by interaction for the entire range of m, and then condensing the vortices not only restore the symmetry, but also gives us an adiabatic evolution from $m > 0$ to $m < 0$ without closing the bulk gap (one can also check that the O(2) vortex has a bosonic statistics, thus it is allowed to condense). Once again, we see that both the boundary and the bulk arguments lead to the same conclusion that the $2d$ $p \pm ip$ TSC is $\mathbb{Z}_8$ classified under interaction.

A3. SCSG phase in $2d$: boundary of $3d$ systems

It is soon discovered that the $3d$ ^{3}He-B TSC with a $\mathcal{T}^2 = -1$ time-reversal symmetry (DIII class) is $\mathbb{Z}_{16}$ classified. Close to the topological-trivial quantum phase transition, the bulk effective Hamiltonian in the free fermion limit is $H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_1\sigma^{11} + \mathrm{i}\partial_2\sigma^{13} + \mathrm{i}\partial_3\sigma^{30} + m\sigma^{20})\chi$ with the $\mathbb{Z}_2^T : \chi \to \mathcal{K}\mathrm{i}\sigma^{12}\chi$ symmetry, where $m > 0$ ($m < 0$) corresponds to the topological (trivial) phase. The $2d$ boundary of a $3d$ ^{3}He-B TSC hosts a gapless Majorana fermion surface mode, described by $H = \frac{1}{2}\int \mathrm{d}^2\boldsymbol{x}\,\chi^\intercal(\mathrm{i}\partial_1\sigma^1 + i\partial_2\sigma^3)\chi$, So if we start with 16 copies of the $3d$ ^{3}He-B TSC, the boundary will host 16 Majorana cones, which can then be driven to the SCSG phase by interaction. The boundary argument proposed in Ref. [17, 34] is actually very similar to the bulk argument in the previous subsection: we can first couple the 16 copies of the $3d$ ^{3}He-B TSC to an O(2) vector, and we manually break the O(2) symmetry by condensing the O(2) vector. Then when and only when there are 16 copies of ^{3}He-B TSC, can the vortex at the boundary be gapped and nondegenerate and have bosonic statistics under interaction. Then this means that one can condense the vortices at the $2d$ boundary to drive the boundary in to the SCSG phase when and only when the flavor number of the system is multiple of 16.

To backup the statement by the bulk argument, we may consider four copies of the $3d$ ^{3}He-B TSC near the $m = 0$ critical point while coupling to an O(3) real boson field $\boldsymbol{n} = (n_1, n_2, n_3)$, as described by $H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_1\sigma^{1100} + \mathrm{i}\partial_2\sigma^{1300} + \mathrm{i}\partial_3\sigma^{3000} + m\sigma^{2000} + n_1\sigma^{1210} + n_2\sigma^{1222} + n_3\sigma^{1230})\chi$. Again, following the defect proliferation argument, we can first order the O(3) field $\boldsymbol{n}$ to locally gap out the bulk criticality at the expense of breaking the $\mathbb{Z}_2^T$ symmetry, then we restore the symmetry by condensing O(3) monopoles of $\boldsymbol{n}$. It is found that each O(3) monopole will trap two Majorana localized modes. So only for 16 copies of the $3d$ ^{3}He-B TSC, the O(3) monopole will trap eight Majorana localized modes and can be therefore trivialized by interaction and safely condense. In fact, one may also consider disordering the O(3) field $\boldsymbol{n}$ by condensing other topological defects, such as vortex rings or domain walls. It turns out that[18] they all reach the same conclusion that only 16 copies of the $3d$ ^{3}He-B TSC can be trivialized by interaction.

One can see that the same pattern of arguments repeats in every dimension. The interaction reduced classification of fermionic TI/TSC states happens in all dimensions, and can be studied systematically by connecting to the bosonic symmetry protected topological states[19].

A4. Application: realizing SCSG phase on SPT boundary

The recent progress in the interaction reduced classification of TI/TSC has led to new proposals[26–28] of constructing SCSG phase based on the classification of SPT states (a generalization of topological insulators). It was pointed out in Ref. [26] that all gauge anomalies are classified by the SPT phases in one higher dimension, and the anomaly-free condition is equivalent to the condition that the SPT state in the bulk must belong to the trivial class, then its unprotected boundary can be driven to the SCSG phase without obstruction. Applying this idea, Ref. [27] argued that the $4d$ TI with SO(10) symmetry has a trivial classification (under interaction), such that the SCSG phase can be realized on its $3d$ boundary with a properly designed interaction. It was further emphasized in Ref. [28], after a thoroughly analysis of the previous attempts towards the SCSG phase, that the interaction must be *well-designed* to meet the "boundary fully gapping (BFG) rules" which exclude all the harmful interactions that could potentially lead to gapless bound-state formation in the mirror sector.

In Ref. [26], the classification of fermionic SPT states was based on the computation of super-cohomology of the symmetry group. The formalism of super-cohomology[46] is very intriguing and elegant, but whether it is the final complete classification or not is an open question. At least it is now known that the cohomology classification of the bosonic SPT states is not complete. Many states beyond cohomology have been found (the first example of such states was proposed in [47]). Thus it is worth trying to understand the classification of interacting fermionic SPT states from all different angles.[48–54]

Moreover in the super-cohomology formalism, there is a fermionic degree of freedom that is completely neutral (invariant) under all symmetry transformations. This would imply that, when we apply super-cohomology to SO(10) GUT, by combining the fermionic and bosonic degrees together, in this theory all representations of SO(10) can be either bosonic or fermionic. For instance, there will be fermionic SO(10) spinonr, fermionic SO(10) vector, fermionic SO(10) adjoint (bound state of the neutral fermion and bosonic excitations) ect. But in SO(10) GUT, SO(10) vectors are always bosons, and SO(10) spinors are always fermions. So the system constructed by super-cohomology approach seems rather different from the actual SO(10) GUT.

On the other hand, due to the technical difficulty of computing high dimension cohomology or super-cohomology of SO(10) group, it is still not mathematically proven whether the SO(10) symmetry protected fermionic state can be trivialized by interaction in $4d$. Therefore we must rely on other physical arguments. In the second part of Ref. [27], it was argued that SO(10) GUT can be driven into the SCSG phase. The argument was based on the fluctuating mass idea, which was first attempted in Ref. [25]. Note that the mass terms of SO(10) GUT form manifold S^9, and S^9 has trivial topological defects in any dimensions lower than 6, so there is no topological term at all for the S^9 target manifold. Thus hopefully we can disorder the mass terms, while keeping the fermions gapped. But in order to guarantee the fermions are gapped, the mass vector can only have very smooth and slow modulation in space-time. However if the mass vector only modulates slowly in the space-time, it is probably still ordered and the symmetry will be broken. For example, consider a 4d classical $\mathrm{O}(N)$ vector model on a 4d cubic (space-time) lattice. At infinitesimal temperature, the vector is already modulating slowly in the space-time, but it is still ordered. When the fluctuation is strong enough to disorder the vector, the vector already has very fast modulation in space-time, and in this case the fermions may have the danger of becoming gapless. In conclusion, the desired SCSG is an intermediate phase, where the mass vector must fluctuate fast enough to prevent ordering, but not too fast to close the fermion gap. The existence of such an intermediate phase still awaits further numerical simulation to confirm.

Motivated by the previous work [26, 27] as reviewed above, in this work we have proposed a different and independent argument for the "interaction trivialized $4d$ bulk topological insulator", which is based on the analysis of "condensation of topological defects" of the mass manifold. Unlike in Ref. [27] which tried to avoid topological defects, here we made use of the topological defects to help us achieve our goal. The success of the "topological defect condensation" argument in all lower dimensional examples has been reviewed in the previous part of this appendix. The advantage of this approach is that it does not assume the "slow modulation" of mass terms (order parameters), and condensing the topological defects will guarantee that the mass order parameter is disordered. As long as the topological defects are gapped out by interaction and non-degenerate, the dual theory for the topological defects already captures all the low energy degrees of freedom, and is a complete description of all the low energy physics.

Using the topological defect condensation argument, we can deduce that both the $\mathrm{SO}(7)\times\mathrm{SO}(3)$ and the $\mathrm{SO}(6)\times\mathrm{SO}(4)$ ($\sim$ Pati-Salam GUT) chiral fermions can be driven to the SCSG phase under interaction. Since both symmetry groups are subgroups of the SO(10) group, so our result also adds a piece of supportive evidence for Ref. [27], though from a very different approach. Moreover our approach has led to a concrete lattice model in the 4d bulk with explicit interacting terms, which can be tested by future numerics.

B. Decomposition and Reconstruction of the Interaction

In Ref. [10], Fidkowski and Kitaev proposed an SO(7) invariant interaction to fully gap out eight local Majorana zero modes. As quoted in Eq. (23), the interaction Hamiltonian contains 14 four-fermion terms. In this appendix, we will provide a Hubbard-Stratonovich decomposition of the Fidkowski-Kitaev (FK) interaction by rewriting the interaction as inner product of fermion bilinear operators. The decomposition potentially allows more efficient numerical simulation (for example the quantum Monte Carlo approach) of the FK interaction in terms of Yukawa-type interactions. The decomposition also allow us to reconstruct many other interactions that has lower symmetry than SO(7) but also gaps out eight local Majorana zero modes with the same unique ground state. These variant interactions provide us more choices to gap out the mirror sector fermions, and will be particularly useful for our purpose of regularizing the GUT on the lattice. The Yukawa-type interaction also naturally extends to higher dimensions which provides a general construction of the interaction that is needed to gap out the gapless fermions in any dimension.

B1. The $0d$ Case: Fidkowski-Kitaev Interaction and its Variants

Let us start form eight Majorana fermion operators χ_i ($i=1,\cdots,8$) defined by $\{\chi_i,\chi_j\}=2\delta_{ij}$, which can be pairwise combined into complex (regular) fermion operators as $f_i=(\chi_{2i-1}+\mathrm{i}\chi_{2i})/2$ for $i=1,\cdots,4$. They act on a

16-dimensional Hilbert space, which admits a set of Fock state basis $|n_1n_2n_3n_4\rangle$ labeled by the fermion occupation numbers $n_i = f_i^\dagger f_i = 0, 1$. The FK interaction is uniquely determined[20] by specifying a reference state $|e_1\rangle$ (the naming convention will be evident later) in the 16-dimensional Hilbert space, which is also the ground state to be stabilized by the interaction,

$$H_{\text{FK}} = -\sum_{i<j<k<l} V_{ijkl}\, \chi_i\chi_j\chi_k\chi_l, \text{ with } V_{ijkl} = \langle e_1|\chi_i\chi_j\chi_k\chi_l|e_1\rangle. \tag{24}$$

In this paper, we choose $|e_1\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$. The ground state $|e_1\rangle$ is chosen to be a symmetric state such that it will not have any fermion bilinear expectation value (not generating any fermion bilinear mass term which breaks the symmetry in general),

$$\langle e_1|\chi_i\chi_j|e_1\rangle = \delta_{ij} \quad \text{for } i, j = 1, \cdots, 8. \tag{25}$$

It can be explicitly verified that for $i < j < k < l$, there are 14 non-zero entries of the interaction vertex tensor V_{ijkl}, and all of them take the value of either $+1$ or -1, i.e. $V_{ijkl} = \pm 1$ if not vanishing. The corresponding 14 four-fermion terms are actually commuting projectors, which single out their common eigen state $|e_1\rangle$ as the ground state of H_{FK} with an energy -14. To see this, we note that $(\chi_i\chi_j\chi_k\chi_l)^2 = 1$ so the eigenvalues of $\chi_i\chi_j\chi_k\chi_l$ are ± 1, then $V_{ijkl} = \pm 1$ implies that $|e_1\rangle$ is the common eigenstate of every four-fermion term in H_{FK}. Moreover, because in general $\chi_i\chi_j\chi_k\chi_l$ and $\chi_{i'}\chi_{j'}\chi_{k'}\chi_{l'}$ must either commute or anticommute with each other (which follows from the Majorana fermion algebra), but since they have a common eigenstate $|e_1\rangle$ then they must commute, so the 14 four-fermion terms are commuting projectors. In the basis that all the projectors are simultaneously diagonalized (which is also an eigen basis of H_{FK}), they must be represented as direct products of four σ^0 or σ^3 matrices,[12] i.e. $V_{ijkl}\chi_i\chi_j\chi_k\chi_l = \pm\sigma^{abcd}$ (if not vanishing) where $a, b, c, d = 0$ or 3. Any 14 such matrices $\pm\sigma^{abcd}$ adding together can only produce at most one eigenstate with eigenvalue -14, so we know that $|e_1\rangle$ must be the unique ground state of H_{FK}. In conclusion, H_{FK} is a nicely designed interaction that can gap out eight Majorana zero modes with a non-degenerate ground state $|e_1\rangle$, on which all the fermion bilinear expectation values vanish.

In fact, the complete set of eigen basis of H_{FK} can be constructed from the ground state $|e_1\rangle$. Depending on the fermion parity $F = (-)^{\sum_{i=1}^4 n_i}$, they can be divided into even ($F = +1$) and odd ($F = -1$) parity states, denoted as $|e_i\rangle$ and $|o_i\rangle$ respectively.

$$|e_i\rangle = \chi_1\chi_i|e_1\rangle, \quad |o_i\rangle = \chi_i|e_1\rangle \quad \text{for } i = 1, \cdots, 8. \tag{26}$$

$|e_i\rangle$ and $|o_i\rangle$ form a set of orthonormal basis for the 16-dimensional Hilbert space, on which the FK interaction is diagonalized

$$H_{\text{FK}} = -14|e_1\rangle\langle e_1| + 2\sum_{i=2}^{8} |e_i\rangle\langle e_i|. \tag{27}$$

The orthogonality of the basis follows from $\langle e_i|e_j\rangle = \langle o_i|o_j\rangle = \langle e_1|\chi_i\chi_j|e_1\rangle = \delta_{ij}$ and $\langle e_i|o_j\rangle = 0$ (due to the different fermion parity). The spectrum of H_{FK} can be explicitly verified by acting Eq. (24) on these basis states. The SO(7) symmetry of the FK interaction[10] is reflected in its spectrum: the ground state $|e_1\rangle$ is a SO(7) scalar, the odd parity states $|o_i\rangle$ ($i = 1, \cdots, 8$) form a SO(7) spinor, and the excited even parity states $|e_a\rangle$ ($a = 2, \cdots, 8$) form a SO(7) vector.

To reveal the SO(7) symmetry explicitly, one may introduce the gamma matrices γ^a,

$$(\gamma^a)_{ij} = \mathrm{i}\langle e_1|\chi_i\chi_j|e_a\rangle = \mathrm{i}\langle e_1|\chi_i\chi_j\chi_1\chi_a|e_1\rangle \quad \text{for } i, j = 1, \cdots, 8 \text{ and } a = 2, \cdots, 8. \tag{28}$$

With our specific choice of $|e_1\rangle$, the explicit matrix form of γ^a reads

$$\gamma^{2,\cdots,8} = (\sigma^{002}, \sigma^{323}, \sigma^{021}, \sigma^{203}, \sigma^{231}, \sigma^{123}, \sigma^{211}), \tag{29}$$

which are also the γ^a matrices in Eq. (9), Eq. (11) and Eq. (13). The SO(7) generators are then given by $S^{ab} = \chi^\intercal s^{ab}\chi = \sum_{i,j=1}^8 \chi_i(s^{ab})_{ij}\chi_j$ where $s^{ab} = \frac{1}{2\mathrm{i}}[\gamma^a, \gamma^b]$ ($a, b = 2, \cdots, 8$). It can be checked that $[H_{\text{FK}}, S^{ab}] = 0$, so that the FK interaction has the SO(7) symmetry indeed. Using the γ^a matrices, the FK interaction can be decomposed as

$$H_{\text{FK}} = -\frac{1}{4!}\sum_{a=2}^{8}(\Phi^a\Phi^a - 16), \text{ with } \Phi^a = \chi^\intercal\gamma^a\chi. \tag{30}$$

To prove this, we expand Eq. (30) into $H_{\mathrm{FK}} = -\frac{1}{4!}\sum_{i,j,k,l}\sum_a(\gamma^a)_{ij}(\gamma^a)_{kl}\chi_i\chi_j\chi_k\chi_l + \text{const.}$, with some constant energy shift. It can be shown that $\sum_{a=2}^{8}(\gamma^a)_{ij}(\gamma^a)_{kl} = \sum_{a=2}^{8}\langle e_1|\chi_i\chi_j|e_a\rangle\langle e_a|\chi_k\chi_l|e_1\rangle = \langle e_1|\chi_i\chi_j\chi_k\chi_l|e_1\rangle = V_{ijkl}$ (for $i \neq j \neq k \neq l$), because $|e_a\rangle$ ($a = 2,\cdots,8$) form a complete set of basis for the two-fermion exited states, thus $\sum_{a=2}^{8}|e_a\rangle\langle e_a|$ is a resolution identity. So $H_{\mathrm{FK}} = -\frac{1}{4!}\sum_{i,j,k,l}V_{ijkl}\chi_i\chi_j\chi_k\chi_l + \text{const.} = -\sum_{i<j<k<l}V_{ijkl}\chi_i\chi_j\chi_k\chi_l$ matches up with Eq. (24) (and the constant energy shift can be fixed by considering the cases when ij and kl coincide). Therefore Eq. (30) is a Hubbard-Stratonovich decomposition of the FK interaction. Note that Φ^a ($a = 2,\cdots,8$) are fermion bilinear operators that rotates like an SO(7) vector, so Eq. (30) is manifestly SO(7) invariant. With this decomposition, we can rewrite the FK interaction in terms of a Yukawa model by introducing the O(7) real boson field ϕ_a such that $H_{\mathrm{FK}} = -\sum_a\phi_a\Phi^a + \frac{1}{2g}\sum_a\phi_a\phi_a$, which may allow more efficient numerical simulations by, for example, the quantum Monte Carlo method.

With the fermion bilinear operator Φ^a, we can reconstruct many other interactions that has a lower symmetry than SO(7), which turns out to be useful for our purpose of designing the appropriate interaction that has the same symmetry as the GUT that we try to regularize. To our knowledge, SO(7) does not appear as a gauge group in the mainstream GUT's, so it worth the effort to explore the variants of the FK interaction with other symmetries. For example, we can take the last six component of Φ^a ($a = 3,\cdots,8$), and construct a SO(6) invariant Yukawa interaction, $H_{\mathrm{int,SO(6)}} = -\frac{1}{4!}\sum_{a=3}^{8}(\Phi^a\Phi^a - 16)$, which is exactly the same interaction as in Eq. (19) up to some constant energy shift, where $\Phi^{3,\cdots,8} = \chi^{\intercal}\gamma^{3,\cdots,8}\chi = f^{\intercal}(\sigma^{32},-\mathrm{i}\sigma^{02},\sigma^{20},-\mathrm{i}\sigma^{23},\sigma^{12},-\mathrm{i}\sigma^{21})f + \text{H.c.}$ can be read out from Eq. (29) straightforwardly. These matrices are the same as the λ^a matrices defined in Eq. (19) up to some rearrangement. This SO(6) invariant interaction also gaps out eight Majorana zero modes with a non-degenerated ground state identical to $|e_1\rangle$. To see this, we start from the representation of the fermion bilinear operator Φ^a in the diagonal basis of H_{FK},

$$\Phi^a = (8\mathrm{i}|e_1\rangle\langle e_a| + \text{H.c.}) - 4\sum_{i,j=1}^{8}|o_i\rangle(\gamma^a)_{ij}\langle o_j|. \tag{31}$$

Then we take the last n components of Φ^a to construct an SO(n) invariant Yukawa interaction,

$$\begin{aligned} H_{\mathrm{int,SO}(n)} &= -\frac{1}{4!}\sum_{a=8-n+1}^{8}(\Phi^a\Phi^a - 16) \\ &= -2n|e_1\rangle\langle e_1| + \frac{2n}{3}\sum_{b=2}^{8-n}|e_b\rangle\langle e_b| + \frac{2(n-4)}{3}\sum_{a=8-n+1}^{8}|e_a\rangle\langle e_a|, \end{aligned} \tag{32}$$

whose energy spectrum is plotted in Fig. 3. As long as $n \geq 2$, the Majorana zero modes are fully gapped with unique ground state. From Eq. (32), we can see the ground state is always $|e_1\rangle$, identical to the ground state of the FK interaction, which will not generate any fermion bilinear expectation value. These conclusions definitely applies to the $n = 6$ SO(6) case, which is the interaction that we used to regularize the Pati-Salam GUT in this paper.

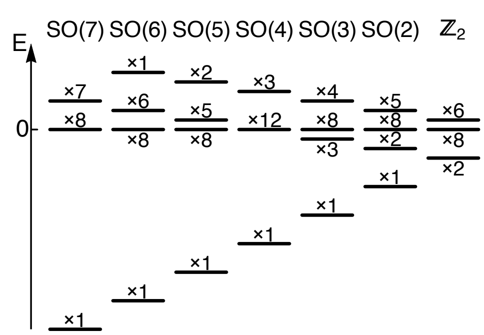

FIG. 3: Energy spectrum of SO(n) invariant Yukawa interaction ($n = 1$ case labeled by $\mathbb{Z}_2$), which is constructed by taking the last n components of Φ^a and coupling them to a n-component real boson field.

At the first glance, the interaction $H_{\mathrm{int}} \sim -\sum_a\Phi^a\Phi^a$ seems to favor the fermion bilinear ordering $\langle\Phi^a\rangle \neq 0$ at the mean field level, which would spontaneously break the symmetry (if applying the interaction to a lattice system), but actually the ordering does not happen. Because the $\langle\boldsymbol{\Phi}\rangle \simeq \boldsymbol{\phi}$ ordered state $|\boldsymbol{\phi}\rangle$ (given by the eigen equation

$(\boldsymbol{\phi}\cdot\boldsymbol{\Phi})|\boldsymbol{\phi}\rangle \simeq |\boldsymbol{\phi}\rangle$ in the even fermion parity sector, and labelled by the unit vector $\boldsymbol{\phi}$ of the ordering direction) has the wave function $|\boldsymbol{\phi}\rangle = (|e_1\rangle - \mathrm{i}\sum_{a=2}^{8}\phi^a|e_a\rangle)/\sqrt{2}$, which is a mixing between the ground state $|e_1\rangle$ and two-fermion excited states $|e_a\rangle$. Although the state $|\boldsymbol{\phi}\rangle$ indeed gains some interaction energy, but judging from the energy spectrum given by Eq. (32) and Fig. 3, $|e_1\rangle$ will gain even more energy than $|\boldsymbol{\phi}\rangle$ as long as $n \geq 2$, thus the ordering does not happen. Physically one may consider Φ^a as competing orders that can not make peace with each other, so they compromise and eventually end up in a quantum superposition state $\int \mathrm{d}\boldsymbol{\phi}|\boldsymbol{\phi}\rangle \simeq |e_1\rangle$ which does not break the symmetry.

In summary, having specified a (desired) symmetric ground state $|e_1\rangle$ in the Hilbert space of eight Majorana fermion modes, we can always find the γ^a matrices by Eq. (28) and use them to construct the Yukawa interaction $H_{\text{int}} \sim -\frac{g}{2}\sum_a(\chi^\intercal\gamma^a\chi)^2$, which, by construction, will single out $|e_1\rangle$ as its unique ground state. In this construction, the symmetry group and other details of the interaction term may vary from one to another, but the flavor number eight for the Majorana fermions (or four for the complex fermions) always stand out. If the flavor number is insufficient, the above construction will cease to work.

B2. Higher Dimensions: Generic Yukawa Interaction

In the above, we have discussed various interactions that can gap out the Majorana zero modes in the $0d$ system (such as on a single site or in a monopole core). The construction can be generalized to higher dimensions to design appropriate interactions that can remove gapless fermion modes. Following the defect proliferation argument elaborated in the main text and in Appendix A, to trivialize a d-dimensional gapless fermion system (if trivializable), we may first couple the fermions to a symmetry-breaking $\mathrm{O}(d)$ vector order parameter $\boldsymbol{n} = (n_1, \cdots, n_d)$, as $H = \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_\mu\alpha^\mu + n_\mu\beta^\mu)\chi$, where α^μ (β^μ) (for $\mu = 1, \cdots, d$) are anticommuting symmetric (antisymmetric) matrices. Then we can condense the order parameter $\boldsymbol{n}$ to gap out the fermions locally, and finally restore the symmetry by proliferating, say, the monopole defects of the $\boldsymbol{n}$ field, meanwhile the interaction must take effect to remove the fermion zero modes in the monopole core, such that the monopole can be safely proliferated. So the interaction in the d-dimensional system must be such designed that it will reduce to the appropriate interaction (as we discussed previously) in the monopole core which is capable of gapping out eight Majorana zero modes. This is our guiding principle to design the interactions in higher dimensions.

Of cause, one may also consider disordering the O(d) order parameter $\boldsymbol{n}$ by proliferating higher dimensional defects, such as $1d$ vortex lines or $2d$ domain wall membranes (if they can be constructed). But as demonstrated in Ref. [18], once the monopole proliferation argument goes through, all the higher dimensional defect proliferation argument will automatically follow. For example, if we try to proliferate the vortex lines, we must design the interaction to gap out the $1d$ gapless fermion modes that reside along the vortex line. Then the problem reduces to its $1d$ version, and we may evoke the defect proliferate argument again, by considering kink proliferation along the vortex line, which will then be exactly equivalent to the monopole proliferation argument. So in the following, we will only focus on the monopole proliferation argument.

Suppose the monopole configuration is given by $n_\mu \sim x_\mu$ around the monopole core (which has been set to the origin), then the fermion zero modes χ in the monopole core are determined as the common eigenstates of a set of eigen equations: $\mathrm{i}\beta^1\alpha^1\chi = \cdots = \mathrm{i}\beta^d\alpha^d\chi = \chi$. Now we define a matrix $M = \prod_{\mu=1}^{d}(\mathrm{i}\beta^\mu\alpha^\mu)$, which will act trivially on the monopole modes $M\chi = \chi$ by construction. So if we consider a fermion bilinear operator $\Phi^a = \chi^\intercal M \otimes \gamma^a\chi$ in the d-dimensional system, then in the monopole core it will reduce to $\Phi'^a = \chi^\intercal\gamma^a\chi$ (as M is effectively set to its eigenvalue $M = 1$), which is exactly the operator that we need to construct the Yukawa interaction in the monopole core. So the general construction is to start with an $\mathrm{SO}(n)$ invariant Yukawa interaction in the monopole core $H_{\text{int}} = -\frac{g'}{2}\sum_{a=1}^{n}(\chi^\intercal\gamma^a\chi)^2$, by reverting the above dimension reduction procedure, we know that the interaction in the d-dimensional system should be

$$H_{\text{int}} = -\frac{g}{2}\sum_{a=1}^{n}\Phi^a\Phi^a = -\frac{g}{2}\sum_{a=1}^{n}(\chi^\intercal M \otimes \gamma^a\chi)^2, \tag{33}$$

in order to induce the desired interaction in the monopole core. This is still an $\mathrm{SO}(n)$ invariant local interaction that can act on each site (or in each unit cell). It shares many similarities with the FK interaction. For example, its has a fully gapped spectrum with a non-degenerated ground state $|G\rangle$, whose leading component is a direct product state of $|e_1\rangle$, i.e. $|G\rangle \sim \otimes_{\alpha=1}^{m}|e_1\rangle_\alpha$ where m is the dimension of the matrix M, and all the fermion bilinear expectation values vanish on $|G\rangle$.

To see this, we need to make a few simplifications. Note that the matrix M is a symmetric matrix by definition, therefore it can always be diagonalized (by orthogonal transformation) to $\sigma^3 \otimes \mathbf{1}$ whose diagonal elements will be denoted as $\eta_\alpha = \pm 1$ ($\alpha = 1, \cdots, m$) with m being the dimension of M. Then the fermion bilinear operator $\Phi^a = \chi^\intercal M \otimes \gamma^a \chi$ can be decomposed as a sum of smaller fermion bilinear operators in the M diagonal basis, i.e. $\Phi^a = \sum_{\alpha=1}^m \eta_\alpha \Phi^a_\alpha$ with $\Phi^a_\alpha = \chi^\intercal_\alpha \gamma^a \chi_\alpha$. So the interaction in Eq. (33) can expanded as $H_{\text{int}} = -\frac{g}{2}\sum_a (\sum_{\alpha=1}^m \eta_\alpha \Phi^a_\alpha)^2 = -\frac{g}{2}(\sum_\alpha \sum_a (\Phi^a_\alpha)^2 + \sum_{\alpha\neq\beta}\eta_\alpha\eta_\beta \sum_a \Phi^a_\alpha \Phi^a_\beta)$. The first term $-\frac{g}{2}\sum_\alpha\sum_a(\Phi^a_\alpha)^2 = -\frac{g}{2}\sum_\alpha [\sum_a (\chi^\intercal \gamma^a \chi)^2]_\alpha$ is simply the sum of Yukawa interactions over the α sectors, which select out $|e_1\rangle_\alpha$ state as the ground state in each α sector, so its ground state will be the direct product of $|e_1\rangle$ states as $|G_0\rangle = \otimes_{\alpha=1}^m |e_1\rangle_\alpha$. Obviously all the fermion bilinear expectation values vanish on $|G_0\rangle$. The second term $-\frac{g}{2}\sum_{\alpha\neq\beta}\eta_\alpha\eta_\beta\sum_a \Phi^a_\alpha\Phi^a_\beta$ serves as an off-diagonal perturbation that mix the $|G_0\rangle$ state with $4k$-fermion excited states ($k = 1, 2, \cdots$). Nevertheless the true ground state $|G\rangle$ of H_{int} will still be dominated by $|G_0\rangle$, as verified by numerics. Also because only $4k$-fermion excited states are involved in the mixing, so all the fermion bilinear expectation values will still vanish on $|G\rangle$, which already implies that $|G\rangle$ is a trivial representation of the SO(n) symmetry. To prove that $|G\rangle$ is the unique ground state, we only need to show that the accidental degeneracy does not occur. To this purpose, we calculate (by exact diagonalization) the ground state energy E_0 (in the even fermion parity sector), the even fermion parity sector first excited state energy E_1, and the odd fermion parity sector lowest-energy state energy E_2 of the interaction Hamiltonian H_{int} in Eq. (33):

$$\begin{aligned} E_0 &= -32g\, m(n+m-1), \\ E_1 &= -32g[m(n+m-1)+n-1], \\ E_2 &= -32g\, m(n+m-2) + 8g(3n-4), \end{aligned} \tag{34}$$

which are indeed the three lowest energy levels of H_{int}. One can see as long as $n \geq 2$, E_1 and E_2 never come into degenerate with E_0. Therefore we have shown that H_{int} has a fully gapped spectrum with a non-degenerated ground state. This conclusion applies to all the interaction Hamiltonians that we constructed in the main text: Eq. (9), Eq. (11), Eq. (13), Eq. (19), Eq. (21), because they all can be written as Eq. (33) (with $n = 7$ or $n = 6$ and m varies).

C. Pati-Salam Model in Majorana Fermion Basis

In this appendix, we conclude the Pati-Salam model in the Majorana fermion basis explicitly, such that the relations among the various symmetry actions and order parameters are clearly exposed. We first introduce the following 128-component Majorana fermion field in the $4d$ bulk

$$\chi = \begin{bmatrix}1\\2\end{bmatrix} \otimes \underset{\text{chirality}}{\begin{bmatrix}L\\R\end{bmatrix}} \otimes \underset{\text{spin}}{\begin{bmatrix}\uparrow\\\downarrow\end{bmatrix}} \otimes \underset{\text{flavor}}{\begin{bmatrix}u\\d\end{bmatrix}} \otimes \underset{\text{color}}{\begin{bmatrix}r\\g\\b\\w\end{bmatrix}} \otimes \underset{\text{particle-hole}}{\begin{bmatrix}\operatorname{Re}\psi\\\operatorname{Im}\psi\end{bmatrix}}. \tag{35}$$

The layer index 1 or 2 labels the fermions that rotate under SU(2)$_1$ or SU(2)$_2$ respectively. In the color sector, r, g, b are the three colors of quarks, and w stands for the lepton. In the particle-hole sector, the complex fermion ψ is written in terms of two Majorana fermion components as $\psi = \operatorname{Re}\psi + \mathrm{i}\operatorname{Im}\psi$. The full effective Hamiltonian in the $4d$ bulk with the coupling to the O(4) and O(6) fields is given by $H = H_{\text{TI}} + H_{\text{O}(4)} + H_{\text{O}(6)}$ (as translated from Eq. (15), Eq. (16) and Eq. (21)),

$$\begin{aligned} H_{\text{TI}} &= \frac{1}{2}\int \mathrm{d}^4\boldsymbol{x}\,\chi^\intercal(\mathrm{i}\partial_1\sigma^{0310000} - \mathrm{i}\partial_2\sigma^{0320002} + \mathrm{i}\partial_3\sigma^{0330000} + \mathrm{i}\partial_4\sigma^{0100000} + m\sigma^{0200000})\chi, \\ H_{\text{O}(4)} &= \frac{1}{2}\int \mathrm{d}^4\boldsymbol{x}\,\chi^\intercal(-n_0\sigma^{1320001} + n_1\sigma^{1321003} - n_2\sigma^{2322001} + n_3\sigma^{1323003})\chi, \\ H_{\text{O}(6)} &= \frac{1}{2}\int \mathrm{d}^4\boldsymbol{x}\,\chi^\intercal(\phi_1\sigma^{0322123} + \phi_2\sigma^{0322203} + \phi_3\sigma^{0322323} + \phi_4\sigma^{3322211} + \phi_5\sigma^{3322021} + \phi_6\sigma^{3322231})\chi. \end{aligned} \tag{36}$$

Hereinafter $\sigma^{ijk\cdots} = \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \cdots$ denotes the tensor product of Pauli matrices. The bulk Hamiltonian H_{TI} has the SU(4) $\times$ SU(2)$_1$ $\times$ SU(2)$_2$ symmetry, given by

$$\begin{aligned} \text{SU}(4) &: \chi \to e^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\rho}}\chi, \\ \text{SU}(2)_1 &: \chi \to e^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\mu}_+}\chi, \\ \text{SU}(2)_2 &: \chi \to e^{\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\mu}_-}\chi, \end{aligned} \tag{37}$$

The 15 generators of SU(4) are represented as $\rho^{ij} = \sigma^{p000ijq}$ with $i,j = 0,1,2,3$ except for $ij = 00$, while $pq = 00$ or 32 is determined by ij to ensure that the generator is antisymmetric, *i.e.* $\rho^{ij\intercal} = -\rho^{ij}$. The generators of $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$ are represented as $\boldsymbol{\mu}_\pm = \frac{1}{2}(\sigma^0 \pm \sigma^3) \otimes (\sigma^{001002}, \sigma^{002000}, \sigma^{003002})$ respectively. The symmetry transformation of the fermion χ determines the symmetry transformation of the O(4) and O(6) fields. The O(4) vector $\vec{n}$ rotates under $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \simeq \mathrm{SO}(4)$, and the O(6) vector $\boldsymbol{\phi}$ rotates under $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$.

The $3d$ boundary of the $4d$ bulk can be considered as the domain wall of the mass term m flipping across $x_4 = 0$. The boundary fermion modes (domain wall fermions) are given by the eigen equation $\mathrm{i}\sigma^{0200000}\sigma^{0100000}\chi = \chi$, which essentially requires to fix $\sigma^{0300000} = 1$. Then the Hamiltonian H_{TI} will be reduced to $H_{3d} = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\chi^\intercal(\mathrm{i}\partial_1\sigma^{010000} - \mathrm{i}\partial_2\sigma^{020002} + \mathrm{i}\partial_3\sigma^{030000})\chi$, which describes the 16 chiral fermions on the $3d$ boundary.

The model Hamiltonian Eq. (36) and symmetry actions Eq. (37) can be reduced to the O(4) monopole core. Suppose monopole configuration is described by $(n_0, n_1, n_2, n_3) \propto (x_1, x_2, x_3, x_4)$ in the vicinity of its core, then the fermion modes localized in the monopole core are given by the following eigen equations (which can be derived from the Schrödinger equation[37]):

$$-\mathrm{i}\sigma^{1320001}\sigma^{0310000}\chi = -\mathrm{i}\sigma^{1321003}\sigma^{0320002}\chi = -\mathrm{i}\sigma^{2322001}\sigma^{0330000}\chi = \mathrm{i}\sigma^{1323003}\sigma^{0100000}\chi = \chi. \tag{38}$$

Eq. (38) can be diagonalized to $\sigma^{3000000}\chi = \sigma^{0300000}\chi = \sigma^{0030000}\chi = \sigma^{0003000}\chi = \chi$ under the following orthogonal transform

$$\chi \to e^{\frac{\mathrm{i}\pi}{4}\sigma^{2030001}} e^{\frac{\mathrm{i}\pi}{4}\sigma^{0012000}} e^{\frac{\mathrm{i}\pi}{4}\sigma^{3202002}} e^{-\frac{\mathrm{i}\pi}{4}\sigma^{0133002}}\chi, \tag{39}$$

which also transforms $\sigma^{0200000} \to -\sigma^{0333002}$, $\sigma^{0322ij3} \to \sigma^{0330ij3}$, $\sigma^{3322ij1} \to \sigma^{3330ij1}$. It is straightforward to see from the diagonalized eigen equations that there are eight solutions, which corresponds to the eight Majorana modes (or four complex fermion modes $f_{1,2,3,4}$ in the main text) localized in the O(4) monopole core. They can be arranged as

$$\chi = \underbrace{\begin{bmatrix} r \\ g \\ b \\ w \end{bmatrix}}_{\text{color}} \otimes \underbrace{\begin{bmatrix} \operatorname{Re} f \\ \operatorname{Im} f \end{bmatrix}}_{\text{particle-hole}}. \tag{40}$$

In the subspace of these localized Majorana modes, the model Hamiltonian is reduced to

$$\begin{aligned} H_{\mathrm{TI}}|_{\mathrm{O(4)\ monopole}} &= \chi^\intercal(-m\sigma^{002})\chi, \\ H_{\mathrm{O(6)}}|_{\mathrm{O(4)\ monopole}} &= \chi^\intercal(\phi_1\sigma^{123} + \phi_2\sigma^{203} + \phi_3\sigma^{323} + \phi_4\sigma^{211} + \phi_5\sigma^{021} + \phi_6\sigma^{231})\chi. \end{aligned} \tag{41}$$

The $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ symmetry is broken by the O(4) monopole. The remaining symmetry in the monopole core is the SU(4) symmetry, whose generators are reduced to $\rho^{ij} = \sigma^{ijk}$ with $i,j = 0,1,2,3$ expect for $ij = 00$, and $k = 0$ or 2 determined by ij to ensure that ρ^{ij} is an antisymmetric matrix. It is then obvious that the localized fermion modes form a fundamental representation of the SU(4) symmetry.

Judging from the reduced Hamiltonian, these localized fermion modes will become zero modes at $m = 0$ (where the bulk topological-trivial phase transition is suppose to occur). However, as discussed in the main text (see Fig. 2(c)) and in appendix B, one can construct an SO(6) invariant Yukawa interaction $H_{\mathrm{int}} = H_{\mathrm{O(6)}}|_{\mathrm{O(4)\ monopole}} + \frac{1}{2g}\boldsymbol{\phi}^2$ to gap out the localized fermion modes for all range of m (including $m = 0$). Since the O(6) Yukawa couplings in the monopole core is originated from the O(6) Yukawa couplings in the bulk Eq. (36), so the corresponding bulk interaction must be given by $H_{\mathrm{int}} = H_{\mathrm{O(6)}} + \frac{1}{2g}\int \mathrm{d}^4\boldsymbol{x}\,\boldsymbol{\phi}^2$. After integrating out the bosons field $\boldsymbol{\phi}$, one obtains exactly the interaction we proposed in Eq. (21) in the main text.

D. Classification of $4d$ Free Fermion Topological Insulators

The topological insulators/superconductors are classified as the fermionic symmetry protected topological (FSPT) states. In this appendix, we fit the various $4d$ topological insulators discussed in the main text into the "10-fold way" classification scheme of free FSPT states.[44, 45, 55–58] In particular, we will focus on the SU(4) and related symmetries, which is important for our discussion in the main text. In the non-interacting limit, the classifications are concluded in Tab. I. It worth mention that interaction may further reduce some of the classifications in the table, as demonstrated in the main text and reviewed in appendix A. The toy model we discussed corresponds to the $\mathrm{U}(1) \times \mathbb{Z}_2$ FSPT state, while the Pati-Salam model corresponds to the $\mathrm{SU}(4) \times \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ FSPT state.

TABLE I: Free fermion classification of some FSPT states in $4d$

Class	Symmetry	Extension Problem	Classifying Space	Classification
A	U(1)	$\mathcal{C}\ell_4 \to \mathcal{C}\ell_5$	$C_4 \cong C_0$	$\mathbb{Z}$
	$\mathrm{U}(1)\times\mathbb{Z}_2$			$\mathbb{Z}\times\mathbb{Z}$
C	SU(2)	$\mathcal{C}\ell_{8,0} \to \mathcal{C}\ell_{8,1}$	$R_{-6} \cong R_2$	$\mathbb{Z}_2$
	$\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$			$\mathbb{Z}_2\times\mathbb{Z}_2$
AII	SU(4)	$\mathcal{C}\ell_{10,0} \to \mathcal{C}\ell_{10,1}$	$R_{-8} \cong R_0$	$\mathbb{Z}$
AI	$\mathrm{SU}(4)\times\mathrm{SU}(2)$	$\mathcal{C}\ell_{9,3} \to \mathcal{C}\ell_{9,4}$	$R_{-4} \cong R_4$	$\mathbb{Z}$
	$\mathrm{SU}(4)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$			$\mathbb{Z}\times\mathbb{Z}$

Let us start from the U(1) FSPT states, which belongs to the symmetry class A. With the U(1) symmetry, the fermion Hamiltonian can be written in the complex basis as $H = c^\dagger(\sum_{i=1}^4 \mathrm{i}\partial_i\Gamma^i + mM)c$, where Γ^i $(i = 1, \cdots, 4)$ and M are anti-commuting matrices. Adding the mass matrix M corresponds to the extension problem $\mathcal{C}\ell_4 \to \mathcal{C}\ell_5$, whose classifying space is $C_4 \cong C_0$, so the free FSPT classification is given by $\pi_0(C_0) \cong \mathbb{Z}$. The $4d$ U(1) FSPT state is also known as the $4d$ quantum Hall (QH) state. We can stack two $4d$ QH states of opposite chiralities together to make a non-chiral $4d$ topolotical insulator (TI), provided an additional $\mathbb{Z}_2$ symmetry which acts as the fermion parity only on one of the chirality. The $\mathbb{Z}_2$ symmetry simply splits the single-particle Hilbert space to two subspaces (according to the ±1 eigen values of the symmetry operator), and in each subspace the problem is reduced to the U(1) FSPT with $\mathbb{Z}$ classification, so putting together, the $\mathrm{U}(1)\times\mathbb{Z}_2$ free FSPT states are $\mathbb{Z}\times\mathbb{Z}$ classified in general. The two $\mathbb{Z}$'s stand for the classification of the non-chiral $4d$ TI and that of the chiral $4d$ QH respectively. The $\mathrm{U}(1)\times\mathbb{Z}_2$ toy model we considered in the main text fits into the non-chiral $\mathbb{Z}$ classification and is hence free from the perturbative anomaly.

Now we turn to the SU(2) FSPT states, which belong to the symmetry class C. In the Majorana basis, the fermion Hamiltonian reads $H = \chi^\intercal(\sum_{i=1}^4 \mathrm{i}\partial_i\Gamma^i + mM)\chi$ where Γ^i $(i = 1, \cdots, 4)$ and M are anti-commuting matrices. For Majorana Hamiltonian, Γ^i and M must also satisfy the symmetry properties that $\Gamma^{i\intercal} = \Gamma^i$ is symmetric and $M^\intercal = -M$ is anti-symmetric. Denote the SU(2) generators as μ^a $(a = 1, 2, 3)$, which (in the Majorana basis) are three anti-commuting and antisymmetric $(\mu^{a\intercal} = -\mu^a)$ matrices. To respect the SU(2) symmetry, the Hamiltonian (the Γ^i and M matrices) must commute with these three generators. All these algebraic relations can be realized in a single Clifford algebra by embedding the matrices in a larger space with auxiliary Pauli matrices as

$$\begin{aligned}
&\sigma^1\otimes\Gamma^i = \alpha^i, && (i = 1, \cdots, 4)\\
&\sigma^2\otimes\mu^a = \alpha^{4+a}, && (a = 1, 2, 3)\\
&\sigma^3\otimes 1 = \alpha^8, &&\\
&\sigma^1\otimes M = \beta^1. &&
\end{aligned} \tag{42}$$

sThen by requiring the symmetric matrices $\alpha^{p\intercal} = \alpha^p$ $(p = 1, \cdots, 8)$ and the antisymmetric matrices $\beta^{1\intercal} = -\beta^1$ to anti-commute with each other, all the algebraic properties of Γ^i, M and μ^a are realized. So adding the mass matrix M corresponds to the extension problem of $\mathcal{C}\ell_{8,0} \to \mathcal{C}\ell_{8,1}$, whose classifying space is $R_{-6} \cong R_2$, therefore the $4d$ SU(2) free FSPT classification is given by $\pi_0(R_2) \cong \mathbb{Z}_2$. If the SU(2) symmetry is enlarged to $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$, the classification will be doubled to $\mathbb{Z}_2\times\mathbb{Z}_2$ correspondingly.

Similar classification approach can be applied to the SU(4) (and SU(4)-related) FSPT states. However, unlike the SU(2) group whose generators are automatically anti-commuting, the 15 generators of the SU(4) group do not always anti-commute with each other. One need to find out the minimal anti-commuting subset among the 15 generators. It is found that $\mathcal{C}\ell_{0,5} \cong \mathbb{C}(4)$ is (one of) the minimal Clifford algebra in which the $\mathfrak{su}(4)$ Lie algebra can be embedded. Denote the generators of $\mathcal{C}\ell_{0,5}$ as λ^a $(a = 1, \cdots, 5)$, which are anti-commuting and antisymmetric $(\lambda^{a\intercal} = -\lambda^a)$ matrices, *e.g* a specific choice may be $\boldsymbol{\lambda} = (\sigma^{102}, \sigma^{200}, \sigma^{312}, \sigma^{320}, \sigma^{332})$. The 15 SU(4) group generators can then be obtained either as λ^a or as $\mathrm{i}\lambda^a\lambda^b$. To respect the SU(4) symmetry, it is sufficient to require the Hamiltonian (the Γ^i and M matrices) to commute with λ^a. All these algebraic relations can be realized in a single Clifford algebra by embedding the matrices in a larger space with auxiliary Pauli matrices as

$$\begin{aligned} &\sigma^1 \otimes \Gamma^i = \alpha^i, && (i = 1, \cdots, 4) \\ &\sigma^2 \otimes \lambda^a = \alpha^{4+a}, && (a = 1, \cdots, 5) \\ &\sigma^3 \otimes 1 = \alpha^{10}, && \\ &\sigma^1 \otimes M = \beta^1. && \end{aligned} \tag{43}$$

Then by requiring the symmetric matrices $\alpha^{p\intercal} = \alpha^p$ $(p = 1, \cdots, 10)$ and the antisymmetric matrices $\beta^{1\intercal} = -\beta^1$ to anti-commute with each other, all the algebraic properties of Γ^i, M and λ^a are realized. So adding the mass matrix M corresponds to the extension problem of $\mathcal{C}\ell_{10,0} \to \mathcal{C}\ell_{10,1}$, whose classifying space is $R_{-8} \cong R_0$ (which belongs to the symmetry class AII), therefore the $4d$ SU(4) free FSPT classification is given by $\pi_0(R_0) \cong \mathbb{Z}$.

The SU(2) symmetry can be added to the the SU(4) FSPT states, and the $\mathbb{Z}$ classification will not change (but the symmetry class does change from AII to AI). With the SU(4) $\times$ SU(2) symmetry, the Clifford algebra embedding scheme can be

$$\begin{aligned} &\sigma^1 \otimes \Gamma^i = \alpha^i, && (i = 1, \cdots, 4) \\ &\sigma^2 \otimes \lambda^a = \alpha^{4+a}, && (a = 1, \cdots, 5) \\ &\sigma^3 \otimes \mu^b = \beta^b, && (b = 1, 2, 3) \\ &\sigma^1 \otimes M = \beta^4, && \end{aligned} \tag{44}$$

where the symmetric matrices $\alpha^{p\intercal} = \alpha^p$ $(p = 1, \cdots, 9)$ and the antisymmetric matrices $\beta^{q\intercal} = -\beta^q$ $(q = 1, \cdots, 4)$ are anti-commuting matrices. So adding the mass matrix M corresponds to the extension problem of $\mathcal{C}\ell_{9,3} \to \mathcal{C}\ell_{9,4}$, whose classifying space is $R_{-4} \cong R_4$ (which belongs to the symmetry class AI), therefore the $4d$ SU(4) $\times$ SU(2) free FSPT classification is given by $\pi_0(R_4) \cong \mathbb{Z}$. If the symmetry is enlarged to SU(4) $\times$ SU(2)$_1$ $\times$ SU(2)$_2$, the classification will be doubled to $\mathbb{Z} \times \mathbb{Z}$. Again, the two $\mathbb{Z}$'s stand for the classification of the non-chiral $4d$ TI and that of the chiral $4d$ QH respectively. The Pati-Salam model fits into the non-chiral $\mathbb{Z}$ classification and is hence free from the perturbative anomaly.

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