Hidden-symmetry decoupling of Majorana bound states in topological superconductors

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Hidden Symmetry Decoupling of Majorana Fermions

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Multiple zero-energy Majorana fermions (MFs) with spatially overlapping wave functions can survive only if their splitting is prevented by an underlying symmetry. Here we show that, in quasi-one-dimensional (Q1D) time reversal invariant topological superconductors (class DIII), a realistic model for superconducting Lithium molybdenum purple bronze ($Li_{0.9}Mo_6O_{17}$) and certain families of organic superconductors, multiple Majorana-Kramers pairs with strongly overlapping wave functions persist at zero energy even in the absence of an easily identifiable symmetry. We find that similar results hold in the case of Q1D semiconductor-superconductor heterostructures (class D) with $t_\perp \ll t$, where $t_\perp$ and t are the transverse and longitudinal hoppings, respectively. Our results, explained in terms of special properties of the Hamiltonian and wave functions, underscore the importance of hidden accidental symmetries in topological superconductors.

Introduction: Topological superconductors are characterized by a bulk superconducting gap and topologically protected gapless edge states[1]. Due to the presence of intrinsic superconducting particle-hole symmetry (PHS) the gapless zero modes constitute Majorana fermions (MFs), characterized by the second quantized operator relation $\gamma^\dagger = \gamma$. In the context of condensed matter, aside from being fascinating non-elementary particles, MFs obey Ising type non-Abelian braiding statistics which is useful in implementing a fault-tolerant topological quantum computer[2]. These emergent excitations are said to be topologically protected, in the sense that their existence and properties are insensitive to many perturbations so long as the system remains gapped. While MFs have not yet been conclusively found in nature, they have been theoretically shown to exist in low dimensional spinless p-wave superconducting systems[3,4] as well as other systems which are similar to them[5–11]. In particular the semiconductor heterostructure scheme has motivated tremendous experimental efforts with a number of recent works claiming to have observed experimental signatures consistent with MFs[12–17], for a review see Ref. [18].

Recent work[19–21] has established that the quadratic Hamiltonians for gapped topological insulators and superconductors can be classified into ten topological symmetry classes each of which is characterized by a topological invariant. The symmetry classification is important as it provides an understanding of the effects of various perturbations on the stability of the protected surface modes such as MFs. For example, recent work[22–31] has proposed time-reversal (TR) invariant topological superconductivity (class DIII) with a $\mathbb{Z}_2$ invariant in a number of systems with intrinsic or proximity induced superconductivity in heterostructures while at the same time an topological classification involving spatial crystalline symmetries has been developed[32,33]. Spin triplet, equal spin pairing (ESP), p-wave superconductivity, which realizes such a TR-invariant topological superconductor[30], is thought to be present in the quasi-one-dimensional (Q1D) transition metal oxide Lithium molybdenum purple bronze $Li_{0.9} Mo_6 O_{17}$ (LiMO) and some organic superconductors[34–39]. These systems posses a distinctly anisotropic electrical conductivity, i.e. the hopping integrals along the crystallographic directions vary as, $t_x \gg t_y \gg t_z$, making them Q1D conductors. Because of its electronic anisotropy, LiMO may be modeled as an array of parallel one dimensional systems coupled by weak transverse hopping (with similar physics possibly being realized in cold fermion systems[40]). As discussed below these systems provide a natural platform to study interaction effects between MFs.

In Q1D multi-chain systems multiple Majorana fermions with spatially overlapping wave functions can remain at zero energy only if their splitting is forbidden by an underlying symmetry. In this work we show that, in Q1D TR-invariant topological superconductors, multiple Majorana-Kramers pairs (MKPs) with strongly overlapping wave functions persist at zero energy even in the absence of an identifiable physical symmetry. We find similar results also for Q1D semiconductor-superconductor heterostructures with spin-orbit (SO) coupling and Zeeman field (class D with $\mathbb{Z}_2$ invariant) with $t_\perp \ll t$, pointing to the existence of a hidden symmetry decoupling of the MFs. To demonstrate this result we start with a strictly 1D (single chain) TR-invariant Kitaev model superconductor (Eq. (1)), modeling the ESP spin-triplet p-wave state proposed to be realized in LiMO. We first note that, in addition to TR-invariance, the model has a chiral as well as a mirror symmetry both of which allow an integer ($\mathbb{Z}$) invariant. In the physically realistic Q1D generalization of this model (with $t_\perp \ll t_x$) the $\mathbb{Z}$ invariant takes arbitrary integer values, allowing multiple MKPs localized at the same end despite wave-function overlap. We show that, even in the absence of such symmetries, multiple MFs can still be protected by symmetries such as spatial reflection. In realistic materials, however, reflection symmetry is expected to be broken by disorder. Remarkably, we find that disorder induced breakdown of reflection symmetry fails to lift the degeneracy of the zero energy modes even with strong wave function overlap. We find very similar results also for Q1D systems in class D. These results, which we explain in terms of special properties of the Hamiltonians and wave functions, underscore the importance of hidden symmetry decoupling of MFs in topological superconductors.

Hamiltonian and equivalent description by chiral and mirror symmetries: We model a one dimensional spin-triplet

topological superconductor by a lattice Hamiltonian which includes nearest neighbor hopping, on-site chemical potential and a general p-wave superconducting order parameter which reads,

$$H^{1D} = \sum_{i,\sigma}[-t(c^{\dagger}_{i+1\sigma}c_{i,\sigma} + H.c.) - \mu c^{\dagger}_{i\sigma}c_{i\sigma}]$$
$$+ \sum_{i,\sigma,\sigma'}[\Delta_{\sigma\sigma'}(c^{\dagger}_{i+1\sigma}c^{\dagger}_{i\sigma'} + H.c.)]. \quad (1)$$

Here $t = t_x$ is the hopping integral between nearest neighbor sites, $i \in [1, N_x]$ is the lattice index and $\sigma = \uparrow, \downarrow$ represents the spin index. Next, we Fourier transform Eq. (1) to study the 1D bulk topological properties. The spin symmetry of Cooper pairing may be classified by the total spin as either singlet ($S = 0$) or triplet ($S = 1$). A general pair potential is expressed compactly in terms of a d-vector as $\Delta_{\alpha\beta}(\mathbf{k}) = [\Delta_s(\mathbf{k}) + \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}](i\sigma_y)_{\alpha\beta}$ with a symmetric singlet component $\Delta_s(\mathbf{k}) = \Delta_s(-\mathbf{k})$, and an antisymmetric triplet d-vector $\mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k})$. For most of this work we consider a pure triplet order parameter with $\Delta_s(\mathbf{k}) = 0$, touching briefly on similar physics involving a singlet term (see Fig. 3).

We now write the bulk Hamiltonian corresponding to Eq. (1) as a momentum space Bogoliubov-de Gennes (BdG) Hamiltonian, $H^{1D} = 1/2 \sum_k \Psi^{\dagger}_k \mathcal{H}_k \Psi_k$. With the Nambu basis $\Psi_k = (c_{k\uparrow}, c_{k\downarrow}, c^{\dagger}_{-k\downarrow}, -c^{\dagger}_{-k\uparrow})^T$ which absorbs the factor $i\sigma_y$ associated with the d-vector, the matrix $\mathcal{H}_k$ then takes the form

$$\mathcal{H}^{1D}_k = (\epsilon(k) - \mu)\sigma_0\tau_z + \mathbf{d}(k) \cdot \boldsymbol{\sigma}\tau_x. \quad (2)$$

Here $k = k_x$ is the 1D crystal-momentum, $\epsilon(k) = -2t\cos(k)$ is the single particle kinetic energy, $\mathbf{d}(k) = \hat{d}|\mathbf{d}(k)| = (d_x, d_y, d_z)\Delta\sin(k)$ is the p-wave order parameter and σ_i, τ_i indicate spin $1/2$ Pauli matrices in the spin and the particle-hole spaces respectively.

Superconducting Hamiltonians observe an intrinsic particle-hole symmetry (PHS) which emerges from the structure of the BdG equations. The BdG Hamiltonian in Eq. (2) satisfies $\Xi\mathcal{H}_k\Xi^{-1} = -\mathcal{H}_{-k}$, where, in this basis the anti-unitary PHS operator reads $\Xi = \sigma_y\tau_y\mathcal{K}$ where $\mathcal{K}$ is the anti-unitary complex conjugation operator. Ξ anti-commutes with the real space representation of $\mathcal{H}^{1D}_k$ and obeys $\Xi^2 = 1$. PHS which relates quasiparticle excitations at $\pm E$ through $\Gamma^{\dagger}_E = \Gamma_{-E}$ is fundamentally important for the formation of Majorana modes which are a special case satisfying $E = 0$. Additionally, $\mathcal{H}^{1D}_k$ obeys the time-reversal symmetry (TRS) relation $\Theta\mathcal{H}_k\Theta^{-1} = \mathcal{H}_{-k}$ with the TR operator $\Theta = \sigma_y\tau_0\mathcal{K}$. The presence of PHS and TRS leads to a unitary chiral symmetry which is simply the product $\Pi = \Xi \cdot \Theta = \sigma_0\tau_y$. When $|\mu| < 2t$ the system is in the topologically non-trivial phase, characterized by a DIII class $\mathbb{Z}_2$ invariant which takes a value $\nu = -1$. This invariant may be viewed as a Kramers polarization and, in the most general case, reduces to Kitaev's Pfaffian invariant for one spin block in the presence of spin rotation symmetry[41]. When $\nu = -1$ unpaired MFs at each end of the wire form topologically protected MKP's. This
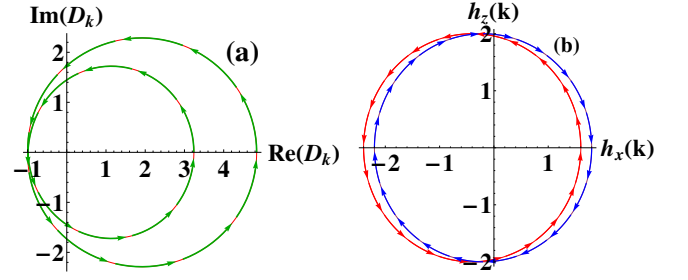

FIG. 1. (Color online) (a) Winding curve for the chiral topological invariant $W = 2$ indicating two topologically protected MFs at each end of a single chain as described by Eq. (2). (b) Mirror topological invariant $\gamma_M = 2$ indicating two MFs at each end of a single chain in Eq. (2). Here we have set the hopping parameter t to unity and used $\mu = .2t, |\Delta| = t$.

explains the robustness of the four zero energy modes in the presence of TR-invariant perturbations.

Additionally, the BdG Hamiltonian Eq. (2) belongs to the topological class BDI, due to a co-existing chiral symmetry given by $\mathcal{S}^{BDI} = \mathcal{O} \cdot \Xi = (\hat{d} \cdot \boldsymbol{\sigma})\tau_y$, which is the product of a TR-like operator $\mathcal{O} = (\hat{d} \cdot \hat{y} + i(\hat{d} \times \hat{y}) \cdot \boldsymbol{\sigma})\mathcal{K}$ with $\mathcal{O}^2 = 1$, and the particle-hole operator Ξ. Note that while the explicit form of this operator is dependent on the orientation of the d-vector, its existence is guaranteed given a superconducting term of the form given in Eq. 2. In $d = 1$ BDI Hamiltonians are classified by a bulk $\mathbb{Z}$ topological winding number invariant W. To calculate the invariant we off-diagonalize the Hamiltonian from Eq. (2) in the basis which diagonalizes $\mathcal{S}^{BDI}$. Writing the determinant of the off-diagonal part in a complex polar form, $D_k = |Det(D_k)|e^{i\theta(k)}$, W is given by[42,43] the number of times $\theta(k)$ winds about the origin as k varies through the 1D Brillouin zone. As can be seen from Fig. [1], panel (a), the invariant takes the value $W = 2$ in the topological phase of Eq. (2) while $W = 0$ in the trivial phase. This chiral symmetry explains the persistence of the zero modes to TR-breaking terms including stray Zeeman fields ($\mathcal{H}^Z = \mathbf{V} \cdot \boldsymbol{\sigma}\tau_0$) perpendicular to the d-vector, while a Zeeman field $\mathbf{V} \parallel \mathbf{d}_k$ breaks the chiral symmetry (since a parallel Zeeman term does not anti-commute with the chiral operator). In the latter case there is no hidden symmetry and the spatially localized Majorana end modes hybridize into conventional gapped modes. A generic SO coupling term aligned in an arbitrary direction in spin space is written $\mathcal{H}^{SO} = \alpha_R\sin(k)\mathbf{a} \cdot \boldsymbol{\sigma}\tau_z$ meaning that $\mathbf{a} \parallel \mathbf{d}_k$ preserves chiral symmetry while a SO term in the plane perpendicular to the d-vector does not respect chiral symmetry.

Recently, mirror symmetry has also been a proposed as a topological protection mechanism for MFs[44]. The one-dimensional Hamiltonian Eq. (2) is invariant ($[\mathcal{M}, \mathcal{H}_k] = 0$) under the mirror symmetry operator $\mathcal{M} = i\hat{d} \cdot \boldsymbol{\sigma}\tau_0$. Because $\mathcal{M}$ and $\mathcal{H}_k$ commute, the Hamiltonian may be expressed in a block diagonal form (we choose to rotate the Hamiltonian by a unitary matrix whose rows are the eigenvectors of the mirror operator $\mathcal{M}$) where each block corresponds to a mirror eigenspace subsector. Each block is writ-

ten $\boldsymbol{h}^{\pm} \cdot \boldsymbol{\sigma}$ where the $\pm$ is the mirror eigenspace index. Explicitly choosing $\hat{d}$ along $\hat{x}$, without loss of generality, we find $\boldsymbol{h}^{\pm} = (\pm\Delta\sin(k), 0, \epsilon_k - \mu)$ such that the mirror winding number invariant in each subsector $C^{\pm}$ is defined in the $(\sigma_x - \sigma_z)$ plane. Each mirror winding curve encloses the origin once, but with opposite helicity, leading to a mirror invariant $\gamma_M = C^+ - C^- = 2$, which is illustrated in Fig. 1.

Multiple chains and multiple Majoranas: A realistic quasi-1D spin triplet superconductor such as LiMO (or quasi-1D TRI systems in cold fermions) may first be modeled as an array of 1D chains coupled by a weak hopping amplitude $t_y \ll t_x$. One may further consider a truly 3D system by stacking 2D arrays and coupling them through a third hopping integral $t_z \ll t_y \ll t_x$. We consider a system which consists of N_y parallel chains, indexed by $l \in [1, N_y]$, coupled only by transverse hopping t_y. The quasi-1D Hamiltonian is a generalization of Eq. (2) given by

$$H^{Q1D} = \sum_{kll'} \Psi^{\dagger}_{kl}(\mathcal{H}^{1D}_k \delta_{l,l'} + \mathcal{H}^{\perp}_{l,l'})\Psi_{kl'}, \qquad (3)$$

where we have used the basis $\Psi_{kl} = (c_{kl\uparrow}, c_{kl\downarrow}, c^{\dagger}_{-kl\downarrow}, -c^{\dagger}_{-kl\uparrow})^T$, and $\mathcal{H}^{\perp}_{l,l'} = -t_{\perp}\sigma_0\tau_z(\delta_{l,l'+1} + \delta_{l,l'-1})$.

We proceed by first examining a double chain setup with $l = 1, 2$ as an illustrative example (the generalization is straighforward). For a two-chain system the Hamiltonian is expressed as 2×2 matrix where every entry is itself a 4×4 matrix (see Eq. 2). This reads,

$$H^{Q1D} = \sum_k (\Psi^{\dagger}_{k,1}, \Psi^{\dagger}_{k,2}) \begin{pmatrix} \mathcal{H}^{1D}_k & -t_{\perp}\sigma_0\tau_z \\ -t_{\perp}\sigma_0\tau_z & \mathcal{H}^{1D}_k \end{pmatrix} \begin{pmatrix} \Psi_{k,1} \\ \Psi_{k,2} \end{pmatrix} \qquad (4)$$

Introducing a new Pauli matrix (ρ) in the double chain Hilbert space allows us to write Eq. (4) compactly as $\mathcal{H}^{1D}_k \rho_0 + t_{\perp}\sigma_0\tau_z\rho_x$. Using this, we may generalize the chiral operator to the double chain space as $\mathcal{S}^{BDI} = \sigma_x\tau_y\rho_0$. We are now able to calculate a generalized multi-chain winding number W counting the number of MFs at each edge which are now localized across both chains. Just as in the single chain case, the magnitude of the chiral invariant $|W|$ is equal to the number of topologically protected MFs present at each end. This is illustrated by the phase diagram presented in Fig. 2 which shows that for small transverse hopping $2 \times N_y = 4$ Majorana modes are present. In general, as long as one can define a chiral and/or mirror invariant, and the transverse hopping is small enough, the number of MFs at each end grows with the size of the sample ($|W| = 2 \times N_y$).

Broken chiral symmetry and the role of reflection symmetry: Let us now investigate the fate of multiple Majoranas in the event of broken chiral and mirror symmetries, which may occur due to intra-chain spin-orbit coupling perpendicular to the d-vector. This modifies the Hamiltonian in Eq. (2) to,

$$\mathcal{H}^{1D}_k \mapsto \mathcal{H}^{1D}_k + \alpha_R \sin(k)\sigma_y\tau_z \qquad (5)$$

altering Eq. (4) accordingly. In the two-chain problem, the two distinct sets of MKPs may interact, each hybridizing to finite energies. We note however, that the Hamiltonian (Eq.
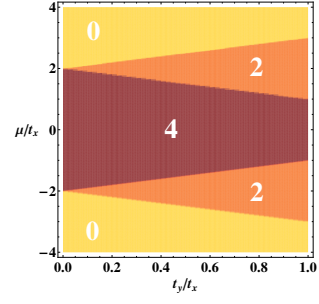

FIG. 2. (Color online) Topological phase diagram for a two channel configuration consisting of two TR-invariant Kitaev chains coupled by weak transverse hopping $t_y \ll t_x$ (with t_x set to unity). For small t_y a broad range of μ accommodates the topological phase indexed by the topological invariant $|W| = 4$. As t_y increases and the splitting between the two sets bands increases topological phases with single Majorana Kramers pairs emerge as seen by the $|W| = 2$ phase.

4) commutes with the spatial reflection operator $\mathcal{R} = \sigma_0\tau_0\rho_x$ which interchanges the chain index, i.e. $\hat{c}_{k_x,1(2)} \mapsto \hat{c}_{k_x,2(1)}$. Writing Eq. (4) in the eigenbasis of $\mathcal{R}$ results in a block-diagonalized form which reads $\mathcal{H}^{1D}_k \rho_0 + t_{\perp}\sigma_0\tau_z\rho_z$. In this form it is clear that transverse hopping modifies the effective chemical potential in two independent bands. Notice also that $[\Theta, \mathcal{R}] = [\Xi, \mathcal{R}] = 0$, so that every diagonal block in the eigenbasis of $\mathcal{R}$ is particle-hole and time-reversal invariant. Because of this invariance each independent, non-interacting block constitutes a DIII topological superconductor hosting a zero energy MKP at each end. The extension of this argument to decouple N_y chains is straightforward. A generalized ρ_x is a totally symmetric $N_y \times N_y$ dimensional matrix given by $\rho_x = (\delta_{l,l'+1} + \delta_{l,l'-1})$ with $l, l' \in (1, 2, ..., N_y)$, that is, the superdiagonal and subdiagonal elements connect nearest neighbor sites are $+1$ and all other matrix elements are zero. The eigenvalues of ρ_x come in pairs of equal magnitude and opposite sign $(\pm\lambda_1, \pm\lambda_2, \pm\lambda_3, ...)$ for N_y even and $(0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3, ...)$ when N_y is odd. In this case the rotated N_y chain Hamiltonian involves a generalized ρ_z. Because ρ_z and ρ_x have the same eigenvalue spectrum, the block diagonal Hamiltonian consists of non-interacting sectors where the chemical potential in sector is modified by $\pm\lambda_i$.

Persistence of multiple Majoranas in the absence of reflection symmetry:

In any realistic system however, reflection symmetry is approximate since some disorder will always be present. The addition of $\delta\mu_i$, with a random magnitude within a normal distribution, to the tight-binding Hamiltonian in Eq.1 (additionally including the Rashba spin-orbit coupling term introduced in the last section, which takes the system from topological class BDI to DIII) effectively models local disorder. As illustrated in Fig. 3, by solving the BdG equations on a double chain system, we find that the presence of on-site disorder minimally affects the bulk spectrum, while the zero-energy modes are insensitive to this perturbation. Note that the Majoranas persist even in the absence of chiral, mirror, and reflection symmetries, each of which are now explicitly broken (note that we
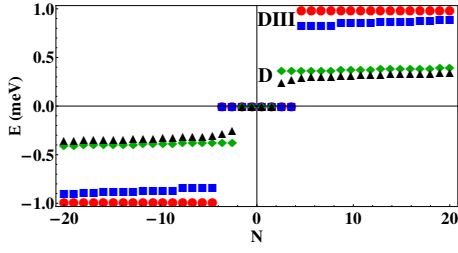
FIG. 3. (Color online) Low energy BdG quasiparticle spectrum for TR-symmetric Kitaev system (class DIII superconductor) for $N_x = 100, N_y = 2$ (red circles) in the absence of chiral and mirror symmetries. The eight MFs (four on each end) are presumably protected from splitting by spatial reflection. Blue squares show same number of protected zero modes in the presence of local chemical potential disorder which breaks spatial reflection. Green diamonds and black triangles show two MFs at each nanowire end for a class D system (again we have $N_y = 2$) with and without spatial reflection respectively. Numerical parameters used in this figure are $t = 1, t_y = 0.1t, \mu = 0.2t, \Delta_s = \Delta_p = t, \alpha_R = 0.2t, V_z = 1.5t$.

take the Zeeman coupling to vanish here and hence the system is in the TR invariant class DIII). We achieve similar results for $N_y > 2$, with the number of Majorana fermions scaling directly with the number of chains existing in the transverse direction. Likewise a TR-breaking class D system based on the semiconducting Majorana wire Hamiltonian given by

$$H^{\text{SMW}} = \sum_{i,\sigma} [-t(c_{i+1\sigma}^\dagger c_{i,\sigma} + H.c.) - \mu c_{i\sigma}^\dagger c_{i\sigma}] \quad (6)$$

$$+ \sum_i [\Delta_s c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + H.c.] + \sum_{i,\sigma,\sigma'} (\boldsymbol{V} \cdot \vec{\sigma})_{\sigma,\sigma'} c_{i\alpha}^\dagger c_{i\sigma'}$$

$$+ \sum_{i,\sigma,\sigma'} [\alpha_R(i\sigma_y)_{\sigma,\sigma'} c_{i\sigma}^\dagger c_{i+1\sigma'} + H.c.]$$

which contains s-wave paring Δ_s, Rashba spin orbit coupling of strength α_R, and a Zeeman splitting $\boldsymbol{V}$ yield similar results for a $N_y = 2$ dimensional chain.

In order to understand the response of the MFs to reflection breaking perturbations we consider first the two-chain Hamiltonian describing the chemical potential imbalance written as,

$$H_k^{Q1D} = \begin{pmatrix} \mathcal{H}_k^{1D} + \delta & -t_\perp \\ -t_\perp & \mathcal{H}_k^{1D} - \delta \end{pmatrix} \quad (7)$$

where $t_\perp$ is understood to be $t_\perp \sigma_0 \tau_z$, $\delta = \delta\mu\sigma_0\tau_z$, and we continue to work in the $(\Psi_{k,1}^\dagger, \Psi_{k,2}^\dagger)$ basis. Note that we still consider $k = k_x$ to be a good quantum number and break reflection symmetry only by introducing a chemical potential imbalance among the chains. Can we systematically block-diagonalize this Hamiltonian with a unitary eigenvalue-conserving transformation that commutes with time reversal and particle-hole symmetries? If yes, MFs will persist in each block due to a 'hidden symmetry' associated with this transformation. Since $\mathcal{H}_k^{1D}$ appears with an identity matrix in chain space, this problem amounts to finding a matrix which diagonalizes the remaining terms leaving $\mathcal{H}_k^{1D}$ invariant.

We search for a hidden unitary transformation in a systematic way by first considering the eigen-decomposed form of the non-diagonal terms in Eq. (7), which we call $A = \delta\rho_z - t_\perp\rho_x$. This is expressed as $A = Q\Lambda Q^{-1}$, where Q is a matrix whose columns are the eigenvectors of A, $\boldsymbol{v}_\pm = 1/(\sqrt{2}N_\pm)\left(-(\delta \pm \sqrt{t_\perp^2 + \delta^2})/t_\perp, 1\right)^T$ and $N_\pm = \sqrt{1 + \frac{\delta^2}{t^2} \pm \frac{\delta\sqrt{t^2+\delta^2}}{t^2}}$ is the normalization constant. Also remember that each entry in Q involves an identity in spin and particle-hole spaces. The unitarity of Q is a direct consequence of the Hermiticity of A. Note that in the limit $\delta \to 0$, this reduces to the eigenbasis of $\mathcal{R}$ which was the reflection transformation operator used in the presence of $\mathcal{R}$. Rotating the full 8×8 Hamiltonian in Eq. (7) by the Q operator we see $Q^{-1}H_k^{Q1D}Q = [-(\epsilon_k - \mu)\sigma_0\tau_x + \alpha_k^R\sigma_y\tau_z + \Delta\sigma_x\tau_x]\rho_0 - \sqrt{t_\perp^2 + \delta\mu^2}\sigma_0\tau_z\rho_z$. The transformed Hamiltonian consists of two non-interacting topological DIII sectors, each block respecting both particle-hole and time reversal symmetries, subject to a modified chemical potential of magnitude $\sqrt{t_\perp^2 + \delta^2}$ and a sign change for the single particle kinetic energy. The commuting hidden symmetry operator associated with the Q transformation is $R' = \left(1 + \frac{\delta\mu^2}{t_\perp^2}\right)^{-1/2}\left(\rho_x - \frac{\delta\mu}{t}\rho_z\right)$. This explains why the multiple MKPs with spatially overlapping wave functions persist even with broken reflection symmetry, as shown in Fig. 3 (given k_x is a good quantum number).

What is the fate of the topological phase that hosts the Majorana modes in the presence of a perturbation H_p that breaks the reflection symmetry and also the translation symmetry along the chains? Our strategy is to identify the generic structure of the Majorana wave functions ϕ_ν based on the symmetries of the system, then calculate the matrix elements $\langle\phi_\nu|H_p|\phi_{\nu'}\rangle$. If all matrix elements are zero, the multiple Majoranas are preserved; otherwise, the perturbation splits the Majorana modes and the systems becomes topologically trivial. Note that virtual transitions to finite energy states do not affect the energy of the zero-modes. This can be seen by writing the Green's function $G = (\omega - H^{Q1D} - H_p)^{-1}$ projected onto the Majorana subspace as $G_{\nu\nu'} = [\omega\delta_{\nu\nu'} - \Sigma_{\nu\nu'}(\omega)]^{-1}$, in terms of the self-energy

$$\Sigma_{\nu\nu'}(\omega) = \sum_n \frac{1}{E_n}\langle\phi_\nu|H_p|\Psi_n\rangle\langle\Psi_n|H_p|\phi_{\nu'}\rangle, \quad (8)$$

where Ψ_n is an eigenstate of energy E_n. Since $|E_n| \geq \Delta_{qp}$, where Δ_{qp} is the quasiparticle gap, $\Sigma_{\nu\nu'} = 0$ in the limit $\Delta_{qp} \to \infty$. Furthermore, any system with a finite quasiparticle gap can be adiabatically connected to the superconductor with infinite Δ_{qp} without crossing a topological phase transition; hence, the two systems belong to the same topological class and have the same number of Majorana modes, i.e. $\Sigma_{\nu\nu'} = 0$ for both. Explicit numerical calculations confirm this result.

To identify the generic form of the Majorana wave function, we make the key observation that H^{Q1D} commutes with the unitary operator $U = \sigma_y\tau_z\rho_0$. The Majorana spinor $\phi_\nu = (u_{\nu\uparrow}, u_{\nu\downarrow}, v_{\nu\downarrow}, -v_{\nu\uparrow})^T$, which satisfies the constraint $v_{\nu\sigma} = u_{\nu\sigma}^* e^{i\varphi}$ due to particle-hole symmetry, has to be an eigenstate of U. Consequently, $u_{\nu\downarrow} = i\lambda u_{m\lambda\uparrow}$, where

$\nu = (m, \lambda)$, $\lambda = \pm 1$ and m takes N_y values to label the multiple Majoranas localized near each end of the system. Finally, time-reversal symmetry requires $u_{m\lambda\uparrow} = u_{m\lambda}(l, i)$ to be a real function of position and chain index satisfying the condition $u_{m+} = u_{m-}$, in addition to the phase condition $\varphi|_{\lambda=+1} = -\varphi|_{\lambda=-1}$. We conclude that the Majorana wave functions have the generic form $\phi_{m\lambda}(l, i) = u_{m\lambda}(l, i)[1, \lambda i, -\lambda i e^{\lambda i \varphi}, -e^{\lambda i \varphi}]^T$. We note that the phase φ takes the value $\varphi = \pi$ in a system with chiral symmetry, i.e. in the absence of spin-orbit coupling ($\alpha^R = 0$). Using the above wavefucntion, it is straightforward to show that all matrix elements $\langle \phi_{m\lambda} | H_p | \phi_{m'\lambda'} \rangle$ of a spin-independent disorder potential $H_p = V_{dis}(l, i) \sigma_0 \tau_z$ vanish; hence, such a perturbation does not destroy the Majoranas as long as the quasiparticle gap is nonzero. This explains the numerical BdG results and persistence of the end MFs presented in Fig. 3.

Conclusion: Multiple Majorana fermions with spatially overlapping wave functions are expected to split and acquire non-zero energies, unless such splitting breaks an underlying symmetry. By working with a realistic model for TR-invariant topological superconductors (class DIII) appropriate for LiMO, we study such interaction effects with multiple Majoranas and show that they can remain protected in topological superconductors even in the absence of an identifiable physical symmetry. We find similar results for Q1D semiconductor-superconductor heterostructures with spin-orbit coupling and Zeeman field (class D). Our results, which we explain in terms of properties of the Hamiltonians and wave functions, underscore the importance of hidden symmetry (which may be broken by additional couplings such spin-depended disorder potential and electron-electron interactions) decoupling of MFs in topological superconductors.

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