Localization length index and subleading corrections in a Chalker-Coddington model: A numerical study

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We calculated numerically the localization length index ν and up to two sub-leading finite size indices in the Chalker-Coddington (CC) network model of the plateau-plateau transitions in the quantum Hall effect. We also carried out fits with logarithmic corrections. The confidence intervals of the four fits for the exponent ν are narrow and overlap. The fit based on one relevant field and one marginal field is slightly more advantageous in comparison to the fits based on a relevant and irrelevant fields. The calculations were carried out by two different programs that produced close results, each one within the error bars of the other.

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I. INTRODUCTION

The computation of critical indices of the plateau-plateau transitions in the quantum Hall effect (QHE) (see for a review[1]) is still an open problem in modern condensed matter physics. According to the pioneering works on localization[2] the dimension two is a marginal dimension, above which delocalization can appear. Exactly at d=2 Levine, Libby and Pruisken[3–5] noticed, that the presence of a topological term in the nonlinear sigma model (NLSM) formulation of the problem may result in the appearance of delocalized states in strong magnetic fields.

The next achievement was reached by Chalker and Coddington[6]. The authors formulated and studied numerically a network model (*CC* model) in a random potential yielding localization-delocalization transitions. The numerical value 2.5 ± 0.5 of the Lyapunov exponent (LE) in the CC model was in good agreement with the experimentally measured localization length index $\nu = 2.4$ in the quantum Hall effect[7].

Various aspects of the CC-model were investigated in a chain of interesting papers: In[8] the model was linked to replicated spin-chains, while in[9–11] its connection to supersymmetric spin-chains was revealed. In[12] an integrable extension of the CC-model was presented. Some links with conformal field theories (CFT) of Wess-Zumino-Witten-Novikov (WZWN) type were presented in[13,14] and[15] where the authors tried to find an appropriate CFT with operator content, which can fit a localization length index around 2.3.

In Refs.[16,17] the authors investigated the multifractal behaviour of the CC model. Both papers reported quartic deviations from the exact quadratic dependence of the multi-fractal indices on the parameter q, which was predicted in Refs.[14,15]. This fact points out that the validity of the simple, supersymmetric WZWN approach to plateau-plateau transitions in the quantum Hall effect is questionable and here we are still far from the application of conformal field theory. Moreover, recently more precise numerical calculations of the localization length index of the CC-model[16,18–20] show values close to 2.61 ± 0.014, which is well far from the most recent experimental value 2.38 ± 0.06[21,22].

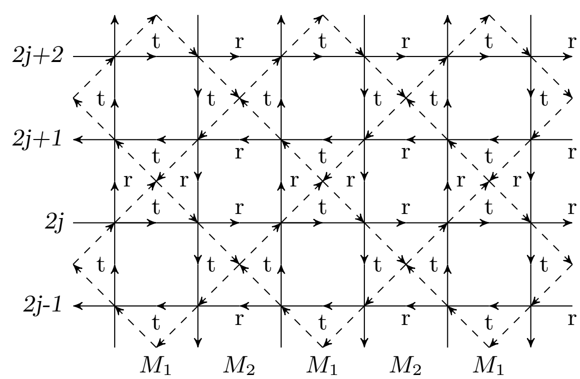

Figure 1. Schematic illustration of the CC network. M_1 and M_2 denote the column transfer matrices as defined in (1) and (2). Multiplication with a column transfer matrix describes the transition of a particle through the corresponding column of the lattice.

Up to now all numerical analyses of finite size scaling in the CC-model[16,18–20] show that the second, irrelevant operator in the model has a scaling dimension very close to the marginal one making the finite size analysis very difficult. Moreover, in [19] it was claimed that the next to leading order finite size resp. width M corrections have $1/\log(M)$-form. In the current publication we provide the theoretical ground and a generalization for such corrections on the basis of a fixed point perturbed by a relevant field and a marginal one. We also explore the possibility of the existence of multiplicative logarithmic corrections arising in the situation of two irrelevant operators having degenerate conformal dimensions.

In order to understand how to modify the CC-model in order to obtain a correlation length index close to the experimental value we first need to understand the continuum limit (CFT) of the model properly. This is the main motivation of our current investigation. The understanding of the leading and the sub-leading operator content and their dimensions will essentially help to identify the model's continuum limit and its possible generalization.

The goal of the current paper is threefold: First we recalculate the localization length index in order to test the results obtained in[16,18–20]. To this end we developed two independent codes to numerically investigate the finite size scaling of the CC-model. Second we explore the possibility of a second irrelevant field in the scaling analysis. Third we address the question of logarithmic corrections to the scaling function.

The paper is organized as follows: In the second chapter we explain the model and describe the principle of our numerical calculations based on LU instead of QR decompositions of matrices. The third chapter presents the details of the data analysis and a study of renormalization group flows resulting in logarithmic corrections. In the fourth chapter we present our results and in the fifth our conclusions.

II. MODEL DESCRIPTION

For the calculation of critical indices we used the transfer-matrix method developed in[23,24]. To calculate the smallest Lyapunov exponent (LE) of the CC-model it is necessary to calculate a product $T_L = \prod_{j=1}^{L} M_1 U_{1j} M_2 U_{2j}$ of layers of transfer matrices $M_1 U_{1j} M_2 U_{2j}$ corresponding to two columns M_1 and M_2 of vertical sequences of 2x2 scattering nodes,

$$M_1 = \begin{pmatrix} B^1 & 0 & \cdots & 0 \\ 0 & B^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B^1 \end{pmatrix} \quad (1)$$

and

$$M_2 = \begin{pmatrix} B^2_{22} & 0 & \cdots & 0 & B^2_{21} \\ 0 & B^2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & B^2 & 0 \\ B^2_{12} & 0 & \cdots & 0 & B^2_{11} \end{pmatrix} \quad (2)$$

with

$$B^1 = \begin{pmatrix} 1/t & r/t \\ r/t & 1/t \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} 1/r & t/r \\ t/r & 1/r \end{pmatrix} \quad (3)$$

where periodic boundary conditions are imposed on M_2. The U-matrices have a simple diagonal form: $[U_{1,2}]_{nm} = \exp(i\alpha_n)\,\delta_{nm}$. Here t and r are the transmission and reflection amplitudes at each node of the regular lattice shown in Fig. 1 which are suitably parameterized by

$$t = \frac{1}{\sqrt{1+e^{2x}}} \quad \text{and} \quad r = \frac{1}{\sqrt{1+e^{-2x}}}. \quad (4)$$

The model parameter x corresponds to the Fermi energy measured from the Landau band center scaled by the Landau band width (so the critical point is $x = 0$). The phases α_n are stochastic variables in the range $[0, 2\pi)$, reflecting the randomness of the smooth electrostatic potential landscape.

We calculated the product of a chain of transfer matrices which contain random parameters. According to the Oceledec theorem[25] the $1/L$ power of the product with a set of random phases used in the transfer matrices has a well-defined set of eigenvalues in the large L limit. The logarithms of the moduli of these eigenvalues are called Lyapunov exponents.

$$\gamma = \lim_{L\to\infty} \frac{\log[T_L T_L^\dagger]}{2L}\ , \quad (5)$$

The smallest positive one of these exponents yields the critical behaviour of the correlation length of the model, i.e. $\xi \sim x^{-\nu}$ where ν is the localization length index.

It is clear, that numerically the infinite limit cannot be calculated. For large systems, the central limit theorem[26] tells us that the Lyapunov exponents have a Gaussian distribution with variance $\sigma_\gamma \sim \sqrt{M/L}$ where L and M are the width and the height of the geometry, i.e. T_L is given by a product of $2L$ many factors of the type M_1 and M_2 which are $2M \times 2M$ matrices.

This means, that by considering a chain of length L we calculate the LE with error $\sim \sqrt{M/L}$. Moreover, if we consider an ensemble of N chains, the variance becomes $\sim \sqrt{M/(LN)}$. Therefore our strategy will be to consider large ensembles of long chains.

We used ensembles of products with length L ranging from 1 000 000 to 5 000 000. The details about our data base can be found in table I.

Calculating these matrix products the naive way is not possible as many entries of the product very soon exceed the size of all available data types. One can overcome this problem by use of the method presented in[23,24], namely, the product can be performed with repeated QR decompositions with unitary Q and upper right triangular matrix R. The rightmost T is QR decomposed. The unitary Q is then multiplied with the next T and the product is decomposed again. Repeating this procedure many times we are in principle left with some Q multiplied with a product of upper right triangular matrices. It appears, that the product of the diagonal entries of the upper triangular matrices are approaching the eigenvalues of the total transfer matrix T_L. Of these numbers we are only interested in those which are close to 1. For details see for instance[27].

M	L	ensemble size	standard deviation	program
20	1 000 000	900	0.0042	Fortran
20	5 000 000	308	0.0019	C++
40	1 000 000	1000	0.0059	Fortran
40	5 000 000	350	0.0027	C++
60	1 000 000	1010	0.0072	Fortran
60	5 000 000	280	0.0033	C++
80	1 000 000	1010	0.0084	Fortran
80	5 000 000	380	0.0038	C++
100	1 000 000	1020	0.0095	Fortran
100	5 000 000	350	0.0043	C++
120	1 000 000	850	0.0105	Fortran
120	5 000 000	300	0.0046	C++
140	1 000 000	1260	0.011	Fortran
140	5 000 000	310	0.0045	C++
160	1 000 000	285	0.012	Fortran
160	5 000 000	220	0.0055	C++
180	1 000 000	240	0.013	Fortran
180	5 000 000	208	0.0059	C++

Table I. This table shows the statistics of the data. For each M we have calculated the Lyapunov exponent with the 13 x-values that divide the interval $[0, 0.08]$ into 12 equal parts. The column 'standard deviation' gives the typical error averaged over all values of x.

In our numerical calculations we found that it is also possible to apply the LU decomposition using a lower triangular matrix L with unit diagonal, a permutation matrix P and an upper triangular matrix U such that A=PL U. Details about the decomposition can be found in[28]. We can now use the LU decomposition analogously to the QR decomposition with PL taking over the role of Q and U the role of R. The calculations with LU decomposition appeared to be faster than those with QR decomposition by a factor of the order of 2. According to the manual of the Intel MKL, which we are using, the number of real number multiplications for the QR and LU decompositions of an (m, n) type complex matrix is $8/3 \cdot n^2(3m - n)$ and $4/3 \cdot n^2(3m - n)$ $(m \geq n)$, respectively. The results of these calculations are equivalent as the final LU form of T_L can be brought into QR form by a single QR decomposition of the lower triangular matrix with practically vanishing contributions to the upper triangular matrix.

Alternatively, we also generated large ensembles of Lyapunov exponents for a given pair of x and M using both the QR and the LU decomposition. For these ensembles we generated histograms. Both histograms are very well described by normal distributions as confirmed by Gaussian fits. The distance of the centres of the Gaussian peaks is by a couple of magnitudes smaller than their widths which in turn agree with the same precision.

III. THE FITTING PROCEDURE

A. One irrelevant field

From the scaling behaviour of the Lyapunov exponent γ near the critical point we expect for finite size systems

$$\gamma \cdot M = \Gamma(M^{1/\nu} u_0, f(M)\, u_1), \tag{6}$$

where $f(M)$ is decreasing with M. Here M is the number of nodes in each column of the lattice, $u_0 = u_0(x)$ is a relevant field and $u_1 = u_1(x)$ the leading irrelevant field. It is common to choose $f(M) = M^y$, $y < 0$. Further it is known, that the relevant field vanishes at the critical point.

On the left hand side of (6) we use the numerical results of (5) for various combinations of the parametrization parameter x and the lattice height M. The right hand side is expanded in a series in x and M and the expansion coefficients are obtained from a fit. Some coefficients in this expansion need not to be taken into account following the symmetry arguments of[18]: If x is replaced by $-x$ we see from (4) that t turns into r and vice versa. Due to the periodic boundary conditions the lattice is unchanged. Therefore the left hand side of (6) is invariant under a sign flip of x. Hence the right hand side must be even in x. That makes $u_0(x)$ and $u_1(x)$ even or odd. For the Chalker Coddington network the critical point is at $x = 0$. This lets us choose $u_0(x)$ odd and $u_1(x)$ even. The fit now should use as few coefficients as possible while reproducing the data as well as possible.

The general idea of expanding the right side of (6) is to expand Γ in the fields u_0 and u_1 yielding

$$\begin{aligned}\Gamma(u_0(x)M^{1/\nu}, u_1(x)M^y) &= \Gamma_{00} + \Gamma_{01}u_1M^y \\ &+ \Gamma_{20}u_0^2M^{2/\nu} + \Gamma_{02}u_1^2M^{2y} \\ &+ \Gamma_{21}u_0^2u_1M^{2/\nu}M^y + \Gamma_{03}u_1^3M^{3y} \\ &+ \Gamma_{40}u_0^4M^{4/\nu} + \Gamma_{22}u_0^2M^{2/\nu}u_1^2M^{2y} + \Gamma_{04}u_1^4M^{4y} + \ldots\end{aligned} \tag{7}$$

and then using expansions of u_0 and u_1 in x as it has been done in[18,19]

$$u_0(x) = x + \sum_{k=1}^{\infty} a_{2k+1}x^{2k+1} \quad \text{and} \quad u_1(x) = 1 + \sum_{k=1}^{\infty} b_{2k}x^{2k}. \tag{8}$$

In (7) all coefficients in the expansion of Γ that would contradict the scaling function being even in x have been dropped. Because of ambiguity in the overall scaling of the fields, the leading coefficient in (8) can be chosen to be 1.

Of course the described expansion is unique. It involves, however, an infinite number of coefficients even when only keeping a finite order in x

$$\begin{aligned}\Gamma = \ &\Gamma_{00} + \sum_{k=1}^{\infty} \Gamma_{0k}\, M^{ky} \\ &+ x^2 \left[\sum_{k=1}^{\infty} k b_2 \Gamma_{0k}\, M^{ky} + M^{2/\nu} \sum_{k=0}^{\infty} \Gamma_{2k} M^{ky} \right] \\ &+ x^4 \left[\sum_{k=1}^{\infty} \left\{ b_4 k + b_2^2 k(k-1)/2 \right\} \Gamma_{0k} M^{ky} \right. \\ &\quad + M^{2/\nu} \sum_{k=1}^{\infty} b_2 k \Gamma_{2k} M^{ky} \\ &\quad \left. + M^{4/\nu} \sum_{k=0}^{\infty} \Gamma_{4k} M^{ky} + M^{2/\nu} \sum_{k=0} a_3 k \Gamma_{2k} M^{ky} \right] \\ &+ O(x^6)\end{aligned} \tag{9}$$

When taking into account a finite number of expansion coefficients Γ_{lk} and a_n, b_m, different fitting procedures can be devised. We choose a finite order of the expansion of Γ in the fields u_0, u_1 (7) and a finite order of the expansion of the fields in terms of x (8).

B. Two irrelevant fields

We also considered the case of two irrelevant fields. This, in analogy to (6), gives

$$\gamma M = \Gamma(M^{1/\nu} u_0, M^{y_1}\, u_1, M^{y_2}\, u_2), \quad y_1, y_2 < 0 \tag{10}$$

With the same reasoning as in the case of one irrelevant field we find that Γ is even in x. Along the lines of the above case we obtain that u_0 is odd and u_1 and u_2 are even in x. Of course Γ is even in x, too. This helps to identify expansion coefficients that are zero like in the case of one irrelevant field. If we expand Γ in the fields u_0, u_1 and u_2 we obtain the following fitting formula:

$$\begin{aligned}\Gamma(u_0(x)M^{1/\nu}, &u_1(x)M^{y_1}, u_2(x)M^{y_2}) = \\ &\Gamma_{000} \\ &+ \Gamma_{010} u_1 M^{y_1} + \Gamma_{001} u_2 M^{y_2} \\ &+ \Gamma_{200} u_0^2 M^{2/\nu} + \Gamma_{020} u_1^2 M^{2y_1} + \Gamma_{002} u_2^2 M^{2y_2} \\ &+ \Gamma_{011} u_1 M^{y_1} u_2 M^{y_2} \\ &+ \Gamma_{030} u_1^3 M^{3y_1} + \Gamma_{003} u_2^3 M^{3y_2} \\ &+ \Gamma_{210} u_0^2 M^{2/\nu} u_1 M^{y_1} + \Gamma_{201} u_0^2 M^{2/\nu} u_2 M^{y_2} \\ &+ \Gamma_{021} u_1^2 M^{2y_1} u_2 M^{y_2} + \Gamma_{012} u_1 M^{y_1} u_2^2 M^{2y_2} \\ &+ \ldots\end{aligned} \tag{11}$$

with the odd relevant field

$$u_0(x) = x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \tag{12}$$

and the even irrelevant fields

$$u_i(x) = 1 + \sum_{k=1}^{\infty} b_{i,2k} x^{2k}, \qquad i = 1, 2 \tag{13}$$

C. Logarithmic corrections

In this section we explore, somewhat heuristically, the consequences of irrelevant fields with special values of the exponent y, either approaching the marginal value 0 or getting close to the value of the exponent of another irrelevant field.

Generally, fit procedures are based on renormalization group (RG) arguments relating the physical properties of a system with couplings $u_0, u_1, \ldots$ and geometrical size M to another such system with couplings $\tilde{u}_0, \tilde{u}_1, \ldots$ and size $\tilde{M}$. Explicit RG relations are usually derived by integrating out degrees of freedom (coarse graining). Let us first summarize the main ideas for the case of a single coupling u. For a relevant (irrelevant) operator with linear β function and RG eigenvalue $y > 0$ (< 0) the relation between the coupling u for size M and $\tilde{u}$ for size $\tilde{M}$ is described by the flow equation $\mathrm{d}u/\mathrm{d}(\log M) = -yu$ with solution $u(M)M^y = u(\tilde{M})\tilde{M}^y$.

In this situation it is convenient to fix an arbitrary size M_0 as the size of a reference system and to translate the coupling u $(= u(M))$ at size M to the coupling $\tilde{u}$ $(= u(M_0))$ at size $\tilde{M} = M_0$. By the very definition of a mass gap γ (inverse correlation length) we have $\gamma M = \tilde{\gamma}\tilde{M}$. The quantity $\gamma\, M$ on the other hand is a function F of u and M: $\gamma\, M = F(u, M)$. But this is independent under the RG flow $\gamma\, M = \tilde{\gamma}\, \tilde{M} = F(\tilde{u}, \tilde{M})$. Now we use $\tilde{u} = u(M_0) = u(M)M^y/M_0^y$ and find $\gamma\, M = F(u(M)M^y/M_0^y, M_0)$. Next we introduce a short-hand for $F(z/M_0^y, M_0) =: \Gamma(z)$ and obtain with $u(M) = u$ the familiar $\gamma\, M = \Gamma(uM^y)$. In case of several fields this reads $\gamma\, M = \Gamma(u_0 M^{y_0}, u_1 M^{y_1}, \ldots)$. The positive value y_0 of the relevant field is related to the critical exponent ν by $y_0 = 1/\nu$.

a) Let us now treat the case of a relevant field ($y_0 = 1/\nu =: y$) and a marginal field ($y_1 = 0$) by closely following[29]

The flow equation for u_1 is described by a β-function that has to be treated in second order. The flow equation reads $\mathrm{d}u_1/\mathrm{d}(\log M) = b u_1^2$ with solution $1/u_1(M) - 1/u_1(M_0) = -b \log(M/M_0)$. Following[29], we may assume and treat for the β-function of u_0 the presence of a contribution $u_0 u_1$ in addition to the linear $y\, u_0$. The flow equation $\mathrm{d}u_0/\mathrm{d}(\log M) = -y u_0 + b_r u_1 u_0$ is solved by $u_0(M)M^y/u_1(M)^{b_r/b} = u_0(M_0)M_0^y/u_1(M_0)^{b_r/b}$. Putting things together like above we find the scaling of the mass gap

$$\gamma\, M = \Gamma\left(\frac{u_0 M^y}{(1 + b u_1 \log(M/M_0))^a}, \frac{u_1}{1 + b u_1 \log(M/M_0)} \right), \tag{14}$$

with a constant $a := b_r/b$ which may be taken as a fit parameter. The default value, however, for the data analysis in this paper is $a = 0$. For finite bu_1 and large M/M_0 the scaling simplifies:

$$\gamma M = \Gamma\left(\frac{u_0 M^y}{(u_1 \log(M/M_0))^a}, \frac{1}{\log(M/M_0)}\right). \qquad (15)$$

b) Next we explore the consequences of one relevant field u_0 with exponent $1/\nu$ and two irrelevant fields u_1, u_2 with exponents $y_1 \sim y_2$. In $\Gamma = \Gamma(u_0 M^{1/\nu}, u_1 M^{y_1}, u_2 M^{y_2})$ we find the last argument close to the middle one $u_2 M^{y_2} = u_2 M^{y_1} + \Delta y\, u_2\, M^{y_1} \log M$ with $\Delta y = y_2 - y_1$. If the function Γ depends regularly on its arguments, the limit $\Delta y \to 0$ yields a scaling function with two scaling fields for the same exponent y. This is just a special case of the previous subsection with $y_1 = y_2$ and we do not need to consider it further. If the dependence on the last two arguments is singular in the difference, let us say of the type $\Gamma = \tilde{\Gamma}(u_0 M^{1/\nu}, u_1 M^{y_1}, (u_2 M^{y_2} - u_1 M^{y_1})/\Delta y)$ then we have two cases:
i) $u_2 - u_1$ is finite and hence $(u_2 - u_1)/\Delta y$ infinite. Therefore in this case we find $\Gamma = \tilde{\Gamma}(u_0 M^{1/\nu}, u_1 M^{y_1}, \infty)$, which is equivalent to the already treated single irrelevant field case.
ii) $u_2 - u_1$ is small and $(u_2 - u_1)/\Delta y$ is finite, let us call it $\tilde{u}_2$. Then we find $\Gamma = \tilde{\Gamma}(u_0 M^{1/\nu}, u_1 M^{y_1}, \tilde{u}_2 M^{y_1} + u_1\, M^{y_1} \log M)$.
The last case allows for a non-trivial occurence of multiplicative logarithmic corrections. Note that this is a little different from an ad hoc ansatz like $\Gamma = \hat{\Gamma}(u_0 M^{1/\nu}, u_1 M^{y_1}, u_2 M^{y_1} \log M)$.

Hence, our ansatz for the scaling function with logarithmic corrections is

$$\gamma M = \Gamma\Big(u_0 M^{1/\nu},\, u_1 M^y,\, u_2 M^y + u_1 M^y \log M\Big) \qquad (16)$$

where u_0 is odd and u_1 and u_2 are even in x. Note that here we cannot normalize the leading coefficients of both u_1 and u_2 to 1 because in the last argument they appear in a sum. In the following we assume u_1 being normalized, but not u_2.

D. Weights and Errors

The fits are performed in several steps. First a weighted non linear least square fit based on a trust region algorithm with specified regions for each parameter is applied. The resulting parameters are used in a further weighted non linear least square fit based on a Levenberg-Marquardt algorithm. Here no limits are imposed on the fit parameters. The last step is repeated until the resulting parameters stop changing.

We use numerical results for the Lyapunov exponent for various combinations of the parameters x (see (4)) and M (see (5)). For a given combination (x, M) we have large ensembles of data from different program implementations and different chain lengths L. We determined for each ensemble the variance by standard estimators. The reciprocal of the variance is then used as the weight in the fit for each data point of the considered ensemble.

E. Evaluation of fits

The next step is the evaluation of the fit results. We present several methods to do this. The most important one is the χ^2 test. χ^2 is given by

$$\chi^2 = \sum_i \frac{(y_i - f_i)^2}{\sigma^2} \qquad (17)$$

where f_i is the value predicted by the fit and y_i the measured value. σ is given by the standard deviation as described in III D. As our fit contains many data points with the same (x, M) coordinates, $\chi^2 = 0$ is not possible, it will be even large due to the huge number of data points. The way to deal with this behaviour is to consider the ratio $\chi^2/$*degrees of freedom*. The expectation value for this ratio is 1 for an ideal fit.

Deviations from 1 are evaluated with the cumulative probability $P(\tilde{\chi}^2 < \chi^2)$ which is the probability of observing – just for statistical reasons – a sample statistic with a smaller χ^2 value than in our fit. A small value of P, i.e. a large value of the complement $Q := 1 - P$ is taken as an indication of a good fit. However, values of P lower than 1/2 indicate problems in the estimation of the error bars of the individual data points.

Another criterion is based on the width of the *confidence intervals*. This quantifies the quality of the prediction for a single parameter. We use 95% confidence intervals which means that for repeated independent generation of the same amount of data and application of the same kind of data analysis the resulting confidence intervals contain the true parameter values in 95% of the cases.

The last criterion we present is the sum of *residuals*. It is given by

$$\text{res} = \sum_i \text{res}_i, \qquad \text{res}_i = y_i - f_i \qquad (18)$$

The sum of residuals should be small compared to the degrees of freedom. The residuals plotted should be noise around zero. If the residuals do not scatter around zero it is to be expected that the fit function is not correct.

IV. RESULTS

In Fig.2 we present the leading Lyapunov exponent for various numbers M of 2×2 blocks in the transfer matrices versus x which measures the deviation of the hopping

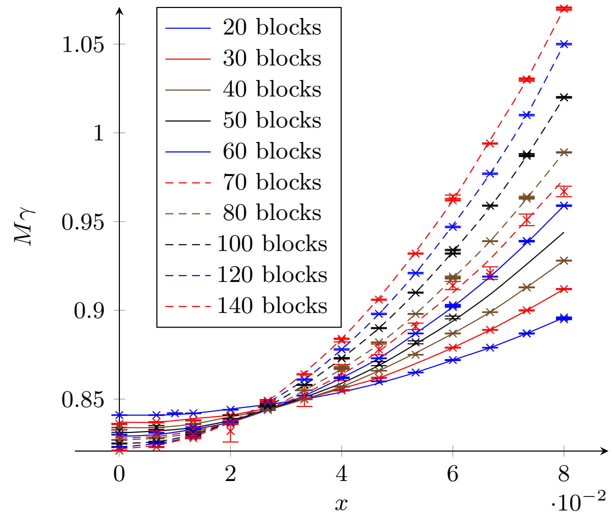

Figure 2. Plot of the smallest eigenvalue of the transfer matrix times M for different block sizes M and in dependence on the distance x from the critical point. The x-values divide the interval $[0, 0.08]$ into 12 equal parts.

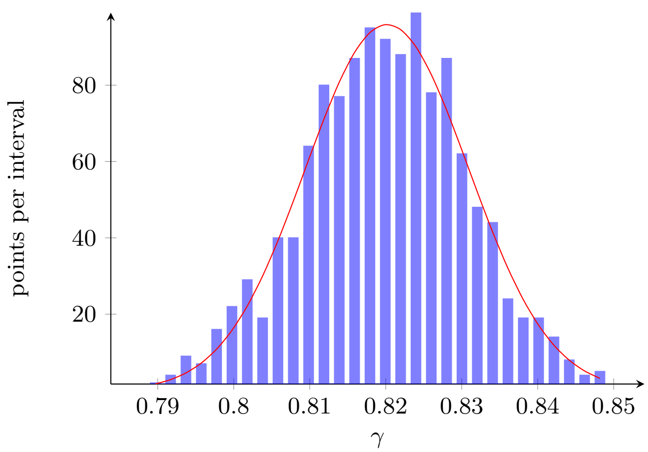

Figure 3. Distribution of Lyapunov exponents in the ensemble of calculations with 1282 elements for chain length $L = 1\,000\,000$, $M = 140$ and $x = 0$.

parameters r and t (4) from their critical value $1/\sqrt{2}$. The corresponding fitting parameters are presented in the table below.

In Fig.3 we present an example of the distribution of Lyapunov exponents with fixed height M, chain length L and x. This distribution defines one data point and its accuracy resp. weight for the fit. Here, we see a Gaussian distribution in full accordance with the central limit theorem[26].

A. One irrelevant field

Our best fitting results have been obtained by expanding Γ up to second order in u_0, and first order in u_1 (7) and expanding u_0 (12) up to the third and u_1 (13) up to the second order in x.

We found the following coefficients and goodness of fit parameters:

Coefficients (confidence bounds 95%):

$\Gamma_{00} =$	0.770	$(0.760, 0.780)$
$\Gamma_{01} =$	0.111	$(0.104, 0.119)$
$\Gamma_{20} =$	0.826	$(0.809, 0.842)$
$a_3 =$	0.980	$(0.762, 1.197)$
$b_2 =$	-1.251	$(-2.612, 0.109)$
$\nu =$	2.566	$(2.554, 2.578)$
$y =$	-0.150	$(-0.174, -0.125)$

Goodness of fit parameters:

χ^2:	37717
degrees of freedom (dof) :	37670
χ^2/dof:	1.0012
P:	0.569
sum of residuals:	-90.038

Here χ^2/dof is close to 1 and the cumulative probability $P = 0.569$ is close to 1/2 marking a good fit result. The sum of residuals is small compared to the degrees of freedom. In a plot the residuals are distributed around zero by eye's measure. All this indicates that the fit is reliable and the data agree with the model equation.

B. Two irrelevant fields

An interesting result is given by the ansatz with two irrelevant fields. By allowing a second irrelevant field accounting for the finite M-dependence we explore the stability of the exponents obtained in the simpler fit above. In fact, we will see that ν is almost unchanged, but the leading irrelevant exponent is changed. In the fit treated here the leading irrelevant exponent is much closer to 0 than above. The fit is done with recursive applications of nonlinear least square fits as described for the case of one irrelevant field.

Our best fits have been obtained for an expansion of Γ, see (11), up to the second order in the fields and an expansion of the fields in x up to first order for u_0 and to second order in u_1 and u_2. For higher order expansions in x the coefficients agree well with the values presented here within their error margins but the new confidence intervals are larger, however the cumulative probability P is a little smaller. This agreement indicates that we have found a good choice for the cut off of the expansion.

We found the following coefficients for (11) and goodness of fit parameters:

Coefficients (confidence bounds 95%) :

$\Gamma_{000} =$	0.4416	$(0.0664, 0.8169)$
$\Gamma_{001} =$	0.3695	$(-0.1332, 0.8722)$
$\Gamma_{002} =$	0.0575	$(-0.2984, 0.4134)$
$\Gamma_{010} =$	0.0093	$(-0.2973, 0.316))$
$\Gamma_{011} =$	− 0.0493	$(-0.388, 0.2895)$
$\Gamma_{020} =$	1.843	$(1.267, 2.418)$
$\Gamma_{200} =$	0.8319	$(0.7234, 0.8535)$
$b_{12} =$	0.692	$(0.8104, 2.77)$
$b_{22} =$	− 0.164	$(-0.4407, 0.1127)$
$\nu =$	2.563	$(2.548, 2.579)$
$y_1 =$	− 0.0204	$(-0.0447, 0.0039)$
$y_2 =$	− 1.151	$(-1.365, -0.9366)$

Goodness of fit parameters :

χ^2 :	37727
degrees of freedom (dof) :	37609
χ^2/dof:	1.00314
P :	0.67
sum of residuals :	−84.8434

Also in this case of two irrelevant fields χ^2/dof is close to 1 and the cumulative probability $P = 0.67$ is close 1/2. The sum of residuals is small compared to the degrees of freedom like in the case of one irrelevant field.

The confidence bounds are reasonably small for ν and for the dimension y_1. Other confidence bounds are wider than in the case of one irrelevant field because there are more parameters and we would need a larger data base to determine them with the same precision.

When comparing this fit with the single irrelevant field fit we see that y_1 does not agree with y as the confidence intervals do not overlap. Less surprising, the exponent y_2 of the second irrelevant field differs significantly in magnitude from y_1. These results lead to doubts whether the first subleading correction is indeed due to an irrelevant operator and the next correction terms are due to irrelevant operators with more negative exponents. In the remainder of this publication we investigate if the numerical findings may better be accounted for by (1) two irrelevant fields with degenerate exponents leading to multiplicative logarithmic corrections, or (2) the presence of a marginal operator.

C. Logarithmic fits

1. *Two degenerate irrelevant fields*

In case of two degenerate dimensions and multiplicative logarithmic corrections, the best results are obtained for an expansion of Γ up to second order in the first argument and first order in the second. The relevant field u_0 is expanded up to first order in x, and the irrelevant fields u_1 and u_2 are simply taken to zeroth order. Our results are :

Coefficients (confidence bounds 95%)

$\Gamma_{000} =$	1.936	$(-30.08, 33.95)$
$\Gamma_{001} =$	− 0.0408	$(-0.548, 0.466)$
$\Gamma_{010} =$	− 0.9663	$(-32.79, 30.86)$
$\Gamma_{200} =$	0.8202	$(0.8151, 0.8253)$
$b_{20} =$	2.244	$(-56.21, 60.7)$
$\nu =$	2.555	$(2.551, 2.56)$
$y =$	− 0.025	$(-0.299, 0.249)$

Goodness of fit parameters :

χ^2 :	37801
degrees of freedom (dof) :	37670
χ^2/dof:	1.0035
P :	0.6845
sum of residuals :	−96.117

As the goodness of fit parameters indicate, this fit is in principle as reliable as the previous ones. The critical exponent ν is in agreement with the other fits. However, the exponent y has a much larger confidence bound than y and y_1 above. Hence we tend to conclude that the current fit scheme is not appropriate.

2. *One marginal field*

In the case of the logarithmic fit based on the ideas of[29] the best results are obtained for an expansion of Γ up to the second order in the in the first argument and first order of the second argument. The relevant field u_0 is expanded up to the third order and u_1 up to the second order in x. The result is as follows :

Coefficients (confidence bounds 95%)

$\Gamma_{00} =$	0.7087	$(0.6888, 0.7285)$
$\Gamma_{01} =$	0.14	$(0.1205, 0.1596)$
$\Gamma_{20} =$	0.8253	$(0.8083, 0.8423)$
$a_3 =$	0.978	$(0.7622, 1.194)$
$b =$	0.0856	$(0.0701, 0.1011)$
$b_2 =$	− 0.69	$(-1.506, 0.1257)$
$\nu =$	2.566	$(2.553, 2.578)$

Goodness of fit parameters:

χ^2:	37716
degrees of freedom (dof) :	37670
χ^2/dof:	1.0012
P:	0.567
sum of residuals:	−90.594

As the goodness of fit parameters show, this is one of our best fits. It is moreover best with respect to stability: The results for different orders of the expansions agree better among each other than in case of the fit schemes above. Hence this fit can be called the most stable one.

Of course, in order to make a strong statement about the existence of logarithmic corrections and particularly about the presence of a marginal field a much larger data basis would be needed. This, however, is currently out of reach.

All results shown in the tables above are based on the data for length $L = 5 \cdot 10^6$. For $L = 1 \cdot 10^6$ similar fits yield results that are absolutely compatible with those shown above. The confidence intervals are larger and the goodness of fit is a little worse for example the cumulative probability P is larger by about 0.1.

V. CONCLUSION AND OUTLOOK

Our result for the localization length index is slightly smaller than the values of the localization length index presented in the recent works[16,18–20]. Still these results agree very well with our result within the $1-\sigma$ confidence bounds.

All of our fits are based on one relevant field and other fields, irrelevant or marginal. The cases with one or two irrelevant fields followed the standard way of data analysis. For the case of just a marginal additional field we presented a derivation of the fit function resulting in logarithmic corrections. Another situation with multiplicative logarithmic corrections was studied for the situation of two irrelevant fields with degenerate dimensions.

The available data base does not allow yet to prove or falsify any of the four scenarios. However, the best result with a minimal number of fit parameters was achieved in the case of one relevant and one marginal field. The value of the critical exponent ν though is the same within the error bars of all presented fits.

With respect to the confidence bounds our result for ν is compatible with the results for this model as known from the literature. Unfortunately it is in disagreement with the experimental value, which is $\nu = 2.38 \pm 0.06$. This does certainly not change by taking into account more irrelevant fields or logarithmic corrections. Our result thus shows in accordance with previous works by the before mentioned authors[16,18–20] the necessity of an essential modification of the CC-model for the description of the plateau-plateau transition in the QHE.

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