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Zero modes, Bosonization and Topological Quantum Order: The Laughlin State in Second Quantization

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We introduce a "second-quantized" representation of the ring of symmetric functions to further develop a purely second-quantized – or "lattice" – approach to the study of zero modes of frustration free Haldane-pseudo-potential-type Hamiltonians, which in particular stabilize Laughlin ground states. We present three applications of this formalism. We start demonstrating how to systematically construct all zero-modes of Laughlin-type parent Hamiltonians in a framework that is free of first-quantized polynomial wave functions, and show that they are in one-to-one correspondence with dominance patterns. The starting point here is the pseudo-potential Hamiltonian in "lattice form", stripped of all information about the analytic structure of Landau levels (dynamical momenta). Secondly, as a by-product, we make contact with the bosonization method, and obtain an alternative proof for the equivalence between bosonic and fermionic Fock spaces. Finally, we explicitly derive the second-quantized version of Read's non-local (string) order parameter for the Laughlin state, extending an earlier description by Stone. Commutation relations between the local quasi-hole operator and the local electron operator are generalized to various geometries.

I. INTRODUCTION

The physics of electrons in a strong, external, magnetic field harbors a great multitude of interesting phases of strongly correlated electrons. For a subset of these phases, representative wave functions can be given that have sufficiently simple analytic properties such that a parent Hamiltonian can be constructed. This includes[1,2] the experimentally relevant Laughlin[3] states, as well as the non-Abelian Moore-Read state,[4] which may explain the plateau at filling factor $\nu = 5/2$,[5] as well as the entire Read-Rezayi series,[6] the Gaffnian[7] state which is presumed critical, as well as a variety of multi-component wave functions.[8,9] Certainly, the dichotomy between analytic wave functions and their parent Hamiltonians is a key component of the theory of fractional quantum Hall states. At the same time, for a great wealth of quantum Hall states that are understood through hierarchical,[1,10] composite fermion,[11] or complementary field theoretical constructions,[12] we do not enjoy the luxury of sufficiently "special" microscopic wave functions that can be seen to be exact eigenstates of suitable local Hamiltonians.

In the more fortunate cases, it is the existence of special analytic "clustering" properties[1,6,13–17] of first quantized wave functions that makes construction of a parent Hamiltonian possible. Interestingly, it has recently been suggested by Haldane that the well-known analytic properties of such wave functions may be deceptive, in the sense that they become meaningless after Landau level projection.[18] Indeed, as a result of such projection, one discards all the degrees of freedom of the problem relating to dynamical momenta (which determine the structure of Landau levels) and keeps only the degrees of freedom associated with guiding centers. The full Hilbert space is isomorphic to the tensor product

$$\mathcal{H}_\pi \otimes \mathcal{H}_\omega, \tag{1}$$

with factors describing degrees of freedom belonging to dynamical momenta (π) and guiding centers (ω), respectively. In principle, the problem of working out the eigenstates of a (projected) interacting Hamiltonian could be naturally reduced to the second factor, were it not for the fortuitous circumstance that looking at the problem in the full Hilbert space with the "right" Landau level structure, ground state wave functions are sometimes seen to have exceptional analytic properties. Often, however, this may not be the case. In view of the above, it may not be surprising that for many quantum Hall states, some as basic as the Jain $\nu = 2/5$ state, no representative wave functions are known with analytic properties that are "special" enough to allow the construction of a suitable parent Hamiltonian. Moreover, "quantum Hall like" Hamiltonians have become fashionable in contexts where the traditional Landau level structure, which provides the basis for these analytic properties, is absent, and is replaced[19] by a basis of Wannier states in the flat "Chern band" (having non-zero Chern number) of a "fractional Chern insulator". There has been much interest recently in suitable flat band systems.[19–37]

In such a context, the traditional analytic wave functions are meaningless, if still useful for the construction of solvable models. Instead, only a manifestly Landau level projected, or "guiding center" representation of quantum many-body states is meaningful. A formalism that naturally implements Landau level (or flat band) projection is obtained by passing to second quantization. For example, for the Haldane pseudo-potentials[1] one obtains second quantized expressions of the form (see, e.g., Ref. 38 for the two-body case in any geometry, or, Ref. 39 for

general n-body generalizations on the cylinder):

$$H_m = \sum_R T_R^{m\,\dagger} T_R^m \,,$$
$$T_R^m = \frac{1}{2} \sum_x \eta_{R,x}^m \, c_{R-x} c_{R+x} \,, \tag{2}$$

where m refers to the mth Haldane pseudo-potential, R and x run over integer and half-integer values, with x constrained via $(-1)^{2x} = (-1)^{2R}$, c_r annihilates a boson (fermion) in the rth Landau level orbital for m even (odd), and the $\eta_{R,x}^m$ are form factors that depend on geometry, and are polynomial in x for the sphere, disk, or cylinder geometry.[38] In the present paper, we will always work in these geometries. Notice that H_m is a separable-potential Hamiltonian.[38]

It is well-known[1] that the $\nu = 1/M$ Laughlin state is the unique zero energy (zero mode) ground state at filling factor $1/M$ of the Hamiltonian

$$H^M = \sum_{\substack{0 \le m < M \\ (-1)^m = (-1)^M}} V_m H_m \,, \tag{3}$$

with positive real numbers V_m. This and other properties of the Laughlin state are well established using the analytic (polynomial) structure of first quantized wave-functions, thus, making use of the embedding of the Landau level(s) into the larger Hilbert space (1), which endows lowest Landau level wave functions with their polynomial character.

In view of the above, one may, however, wonder if such embedding is *necessary* to understand the (zero mode) properties of the Hamiltonian (2), (3) when given in the above, manifestly projected, second-quantized form. Understanding the problem without the introduction of spurious (as far as interactions are concerned) degrees of freedom may not only be pleasing from a mathematical point of view, but the abandonment of a manifestly polynomial wave-function structure requires the development of a new route of attack that may well be beneficial in the broader context of constructing quantum Hall parent Hamiltonians. Moreover, one observes that written in the form (2), the Hamiltonian belongs to a particular breed of frustration-free lattice Hamiltonians, which have generally attracted much interest in recent years.[40–50] We call this class of generalized pseudo-potential systems, *Loop-algebraic Hamiltonians* because of the underlying algebra the operators T_R^m satisfy.[38]

The connection with frustration-free lattice Hamiltonians may not be surprising, in view of the recently discovered matrix-product structure of the Laughlin state.[51,52] Unlike in other well-known examples of frustration-free models, the H_m's in (2) are not strictly finite ranged in the lattice basis. It would thus seem that studying the existence and properties of zero modes of general Hamiltonians of the form (2), (3), with generic, not necessarily short-ranged coefficients $\eta_{R,x}^m$, is a much harder problem than for ordinary, finite-range frustration-free models. Nonetheless, some general mathematical statements have been obtained in Ref. 38. Under quite general circumstances, which we shall not repeat here but which apply for the $\eta_{R,x}^m$ corresponding to Haldane pseudo-potentials, any zero mode of the Hmiltonian (3) can be "inward squeezed" from one or several partitions satisfying a "generalized Pauli-principle".[14,15] We review details below. This statement generalizes a fact that is known for many quantum Hall wave functions, especially, those with polynomial wave functions characterized by certain clustering properties,[14,15] to zero modes of essentially *any* Hamiltonian of the form (2), (3) (with natural generalizations to higher-body terms). This includes the lattice Hamiltonians constructed in Refs. 53 and 54, whose ground states are not described by polynomials with nice clustering properties. For the Hamiltonian (2), (3), one then immediately obtains[38] that there are no zero modes at filling factors $\nu > 1/M$, and that there is at most one zero mode at filling factor $\nu = 1/M$ (assuming the topology of the sphere, cylinder, or disk). A zero mode at $\nu = 1/M$ would thus always have an "incompressible character" for the given class of Hamiltonians, in the sense that at finite system size, a state at $\nu > 1/M$ necessarily has positive energy (although we make no statement here about the thermodynamic limit). For the purpose of this paper, it is beneficial to use the term "incompressible" in the above sense.

Specializing to the case of Haldane peudo-potentials and taking, for now, the existence of an incompressible zero mode at $\nu = 1/M$ as given (which then of course is just the $1/M$-Laughlin state), second quantized operators have been identified[38] that generate new zero modes, at lower filling factor $\nu < 1/M$, when acting on any given zero mode. These operators depend on a positive integer d, and with normalization conventions that belong to the infinitely "thick" cylinder geometry and that we will review below, can be written as

$$\mathcal{O}_d = \sum_r c_{r+d}^\dagger c_r \,. \tag{4}$$

The operators $\mathcal{O}_d$ generate a commutative unital algebra $\mathcal{A}$. Once it is known that: 1) there is a special "incompressible" zero mode $|\psi_{1/M}\rangle$ at filling factor $1/M$, and 2) *every* zero mode can be written in the form $\hat{a}|\psi_{1/M}\rangle$ with $\hat{a} \in \mathcal{A}$ (i.e., $\hat{a}$ a linear combination of products of $\mathcal{O}_d$'s), then we can say that every property of zero modes known from the first quantized analytic wave function approach can also be understood from a second-quantized or "lattice" point of view, which is manifestly "guiding-center" and for which the starting point is given by the operators T_R^m of Eq. (2). The first statement, the existence of a state at filling factor $1/M$ that is annihilated by all operators T_R^m with $m < M$ and m even (odd) for M even (odd), is demonstrated in a purely second quantized approach by work in parallel, Ref. 55. This is done by giving a recursive definition for the Laughlin state in the lattice basis, and demonstrating that it has the zero mode property, making no reference to first

quantized polynomial wave functions. Here we will prove the second statement in a similarly wave-function free approach, thus completing the program of describing zero modes of Haldane pseudo-potentials and related Loop-algebraic Hamiltonians (stabilizing the Laughlin state) in an intrinsically second-quantized language.

We point out that application of the operator $\mathcal{O}_d$ corresponds[38] to the multiplication of certain symmetric polynomials (power-sum symmetric polynomials) in first quantization, though we will not make direct use of this fact. The zero mode excitations created in this way may be viewed as edge excitations, although they may of course live arbitrarily deep in the bulk of the system (in particular, for large d, and are then better thought of as quasi-hole excitations). Indeed, there is an obvious connection between the form of the operators $\mathcal{O}_d$, and the boson operators in the traditional bosonization scheme (even though the operators c_r may themselves be bosonic or fermonic here, depending on the situation). It is thus worth emphasizing that all our results apply to finite systems of arbitrary size, and require no long wavelength (effective field theory) limit to be taken.

The remainder of the paper is organized as follows. We derive the completeness of the zero modes generated by the operators $\mathcal{O}_d$ in Secs. II and III, We then proceed to make contact with the standard bosonization scheme in Sec. IV, by observing that as a byproduct of our results, one obtains an alternative proof of a standard theorem of the bosonization method, which states the equivalence of the fermonic and the bosonic Hilbert space.

Finally, in Sec. V, we give yet another application of our second-quantization formalism, which is to express the local quasi-hole operator explicitly in terms of the second quantized electron operator. The quasi-hole operator is the central ingredient to Read's non-local (string) order parameter of Laughlin quantum Hall states.[56] We extend the commutation relations between this operator and the local electron operator given by Stone[57] for the disk geometry to the cylinder and the sphere.

At the technical level, a main contribution of the present work is the representation of the ring of universal symmetric functions through an embedding into the algebra of canonical commutation- or anti-commutation-relations. This embedding plays a crucial role in all of our results, and maps alternatively power-sum or elementary symmetric polynomials to fairly simple expressions in terms of fermionic or bosonic ladder operators, obeying well-known non-trivial relations of the Newton-Girard type. We believe that this mapping could be of quite general value beyond the applications given here.

II. DEFINITIONS AND PRELIMINARY CONSIDERATIONS

A. Statement of the problem

We will, in the following, work with a Hilbert space defined by a half-infinite lattice of orbitals ϕ_r labeled by an integer $r \geq 0$. This may describe the lowest Landau level of a system with either disk geometry, or a half-infinite cylinder, or, if an additional upper cutoff is introduced, a sphere. Zero-modes of the Laughlin state parent Hamiltonians in any two of these geometries are in one-to-one correspondence (up to the aforementioned cutoff, if one of the geometries is the sphere). In second quantization, these zero modes differ only by normalization factors assigned to each basis state,[38] as we review in Sec. V. Here, we will give all operators using the normalization conventions of the infinite-radius cylinder, which are simplest. Fixing a certain value for M, zero modes can then be characterized as states annihilated by all operators[38]

$$Q_R^m = \sum_{\substack{x \\ (-1)^{2x}=(-1)^{2R}}} x^m \, c_{R-x} c_{R+x} \, , \tag{5}$$

where the c_r, $c_r^\dagger$ are bosonic (fermionic) ladder operators when M is even (odd), m runs over even (odd) values satisfying $0 \leq m < M$, R runs over non-negative half-integer values, x runs over all values such that $R \pm x$ is integer, and we use the convention $c_r = c_r^\dagger = 0$ for $r < 0$. The operators Q_R^m may be thought of as linear combinations of the T_R^m in Eq. (2) in the limit of an infinitely thick cylinder. These linear combinations are taken for convenience to yield the simple monomial factors of Eq. (5). In general, the zero mode condition can be stated in terms of the original operators T_R^m or any linearly independent combination thereof. We will thus be interested in finding *all* states $|\psi\rangle$ in the Hilbert space that are characterized by the following algebraic property:

$$\begin{aligned} Q_R^m |\psi\rangle = 0 \quad & \forall \; R = 0, \frac{1}{2}, \ldots , \\ & \forall \; 0 \leq m < M, (-1)^m = (-1)^M \, , \end{aligned} \tag{6}$$

which we will refer to as the *zero mode property* (for fixed M). These states $|\psi\rangle$ constitute the low-energy subspace of the Hilbert space $\mathcal{H}_\omega$. We define the filling factor of any zero mode $|\psi\rangle$ of N particles as

$$\nu = \frac{N-1}{r_{\mathsf{max}}} \, , \tag{7}$$

where r_{max} is the highest orbital index among the orbitals occupied in the zero mode, i.e. $r_{\mathsf{max}} = \max \{ r | \langle \psi | c_r^\dagger c_r | \psi \rangle \neq 0 \}$.

For a much more general class of problems defined through general deformations of the operators (5), it is known[38] that zero modes can exist only for filling factor $\nu \leq 1/M$. It is further known, for the same general class

of problems studied in Ref. 38, that *if* there exists a zero mode at filling factor $1/M$, it is unique, and is of the form

$$|\psi_{1/M}\rangle = |\tilde{\psi}_{1/M}\rangle + \sum_{\lambda}{}' \, C_\lambda |\lambda\rangle, \tag{8}$$

where $\sum_\lambda{}'$ excludes the term $\lambda = \tilde{\psi}_{1/M}$, and

$$|\tilde{\psi}_{1/M}\rangle = |1\underbrace{0\ldots0}_{M-1}1\underbrace{0\ldots0}_{M-1}10\ldots\rangle \tag{9}$$

is the "thin cylinder/torus pattern"[58–72] or "root partition",[14,15,38,73,74] where 1s and 0s denote occupation numbers of the orbitals created by $c_r^\dagger$, with 1s separated by $M-1$ zeros. The $|\lambda\rangle$'s denote other occupation number eigenstates. As the occupation number eigenstates form a basis of the Hilbert space, the statement of Eq. (8) becomes non-trivial only with the additional information that $C_\lambda \neq 0$ only for occupation number configurations λ that are dominated, in the usual sense,[58] to be reviewed below, by the configuration $|\tilde{\psi}_{1/M}\rangle$.

For the problem at hand, with Q_R^m defined as given in Eq. (5), it is further known that a unique zero mode exists at filling factor $1/M$. This is simply the $1/M$-Laughlin state $|\psi_{1/M}\rangle$. We note that the existence of such a state can, if desired, be derived solely from the zero mode condition (6), and algebraic properties of the operators Q_R^m, i.e., in the algebraic, wave-function-free language preferred here (See work in parallel[55]). Our goal here will be to proof the following statement:

Theorem 1. *Every zero mode is the linear combination of states given by products of operators $\mathcal{O}_d$ ($d > 0$) acting on the special zero mode $|\psi_{1/M}\rangle$.*

In slightly more technical terms, consider the algebra $\mathcal{A}$ generated by the (commuting) operators $\mathcal{O}_d$, $d > 0$. Then we have that the zero mode subspace (of $\mathcal{H}_\omega$) Z is obtained by applying all elements $\hat{a}$ of $\mathcal{A}$ to the highest filling factor zero mode $|\psi_{1/M}\rangle$:

$$Z = \mathcal{A}|\psi_{1/M}\rangle = \{\hat{a}|\psi_{1/M}\rangle \ , \ \text{with} \ \ \hat{a} \in \mathcal{A}\} \, . \tag{10}$$

B. A second-quantized representation for the ring of symmetric polynomials

While our goal is here to establish techniques to proof Theorem 1 in an intrinsically second-quantized fashion, it would be remiss if we did not make contact with the usual first quantized procedure every now and then, for reasons of transparency and pedagogy. The key property of the operators $\mathcal{O}_d$ is that they produce new zero modes at lower filling factor and same particle number when acting on given zero modes. This is a simple consequence of the commutation relations[38] between the $\mathcal{O}_d$ and the Q_R^m, or the original T_R^m, Eq. (2). However, it also follows from the fact that the action of the operator $\mathcal{O}_d$ on a given state, in first quantized language, is to multiply the wave function with "power-sum" symmetric polynomials,

$$p_d = \sum_{i=1}^{N} z_i^d \, . \tag{11}$$

The identification of zero modes with symmetric polynomials has a long tradition.[75–77] As emphasized, here we wish to bypass this language entirely. Instead we seek a way to establish a one-to-one correspondence between zero modes and dominance patterns satisfying certain rules[14,15,58,60,74,78] that follows directly from the algebraic zero mode definition, Eq. (6). It turns out, however, that a direct proof of Theorem 1 as stated above is quite cumbersome. Our strategy is to restate the problem in terms of a different set of zero modes generating operators, namely those that correspond, in first quantization, to the multiplication of wave functions with *elementary* symmetric polynomials:

$$s_d = \sum_{1 \leq i_1 < \ldots < i_d \leq N} z_{i_1} z_{i_2} \cdots z_{i_d} \, . \tag{12}$$

We postulate, and show later, that this is facilitated by the following operators:

$$e_d = \frac{1}{d!} \sum_{r_1 \ldots r_d} c^\dagger_{r_1+1} c^\dagger_{r_2+1} \cdots c^\dagger_{r_d+1} c_{r_d} \cdots c_{r_2} c_{r_1} \, . \tag{13}$$

For pedagogical purposes and to develop intuition, let us illustrate the action of the operators $\mathcal{O}_d$ and e_d when $d = 2$ on a simple state, such as the root partition $|\tilde{\psi}_{1/3}\rangle = |1001001001000\ldots\rangle$, with $N = 4$ and $r_{\mathsf{max}} = 9$

$$\begin{aligned}\mathcal{O}_2|\tilde{\psi}_{1/3}\rangle = \ & |0011001001000\ldots\rangle + |1000011001000\ldots\rangle \\ & + |1001000011000\ldots\rangle + |1001001000010\ldots\rangle,\end{aligned} \tag{14}$$

$$\begin{aligned}e_2|\tilde{\psi}_{1/3}\rangle = \tfrac{1}{2!}(&|0100101001000\ldots\rangle + |0101000101000\ldots\rangle \\ + \ & |1000100101000\ldots\rangle + |0101001000100\ldots\rangle \\ + \ & |1000101000100\ldots\rangle + |1001000100100\ldots\rangle).\end{aligned} \tag{15}$$

In this work, we wish to avoid making direct contact with the polynomial language. Rather, we want make sure that the logic we follow, and generality of our results, are entirely independent of the analytic polynomial wave functions. Therefore, we will *not* proceed by showing any connection between Eqs. (12) and (13), though it would not be difficult to do that. Rather, we seek to directly establish the algebraic relations between the operators (4) and (13). These must be the same as those between the associated symmetric polynomials, if the aforementioned associations are correct. These are the Newton-Girard *operator* relations: ($e_0 = 0$ and $e_1 = \mathcal{O}_1$)

$$d\, e_d + \sum_{k=1}^{d} (-1)^k \mathcal{O}_k e_{d-k} = 0 \, . \tag{16}$$

That the $\mathcal{O}_d$ and e_d are indeed so related is shown in Appendix A. Consider again the simple example above with $d = 2$. The Newton-Girard operator relations imply that $2e_2 = \mathcal{O}_1^2 - \mathcal{O}_2$, and it is easy to see that the action of $\mathcal{O}_1^2$ on $|\tilde{\psi}_{1/3}\rangle$ does indeed produce every term in Eq. (14), as well as *twice* every term in Eq. (15), indeed satisfying the required relations.

The relations (16) show in particular that, by induction, every e_d is expressible in terms of $\mathcal{O}_d$'s, and vice versa. From this fact, we in particular infer that the e_d's must have the following in common with the $\mathcal{O}_d$'s (for which these statements are already known[38]):

- The e_d's all commute.
- When acting on zero modes, the e_d's generate new zero modes.
- The e_d's generate the same algebra $\mathcal{A}$ as the $\mathcal{O}_d$'s.

From the last statement, we see that in Theorem 1, we may replace $\mathcal{O}_d$ with e_d without changing the meaning. It is in this form that we will prove Theorem 1.

It is a remarkable property of the Newton-Girard relations that they do not depend on the number of variables N, even though the polynomials $\mathcal{O}_d$ and s_d do (where the latter vanish for $d > N$). Related to that, one can define the ring of "universal symmetric functions",[79] which may be thought of as the limit $N \to \infty$ of the ring of symmetric polynomials for fixed N. Our operators $\mathcal{O}_d$ and e_d likewise do not depend on N, and should thus be though of as representations of the respective generators of the universal symmetric functions ring. This is one of the benefits of the second quantized approach developed here. Note that indeed the e_d automatically annihilate any state of $N < d$ particles.

C. Partitions and dominance

We may expand general many-body states in the lowest Landau level as

$$|\psi\rangle = \sum_\lambda C_\lambda |\lambda\rangle \,, \tag{17}$$

where the $|\lambda\rangle$ denote a basis of occupation number eigenstates, with respect to an appropriate lowest Landau level basis of single particle states. The latter will always be chosen as eigenstates of some guiding-center or momentum quantum number. One way to denote a ket $|\lambda\rangle$ is therefore through a partition $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_N$ for bosons, or $\ell_1 > \ell_2 > \ldots > \ell_N$ for fermions, where ℓ_i denotes the orbital index of the ith particle. Since the Hamiltonian conserves momentum, when discussing zero modes we may assume that

$$L = \ell_1 + \ldots + \ell_N \tag{18}$$

is the same for all kets $|\lambda\rangle$ contribution to Eq. (17). We will also refer to the kets $|\lambda\rangle$ as partitions of the number L. There is a standard notion of "dominance" for partitions whose utility in the context of quantum Hall states has been noted previously.[14,15] We say that $|\lambda\rangle$ dominates $|\lambda'\rangle$, or $|\lambda\rangle \geq |\lambda'\rangle$, if $|\lambda\rangle = |\lambda'\rangle$ or $|\lambda'\rangle$ can be generated from $|\lambda\rangle$ (up to normalization factors or phases) by repeated application of inward "squeezing operations":[58]

$$c_{r_1}^\dagger c_{r_2}^\dagger c_{r_2+d} c_{r_1-d}, \quad r_1 \leq r_2,\ d > 0\,. \tag{19}$$

An equivalent, and technically more useful characterization of dominance is given by

$$\begin{aligned} &\{\ell_i\}_{i=1\ldots N} \geq \{\ell_i'\}_{i=1\ldots N} \\ \Leftrightarrow\ &\sum_{i=1}^n \ell_i \geq \sum_{i=1}^n \ell_i' \quad \text{for all } n = 1 \ldots N. \end{aligned} \tag{20}$$

We we always refer to this definition here.

A special role is further played by partitions satisfying a "generalized Pauli principle":[14,15] Here, we will say that a partition $|\lambda\rangle$ satisfies the M-Pauli-principal if no more than 1 particle is present in any M-consecutive orbitals, or $\ell_i \geq \ell_{i-1} + M$. In particular, the state (8) satisfies the M-Pauli principle, and is the lowest-L state that does so for given particle number N. We will also say that a general superposition (17) is dominated by a partition $|\lambda_0\rangle$ if $C_{\lambda_0} \neq 0$ and $|\lambda_0\rangle \geq |\lambda\rangle$ for every λ with $C_\lambda \neq 0$. Note that we require that the dominant partition $|\lambda_0\rangle$ appears with nonzero coefficient also, which is not always the case in the literature. Thus, in particular, the statement made in Eq. (8) and below is that $|\psi_{1/M}\rangle$ is dominated by $|\tilde{\psi}_{1/M}\rangle$.

Our strategy for proving Theorem 1 is now the following. It will turn out that, using general results of Ref. 38, Theorem 1 can be obtained as a corollary of the following:

Theorem 2.

For any partition $|\lambda\rangle$ satisfying the M-Pauli-principle, there is a zero mode $|\psi_\lambda\rangle$ that is dominated by $|\lambda\rangle$ and that is of the form

$$|\psi_\lambda\rangle = \prod_{\alpha=1}^k e_{d_\alpha} |\psi_{1/M}\rangle \,. \tag{21}$$

In particular, $|\psi_\lambda\rangle$ is obtained by applying a special element of the algebra $\mathcal{A}$ to the "incompressible" zero mode $|\psi_{1/M}\rangle$. Note again that the order of operators in Eq. (21) does not matter.

III. PROOFS

We will proof Theorem 2 first. Observe that the action of the operator e_d on the partition $|\lambda\rangle$ is just to promote the orbital indices of d particles by 1, in all possible ways. It is thus useful to consider the following decomposition of the operator e_d, where we consider fermions first:

$$e_d = \sum_{\substack{S \subset 2^{\bar{N}} \\ |S| = d}} \mathfrak{e}_S \quad \text{(fermions)}. \tag{22}$$

Here, for an integer N, the bar denotes the set $\bar{N} = \{1, \ldots, N\}$, $2^{\bar{N}}$ denotes the set of all subsets of $\bar{N}$, and the sum goes over all such subsets of d elements. $\mathfrak{e}_S$ then is an operator that acts of the partition $|\lambda\rangle$ by promoting the particles corresponding to the subset S, where this correspondence is established by ordering particles according to their orbital index in $|\lambda\rangle$. Hence, if $|\lambda\rangle$ corresponds to the partition $\{\ell_i\}_{i=1\ldots N}$ and $\mathfrak{e}_S|\lambda\rangle$ is a partition made from the numbers ℓ_i for $i \notin S$ and $\ell_i + 1$ for $i \in S$, and for fermions, the state will be annihilated if this would lead to double occupancies. We may choose phase conventions for the $|\lambda\rangle$-basis such that $\mathfrak{e}_S|\lambda\rangle$ is always either zero or equal to another basis state $|\lambda'\rangle$.

In particular, we may write

$$e_d = \mathfrak{e}_{\bar{d}} + \sum_{\substack{S\subset 2^{\bar{N}} \\ |S|=d \\ S\neq\bar{d}}} \mathfrak{e}_S \quad \text{(fermions)}, \tag{23a}$$

where $\mathfrak{e}_{\bar{d}}$ is an operator that promotes the d particles of highest orbital index, by one orbital index (and never annihilates a partition $|\lambda\rangle$). Note that the operators $\mathfrak{e}_{\bar{d}}$ commute, just as the e_d do.

The situation is not really more complicated for bosons, but for accuracy, we should write

$$e_d = \mathfrak{e}_{\bar{d}} b_{\bar{d}} + \sum_{\substack{S\subset 2^{\bar{N}} \\ |S|=d \\ S\neq\bar{d}}} \mathfrak{e}_S b_S\,, \quad \text{(bosons)}, \tag{23b}$$

where the b_S are positive operators that act diagonally on the basis of occupation number eigenstates, and multiply state with multiple occupancies by necessary combinatorial factors. These factors are necessary, since we still insist that $\mathfrak{e}_S$ acting on $|\lambda\rangle$ gives another basis state $|\lambda'\rangle$, with unit coefficient (no annihilation occurs for bosons). When acting on a $|\lambda\rangle$ with multiple occupancies, there are different subsets S that have the same effect on $|\lambda\rangle$, and are then associated with the same terms in Eq. (13). The resulting ambiguity in the operators b_S can be resolved arbitrarily. The point is not worth further elaborating, since no such ambiguities exist whenever $\mathfrak{e}_{\bar{d}}$ acts on a state $|\lambda\rangle$ without double occupancies, in which case we always have $b_{\bar{d}}|\lambda\rangle = |\lambda\rangle$. This is in particular the case for any $|\lambda\rangle$ satisfying the M-Pauli-principle. We may thus proceed without distinguishing between fermions and bosons.

Theorem 2 now follows from the following simple facts:

$$|\lambda\rangle \geq |\lambda'\rangle \Leftrightarrow \mathfrak{e}_{\bar{d}}|\lambda\rangle \geq \mathfrak{e}_{\bar{d}}|\lambda'\rangle \tag{24a}$$

$$\mathfrak{e}_{\bar{d}}|\lambda\rangle \geq \mathfrak{e}_S|\lambda\rangle \quad \text{for } |S| = d. \tag{24b}$$

Note that by definition of the operators $\mathfrak{e}_S$, both sides of Eq. (24b) are basis states and can be identified with partitions, such that the relation is meaningful, *except* where the right hand side vanishes, in which case we take the relation to be satisfied by convention. The simple proofs are relegated to Appendix B. Useful consequences of Eqs. (24) are

$$|\lambda\rangle \geq |\lambda'\rangle \Rightarrow \mathfrak{e}_{\bar{d}}|\lambda\rangle \geq \mathfrak{e}_S|\lambda'\rangle \quad \text{for } |S| = d, \tag{25}$$

which uses the transitivity of the dominance relation, and

$$\begin{aligned} \text{If } |\lambda\rangle \geq |\lambda'\rangle \text{ and } 0 \neq \mathfrak{e}_S|\lambda'\rangle \geq \mathfrak{e}_{\bar{d}}|\lambda\rangle\,, \ |S| = d\,, \\ \text{then } \quad |\lambda\rangle = |\lambda'\rangle \text{ and } \mathfrak{e}_S|\lambda'\rangle = \mathfrak{e}_{\bar{d}}|\lambda\rangle\,. \end{aligned} \tag{26}$$

The last follows from Eq. (25), the transitivity and antisymmetry of the dominance relation, and the '$\Leftarrow$' direction of Eq. (24a).

From these facts, and the knowledge that $|\psi_{1/M}\rangle$ is dominated by $|\tilde{\psi}_{1/M}\rangle$, one immediately obtains that the right hand side of Eq. (21) is dominated by the first term on the right hand side of the following

$$\prod_{\alpha=1}^{k} e_{d_\alpha}|\psi_{1/M}\rangle = \prod_{\alpha=1}^{k} \mathfrak{e}_{\overline{d_\alpha}}|\tilde{\psi}_{1/M}\rangle + \text{subdominant}. \tag{27}$$

The proof is by simple induction in k, and uses only Eqs. (25), (26). Suffice it to give details for $k = 1$. We act with Eq. (23a) or Eq. (23b), whichever applies, on Eq. (8). Since the first term in Eq. (8) dominates all others, Eq. (25) immediately implies that every term generated in the action of Eq. (23) on Eq. (8) is dominated by the first term in Eq. (27). We must also make sure that contributions to this term cannot cancel. By Eq. (26), they cannot be canceled by any contributions coming from the sum in Eq. (8). However, contributions from the first term cannot cancel either, since, for any $|\lambda\rangle$ without double occupancies, it is easy to see that different operators $\mathfrak{e}_S$ generate different states $\mathfrak{e}_S|\lambda\rangle$. In particular, that is the case if $|\lambda\rangle$ satisfies the M-Pauli principle, as the first term in Eq. (8) or in Eq. (27) (see below) does.

Now, suppose $|\lambda\rangle$ is a partition that satisfies the M-Pauli-principle. Then $\mathfrak{e}_{\bar{d}}$ acts on $|\lambda\rangle$ by squeezing in another zero to the left of the particle that is the dth particle from the right, by means of right-pushing this particle and every particle to its right by one orbital. We can think of any $|\lambda\rangle$ satisfying the M-Pauli-principle as consisting of N compartments of zeros separated by 1s, where the dth compartment is to the left of the dth particle, counted right to left, and has at least $M - 1$ zeros, except for $d = N$th compartment, which may be devoid of zeros. Then the operator $\mathfrak{e}_{\bar{d}}$ just fills another zero into the dth compartment. Given that $|\tilde{\psi}_{1/M}\rangle$ is the densest partition satisfying the M-Pauli principle (all compartments have the minimum possible number of zeros), it is clear that every $|\lambda\rangle$ satisfying the M-Pauli-principle can be written in the form displayed by the first term on the right hand side of Eq. (27). This completes the proof of Theorem 2.

For Theorem 1, we now consider the operator

$$S = \sum_{r\geq 0} r^2 c_r^\dagger c_r\,. \tag{28}$$

When working in the cylinder geometry, this can be thought of as the generator of changes in the cylinder's radius.[80] It is easy to see that all partition states $|\lambda\rangle$ are eigenstates of S, and that inward squeezing always lowers the value of S. So, if by S_λ we denote the S-eigenvalue of $|\lambda\rangle$, $|\lambda\rangle \geq |\lambda'\rangle$ implies $S_\lambda \geq S_{\lambda'}$, with equality in the latter only for equality in the former. For an arbitrary ket $|\psi\rangle$, we denote by $S(\psi)$ the largest S-value of any partition contributing to the expansion (17) of the state. Hence if $|\psi\rangle$ is dominated by λ, then $S(\psi) = S_\lambda$. It is for this reason that the dominant partition also coincides with the thin cylinder limit of the state.

We now fix the particle number N, and prove Theorem 1 by induction over the possible values of $S(\psi)$ for zero modes $|\psi\rangle$. The general results of Ref. 38 show that every partition $|\lambda'\rangle$ contributing to a zero mode $|\psi\rangle$ can be obtained from a partition $|\lambda\rangle$ (not necessarily always the same) via inward squeezing, where $|\lambda\rangle$ contributes to ψ *and* satisfies the M-Pauli-principle. Then, by Theorem 2, the possible values $S(\psi)$ for zero modes are exactly the values S_λ for $|\lambda\rangle$ satisfying the M-Pauli-principle. The lowest such S_λ is uniquely obtained for $|\lambda\rangle = |\tilde{\psi}_{1/M}\rangle$. A corresponding zero mode with $S(\psi) = S_{\tilde{\psi}_{1/M}}$ must then be dominated by $|\tilde{\psi}_{1/M}\rangle$, and the unique zero mode for which this is true is denoted $|\psi_{1/M}\rangle$ in Eq. (8). The uniqueness can be followed from the fact that if two different zero modes were dominated by $|\tilde{\psi}_{1/M}\rangle$, we could make a non-trivial linear combination that is also a zero mode and is not dominated by any partition satisfying the M-Pauli-principle, in violation of the general rules for zero modes found in Ref. 38. Thus, for the smallest possible S-value $S(\psi) = S_{\tilde{\psi}_{1/M}}$, the statement of Theorem 1 is clearly correct.

The induction step proceeds similarly, but makes further use of Theorem 2. Suppose Theorem 1 has been shown for all zero modes $|\psi\rangle$ with $S(\psi) < S$, for some S. Now consider a zero mode $|\psi\rangle$ with $S(\psi) = S$. Then we can write

$$|\psi\rangle = \sum_{i=1}^{n} a_i |\lambda_i\rangle + \sum_{\lambda'} C_{\lambda'} |\lambda'\rangle \,, \tag{29}$$

with $S_{\lambda_i} = S$, $a_i \neq 0$, and for every $|\lambda'\rangle$ with $C_{\lambda'} \neq 0$, $S_{\lambda'} < S$. Invoking again the aforementioned general results[38] the $|\lambda_i\rangle$ must each be dominated by a partition that satisfies the M-Pauli-principle, and that has non-zero coefficient in Eq. (29). However, no term in the second sum can dominate any term in the first, since the latter have larger S-value. It follows, then, that all the λ_i must themselves satisfy the M-Pauli principle. If we now choose zero modes $|\psi_{\lambda_i}\rangle$ as in Theorem 2, then

$$|\psi\rangle - \sum_{i=1}^{n} a_i |\psi_{\lambda_i}\rangle \tag{30}$$

is a zero mode, and the coefficients of the partitions $|\lambda_i\rangle$ cancel. Then all partitions $|\lambda'\rangle$ contributing to Eq. (30) have $S_{\lambda'}$ less than our given S. This is so since it is true for every term in the second sum of Eq. (29), and also for every partition contributing to $|\psi_{\lambda_i}\rangle$ *except* $|\lambda_i\rangle$, since $|\psi_{\lambda_i}\rangle$ is dominated by $|\lambda_i\rangle$. By induction, the statement of Theorem 1 then applies to the zero mode (30). But then it also applies to $|\psi\rangle$, since it does apply to each $|\psi_{\lambda_i}\rangle$ individually, according to Eq. (21).

The same inductive procedure in S can be used to show that the states $|\psi_\lambda\rangle$ defined in Theorem 2 are linearly independent, and are therefore a basis of all zero modes. This establishes the well-known one-to-one correspondence between Laughlin-like zero modes and patterns satisfying the M-Pauli principal, which was first obtained by thin cylinder[58,60,78] methods as well as Jack polynomial methods and its generalization to fermions[14,15,74] (see also Ref. 81). Note that while $|\lambda\rangle$ dominates $|\psi_\lambda\rangle$ defined in Eq. (21), it is not necessarily the only partition satisfying the M-Pauli-principle contributing to this state. By forming new linear combinations, we may define a new zero mode basis $|\phi_\lambda\rangle$ such that $|\lambda\rangle$ is the only partition satisfying the M-Pauli-principle contributing to $|\lambda\rangle$. This requirement uniquely defines $|\phi_\lambda\rangle$, up to normalization, by an argument analogous to that of the uniqueness of a zero mode dominated by $|\tilde{\psi}_{1/M}\rangle$ given above. To show the existence of such a basis $|\phi_\lambda\rangle$, one again uses Theorem 2 and induction in S. We thus note the following corollary to Theorem 2:

Theorem 2'.

For any partition $|\lambda\rangle$ satisfying the M-Pauli-principle, there is a state

$$|\phi_\lambda\rangle = \hat{a}|\psi_{1/M}\rangle \,, \quad \textit{with } \hat{a} \in \mathcal{A}, \tag{31}$$

which is dominated by $|\lambda\rangle$, with the additional property that $\langle\lambda'|\phi_\lambda\rangle = 0$ for every $|\lambda'\rangle$ satisfying the M-Pauli principle other than $|\lambda\rangle$.

Finally, the linear independence of the states in Eq. (21) shows that the e_d generate the commutative algebra $\mathcal{A}$ freely. This is not surprising, given the relation with polynomial rings that we know of, but did not need to make use of so far.

IV. RELATION WITH BOSONIZATION

We now recall that our original motivation was to answer questions about the operators $\mathcal{O}_d$, Eq. (4), in particular the question of the completeness of the zero modes they generate when acting on $|\psi_{1/M}\rangle$. We have answered this question positively. The $\mathcal{O}_d$ are in some sense more interesting physically than their counterparts e_d, since they are single particle operators. One cannot help but noticing that, especially with the "thick cylinder" normalization convention chosen here,[82] operators similar to (4) appear in every bosonization dictionary (even though the operators c_r, $c_r^\dagger$ create fermions only for M odd). They represent density modes, which in the quantum Hall context are naturally associated with edge

excitations.[76] The one-dimensional edge theory of a two-dimensional bulk system is necessarily a *long-wavelength effective theory*, since at higher quantum numbers, the excitations penetrate deeper and deeper into the bulk, removing themselves from the edge (and are, for quantum Hall states, more properly thought of as quasi-holes). It is thus remarkable that in the present context, the operators (4) appear as generators of exact eigenstates of a microscopic bulk theory (Loop-algebraic Hamiltonian) that does not require any thermodynamic limit to be taken. Similar observations can of course be made at the level of polynomial wave functions,[75,77,83] though we regard it as additional benefit that "bosonic mode operators" of the form (4) can be given an exact meaning, with their relation to the microscopic electron operator absolutely explicit, without the need to take any large N limit or apply a *normal ordering* operation. This, in our opinion, makes the correspondence between edge and bulk physics of the Laughlin states particularly lucid.

As one application of our results to the bosonization method, recall that every bosonization scheme needs to address the equivalence of the fermionic and bosonic Hilbert spaces. (Notice, however, that the bosonization scheme we are referring to is performed on an angular momentum lattice [not a real space lattice], and for $M > 1$ the Hilbert space equivalence applies only to the zero mode subspace Z of the total Hilbert space.) Usually, this is done either through partition functions,[84] or through Schur functions[83]. If we apply the results of the preceding sections to the special case $M = 1$, we obtain an alternative proof of this statement. In this case, the M-Pauli-principle is just the standard Pauli principle for fermions. Every fermionic occupancy eigenstate $|\lambda\rangle$ satisfies the $M = 1$ Pauli principle. Consistent with that, the Hamiltonian (2) vanishes identically in this case, and the subspace of zero modes Z is really the entire fermionic Fock space. Indeed, in the light of the above the statement of Theorem 2' just becomes, in this special case, that every N-particle fermionic occupation number eigenstate $|\lambda\rangle$ can be written as the N-particle "vacuum" state $|\psi_{M=1}\rangle = |1111\ldots100000\ldots\rangle$, acted upon by a linear combination of products of operators $\mathcal{O}_d$. For, since we can drop the qualifying statement "satisfying the M-Pauli-principle" for $M = 1$, it follows that $|\phi_\lambda\rangle = |\lambda\rangle$. This is just the statement that the subspace generated by the operators $\mathcal{O}_d$ when acting on the N-particle "vacuum" is just the entire N-particle sector of the fermionic Fock space. Note that some authors like to set up the bosonization scheme starting with large but finite N, taking the limit $N \to \infty$ only at the end.[85] It is then useful to know that the identity between the two Hilbert spaces is already exact for finite N. This is not obvious from the usual partition function technique.[84] It does, however, also follow from Stone's Schur function method.[83] Our approach is more naturally related to the latter, although we avoid the language and apparatus of symmetric polynomial rings (with the exception of the Newton-Girard operator relations, which, nevertheless, do not make explicit reference to polynomials). In its general, $M \geq 1$ form, however, our approach gives an exact treatment of zero modes in the correlated Laughlin states while making the relation between generators of such zero modes and the second quantized electron operator explicit. Another direct application of this explicit relation is given in the following section.

V. RELATION TO THE STRING ORDER PARAMETER

A problem of great conceptual importance is to understand the relationship between the topological order in fractional quantum Hall liquids and more conventional orders. In addition, it is desirable to characterize this type of hidden order to embed it in the more general context of topological orders in quantum matter, probably helping to unveil a so elusive classification. Much insight has been drawn in particular from parallels with the off-diagonal long range order in superfluids, where, in the quantum Hall context, the corresponding order parameter is necessarily non-local. In the approach pioneered by Read,[56] such non-locality comes about by multiplying the local electron creation operator with powers of the non-local "quasi-hole operator". In the preceding sections, we have avoided any contact between our second quantized formalism and the language of first-quantized analytic wave functions. In this section, we breach this barrier, in order to give another application of our results. The quasi-hole operator can be defined, for the disk geometry, as the operator multiplying wave functions in the lowest Landau level by the function[3]

$$U(\zeta) = \prod_{i=1}^{N} (z_i - \zeta)\,. \tag{32}$$

Here, $z_i = x_i + \mathrm{i}y_i$ ($\bar{z}_i = x_i - \mathrm{i}y_i$) represents the position of the electron i in the complex plane, and ζ is the location of the quasi-hole in complex-variable notation.

The problem of giving a second quantized form to this operator has been posed and analyzed by Stone.[57] There, the operator corresponding to Eq. (32) is defined abstractly, through its intertwining relation with the electron field creation operator $\psi(z)^\dagger$,[86]

$$\hat{U}(\zeta)\psi(z)^\dagger = (2\partial_{\bar{z}} + \frac{1}{2}z - \zeta)\psi(z)^\dagger \hat{U}(\zeta) \quad \text{(disk)} \tag{33a}$$

valid for the disk geometry, and the additional requirement

$$\hat{U}(\zeta)|0\rangle = |0\rangle\,. \tag{33b}$$

In the equation above $2\partial_{\bar{z}} = \partial_x + \mathrm{i}\partial_y$. Equations (33) allow one to work out the effect of the operator $\hat{U}(\zeta)$ on arbitrary states created out of the vacuum by means of products of electron creation operators. It can be argued[57] that this action agrees with that defined by

the multiplication of first quantized wave functions with the function given in Eq. (32). However, the relation of the operator $\hat{U}(\zeta)$ with the second quantized electron creation operator remains somewhat obscure. Here we specify this relation explicitly, which is needed in order to specify Read's non-local (string) order parameter $\hat{K}(\zeta)$.

It is easy to multiply out the product in Eq. (32), giving

$$U(\zeta) = \sum_{d=0}^{N} (-\zeta)^{N-d} s_d \,, \tag{34}$$

where s_d is the elementary symmetric polynomial defined in Eq. (12). In the above, we have identified explicitly the second quantized operators that facilitate multiplication with elementary symmetric polynomials, and expressed them through electron creation and annihilation operators. This gives

$$\hat{U}(\zeta) = \sum_{d=0}^{N} (-\zeta)^{N-d} e_d \,, \tag{35}$$

where the e_d are inferred from Eq. (13), except that we must now work with the normalization conventions of the disk geometry. This corresponds to applying a similarity transformation $e_d \to e^{\mathfrak{S}} e_d e^{-\mathfrak{S}}$ to all of the operators, which is defined via[38]

$$e^{\mathfrak{S}} c_r e^{-\mathfrak{S}} = \mathcal{N}_r^{-1} c_r, \quad e^{\mathfrak{S}} c_r^\dagger e^{-\mathfrak{S}} = \mathcal{N}_r c_r^\dagger \,, \tag{36}$$

and $\mathcal{N}_r$ is a factor related to the normalization constant of single particle orbitals for the geometry in question. For the disk geometry, we have $\mathcal{N}_r = \sqrt{2\pi 2^r r!}$. In this section, by e_d we will always mean the operators that have undergone the appropriate similarity transformation in question (and continue to represent the multiplication with elementary symmetric polynomials in the first quantized picture of this geometry). We denote the original operators defined in Eq. (13) by E_d, which correspond to the geometry of an infinitely thick cylinder. The E_d satisfy the commutation relation

$$[E_d, c_r^\dagger] = c_{r+1}^\dagger E_{d-1} \,, \tag{37}$$

which, after conjugation with $e^{\mathfrak{S}}$, becomes

$$[e_d, c_r^\dagger] = \frac{\mathcal{N}_{r+1}}{\mathcal{N}_r} c_{r+1}^\dagger e_{d-1} \,, \tag{38}$$

or, for the disk geometry

$$[e_d, c_r^\dagger] = \sqrt{2r+2}\, c_{r+1}^\dagger e_{d-1} \quad \text{(disk)}. \tag{39}$$

It is advantageous to rewrite Eq. (35) as

$$\hat{U}(\zeta) = (-\zeta)^{\hat{N}} \sum_{d=0}^{\infty} (-\zeta)^{-d} e_d \,, \tag{40}$$

where $\hat{N}$ is the particle number operator, and we have used the fact that e_d annihilates states with particle number $N < d$. Hence the operator depends explicitly on particle number only through the trivial pre-factor. The latter is not all that important when working within a subspace of constant particle number, but is crucial in the following intertwining relation,

$$\hat{U}(\zeta) c_r^\dagger = -\zeta c_r^\dagger \hat{U}(\zeta) + \sqrt{2r+2}\, c_{r+1}^\dagger \hat{U}(\zeta) \quad \text{(disk)}, \tag{41}$$

which follows straightforwardly from Eqs. (39) and (40). Eq. (41) allows us to show directly that the second quantized quasi-hole operator, which we have explicitly defined in terms of the electron operator in Eq. (40), does indeed have the property (33a), as originally conjectured by Stone.[57] Here we prove relation (33a) and extend it to arbitrary geometries. To this end, we write out the mode expansion of the field operator $\psi^\dagger(z)$ in terms of $c_r^\dagger$s:[86]

$$\psi^\dagger(z) = \sum_{r=0}^{\infty} \phi_r^*(z) c_r^\dagger \,, \tag{42}$$

where, for the disk, the single particle orbitals are

$$\phi_r(z) = \frac{1}{\sqrt{2\pi 2^r r!}} z^r e^{-|z|^2/4} \quad \text{(disk)}. \tag{43}$$

With the use of Eq. (41), one obtains

$$\hat{U}(\zeta)\psi^\dagger(z) = -\zeta \psi^\dagger(z)\hat{U}(\zeta) + \sum_{r=1}^{\infty} \frac{\mathcal{N}_r}{\mathcal{N}_{r-1}} \phi_{r-1}^*(z) c_r^\dagger \, \hat{U}(\zeta) \,, \tag{44}$$

where we have again identified $\sqrt{2r}$ with $\frac{\mathcal{N}_r}{\mathcal{N}_{r-1}}$ for greater generality. We now note that indeed Eq. (33a) follows simply from the fact that

$$D = 2\partial_{\bar{z}} + \frac{1}{2} z \quad \text{(disk)} \tag{45}$$

is a differential operator that satisfies

$$D\phi_0^*(z) = 0 \tag{46a}$$

and

$$D\phi_r^*(z) = \frac{\mathcal{N}_r}{\mathcal{N}_{r-1}} \phi_{r-1}^*(z) \quad \text{for } r > 0. \tag{46b}$$

We also note that the operator (40) trivially satisfies Eq. (33b), which completes the demonstration that our expression (40) satisfies the axiomatic properties of Ref. 57.

As a further application, we generalize Eqs. (33a) and (41) to the cylinder and sphere geometries. We begin with Eq. (41) which is useful in practical computations. This is trivial, since all we have to do is to identify the factor $\sqrt{2r+2}$ once again with the ratio $\mathcal{N}_{r+1}/\mathcal{N}_r$ of normalization constants of the corresponding single particle orbitals. For the cylinder of finite radius $R_y = 1/\kappa$, we use the standard (Landau) gauge where

$$\phi_r(z) = (4\pi^3)^{-1/4} \sqrt{\kappa} e^{-\frac{1}{2}\kappa^2 r^2} \xi^r e^{-\frac{1}{2}x^2} \quad \text{(cylinder)}, \tag{47}$$

with $\xi = e^{\kappa z}$ playing the role of the variable in the polynomial part of wave functions. The constant $\mathcal{N}_r$ is the inverse of the factor multiplying the monomial (here ξ^r),[38] where r-independent factors can be dropped as only ratios of these constants matter in the following. We may hence choose $\mathcal{N}_r = \exp(\frac{1}{2}\kappa^2 r^2)$ for the cylinder. This then yields

$$\hat{U}(\zeta)c_r^\dagger = -\zeta c_r^\dagger \hat{U}(\zeta) + e^{\kappa^2(r+\frac{1}{2})} c_{r+1}^\dagger \hat{U}(\zeta) \quad \text{(cylinder)}, \tag{48}$$

where, for the cylinder, ζ is related to the complex quasihole coordinate as ξ is to z.

For the spherical geometry, we follow Ref. 77 in identifying a sphere of radius R with the infinite plane with complex coordinate z via stereographic projection. As described there, this leads to normalized single particle orbitals of the form

$$\phi_r(z) = \frac{1}{(2R)^{1+r}} \sqrt{\frac{N_\Phi+1}{4\pi}} \binom{N_\Phi}{r}^{\frac{1}{2}} \frac{z^r}{(1+\frac{|z|^2}{4R^2})^{1+\frac{N_\Phi}{2}}} \qquad r = 0 \ldots N_\Phi \text{ (sphere)}, \tag{49}$$

with N_Φ being the number of magnetic flux quanta piercing the sphere. Following the same reasoning as for the cylinder, this leads to $\mathcal{N}_r = (2R)^{1+r}\binom{N_\Phi}{r}^{-\frac{1}{2}}$, giving

$$\hat{U}(\zeta)c_r^\dagger = -\zeta c_r^\dagger \hat{U}(\zeta) + 2R\sqrt{\frac{r+1}{N_\Phi - r}} c_{r+1}^\dagger \hat{U}(\zeta) \quad \text{(sphere)}. \tag{50}$$

It may be of some interest to also generalize Eq. (33a) to the other geometries, the starting point being the general Eq. (44). We will see that the local character of Eq. (33a) is somewhat lost for a finite-radius cylinder or sphere, and is recovered only in the large-radius limit. The generality of the discussion following Eq. (44) implies that all we need to do is to find a differential operator D that satisfies Eqs. (46), for the appropriate orbitals and normalization factors. We then have

$$\hat{U}(\zeta)\psi(z)^\dagger = (D-\zeta)\psi(z)^\dagger \hat{U}(\zeta)\,. \tag{51}$$

For the cylinder, it is more convenient to let the orbital index r roam over all integers, including negative. This leaves the relevant commutation relations intact. (One only looses the equivalence between the zero mode counting of the disk and cylinder geometries, which mattered in Sec. III but not in the present context.) Thus, for the cylinder, we abandon Eq. (46a) and only seek to enforce Eq. (46b). The latter essentially says that D must be a magnetic translation (as the orbitals magnetically translate into one another on the cylinder), except for the trivial normalization factor. Alternatively, it is elementary to see that $D = e^{-\kappa(\bar{z}+\kappa)} e^{\kappa(2\partial_z + \frac{z}{2} + \frac{\bar{z}}{2})}$ has all desired properties, where the derivative produces a factor of $e^{2r\kappa^2}$ when acting on ϕ_r, and the following term $\frac{z}{2}+\frac{\bar{z}}{2}$ only serves to cancel the derivative-action on $e^{-\frac{1}{2}x^2}$. Thus we have

$$\hat{U}(\zeta)\psi(z)^\dagger = (e^{-\kappa(\bar{z}+\kappa)} e^{\kappa(2\partial_z + \frac{z}{2} + \frac{\bar{z}}{2})} - \zeta)\psi(z)^\dagger \hat{U}(\zeta) \quad \text{(cylinder)}. \tag{52}$$

For the sphere, rather than dropping the orbital cutoff at $r = 0$ as for the cylinder, we must enforce an additional cutoff at $r = N_\Phi$, due to the finite dimensionality of the Hilbert space. We may do so by using the convention

$$c_r = c_r^\dagger = 0 \quad \text{for } r > N_\Phi. \tag{53}$$

One may see that the commutation relation (38) then still holds even for $r = N_\Phi$. Then we may seek an operator D satisfying Eqs. (46), and note that the operator

$$\Delta = \frac{4R^2}{N_\Phi}\left(\partial_{\bar{z}} + \frac{z}{4R^2}(1+\frac{|z|^2}{4R^2})^{-1}(1+\frac{N_\Phi}{2})\right) \tag{54}$$

satisfies Eq. (46a) and also Eq. (46b) for the special case of $r = 1$. Again, the terms following the derivative only serve to cancel its action on the non-holomorphic part in Eq. (49). Thus, Δ also satisfies

$$\begin{aligned}\Delta^m \phi_r^*(z) &= (\frac{4R^2}{N_\Phi})^m m! \binom{r}{m} \bar{z}^{-m} \phi_r^*(z) \\ &= \frac{(2R)^{2m-1}}{N_\Phi^m} m! \binom{r}{m} \sqrt{\frac{N_\Phi - r + 1}{r}} \bar{z}^{-(m-1)} \phi_{r-1}^*(z)\end{aligned} \quad \text{(sphere)}. \tag{55}$$

In particular, $\Delta^m \phi_r^*(z) = 0$ for $m > r$. These observations motivate the following Ansatz:

$$D = \sum_{r=1}^{N_\Phi} a_r\, \bar{z}^{r-1} \Delta^r \quad \text{(sphere)} \tag{56}$$

with $a_1 = 1$. Plugging this into Eq. (46b) for each r leads to the relations

$$(N_\Phi - r + 1)(r-1)! \sum_{m=1}^{r} \frac{1}{(r-m)!} \frac{(2R)^{2m-2}}{N_\Phi^m} a_m = 1\,, \tag{57}$$

from which the a_r may be determined recursively. It turns out that the unique solution to these equations can be given as

$$a_r = \frac{N_\Phi^r}{(2R)^{2(r-1)}} \frac{(N_\Phi - r)!}{N_\Phi!}\,. \tag{58}$$

Equation (51), along with Eqs. (56) and (58), then generalize Eq. (33a) to the sphere. It is clear from Eq. (58) that in the limit of a large sphere, higher derivatives are unimportant in Eq. (56). In particular, in the limit $N_\Phi \to \infty$ with N_Φ/R^2 fixed and equal to 2, one recovers the equation derived for the disk, Eq. (33a), as expected.[77]

In closing this section, we make two remarks. Notice, first, that the state

$$\hat{U}(\zeta)|\psi_{1/M}\rangle \tag{59}$$

is also a zero mode of a Haldane pseudo-potential-type Hamiltonian. The quasi-hole operator $\hat{U}(\zeta)$ preserves the number of particles N but changes the number of fluxes of the incompressible state $|\psi_{1/M}\rangle$. Second, our approach allows us to explicitly express Read's non-local (string) order parameter

$$\hat{K}(\zeta) = \psi(\zeta)^\dagger \hat{U}(\zeta)^M \tag{60}$$

in second-quantized form, and thus can be used to study off-diagonal long-range order in Haldane pseudo-potential-type systems

$$\lim_{|\zeta-\zeta'|\to\infty} \langle\psi_{1/M}|\hat{K}(\zeta)^\dagger \hat{K}(\zeta')|\psi_{1/M}\rangle \to \mathsf{constant}\ . \tag{61}$$

VI. DISCUSSION AND CONCLUSION

In this paper, we have developed a representation of the ring of symmetric functions using bosonic or fermionic ladder operators satisfying canonical commutation relations. In particular, one set of operators we constructed can be understood as affecting states by multiplying associated first quantized wave functions in the lowest Landau level with elementary symmetric polynomials. Our primary motivation for constructing this representation is to provide tools for an alternative, polynomial-free or second quantized approach to represent physics in the lowest Landau level. We have given three concrete and loosely connected applications. First, we have given an independent proof that our operator algebra generates all zero modes of the class of Haldane pseudo-potential Hamiltonians whose highest filling factor ground states are the $1/M$-Laughlin states. Together with Ref. 55, this completes a program[38] that allowes us to reproduces all known properties of these zero-modes that are usually derived using analytic polynomial wave functions through an alternative, second quantized formalism. The starting point here are the second quantized "lattice" versions of these pseudo-potential Hamiltonians that are special instances of frustration free but *infinitely*-ranged lattice models. We believe that our results will impact the more general development of such models which may have important applications in fractional Chern insulators.[19–36] In many ways our approach can be thought of as the *microscopic* bosonization of the lattice pseudo-potential Hamiltonians. Here, "microscopic" means that the relation between boson modes creating exact zero energy eigenstates on the one hand and the microscopic, second-quantized electron creation operator on the other is completely manifest. It may thus be no surprise that our approach also leads to an alternative proof of one of the central results in bosonization, the equivalence of the bosonic and the fermionic Fock spaces. As a third application, we have extended some earlier results by Stone[57] on the second quantized version of Read's order parameter[56], or its "string factor" that creates local quasi-holes. We have given the explicit expression of this operator to the local electron creation operator and generalized its commutation relation with the latter to the cylinder and spherical geometries. We believe that the extension of our results to other models –both old and new– will be an interesting direction for the future.

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Appendix A: Newton-Girard operator relations between operators $\mathcal{O}_d$ ans e_d

In this Appendix we show explicitly that the operators e_d and $\mathcal{O}_d$ satisfy the *Newton-Girard operator* relations. Equations (4) and (13) are reproduced here for convenience as follows:

$$\mathcal{O}_d = \sum_r c^\dagger_{r+d} c_r\,, \tag{A1}$$

$$e_d = \frac{1}{d!}\sum_{r_1\ldots r_d} c^\dagger_{r_1+1} c^\dagger_{r_2+1}\cdots c^\dagger_{r_d+1} c_{r_d}\cdots c_{r_2} c_{r_1}\,, \tag{A2}$$

The operators e_d satisfy the commutation relations (37) with electron creation operators. Here we need the analogous relation for electron annihilation operators:

$$[e_d, c_r] = -e_{d-1} c_{r-1}\,. \tag{A3}$$

From the definition (A2), we also infer the following recursion relation,

$$e_d = \frac{1}{d}\sum_r c^\dagger_{r+1} e_{d-1} c_r\,, \tag{A4}$$

where again $e_0 = 1$. Now using the commutator (A3) we can write this as:

$$\begin{aligned} e_d &= \frac{1}{d}\sum_r c^\dagger_{r+1}(c_r e_{d-1} - e_{d-2}c_{r-1}) \\ &= \frac{1}{d}(\mathcal{O}_1 e_{d-1} + (-1)\sum_r c^\dagger_{r+1} e_{d-2} c_{r-1})\,. \end{aligned} \tag{A5}$$

By repeatedly applying the last step to the sum in the second line, we arrive at the following:

$$e_d = -\frac{1}{d}\sum_{k\geq 1}(-1)^k \mathcal{O}_k e_{d-k}\,, \tag{A6}$$

which is equvalent to the *Newton-Girard* relation (16).

Appendix B: Proof of Eqs. (24)

We consider Eq. (24a) first, here reproduced as

$$|\lambda\rangle \geq |\lambda'\rangle \Leftrightarrow \mathfrak{e}_{\bar{d}}|\lambda\rangle \geq \mathfrak{e}_{\bar{d}}|\lambda'\rangle\,. \tag{B1}$$

We identify $|\lambda\rangle$, $|\lambda'\rangle$ with partitions $\{\lambda_i\}_{i=1\ldots N}$, $\{\lambda'_i\}_{i=1\ldots N}$, respectively. Then $\mathfrak{e}_{\bar{d}}|\lambda\rangle$, $\mathfrak{e}_{\bar{d}}|\lambda'\rangle$, respectively, correspond to partitions $\{\ell_i\}_{i=1\ldots N}$, $\{\ell'_i\}_{i=1\ldots N}$, where

$$\begin{aligned} \ell_i = \lambda_i + 1\,,\ \ \ell'_i = \lambda'_i + 1 \quad &\text{for } i = 1\ldots d,\\ \ell_i = \lambda_i\,,\ \ \ell'_i = \lambda'_i \quad &\text{for } i = d+1\ldots N. \end{aligned} \tag{B2}$$

This clearly implies

$$\sum_{i=1}^{n}(\ell_i - \ell'_i) = \sum_{i=1}^{n}(\lambda_i - \lambda'_i) \quad \text{for all } n = 1\ldots N, \tag{B3}$$

and thus, by the criterion for dominance (20), Eq. (B1) follows.

Now for Eq. (24b), reproduced here as

$$\mathfrak{e}_{\bar{d}}|\lambda\rangle \geq \mathfrak{e}_S|\lambda\rangle \quad \text{if } |S| = d, \tag{B4}$$

we reiterate once more that, for fermions, we consider this equation automatically satisfied if the right hand side vanishes (the left hand side can never vanish). Let again $|\lambda\rangle$, $\mathfrak{e}_{\bar{d}}|\lambda\rangle$ be associated with partitions $\{\lambda_i\}_{i=1\ldots N}$, , $\{\ell_i\}_{i=1\ldots N}$, respectively, and let $\mathfrak{e}_S|\lambda\rangle$ be associated with a partition $\{\ell'_i\}_{i=1\ldots N}$. Let

$$\begin{aligned} \delta_i = 1 \quad &\text{for } i = 1\ldots d,\\ \delta_i = 0 \quad &\text{for } i = d+1\ldots N. \end{aligned} \tag{B5}$$

Then

$$\ell_i = \lambda_i + \delta_i\,. \tag{B6}$$

Furthermore, we can define numbers s_i, $i = 1\ldots N$ with

$$s_i = 1 \quad \text{for } i \in S, \quad s_i = 0 \quad \text{for } i \notin S, \tag{B7}$$

i.e., s_i is the characteristic function of the set S, such that

$$\ell'_i = \lambda_i + s_i\,. \tag{B8}$$

We must convince ourselves that the ℓ'_i defined by the above equation may still be assumed to satisfy $\ell'_i \geq \ell'_{i+1}$, as long as the λ_i satisfy the analogous relation. This can only become problematic for bosons, if $\lambda_i = \lambda_{i+1}$ and $s_i = 0$, $s_{i+1} = 1$. However, we may then instead consider a new sequence of numbers s'_i which is identical to s_i, except with the entries for i and $i+1$ swapped. This is then the characteristic function of a set S' different from S, where, however, $\mathfrak{e}_S$ and $\mathfrak{e}_{S'}$ have the same effect on $|\lambda\rangle$. For this reason, we may assume without loss of generality that Eq. (B8) leads to a monotonously decreasing set of numbers, as is must for the ℓ'_i to define a partition. Then the dominance criterion $\sum_{i=1}^{n}\ell_i \geq \sum_{i=1}^{n}\ell'_i$ is equivalent to

$$\sum_{i=1}^{n}\delta_i \geq \sum_{i=1}^{n} s_i\,, \tag{B9}$$

which, using the fact that $|S| = d$, evidently follows for all $n \leq N$ from Eqs. (B5), (B7).

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