Dynamical spin structure factor of one-dimensional interacting fermions

Vladimir A. Zyuzin and Dmitrii L. Maslov

Phys. Rev. B **91**, 081102 — Published 4 February 2015

DOI: 10.1103/PhysRevB.91.081102

Dynamical spin structure factor of one-dimensional interacting fermions

Vladimir A. Zyuzin and Dmitrii L. Maslov
Department of Physics, University of Florida, P.O. Box 118440, Gainesville, Florida 32611-8440, USA

We revisit the dynamic spin susceptibility, $\chi(q,\omega)$, of one-dimensional interacting fermions. To second order in the interaction, backscattering results in a logarithmic correction to $\chi(q,\omega)$ at $q \ll k_F$, even if the single-particle spectrum is linearized near the Fermi points. Consequently, the dynamic spin structure factor, $\mathrm{Im}\chi(q,\omega)$, is non-zero at frequencies above the single-particle continuum. In the boson language, this effect results from the marginally irrelevant backscattering operator of the sine-Gordon model. Away from the threshold, the high-frequency tail of $\mathrm{Im}\chi(q,\omega)$ due to backscattering is larger than that due to finite mass by a factor of k_F/q. We derive the renormalization group equations for the coupling constants of the g-ology model at finite ω and q and find the corresponding expression for $\chi(q,\omega)$, valid to all orders in the interaction but not in the immediate vicinity of the continuum boundary, where the finite-mass effects become dominant.

<u>*Introduction*</u> Bosonization is the most common way to describe one-dimensional (1D) interacting fermions.[1] If the lattice effects are not essential, an exact correspondence between the fermion and fermion-hole (boson) operators maps the charge sector of the system onto a gas of free bosons [the Tomonaga-Luttinger liquid (TLL)]. The spin sector, however, is not free but maps onto the sine-Gordon model. The non-Gaussian (cosine) term of this model results from backscattering of fermions with opposite spins. If the interaction between the original fermions is repulsive, the backscattering term represents a marginally irrelevant operator and is renormalized down to zero at the fixed point, where the spin sector also becomes free. At intermediate energy scales, such marginally irrelevant operators lead to logarithmic renormalizations of the observables.[2] Since the original paper by Dzyaloshinskii and Larkin (DL),[3] it has been known that the backscattering operator gives rise to the logarithmic temperature (or external magnetic field) corrections to the static spin susceptibility. In Refs. 4 and 5, it was shown that the static spin susceptibility also depends logarithmically on the external momentum q at small q. In addition, both the spin- and charge susceptibilities at $2k_F$ acquire multiplicative logarithmic renormalizations.[1,6]

In this work, we focus on *dynamics* of the long-wavelength part of the spin response. First, we need to outline the differences between the charge and spin sectors. As charge bosons are free at all energies, the dynamical charge structure factor (the imaginary part of the charge susceptibility at finite frequency, ω, and momentum, q) is a delta function centered at the boson dispersion, $\omega = v_c q$, which is represented by a straight line in Fig. 1A. This result differs from that for free fermions only in that the Fermi velocity, v_F, is replaced by the renormalized charge velocity, v_c. A non-zero width of the charge structure factor appears only if one goes beyond the TLL paradigm by taking into account finite curvature (inverse mass) of the fermion spectrum. In the pioneering paper[7] and subsequent work (for a review, see Ref. 8), it was shown that the combined effect of the curvature and interactions results in many new features in the charge structure factor, such as edge singularities at the boundaries of the continuum and the high-frequency tail both of which are absent for free fermions.

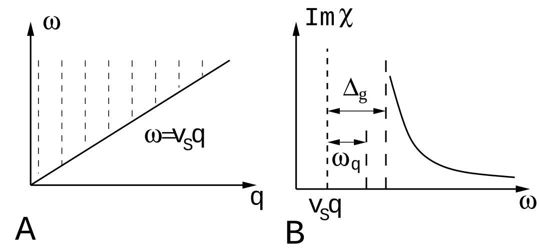

FIG. 1. A. Particle-hole excitations at small momenta and frequency in 1D fermion system. The line $\omega = v_F q$ corresponds to the continuum. The hatched area is the domain of incoherent spin excitations arising due to backscattering processes. B. Schematic frequency dependence of the DSSF at given momentum q. Renormalization group flows into the strong coupling regime at frequency Δ_g [Eq. (16)] above the threshold. At distance $\omega_q = q^2/m$ to the threshold, the effect of finite mass becomes more important than that of backscattering.

As far as the spin channel is concerned, it is common to treat the backscattering operator via renormalization group (RG). As the fixed point corresponds to free bosons, the formal result for the dynamical spin structure factor (DSSF) at the fixed point is also a delta function with v_c replaced by the spin velocity, v_s. The problem with this argument is that DSSF is measured at finite ω and q and thus away from the fixed point. Therefore, its dependences on ω and q must reflect the flow at finite rather than infinite RG time.

In this paper, we revisit the DSSF of a 1D interacting fermion system. Besides being of a fundamental interest on its own, the DSSF is relevant for a number of experiments in both condensed-matter and cold-atom systems, such as inelastic neutron and spin-resolved X-ray spectroscopies, nuclear magnetic resonance, spin Coulomb drag,[9] etc. First, we show by direct perturbation theory that the logarithmic renormalization of the dynamical spin susceptibility occurs in a Lorentz-invariant way, via a $\ln(v_F^2 q^2 - \omega^2)$ term. This already implies that, in contrast to the free-fermion case, the DSSF is non-

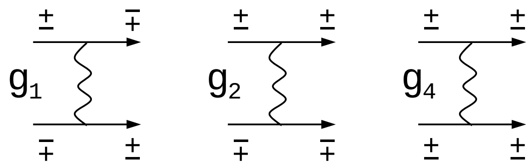

FIG. 2. Scattering amplitudes of the fermion-fermion interaction $g_{1,2,4}$.

zero in the entire sector $|\omega| > v_F|q|$ (the hatched region in Fig. 1A). In contrast to the charge sector, this high-frequency tail occurs even without taking into account the finite-curvature effects. Next, we collect all leading logarithmic terms by using RG for the spin vertex. Our final result for the spin susceptibility reads

$$\chi(q,\omega) = \chi_0 \frac{v_s v_F q^2}{(v_s q)^2 - \omega^2 - i\delta} \times \left\{ 1 + \frac{g_1}{2} \frac{1}{1 + \frac{g_1}{2} \ln\left[\frac{(v_s\Lambda)^2}{(v_s q)^2 - \omega^2 - i\delta}\right]} \right\}, \tag{1}$$

where $\chi_0 = 2\mu_B^2/\pi v_F$ is the spin susceptibility of 1D free fermions, μ_B is the Bohr magneton, g_1 is the (dimensionless) backscattering amplitude, Λ is the ultraviolet cutoff, $\delta = \mathrm{sgn}\omega 0^+$, and

$$v_s = v_F\sqrt{1-g_1} \tag{2}$$

is the spin velocity.[10,11] At $\omega = 0$ and to second order in g_1, Eq. (1) reduces to the $\ln q$ correction of Refs. 4 and 5. Also, setting $\omega = 0$ and replacing $v_F q$ by either temperature (T) or the Zeeman energy in the external magnetic field (H), we reproduce the DL result.[3] To second order in g_1, our result is in line with those of Refs. 14 and 15 for a spin-1/2 Heisenberg antiferromagnetic chain. In that case, the high-frequency tail of the DSSF occurs due to a marginally irrelevant operators arising from umklapp scattering. The profile of $\mathrm{Im}\chi(q,\omega)$ is shown schematically in Fig. 1B. Sufficiently close to the threshold, the divergence in $\mathrm{Im}\chi(q,\omega)$ must be regularized by the finite-curvature effects–see below for a more detailed discussion of the crossover between the high-frequency tail and threshold singularities.

Feynman diagrams in the fermion language. As usual, we linearize the fermion spectrum near the Fermi energy and decompose the fermion operators into the left- and right-movers, so that the Hamiltonian of a free system is

$$H = \sum_{s=\pm,\alpha} \int dp\ \xi_s(p) c^\dagger_{s,\alpha}(p) c_{s,\alpha}(p), \tag{3}$$

where $\xi_\pm(p) = \pm v_F p$, with p being a deviation from the Fermi momenta, v_F the Fermi velocity, and $c_{\pm,\alpha}(p)$ the right/left-moving fermion with spin α. Linearization presumes that the fermion momenta are bounded by $\pm\Lambda_F$. Fermions interact via a short-range and SU(2)-invariant potential $U(q)$, parameterized by three (dimensionless) scattering amplitudes, g_1, g_2, and g_4, which are defined in Fig. 2. To first order in U, $g_2 = g_4 = U(0)/\pi v_F$ and $g_1 = U(2k_F)/\pi v_F$. We assume that the Fermi momentum is not commensurate with the lattice and thus neglect umklapp scattering.

We now calculate the spin susceptibility at small but finite ω_m and q via a perturbation theory in the coupling constants g_1, g_2, and g_4. The free spin susceptibility–diagram A in Fig. 3– is given by

$$\chi^{(0)}(q,\omega_m) = \chi_0 \frac{(v_F q)^2}{\omega_m^2 + (v_F q)^2}, \tag{4}$$

where ω_m is the Matsubara frequency. Upon analytic continuation to real frequencies ($\omega_m \to -i\omega + 0^+$), the imaginary part of the susceptibility is $\propto q^2\delta(\omega^2 - v_F^2 q^2)$, which corresponds to well-defined spin excitations.

The first-order corrections are given by diagrams 3B and 3C. It can be shown [see Supplementary Material (SM)] that diagrams with g_4 sum up to zero, while diagrams with g_2 cannot be constructed at this order. For the backscattering contribution, we obtain

$$\delta\chi^{(1)}(q,\omega_m) = g_1\chi_0 \left[\frac{(v_F q)^2}{\omega_m^2 + (v_F q)^2}\right]^2. \tag{5}$$

To second order in the interaction, there are five nontrivial diagrams for the spin susceptibility, presented in Fig. 3D-H. Calculations show that all the $g_1 g_2$ terms from diagrams 3F-H sum up to zero. All the g_4^2 terms from all the second-order diagrams sum up to zero as well. Finally, the g_2^2 terms from diagrams 3D and 3E cancel each other. What remains is the backscattering, g_1^2 contribution from diagrams 3D and 3E. In the leading logarithmic approximation, we find

$$\delta\chi^{(2)}(q,\omega_m) = -\frac{1}{4} g_1^2 \chi_0 \frac{(v_F q)^2}{\omega_m^2 + (v_F q)^2} \ln\left[\frac{(v_F\Lambda)^2}{\omega_m^2 + (v_F q)^2}\right], \tag{6}$$

where Λ is the cutoff imposed on the interaction. At $\omega_m = 0$, Eq. (6) reduces to the result of Refs. 4 and 5. Upon analytic continuation, the logarithmic factor gives rise to a non-zero DSSF at $|\omega| > v_F|q|$. In the next section, we will employ RG to calculate the spin susceptibility to first order in the renormalized vertex. We outline the resulting expression for the spin susceptibility:

$$\chi(q,\omega_m) = \chi_0 \left\{ \frac{(v_F q)^2(1 - g_1/2)}{\omega_m^2 + (v_F q)^2} + g_1 \frac{(v_F q)^4}{\left[\omega_m^2 + (v_F q)^2\right]^2} + \frac{g_1}{2} \frac{(v_F q)^2}{\omega_m^2 + (v_F q)^2} \frac{1}{1 + \frac{g_1}{2}\ln\left[\frac{(v_F\Lambda)^2}{\omega_m^2 + (v_F q)^2}\right]} \right\} \tag{7}$$

where a factor of $-g_1/2$ in the numerator of the first term was added to compensate for the first-order contribution from the third term. The first and second terms in Eq. (7) can be combined into an expansion of $\chi_{\mathrm{f}}(q,\omega_m) \equiv \chi_0 v_s v_F q^2 / \left[\omega_m^2 + (v_s q)^2\right]$ to order g_1, which

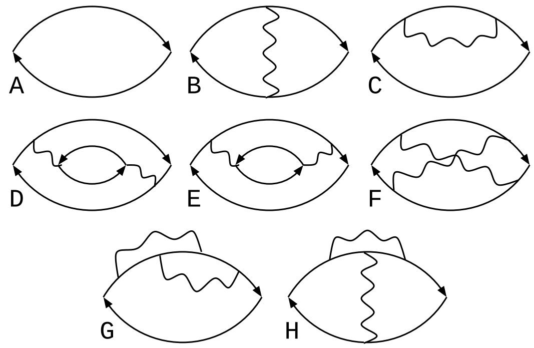

FIG. 3. Feynman diagrams for the spin susceptibility to second order in the interaction (wavy line). The Aslamazov-Larkin diagrams (not shown) vanish identically by SU(2) symmetry. The second-order RPA diagram (also not shown), obtained from diagram B by inserting one more wavy line parallel to the first one, does not contain a logarithmic singularity and is thus ignored.

is the spin susceptibility of free bosons at the fixed point, where the Luttinger parameter of the spin channel is renormalized to unity. We surmise that all non-logarithmic terms can be absorbed into $\chi_{\mathrm{f}}(q,\omega_m)$. In the last term of Eq. (7), we cannot distinguish between v_s and v_F within the leading logarithmic approximation. However, guided by the general RG principle, we conjecture that v_F in the last term must be replaced by v_s as well. Performing these replacements, we obtain (after analytic continuation) the result announced in Eq. (1). Next, we are going to show that RG does indeed reproduce the logarithmic part of the result in Eq. (7).

Renormalization group. From now on, we neglect the g_4 processes, as they do not flow under RG. The interaction vertex, shown graphically in Fig. 4A, can be decomposed into the spin and charge parts as

$$\frac{1}{\pi v_F}\Gamma^{\alpha\beta}_{\mu\eta} = -\frac{1}{2}\gamma_1\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\mu\eta} - \left(\frac{1}{2}\gamma_1 - \gamma_2\right)\delta_{\alpha\beta}\delta_{\mu\eta}, \quad (8)$$

where $\gamma_{1,2}$ are the renormalized back- and forward-scattering amplitudes. As these vertices will be used to find the spin susceptibility at finite ω and q, we will need to know them away from the Fermi surface along both the energy and momentum axes. The equations for γ_1 and γ_2, derived in SM and shown graphically in Fig. 4D, are of the standard form[1,16,17]

$$\frac{d\gamma_1(\ell)}{d\ell} = -\gamma_1^2(\ell), \quad \frac{d\gamma_2(\tilde{\ell})}{d\tilde{\ell}} = -\frac{\gamma_1^2(\tilde{\ell})}{2}, \quad (9)$$

except that the RG times, ℓ and $\tilde{\ell}$, are different. The vertex γ_1, which is renormalized in the particle-hole channel, evolves with $\ell \equiv \ln\left[\frac{v_F\Lambda}{\sqrt{\Omega_n^2+(v_FQ)^2}}\right]$, where Ω_n and Q are the Matsubara frequency and momentum transfers through the vertex, correspondingly. The vertex γ_2, which is renormalized in the particle-particle channel, depends on the total incoming frequency and momenta.

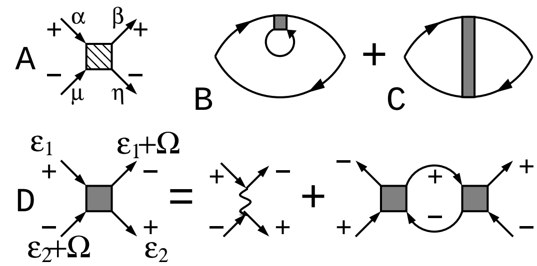

FIG. 4. A. Interaction vertex given by Eq. (8). B,C. Spin susceptibility to first order in renormalized vertex. D. Graphical equation for the backscattering amplitude γ_1.

The meaning of Eq. (9) is that one first solves for γ_1 as a function of ℓ and then replaces ℓ by $\tilde{\ell}$ to find γ_2. The initial values are $\gamma_i(\ell=0)=g_i$, $i=1,2$. (Details of the derivation are given in SM.) Solving Eq. (9), one obtains

$$\gamma_1(\ell) = \frac{g_1}{1+g_1\ell}, \quad \gamma_1(\tilde{\ell}) - 2\gamma_2(\tilde{\ell}) = \text{const.} \quad (10)$$

We are now in a position to calculate the renormalized spin susceptibility at finite ω_m and q, using γ_1 as an effective interaction. To first order in γ_1, there are only two diagrams for the spin susceptibility: diagrams B and C in Fig. 4. Combining the two diagrams together and using Eq. (10), we obtain renormalized the part of the spin susceptibility along with the linear in g_1 term (see SM for details of the calculations):

$$\begin{aligned}\chi^{(\ln)} &= -4\left[\frac{(v_Fq)^2}{|\bar{\omega}|^4}\right]\int\frac{dQd\Omega_n}{(2\pi)^2}\ln\left[\frac{(v_F\Lambda)^2}{|\bar{\Omega}+\bar{\omega}|^2}\right]\gamma_1(\ell)\\ &= \chi_0\frac{g_1}{2}\frac{(v_Fq)^2}{\omega_m^2+(v_Fq)^2}\frac{1}{1+\frac{g_1}{2}\ln\left[\frac{(v_F\Lambda)^2}{\omega_m^2+(v_Fq)^2}\right]}, \quad (11)\end{aligned}$$

where $\bar{\Omega} = \Omega_n + iv_FQ$ and $\bar{\omega} = \omega_m + iv_Fq$. This result indeed coincides with the last term in Eq. (7), and thus the conjecture leading to Eq. (1) is proven.

Equation (1) is the central result of this paper, which shows that DSSF is non-zero at $|\omega| > v_F|q|$. Away from the free-boson pole, the DSSF is given by

$$\mathrm{Im}\chi(q,\omega) = \chi_0\frac{\pi g_1^2}{4}\frac{v_Fv_sq^2}{\omega^2-(v_sq)^2}\frac{\mathrm{sgn}\omega}{\left\{1+\frac{g_1}{2}\ln\left[\frac{(v_s\Lambda)^2}{\omega^2-(v_sq)^2}\right]\right\}^2+\left[\frac{\pi g_1}{2}\right]^2}. \quad (12)$$

The conditions for the validity of Eq. (12) near the threshold are discussed before the concluding paragraph of this paper.

Fermi liquid in 1D. Putting $\omega_m = 0$, replacing $v_Fq \to \max\{T,\mu_BH\}$ under the logarithm, and expanding back the non-logarithmic term to order g_1 in Eq. (1), we reproduce the DL result for the static spin susceptibility:[3]

$$\chi(T,H) = \chi_0\left\{1+\frac{g_1}{2}\left[1+\frac{1}{1+g_1\ln\left[\frac{v_F\Lambda}{\max(T,\mu_BH)}\right]}\right]\right\}. \quad (13)$$

We now show that the DL result can be reproduced by the Fermi-liquid (FL) theory, if one replaces the FL parameters by running values of the scattering amplitudes. We recall the FL expression for the spin susceptibility[18]

$$\chi = \chi_0 \frac{m^*}{m} \frac{1}{1+\Gamma_s \frac{m^*}{m} Z^2}, \tag{14}$$

where $m^* = m\left[1-\partial_\omega \Sigma(\omega,k)\right]/\left[1+\partial_k\Sigma(\omega,k)/v_F\right]|_{\omega,k=0}$ is the effective mass, $Z = \left[1-\partial_\omega\Sigma(\omega,k)\right]^{-1}|_{\omega,k=0}$ is the quasiparticle residue, and Γ_s is the spin part of the interaction vertex. Since the original interaction is static, it gives a frequency-independent self-energy to first order. Therefore, the Z-factor is not renormalized to this order, while the momentum dependence of the self-energy amounts to renormalization of the effective mass $m^* = m(1+g_1/2)$ both for right- and left-moving fermions. According to Eq. (8), $\Gamma_s = -\gamma_1/2$, where γ_1 is given by Eq. (10) with $\ell = \ln\left[\frac{v_F\Lambda}{\max(T,\mu_B H)}\right]$ in the static case. Substituting these m^* and Γ_s into Eq. (14) and expanding to first order in γ_1, we reproduce the DL result, Eq. 13. Although the FL theory is not valid in $D=1$, it is still valid in $D=1+\varepsilon$ with $\varepsilon\to 0^+$. Our example shows that a logarithmic flow of the scattering amplitude with ε reproduces the correct result even in $D=1$.

Another interesting feature of the DL result is that the fixed-point value $\chi=\chi_0(1+g_1/2)$ is renormalized by g_1. This seems to contradict RG because g_1 flows to zero at the fixed point. In fact, this means that not all the coupling constants in the perturbative result should be replaced by their running values: some remain at their bare values evaluated at the ultraviolet cutoff. This is an example of the "anomaly" frequently encountered in massless field-theoretical models.[19] Another example of such an anomaly is the specific heat of 1D fermions.[10]

Connection to bosonization. The fermion language is not a preferred one: one can equally well obtain the same results in the boson language. However, one has to be careful about not taking the limit of local interaction too soon. Assuming a non-local interaction potential $U(x-x')$, we bosonize the Hamiltonian, treating the interaction as local in the Gaussian part as non-local in the non-Gaussian part. For clarity, we distinguish between the backscattering amplitudes of fermions with the same and opposite spins, $g_{1||}$ and $g_{1\perp}$, correspondingly. This yields[10]

$$\begin{aligned} H &= \frac{1}{2\pi}\sum_{i=c,s}\int dx\left[\frac{v_i}{K_i}(\partial_x\phi_i)^2 + v_iK_i(\partial_x\theta_i)^2\right] \\ &+ \frac{\Lambda_F^2}{2\pi^2}\sum_{\alpha=\pm}\int\int dxdx'\ \cos\left\{\sqrt{2}\left[\phi_s(x)+\alpha\phi_s(x')\right]\right\} \\ &\times\cos\left\{\sqrt{2}\left[\phi_c(x)-\phi_c(x')\right]+2k_F(x-x')\right\}U(x-x'), \end{aligned} \tag{15}$$

where $\phi_{c(s)}$ and $\theta_{c(s)}$ are usual position and momentum boson fields in the charge (c) and spin (s) sectors, $v_s=v_F$, $K_s=1$, and explicit expressions for v_c and K_c are given in SM. If the local limit is taken also in the non-Gaussian part, we obtain a spin-charge separated sine-Gordon model with the coupling $g_{1\perp}$ in the cosine term and $v_s = v_F\sqrt{1-g_{1\|}}$, and $K_s = 1/\sqrt{1-g_{1\|}}$. With Hamiltonian (15), one can construct a perturbation theory for the spin susceptibility $\chi(x,\tau) = -\langle T_\tau \partial_x\phi_s(x,\tau)\partial_x\phi_s(0,0)\rangle$. In doing so, we reproduce the same diagrams for the spin susceptibility as in the fermion approach. The first-order term of the fermion approach, Eq. (5), is reproduced correctly only if one keeps non-local interaction in the non-Gaussian term. The reason is that a part of this result comes from mass renormalization (diagram C in Fig. 3), which is absent to first order in the local interaction. Starting from second order, one can take the local limit. The results of the boson and fermion approaches are identical, as they should be. In particular, the second-order result for $\chi(q,\omega)$ [Eq. (6)] can obtained by expanding the partition function of the sine-Gordon model to second order in the backscattering operator, as it was done in Ref. 15 for the umklapp operator. In our case, this gives $\mathrm{Im}\chi(q,\omega)\propto g_{1\perp}^2 q^2(\omega^2-v_s^2q^2)^{2K_s-3}$ which, upon expanding near weak coupling ($K_s\approx 1+g_{1||}/2$) and setting $g_{1\perp}=g_{1||}$, reproduces Eq. (12) to third order in g_1.

Finite-mass effects. Taking into account finite curvature of the fermion dispersion is the only way to smear the delta-function singularity in the dynamic *charge* structure factor.[7,8] In the spin sector, there are two competing effects that lead to a non-zero DSSF outside the continuum: the marginally irrelevant backscattering operator and finite curvature. For massive fermions with dispersion $p^2/2m$, the effect of finite curvature becomes important when the distance to the threshold, $\Delta\equiv|\omega|-v_s|q|$, becomes comparable to $\omega_q\equiv q^2/m$. On the other hand, the RG flow develops at $\Delta\lesssim\Delta_g$, where

$$\Delta_g \equiv (v_s\Lambda^2/q)\exp(-2/g_1). \tag{16}$$

The results of this paper are valid the RG flow develops before the curvature becomes important, i.e., if $\Delta_g\gg\omega_q$ or $q\ll(mv_s\Lambda^2)^{1/3}\exp(-2/3g_1)$ (see Fig. 1).

Even in this case, the RG result (1) diverges at the threshold $|\omega|=v_s|q|$ and the divergence must be regularized by finite curvature. A detailed analysis of matching between the threshold singularities due to finite mass (Ref. 20) with our RG result is outside the scope of this paper. However, one can compare the high-frequency tails due to each of these effects. For $\Delta\gg\Delta_g$, the RG result reduces to the second-order one; up to a numerical coefficient

$$\mathrm{Im}\chi(q,\omega)\sim\chi_0 g_1^2\frac{v_F^2q^2}{\omega^2-v_F^2q^2}\Theta(|\omega|-v_F|q|). \tag{17}$$

On the other hand, the high-frequency tail for spineless fermions arising from finite mass was calculated via an expansion in $1/m$ in Refs. 7, 15, and 22 For massive fermions, however, the difference in the $1/m$ expansions of the charge and spin susceptibilities amounts only to a numerical coefficient. Up to this coefficient, we can simply borrow the result of Refs. 15, 21, and 22:

$$\mathrm{Im}\chi_{1/m}(q,\omega) \sim g^2 \left(\frac{q}{mv_F}\right)^2 \frac{v_F^2 q^2}{\omega^2 - v_F^2 q^2}\Theta(|\omega| - v_F|q|), \tag{18}$$

where g is some dimensionless coupling constant. Comparing Eqs. (17) and (18), we see that the high-frequency tail due to a marginally irrelevant operator is larger than that due to finite mass by factor of $mv_F/q \gg 1$.

Conclusions. We have studied the dynamical spin structure factor (DSSF) of one dimensional interacting fermions for small momenta ($q \ll k_F$). In contrast to the charge structure factor, the DSSF is non-zero above the continuum even in a model with linearized fermion spectrum due to the effect of a marginally irrelevant backscattering operator. We found the DSSF by direct perturbation theory in the fermion language, supplemented by RG. The high-frequency tail due to backscattering is larger than that due to finite mass by a factor of $mv_F/q \gg 1$. One immediate consequence of our result is the non-analytic temperature dependence of the nuclear spin relaxation rate $T_1^{-1} \propto T \int dq \frac{\mathrm{Im}\chi(q,\omega)}{\omega}\Big|_{\omega\to 0}$ resulting from the region $q \ll k_F$. Logarithmic renormalization of $\chi(q,\omega)$ modifies the Korringa law as $1/T_1 \propto T/\ln(v_F\Lambda/T)$, which is line with the logarithmic corrections to the FL theory discussed early in this paper. At weak coupling, renormalization from the $q = 0$ region is comparable to the usually considered $2k_F$ contribution.[1]

We are grateful to A. V. Chubukov, S. Maiti, C. Reeg, O. A. Starykh, and especially to L. I. Glazman for stimulating discussions. This work was supported by the National Science Foundation via grant NSF DMR-1308972.

[1] T. Giamarchi, *Quantum Physics in One Dimension*, (Oxford University Press, New York, 2003).

[2] J.L. Cardy, J. Phys. A: Math. Gen. **19**, L1093 (1986).

[3] I.E. Dzyaloshinskii and A.I. Larkin, Sov. Phys. JETP **34**, 422 (1972).

[4] D. Belitz, T.R. Kirkpatrick, and T. Vojta, Phys. Rev. B **55**, 9452 (1997).

[5] A.V. Chubukov and D.L. Maslov, Phys. Rev. B **68**, 155113 (2003).

[6] T. Giamarchi and H. J. Schulz, Phys. Rev. B **39**, 4620 (1989).

[7] M. Pustilnik, M. Khodas, A. Kamenev, and L.I. Glazman, Phys. Rev. Lett. **96**, 196405 (2006).

[8] A. Imambekov, T.L. Schmidt, and L.I. Glazman, Rev. Mod. Phys. **84**, 1253 (2012).

[9] I. D'Amico and G. Vignale, Phys. Rev. B **62**, 4853 (2000); M. Polini and G.Vignale, Phys. Rev. Lett. **98**, 266403 (2007).

[10] A.V. Chubukov, D.L. Maslov, and R. Saha, Phys. Rev. B **77**, 085109 (2008).

[11] Our expression for the spin velocity differs from that in Ref. 1: $\tilde{v}_s = \sqrt{1-(g_1/2)^2}$. The difference is due to the fact that the commonly used bosonization scheme ignores a subtlety arising from backscattering of fermions with the same spin. Taking this effect into account leads to our result for v_s (see SM for details). To first order in g_1, our result coincides with a weak-coupling expansion of the exact Bethe-ansatz solution.[12,13] and ensures that the expressions for spin susceptibility, derived in the fermion and boson languages, coincide. To second order, the correspondence with the Bethe-ansatz solution is restored by taking into account "high-energy" contributions to the interaction vertices.[13]

[12] H. Frahm and M. P. Pfanmüller, Phys. Lett. A **204**, 347 (1995).

[13] A. V. Chubukov, D. L. Maslov, and F. H. L. Essler, Phys. Rev. B **77**, 161102(R) (2008).

[14] S. Eggert, I. Affleck, and M. Takahashi, Phys. Rev. Lett. **73**, 332 (1994).

[15] R. G. Pereira, J. Sirker, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, and I. Affleck, Phys. Rev. Lett. **96**, 257202 (2006); J. Stat. Mech. **2007**, 08022 (2007).

[16] J. Sólyom, Adv. Phys. **28**, 201 (1979)

[17] D.L. Maslov, in *Nanophysics: Coherence and Transport*, edited by H. Bouchiat, Y. Gefen, G. Montambaux, and J. Dalibard, Les Houches, Session LXXXI, 2004 (Elsevier, 2005), p. 1; arXiv:cond-mat/0506035.

[18] E.M. Lifshits and L.P. Pitaevskii, *Statistical Physics, Part 2: Theory of the Condensed State*, (Pergamon Press, New York, 1980).

[19] S. B. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and its Applications*, (Princeton University Press, Princeton, 1972).

[20] T. L. Schmidt, A. Imambekov, and L. I. Glazman, Phys. Rev. B**82** 245014 (2010).

[21] M. Pustilnik, E. G. Mishchenko, L. I. Glazman, and A. V. Andreev, Phys. Rev. Lett. **91**, 126805 (2003).

[22] S. Teber, Eur. Phys. J. B **52**, 233 (2006).