Universal topological data for gapped quantum liquids in three dimensions and fusion algebra for non-Abelian string excitations

Heidar Moradi and Xiao-Gang Wen

Phys. Rev. B **91**, 075114 — Published 17 February 2015

DOI: 10.1103/PhysRevB.91.075114

Universal Topological Data for Gapped Quantum Liquids in Three Dimensions and Fusion Algebra for Non-Abelian String Excitations

Heidar Moradi[1] and Xiao-Gang Wen[1, 2]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5 Canada*
[2]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Dated: January 13, 2015)

Recently we conjectured that a certain set of universal topological quantities characterize topological order in any dimension. Those quantities can be extracted from the universal overlap of the ground state wave functions. For systems with gapped boundaries, these quantities are representations of the mapping class group $\mathtt{MCG}(\mathcal{M})$ of the space manifold $\mathcal{M}$ on which the systems lives. We will here consider simple examples in three dimensions and give physical interpretation of these quantities, related to fusion algebra and statistics of particle and string excitations. In particular, we will consider dimensional reduction from 3+1D to 2+1D, and show how the induced 2+1D topological data contains information on the fusion and the braiding of non-Abelian string excitations in 3D. These universal quantities generalize the well-known modular S and T matrices to any dimension.

I. INTRODUCTION

For more than two decades exotic quantum states[1–12] have attracted a lot attention from the condensed matter community. In particular gapped systems with nontrivial topological order,[13–15] which is a reflection of long-range entanglement[16] of the ground state, have been studied intensely in $2+1$ dimensions. Recently, people started to work on a general theory of topological order in higher than $2+1$ dimensions.[17–21]

In a recent work Ref. 19, we conjectured that for a gapped system on a d-dimensional manifold $\mathcal{M}$ of volume V with the set of degenerate ground states $\{|\psi_\alpha\rangle\}_{\alpha=1}^N$ on $\mathcal{M}$, we have the following overlaps

$$\langle\psi_\alpha|\hat{O}_A|\psi_\beta\rangle = e^{-\alpha V+o(1/V)} M^A_{\alpha,\beta}, \tag{1}$$

where $\hat{O}_A$ are transformations on the wave functions induced by the automorphisms $A:\mathcal{M}\to\mathcal{M}$, α is a non-universal constant and M^A is a universal matrix up to an overall $U(1)$ phase. Here M^A form a projective representation of the automorphism group $\mathtt{AMG}(\mathcal{M})$, which is robust against *any* local perturbations that do not close the bulk gap.[15,22] In Ref. 19 we conjectured that such projective representations for different space manifold topologies fully characterize topological orders with finite ground state degeneracy in any dimension. Furthermore, we conjectured that projective representations of the mapping class groups $\mathtt{MCG}(\mathcal{M}) = \pi_0[\mathtt{AMG}(\mathcal{M})]$ classify topological order with gapped boundaries.[15,22] These quantities can be used as order parameters for topological order and detect transitions between different phases.[23]

In this paper we will study these universal quantities further in 3-dimensions for one of the most simple manifolds, the 3-torus $\mathcal{M}=T^3$. The mapping class group of the 3-torus is $\mathtt{MCG}(T^3)=SL(3,\mathbb{Z})$. This group is generated by two elements of the form[24]

$$\hat{\tilde{S}} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \qquad \hat{\tilde{T}} = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}. \tag{2}$$

These matrices act on the unit vectors by $\hat{\tilde{S}}:(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\mapsto(\hat{\boldsymbol{z}},\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$ and similarly $\hat{\tilde{T}}:(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\mapsto(\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$. Thus $\tilde{S}$ corresponds to a rotation, while $\tilde{T}$ is shear transformation in the xy-plane.

In this paper, we will study the $SL(3,\mathbb{Z})$ representations generated by a very simple class of $\mathbb{Z}_N$ models in detail and then consider models for any finite group G, which are 3-dimensional versions of Kitaevs quantum double models[25]. One can also generalize into twisted versions of these based on the group cohomology $H^4(G,U(1))$ by direct generalization of Ref. 26 into 3+1D, which has been done for some simple groups in Ref. 21 and 27.

We will consider dimensional reduction of a 3D topological order $\mathcal{C}^{3D}$ to 2D by making one direction of the 3D space into a small circle. In this limit, the 3D topologically ordered states $\mathcal{C}^{3D}$ can be viewed as several 2D topological orders $\mathcal{C}^{2D}_i$, $i=1,2,\cdots$ which happen to have degenerate ground state energy. We denote such a dimensional reduction process as

$$\mathcal{C}^{3D} = \bigoplus_i \mathcal{C}^{2D}_i. \tag{3}$$

We can compute such a dimensional reduction using the representation of $SL(3,\mathbb{Z})$ that we have calculated.

We consider $SL(2,\mathbb{Z})\subset SL(3,\mathbb{Z})$ subgroup and the reduction of the $SL(3,\mathbb{Z})$ representation R^{3D} to the $SL(2,\mathbb{Z})$ representations R^{2D}_i:

$$R^{3D} = \bigoplus_i R^{2D}_i. \tag{4}$$

We will refer to this as branching rules for the $SL(2,\mathbb{Z})$ subgroup. The $SL(3,\mathbb{Z})$ representation R^{3D} describes the 3D topological order $\mathcal{C}^{3D}$ and the $SL(2,\mathbb{Z})$ representations R^{2D}_i describe the 2D topological orders $\mathcal{C}^{2D}_i$. The decomposition (4) gives us the dimensional reduction (3).

Let us use $\mathcal{C}_G$ to denote the topological order described by the gauge theory with the finite gauge group G. Using the above result, we find that

$$\mathcal{C}_G^{3D} = \bigoplus_{n=1}^{|G|} \mathcal{C}_G^{2D} \tag{5}$$

for Abelian G where $|G|$ is the number of the group elements. For non-Abelian group G

$$\mathcal{C}_G^{3D} = \bigoplus_{C} \mathcal{C}_{G_C}^{2D} \tag{6}$$

where $\bigoplus_C$ sums over all different conjugacy classes C of G, and G_C is a subgroup of G which commutes with an element in C. The results for $G = \mathbb{Z}_N$ were mentioned in our previous paper.[19]

We also found that the reduction of $SL(3,\mathbb{Z})$ representation, eqn. (4), encodes all the information about the three-string statistics discussed in Ref. 20 for Abelian groups. For non-Abelian groups, we will have a "non-Abelian" string braiding statistics and a non-trivial string fusion algebra. We also have a "non-Abelian" three-string braiding statistics and a non-trivial three-string fusion algebra. Within the dimension reduction picture, the 3D strings reduces to particles in 2D, and the (non-Abelian) statistics of the particles encode the (non-Abelian) statistics of the strings.

II. $\mathbb{Z}_N$ MODEL IN 3-DIMENSIONS

In this section we will define and study the excitations of a $\mathbb{Z}_N$ model in detail[28] and compute the 3-torus universal matrices, eq. (1).

Consider a simple cubic lattice with a local Hilbert space on each link isomorphic to the group algebra of $\mathbb{Z}_N$, $\mathcal{H}_i \approx \mathbb{C}[\mathbb{Z}_N] \approx \mathbb{C}^N \approx \mathrm{span}_{\mathbb{C}}\{|\sigma\rangle | \sigma \in \mathbb{Z}_N\}$. Give the links on the lattice an orientation as in figure 1 and let there be a natural isomorphism $\mathcal{H}_i \xrightarrow{\sim} \mathcal{H}_{i^\star}$ for link i and its reversed orientation $i^\star$ as $|\sigma_i\rangle \mapsto |\sigma_{i^\star}\rangle = |-\sigma_i\rangle$. Let this basis be orthonormal. Define two local operators

$$Z_i|\sigma_i\rangle = \omega^{\sigma_i}|\sigma_i\rangle, \qquad X_i|\sigma_i\rangle = |\sigma_i - 1\rangle,$$

where $\omega = e^{\frac{2\pi i}{N}}$. These operators have the important commutation relation $X_i Z_i = \omega Z_i X_i$. Note that these operators are unitary and satisfy $X_i^N = Z_i^N = 1$. For each lattice site s and plaquette p define

$$A_s = \prod_{i\in s_+} Z_i \prod_{j\in s_-} Z_j^\dagger, \quad B_p = \prod_{i\in\partial p_+} X_i^\dagger \prod_{j\in\partial p_-} X_j.$$

Here s_+ is the set of links pointing into s, while s_- is the set of links pointing away from s. B_p creates a string around plaquette p with orientation given by the normal direction using the right hand thumb rule. Then $\partial p_\pm$ are the set of links surrounding plaquette p with the same or opposite orientation as the lattice. One can directly check that all these operators commute for all sites and plaquettes.

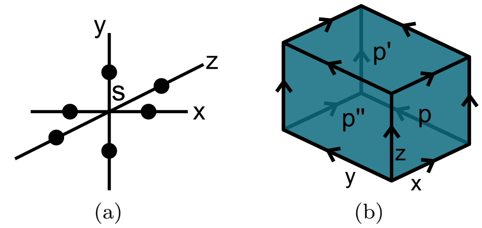


FIG. 1. (a) Lattice site of 3D cubic lattice. A_s act on spins connected to site s. (b) 2D plaquettes. B_p acts on the four spins surrounding p. Choose a righthanded (x, y, z) frame, and let all links be oriented wrt. to these directions. This associates a natural orientation to $2D$ plaquettes on the dual lattice.

We can now define the $\mathbb{Z}_N$ model by the Hamiltonian

$$H_{3D,\mathbb{Z}_N} = -\frac{J_e}{2}\sum_s \left(A_s + A_s^\dagger\right) - \frac{J_m}{2}\sum_p \left(B_p + B_p^\dagger\right),$$

where we will assume $J_e, J_m \geq 0$ throughout. Since $\mathrm{eigen}(A_s + A_s^\dagger) = \{2\cos(\frac{2\pi}{N}q)\}_0^{N-1}$, and the similar for $B_p + B_p^\dagger$, the ground state is the state satisfying

$$A_s|GS\rangle = |GS\rangle, \qquad B_p|GS\rangle = |GS\rangle, \tag{7}$$

for all s and p. We can easily construct hermitian projectors to the state with eigenvalue 1 for all vertices and plaquettes

$$\rho_s = \frac{1}{N}\sum_{k=0}^{N-1} A_s^k, \qquad \rho_p = \frac{1}{N}\sum_{k=0}^{N-1} B_p^k.$$

The ground state is thus $|GS\rangle = \prod_s \rho_s \prod_p \rho_p |\psi\rangle$, for any reference state $|\psi\rangle$ such that $|GS\rangle$ is non-zero. For the choice $|\psi\rangle = |00\ldots0\rangle \equiv |0\rangle$, the ρ_s is trivial and the ground state is thus

$$|GS\rangle = \prod_p \left(\frac{1}{N}\sum_{k=0}^{N-1} B_p^k\right)|0\rangle = \mathcal{N}\sum_{\mathbb{Z}_N \text{ string nets}} |\text{loops}\rangle.$$

The first condition in equation (7) requires that the ground state consists of $\mathbb{Z}_N$ string-nets, while the second requires that these appear with equal superpositions. Note that if we had used eigenstates of X_i instead, we would find that the ground state is a membrane condensate on the dual lattice.

1. *String and Membrane Operators*

Now let l_{ab} denote a curve on the lattice from site a to b, with the orientation that it points from a to b. And let $\Sigma_{\mathcal{C}}$ denote an oriented surface on the dual lattice with $\partial\Sigma_{\mathcal{C}} = \mathcal{C}$. Using these, define string and membrane operators

$$W[l_{ab}] = \prod_{i\in l_{ab}^-} X_i \prod_{j\in l_{ab}^+} X_j^\dagger, \quad \Gamma[\Sigma_{\mathcal{C}}] = \prod_{i\in\Sigma_{\mathcal{C}}^-} Z_i^\dagger \prod_{j\in\Sigma_{\mathcal{C}}^+} Z_j.$$

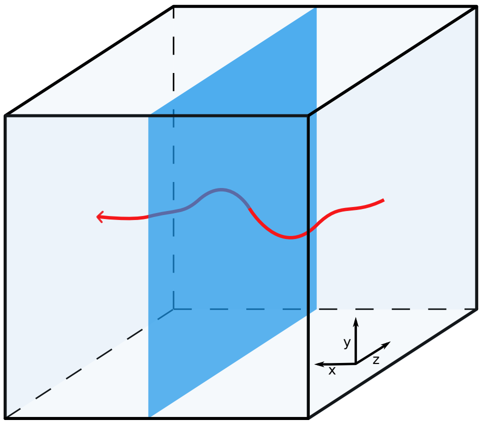


FIG. 2. The cube represents the 3-torus T^3, where the sides are appropriately identified. The red string represents l_x, a closed non-contractable loop wrapping around the x-cycle of the torus (orientation along the x-axis). Similarly two other non-contractable strings, l_y and l_z can be defined. The blue surface Σ_x (orientation of normal along x-axis), is a non-contractable surface with topology of T^2. Similarly Σ_y and Σ_z surfaces can be defined.

Again $l^{\pm}_{ab}$ and $\Sigma^{\pm}_{\mathcal{C}}$ are defined wrt. the orientation of the lattice. Note that $B_p = W[\partial p]$, where ∂p denotes a closed loop around plaquette p with right hand thumb rule orientation wrt. the normal direction. Similarly, $A_s = \Gamma[\text{star}(s)]$, where star$(s)$ is the closed surface on the dual lattice surrounding site s with inward orientation.

It is clear that the following operators commute

$$\Big[W[l_{ab}], B_p\Big] = 0, \quad \forall p, \quad \text{and} \quad \Big[\Gamma[\Sigma_{\mathcal{C}}], A_s\Big] = 0, \quad \forall s.$$

Furthermore it is easy to show that

$$\Big[W[l_{ab}], A_s\Big] = 0, \quad s \neq a, b, \qquad \Big[\Gamma[\Sigma_{\mathcal{C}}], B_p\Big] = 0, \quad p \notin \mathcal{C},$$

while

$$A_a W[l_{ab}] = \omega^{-1}\, W[l_{ab}] A_a, \qquad A_b W[l_{ab}] = \omega\, W[l_{ab}] A_b,$$

and

$$B_p \Gamma[\Sigma_{\mathcal{C}}] = \omega^{\pm 1}\, \Gamma[\Sigma_{\mathcal{C}}] B_p, \quad p \in \mathcal{C},$$

where $\pm$ depends on orientation of $\Sigma_{\mathcal{C}}$.

2. *Ground States on 3-Torus*

The ground state degeneracy depends on the topology of the manifold on which the theory is defined, take for example the 3-torus T^3. Let l_x, l_y and l_z be non-contractible loops along the three cycles on the lattice, with the orientation of the lattice. Similarly, let Σ_x, Σ_y and Σ_z be non-contractible surfaces along the three-directions, with the orientation of the dual lattice (see figure 2). We can define the operators

$$W_i \equiv W[l_i] = \prod_{j \in l_i} X_j^\dagger, \quad \Gamma_i \equiv \Gamma[\Sigma_i] = \prod_{j \in \Sigma_i} Z_i, \quad i = x, y, z.$$

These operators have the commutation relations

$$W_i \Gamma_i = \omega^{-1}\, \Gamma_i W_i, \qquad i = x, y, z. \tag{8}$$

We can thus find three commuting (independent) non-contractible operators to get N^3 fold ground state degeneracy. For example $|\alpha, \beta, \gamma\rangle = (W_x)^\alpha (W_y)^\beta (W_z)^\gamma |GS\rangle$, where $\alpha, \beta, \gamma = 0, \ldots, N-1$. This basis correspond to eigenstates of the surface operators $\Gamma_i |\alpha_1, \alpha_2, \alpha_3\rangle = \omega^{\alpha_i} |\alpha_1, \alpha_2, \alpha_3\rangle$. Note that on the torus we get the extra set of constraints $\prod_s A_s = 1$, $\prod_p B_p = 1$. Let G be the group generated by B_p for all p, modulo $B_p B_{p'} = B_{p'} B_p$, $B_p^N = 1$ and $\prod_p B_p = 1$. Furthermore define the groups $G_{\alpha\beta\gamma} \equiv (W_x)^\alpha (W_y)^\beta (W_z)^\gamma G$, then we can write the ground states as

$$|\alpha, \beta, \gamma\rangle = \frac{1}{\sqrt{|G_{\alpha\beta\gamma}|}} \sum_{g \in G_{\alpha\beta\gamma}} |g\rangle,$$

where $|g\rangle \equiv g|0\rangle$.

In 2D, the quasiparticle basis corresponds to the basis in which there is well-defined magnetic and electric flux along one cycle of the torus. We can try to do the same in three-dimensions. Γ_x, W_y, W_z all commute with each other and we can consider the basis which diagonalizes all of them. This basis is given by

$$|\psi_{abc}\rangle = \frac{1}{N} \sum_{\beta\gamma} \omega^{-\beta b - \gamma c} |a, \beta, \gamma\rangle, \tag{9}$$

where $a, b, c = 0, \ldots, N-1$. These are clearly eigenstates of Γ_x, and furthermore we have that $W_y |\psi_{abc}\rangle = \omega^b |\psi_{abc}\rangle$ and $W_z |\psi_{abc}\rangle = \omega^c |\psi_{abc}\rangle$. This basis is a 3D version of minimum entropy states (MES).[29]

3. *Excitations*

Now lets go back to, say, this theory on S^3 and look at elementary excitations of our model. An excitation correspond to a state in which the conditions (7) are violated in a small region. Using the string operators, we can create a pair of particles by $|-q_e, q_e\rangle = W[l_{ab}]^{q_e} |GS\rangle$ with the electric charges

$$A_a |-q_e, q_e\rangle = \omega^{-q_e} |-q_e, q_e\rangle, \quad A_b |-q_e, q_e\rangle = \omega^{q_e} |-q_e, q_e\rangle.$$

This excitation has an energy cost of $\Delta E_{\text{particles}} = 2J_e[1 - \cos(\frac{2\pi}{N} q_e)]$. Furthermore we have oriented string excitations by using the membrane operators $|\mathcal{C}, q_m\rangle = \Gamma[\Sigma_{\mathcal{C}}]^{q_m} |GS\rangle$, with the magnetic flux

$$B_p |\mathcal{C}, q_m\rangle = \omega^{\pm q_m} |\mathcal{C}, q_m\rangle, \quad p \in \mathcal{C},$$

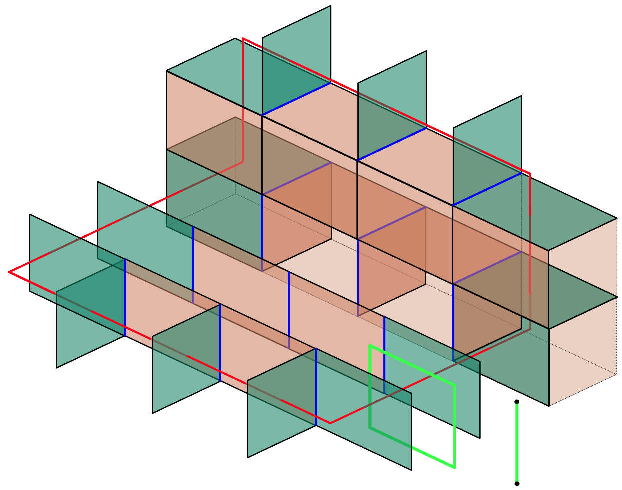


FIG. 3. String and particle excitations. The red curve is the boundary of a membrane on the dual lattice and correspond to a string excitation. The blue links are the ones affected by the membrane operator and the green plaquettes are the ones on which B_p can measure the presence of the string excitation. The green line correspond to a string operator on the lattice, in which the end point are particles. Mutual statistics between strings and particles can be calculated by creating a particle-antiparticle pair from the vacuum, moving one particle around the string excitation and annihilating the particles.

where the $\pm$ depend on the orientation of $\mathcal{C}$. This excitation comes with the energy penalty $\Delta E_{\text{string}} = \text{Lenght}(\mathcal{C}) J_m[1-\cos(\frac{2\pi}{N}q_m)]$.

One can easily show that all the particles have trivial self and mutual statistics, and the same with the strings. Mutual statistics between particles and strings can be non-trivial however, taking a charge q_e particle through a flux q_m string gives the anyonic phase $\omega^{\pm q_e q_m}$, where the $\pm$ depend on the orientations. See figure 3.

III. REPRESENTATIONS OF $\texttt{MCG}(T^3) = SL(3,\mathbb{Z})$

Let us now go back to T^3 and consider the universal quantities as defined in (1). In the $|\alpha,\beta,\gamma\rangle$ basis, the representation of the $SL(3,\mathbb{Z})$ generators (2) is given by

$$\tilde{S}_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \delta_{\alpha,\beta'}\delta_{\beta,\gamma'}\delta_{\gamma,\alpha'}, \tag{10}$$

and

$$\tilde{T}_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \delta_{\alpha,\alpha'}\delta_{\beta,\alpha'+\beta'}\delta_{\gamma,\gamma'}. \tag{11}$$

In the 3D quasiparticle basis (9) these are given by

$$\tilde{S}_{abc,\bar{a}\bar{b}\bar{c}} = \frac{1}{N}\delta_{b,\bar{c}}e^{\frac{2\pi i}{N}(\bar{a}c - a\bar{b})}, \quad \tilde{T}_{abc,\bar{a}\bar{b}\bar{c}} = \delta_{a,\bar{a}}\delta_{b,\bar{b}}\delta_{c,\bar{c}}e^{\frac{2\pi i}{N}ab}.$$

For example in the simplest case $N=2$, which is the 3D Toric code, we have

$$\tilde{T} = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& -1 \end{pmatrix},$$

and

$$\tilde{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

4. *Interpretation of $\tilde{T}$*

These matrix elements in this particular ground state basis, actually contain some physical information about statistics of excitations. In order to see this, we can associate a collection of excitations to each ground state on the 3-torus.

First cut the 3-torus along the x-axis such that it now has two boundaries. We can measure the presence of excitations on the boundary using the operators Γ_x, W_y and W_z. First take the state with no particle, $|\mathbf{1}\rangle = \frac{1}{N}\sum_{\beta\gamma}|\beta,\gamma\rangle$, in which all operators have eigenvalue 1. Here $|\beta,\gamma\rangle$ are states with β and γ non-contractible electric loops along the y and z axis, respectively. Now add excitations on the boundary using open string and membrane operators (see fig. 4) $|e_a\rangle = (W[l_{12}])^a|\mathbf{1}\rangle$, $|m_{y,c}\rangle = (\Gamma[\Sigma_{\mathcal{C}_y}])^c|\mathbf{1}\rangle$, $|m_{z,b}\rangle = (\Gamma[\Sigma_{\mathcal{C}_z}])^b|\mathbf{1}\rangle$, $|e_a m_{y,c}\rangle = (W[l_{12}])^a(\Gamma[\Sigma_{\mathcal{C}_y}])^c|\mathbf{1}\rangle$, $|e_a m_{z,b}\rangle = (W[l_{12}])^a(\Gamma[\Sigma_{\mathcal{C}_z}])^b|\mathbf{1}\rangle$, $|m_{y,c}m_{z,b}\rangle = (\Gamma[\Sigma_{\mathcal{C}_y}])^c(\Gamma[\Sigma_{\mathcal{C}_z}])^b|\mathbf{1}\rangle$ and $|e_a m_{y,c}m_{z,b}\rangle = (W[l_{12}])^a(\Gamma[\Sigma_{\mathcal{C}_y}])^c(\Gamma[\Sigma_{\mathcal{C}_z}])^b|\mathbf{1}\rangle$, where $a,b,c = 1,\ldots,N-1$. Or more compactly, $|e_a m_{y,c}m_{z,b}\rangle$, where $a,b,c = 0,\ldots,N-1$. Here l_{12} is a curve from one edge to the other, $\Sigma_{\mathcal{C}_y}$ is a membrane between edges wrapping along the y-cycle and $\Sigma_{\mathcal{C}_z}$ is a membrane between edges wrapping along z-cycle. All these have the same orientation as the (dual) lattice. These states have well-defined electric and magnetic flux wrt. Γ_x, W_y and W_z. Here m_y and m_z correspond to the strings on the boundaries, wrapping around the y and z cycles, respectively.

If we now glue the two boundaries together, we see that for each of these excitations we have a 3-torus ground

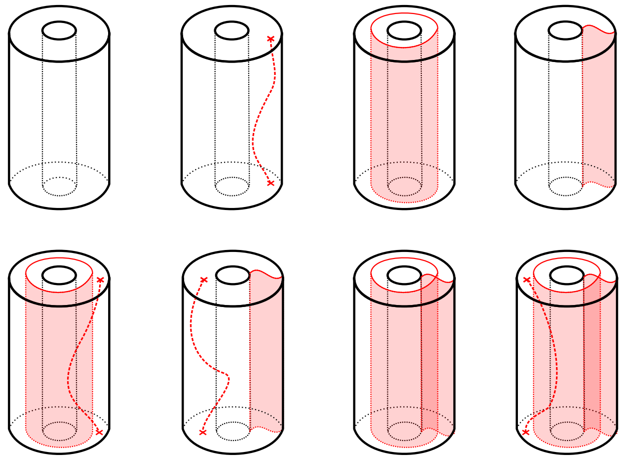


FIG. 4. The result of cutting open the 3-torus along the x-axis, can be represented by a hollow solid cylinder where the inner and outer surfaces are identified, but there are two boundaries along x. In the above, the compactified direction is y and the radial direction is z, while the open direction is x. We can see the N^3 possible excitations on the boundaries which give rise to 3-torus ground states uppon gluing. The four first states correspond to $|\mathbf{1}\rangle$, $|e_a\rangle$, $|m_{y,c}\rangle$ and $|m_{z,b}\rangle$.

state

$$\begin{aligned} |\mathbf{1}\rangle &= |\psi_{000}\rangle, & |e_a m_{1,c}\rangle &= |\psi_{a0c}\rangle, \\ |e_a\rangle &= |\psi_{a00}\rangle, & |e_a m_{2,b}\rangle &= |\psi_{ab0}\rangle, \\ |m_{1,c}\rangle &= |\psi_{00c}\rangle, & |m_{1,c} m_{2,b}\rangle &= |\psi_{0bc}\rangle, \\ |m_{2,b}\rangle &= |\psi_{0b0}\rangle, & |e_a m_{1,c} m_{2,b}\rangle &= |\psi_{abc}\rangle. \end{aligned}$$

We can add other string excitations on the boundary, however they will not give rise to new 3-torus ground states after gluing. We thus see a generalization of the situation in 2D, where there is a direct relation between number of excitation types and GSD on the torus.

Now lets to back to the open boundaries, and consider making a 2π twist of one of the boundaries, which will give some kind of 3D analogue of *topological spin*. It can be seen that most states will be invariant under such an operation by appropriately deforming and reconnecting the string and membrane operators. For example $|e_a\rangle \to |e_a\rangle$, which implies that the particles e_a are bosons. However we pick up a factor of ω^{ab} for $|e_a m_{2,b}\rangle$ and $|e_a m_{1,c} m_{2,b}\rangle$, since the string corresponding to particle e_a has to cross the membrane corresponding to $m_{2,b}$. Physically this is a consequence of mutual statistics of the particle and string excitation. We can consider these as 3D analogue of topological spin.

Now notice that this operation precisely corresponds to the $\tilde{T}$ Dehn twist on the 3-torus by gluing the boundaries (see fig.5). Thus $\tilde{T}$, as calculated from the ground state, should contain information about statistics of excitations. Writing $\tilde{T}_{abc,\bar{a}\bar{b}\bar{c}} = \delta_{a,\bar{a}}\delta_{b,\bar{b}}\delta_{c,\bar{c}} e^{\frac{2\pi i}{N}ab} \equiv \delta_{a,\bar{a}}\delta_{b,\bar{b}}\delta_{c,\bar{c}}\tilde{T}_{abc}$, we get the following 3D *topological spins*

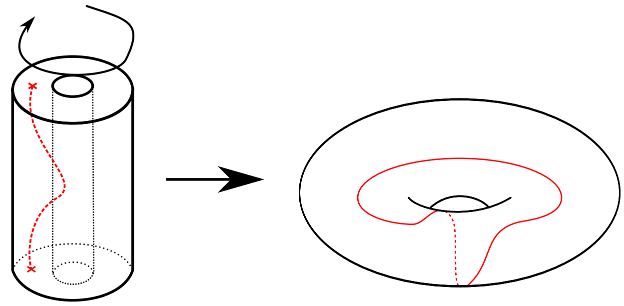


FIG. 5. The Dehn twist $\tilde{T}$ is along the $x-y$ plane, thus it is natural to think of T^3 as a solid hollow 2-torus where the inner and outer boundaries are identified, here the thickened direction is z. In this picture, we can think of $\tilde{T}$ just as a usual Dehn twist of a 2-torus.

$$\begin{aligned} \tilde{T}_{\mathbf{1}} &= \tilde{T}_{000} = 1, & \tilde{T}_{e_a} &= \tilde{T}_{a00} = 1, \\ \tilde{T}_{m_{1,c}} &= \tilde{T}_{00c} = 1, & \tilde{T}_{m_{2,b}} &= \tilde{T}_{0b0} = 1, \\ \tilde{T}_{e_a m_{1,c}} &= \tilde{T}_{a0c} = 1, & \tilde{T}_{e_a m_{2,b}} &= \tilde{T}_{ab0} = e^{\frac{2\pi i}{N}ab}, \\ \tilde{T}_{m_{1,c} m_{2,b}} &= \tilde{T}_{0bc} = 1, & \tilde{T}_{e_a m_{1,c} m_{2,b}} &= \tilde{T}_{abc} = e^{\frac{2\pi i}{N}ab}. \end{aligned}$$

This exactly match the properties of the excitations. Thus the universal quantity $\tilde{T}$ calculated from the ground state alone, contain direct physical information about statistics of excitations in the system. Note that elements like $\tilde{T}_{m_{1,c}m_{2,b}}$ can be non-trivial in theories with non-trivial string-string statistics.

5. 3D → 2D Dimensional Reduction

We can actually relate these universal quantities to the well-known S and T matrices in two dimensions. Consider now the $SL(2,\mathbb{Z})$ subgroup of $SL(3,\mathbb{Z})$ generated by

$$\hat{T}^{yx} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{S}^{yx} \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

One can directly compute the representation of this subgroup for the above $\mathbb{Z}_N$ model, which is given by

$$S^{yx}_{abc,\bar{a}\bar{b}\bar{c}} = \frac{1}{N}\delta_{c,\bar{c}} e^{-\frac{2\pi i}{N}(a\bar{b}+\bar{a}b)}, \quad T^{yx}_{abc,\bar{a}\bar{b}\bar{c}} = \delta_{a,\bar{a}}\delta_{b,\bar{b}}\delta_{c,\bar{c}} e^{\frac{2\pi i}{N}ab}.$$

Note that $S^{3D}_{\mathbb{Z}_N} = \bigoplus_{n=1}^{N} S^{2D}_{\mathbb{Z}_N}$ and $T^{3D}_{\mathbb{Z}_N} = \bigoplus_{n=1}^{N} T^{2D}_{\mathbb{Z}_N}$. In particular, for the toric code $N=2$ we have

$$S^{yx} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & & & & 1 & 1 & 1 & 1 \\ & & & & 1 & 1 & -1 & -1 \\ & & & & 1 & -1 & 1 & -1 \\ & & & & 1 & -1 & -1 & 1 \end{pmatrix},$$

and

$$T^{yx} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix}.$$

These N blocks are distinguished by eigenvalues of W_z. Consider the 2D limit of the three-dimensional $\mathbb{Z}_N$ model where the x and y directions are taken to be very large compared to the z direction. In this limit a non-contractible loop along the z-cycle becomes very small and the following perturbation is essentially local

$$H = H_{3D,\mathbb{Z}_N} - \frac{J_z}{2}\left(W_z + W_z^\dagger\right), \qquad (13)$$

where W_z creates a loop along z. Since this perturbation commutes with the original Hamiltonian, besides the conditions (7) the ground state must also satisfy $W_z|GS\rangle = |GS\rangle$. Thus the N^3-fold degeneracy is not stable in the 2D limit and the N^2 remaining ground states are now $|2D, a, b\rangle \equiv |\psi_{ab0}\rangle$. The gap to the state $|\psi_{abc}\rangle$ is $\Delta E_c = J_c[1 - \cos(\frac{2\pi}{N}c)]$.

It is easy to see that S_{yx} and T_{yx} on this set of ground states exactly correspond the two dimensional $\mathbb{Z}_N$ modular matrices and can be used to construct the corresponding UMTC. Thus the 3D $\mathbb{Z}_N$ model and our universal quantities exactly reduce to the 2D versions in this limit. Furthermore, the 3D quasiparticle basis also directly reduce to the 2D quasiparticle basis.

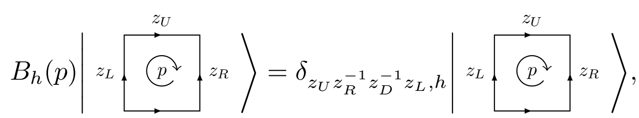
IV. QUANTUM DOUBLE MODELS IN THREE-DIMENSIONS

In this section we will construct exactly soluble models in three-dimensions for any finite group G. These are nothing but a natural generalization of Kitaev's quantum double models[30] to three-dimensions and are closely related to discrete gauge theories with gauge group G. These models will have the above $\mathbb{Z}_N$ models as a special case, but formulated in a slightly different way.

Consider a simple cubic lattice[31] with the orientation used above. Let there be a Hilbert space $\mathcal{H}_l \approx \mathbb{C}[G]$ on each link l, where G is a finite group, and let there be an isomorphism $\mathcal{H}_l \xrightarrow{\sim} \mathcal{H}_{l^\star}$ for the link l and its reverse orientation $l^\star$ as $|g_l\rangle \mapsto |g_{l^\star}\rangle = |g_l^{-1}\rangle$. Furthermore let the natural basis of the group algebra be orthonormal. The following local operators will be useful

$$\begin{aligned} L_+^g|z\rangle &= |gz\rangle, & T_+^h|z\rangle &= \delta_{h,z}|z\rangle, \\ L_-^g|z\rangle &= |zg^{-1}\rangle, & T_-^h|z\rangle &= \delta_{h^{-1},z}|z\rangle. \end{aligned}$$

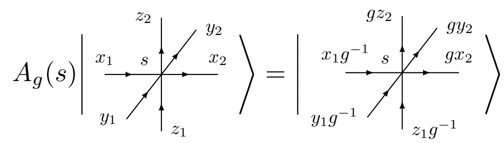
To each two dimensional plaquette p, associate an orientation wrt. to the lattice orientation using the right-hand rule. For such a plaquette, define the following operator



and similar for other orientations of plaquettes. Note that the order of the product is important for non-Abelian groups. To each lattice site s, define the operator

$$A_g(s) = \prod_{l_-} L_-^g(l_-) \prod_{l_+} L_+^g(l_+),$$

where l_- are the set of links pointing into s while l_+ are the links pointing away from s. In particular we have that



From these we have two important operators

$$A(s) = \frac{1}{|G|}\sum_{g\in G} A_g(s),$$

and $B(p) \equiv B_1(p)$, where $1 \in G$ is the identity element. One can show that both these operators are hermitian projectors. Furthermore one can check that they all commute together

$$\begin{aligned} \big[A(s), B(p)\big] &= 0, & \forall s, p, \\ \big[B(p), B(p')\big] &= 0, & \forall p, p', \\ \big[A(s), A(s')\big] &= 0, & \forall s, s'. \end{aligned}$$

We can now define the Hamiltonian of the three-dimensional quantum double model as

$$H = -J_e \sum_s A(s) - J_m \sum_p B(p). \qquad (14)$$

Since the Hamiltonian is just a sum of commuting projectors, the ground states of the system must satisfy

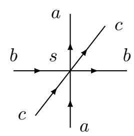
$$A(s)|GS\rangle = B(p)|GS\rangle = |GS\rangle,$$

for all s and p. The ground state can be constructed using the following hermitian projector $\rho_{GS} = \prod_s A(s) \prod_p B(p)$. If we take as reference state $|1\rangle = |1_{l_1} 1_{l_2} \ldots\rangle$, we can write

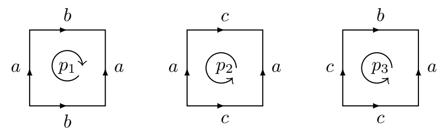
$$|GS\rangle = \rho_{GS}|1\rangle = \prod_s A(s)|1\rangle.$$

A. Ground states on T^3

The easiest way to construct the ground states on the three-torus is to consider the minimal torus, which is just a single cube where the boundaries are identified. The minimal torus has one site s



and three plaquettes p_1, p_2, p_3



One can readily show that the subspace $\mathcal{H}^{B=1}$ satisfying $B(p)|GS\rangle \overset{!}{=} |GS\rangle$ for $p = p_1, p_2, p_3$, is spanned by the vectors $|a, b, c\rangle$ such that $ab = ba$, $bc = cb$ and $ac = ca$. The last condition is $A(s)|GS\rangle = |GS\rangle$ where on the basis vectors

$$A(s)|a, b, c\rangle = \frac{1}{|G|} \sum_{g\in G} |gag^{-1}, gbg^{-1}, gcg^{-1}\rangle.$$

In the case of Abelian groups G, this condition is clearly trivial and then we have $GSD = |G|^3$. In general we can find the ground state degeneracy by taking the trace of the projector $A(s)$ in $\mathcal{H}^{B=1}$. This is given by

$$\begin{aligned} GSD &= \sum_{\{a,b,c\}} \langle a, b, c|A(s)|a, b, c\rangle \\ &= \frac{1}{|G|} \sum_{g\in G} \sum_{\{a,b,c\}} \delta_{ag,ga}\delta_{bg,gb}\delta_{cg,gc}, \end{aligned}$$

where $\{a, b, c\}$ is triplets of commuting group elements. One can actually easily check that the following vectors span the ground state subspace

$$|\psi_{[a,b,c]}\rangle = \frac{1}{|G|} \sum_{g\in G} |gag^{-1}, gbg^{-1}, gcg^{-1}\rangle, \tag{15}$$

where $[a, b, c] = \{(\tilde{a}, \tilde{b}, \tilde{c}) \in G \times G \times G \,|\, (\tilde{a}, \tilde{b}, \tilde{c}) = (gag^{-1}, gbg^{-1}, gcg^{-1}), g \in G\}$ is the three-element conjugacy class and a, b, c are representatives of the class.

B. 3D $\tilde{S}$ and $\tilde{T}$ matrices and the $SL(2,\mathbb{Z})$ subgroup

We can now readily compute the overlaps (1) for the above model for any group G. We find the following representations of $\texttt{MCG}(T^3) = SL(3,\mathbb{Z})$

$$\tilde{S}_{[a,b,c],[\bar{a},\bar{b},\bar{c}]} = \langle\psi_{[a,b,c]}|\, \tilde{S}\, |\psi_{[\bar{a},\bar{b},\bar{c}]}\rangle = \delta_{[a,b,c],[\bar{b},\bar{c},\bar{a}]}$$

and

$$\tilde{T}_{[a,b,c],[\bar{a},\bar{b},\bar{c}]} = \langle\psi_{[a,b,c]}|\, \tilde{T}\, |\psi_{[\bar{a},\bar{b},\bar{c}]}\rangle = \delta_{[a,b,c],[\bar{a},\bar{a}\bar{b},\bar{c}]},$$

since $\tilde{S}|\psi_{[a,b,c]}\rangle = |\psi_{[b,c,a]}\rangle$ and $\tilde{T}|\psi_{[a,b,c]}\rangle = |\psi_{[a,ab,c]}\rangle$.

Once again we can consider the subgroup $SL(2,\mathbb{Z}) \subset SL(3,\mathbb{Z})$ generated by (12). The representation of this subgroup can be directly computed and is given by

$$S^{yx}_{[a,b,c],[\bar{a},\bar{b},\bar{c}]} = \langle\psi_{[a,b,c]}|\, S^{yx}\, |\psi_{[\bar{a},\bar{b},\bar{c}]}\rangle = \delta_{[a,b,c],[\bar{b},\bar{a}^{-1},\bar{c}]}$$

and

$$T^{yx}_{[a,b,c],[\bar{a},\bar{b},\bar{c}]} = \langle\psi_{[a,b,c]}|\, T^{yx}\, |\psi_{[\bar{a},\bar{b},\bar{c}]}\rangle = \delta_{[a,b,c],[\bar{a},\bar{a}\bar{b},\bar{c}]}.$$

Note that since c is not independent of a and b, in general we don't have the decomposition $S^{3D}_G = \bigoplus_{n=1}^{|G|} S^{2D}_G$ and $T^{3D}_G = \bigoplus_{n=1}^{|G|} T^{2D}_G$, unless the group is Abelian.

C. Branching Rules and Dimensional Reduction

With the above formulas, we can directly compute the $\tilde{S}$ and $\tilde{T}$ generators for any group G. In the limit where one direction of the 3-torus is taken to be very small, we can view the 3D topological order as several 2D topological orders.

The branching rules (3) for the dimensional reduction can be directly computed by studying how a representation of $SL(3,\mathbb{Z})$ decomposes into representations of the subgroup $SL(2,\mathbb{Z}) \subset SL(3,\mathbb{Z})$. For example, for some of the simplest non-Abelian finite groups we find the branching rules

$$\begin{aligned} \mathcal{C}^{3D}_{S_3} &= \mathcal{C}^{2D}_{S_3} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_3} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_2}, \\ \mathcal{C}^{3D}_{D_4} &= 2\,\mathcal{C}^{2D}_{D_4} \oplus 2\,\mathcal{C}^{2D}_{D_2} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_4}, \\ \mathcal{C}^{3D}_{D_5} &= \mathcal{C}^{2D}_{D_5} \oplus 2\,\mathcal{C}^{2D}_{\mathbb{Z}_5} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_2}, \\ \mathcal{C}^{3D}_{S_4} &= \mathcal{C}^{2D}_{S_4} \oplus \mathcal{C}^{2D}_{D_4} \oplus \mathcal{C}^{2D}_{D_2} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_4} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_3}. \end{aligned}$$

In general we find the following branching in the dimensional reduction $\mathcal{C}^{3D}_G = \bigoplus_C \mathcal{C}^{2D}_{G_C}$, where $\bigoplus_C$ sums over all different conjugacy classes C of G, and G_C is the centralizer subgroup of G for some representative $g_C \in C$. Similar to the $G = \mathbb{Z}_N$ case above (13), the degeneracy between the different sectors can be lifted by a perturbation creating Wilson loops along the small non-contractible cycle of T^3, which is essentially a local perturbation in the 2D limit.

We like to remark that the above branching result for dimensional reduction can be understood from a "gauge symmetry breaking" point of view. In the dimensional reduction, we can choose to insert gauge flux through the small compactified circle. The different choices of the gauge flux is given by the conjugacy classes C of G. Such gauge flux break the "gauge symmetry" from G to G_C. So, such a compactification leads to a 2D gauge theory with gauge group G_C and reduces the 3D topological order $\mathcal{C}^{3D}_G$ to a 2D topological order $\mathcal{C}^{2D}_{G_C}$. The different choices of gauge flux lead to different degenerate 2D topological ordered states, each described by $\mathcal{C}^{2D}_{G_C}$ for a certain G_C. This gives us the result eqn. (6). It is quite interesting to see that the branching (4) of the representation of the mapping class group $SL(3,\mathbb{Z}) \to SL(2,\mathbb{Z})$ is closely related to the "gauge symmetry breaking" in our examples.

In order to gain a better understanding of the information contained in these branching rules, we will consider a simple example.

V. EXAMPLE: $G = S_3$

A. Two-Dimensional $D(S_3)$

Let us consider the simplest non-Abelian group $G = S_3$. Let us first recall the 2D quantum double models. The excitations of these models are given by irreducible representations of the Drinfeld Quantum Double $D(G)$. The states can be labelled by $|C,\rho\rangle$, where C denote a conjugacy class of G while ρ is a representation of the centralizer subgroup $G_C \equiv Z(a) = \{g \in G | ag = ga\}$ of some element in $a \in C$ (note that $Z(a) \approx Z(gag^{-1})$).

The symmetric group $G = S_3$ consists of the elements $\{(), (23), (12), (123), (132), (13)\}$, where $(\dots)$ is the standard notation for cycles (cyclic permutations). There are three conjugacy classes $A = \{()\}$, $B = \{(12), (13), (23)\}$ and $C = \{(123), (132)\}$, with the corresponding centralizer subgroups $G_A = S_3$, $G_B = \mathbb{Z}_2$, $G_C = \mathbb{Z}_3$. The number of irreducible representations for each group is equal to the number of conjugacy classes, 3 for G_A and G_C while 2 for G_B. For simplicity we will label the particles corresponding to the three different conjugacy classes by $(\mathbf{1}, A^1, A^2)$, (B, B^1) and (C, C^1, C^2). Here the particles without a superscript, B and C, are pure fluxes (trivial representation), A^1 and A^2 are pure charges (trivial conjugacy class), while B^1, C^1 and C^2 are charge-flux composites. The fusion rules for the two-dimensional $D(S_3)$ model is given in table I.

B. Three-Dimensional $G = S_3$ Model

In three dimensions, the S_3 model has two point-like topological excitations, which are pure charge excitations that can be labelled by A^1_{3D} and A^2_{3D}. Here A^1 is the one-dimensional irreducible representation of S_3 and A^2 the two-dimensional irreducible representation of S_3. Under the dimensional reduction to 2D, they become the 2D charge particles labelled by A^1 and A^2. The S_3 model also has two string-like topological excitations, labelled by the non-trivial conjugacy classes B_{3D} and C_{3D}. Under the dimensional reduction to 2D, they become the 2D particles with pure fluxes described by B and C. (For details, see the discussion below.) We can also add a 3D charged particle to a 3D string and obtain a so called mixed string-charge excitation. Those mixed string-charge excitations are labelled by B^1_{3D}, C^2_{3D}, and C^3_{3D}, and, under the dimensional reduction, become the 2D particles B^1, C^2, and C^3 (see Table I).

We like to remark that, since a 3D string carries gauge flux described by a conjugacy class B or C, the S_3 "gauge symmetry" is broken down to $G_B = \mathbb{Z}_2$ on the B_{3D} string, and down to $G_C = \mathbb{Z}_3$ on the C_{3D} string.

Under the symmetry breaking $S_3 \to \mathbb{Z}_2$, the two irreducible representations A^1 and A^2 of S_3 reduce to the irreducible representations 1 and e of $\mathbb{Z}_2$: $A^1 \to e$ and $A^2 \to 1 \oplus e$. Thus fusing the S_3 charge A^1_{3D} to a B_{3D} string give us the mixed string-charge excitation B^1_{3D}. But fusing the S_3 charge A^2_{3D} to a B_{3D} string gives us a composite mixed string-charge excitation $B_{3D} \oplus B^1_{3D}$. (The physical meaning of the composite topological excitations $B_{3D} \oplus B^1_{3D}$ is explained in Ref. 32.) So fusing the two non-trivial S_3 charges to a B_{3D} string only give us one mixed string-charge excitation B^1_{3D}.

Under the symmetry breaking $S_3 \to \mathbb{Z}_3$, the two irreducible representations A^1 and A^2 of S_3 reduce to the irreducible representations 1, e_1 and e_2 of $\mathbb{Z}_3$: $A^1 \to 1$ and $A^2 \to e_1 \oplus e_2$. Thus fusing the S_3 charge A^1 to a C_{3D} string still gives us the string excitation C_{3D}. But fusing the S_3 charge A^2_{3D} to a C_{3D} string gives us a composite mixed string-charge excitation $C^1_{3D} \oplus C^2_{3D}$. So fusing the two non-trivial S_3 charges to a C string give us two mixed string-charge excitations C^1_{3D} and C^2_{3D}. We see that the fusion between point S_3 charges and the strings is consistent with fusion of the corresponding 2D particles.

Now, we would like to understand the fusion and braiding properties of the 3D strings B_{3D} and C_{3D}. To do that, let us consider the dimension reduction $\mathcal{C}^{3D}_{S_3} = \mathcal{C}^{2D}_{S_3} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_3} \oplus \mathcal{C}^{2D}_{\mathbb{Z}_2}$. Let us choose the gauge flux through the small compactified circle to be B. In this case $\mathcal{C}^{3D}_{S_3} \to \mathcal{C}^{2D}_{\mathbb{Z}_2}$. $\mathcal{C}^{2D}_{\mathbb{Z}_2}$ is a $\mathbb{Z}_2$ topological order in 2D and contains four particle-like topological excitations $\mathbf{1}$, e, m, f, where $\mathbf{1}$ is the trivial excitations. e is the $\mathbb{Z}_2$ charge and m the $\mathbb{Z}_2$ vortex, which are both bosons. f is the bound state of e and m which is a fermion. The trivial 2D excitation $\mathbf{1}$ comes from the trivial 3D excitation $\mathbf{1}_{3D}$, and the $\mathbb{Z}_2$ charge e comes from the 3D charge excitation A^1. The 3D string excitations B and B^1, wrapping around the small compactified circle, give rise to two particle-like excitations in 2D – the $\mathbb{Z}_2$ vortex m and the fermion f. In the dimensional reduction, the gauge flux B through the small compactified circle forbids the 3D string excitations C_{3D}, C^1_{3D}, and C^2_{3D} to wrap around the small compactified circle. So there is no 2D excitations that correspond to the 3D string excitations C_{3D}, C^1_{3D}, and C^2_{3D}. Because of the symmetry breaking $S_3 \to \mathbb{Z}_2$ caused by the gauge flux B, the 3D particle A^2_{3D} reduces to $\mathbf{1} \oplus e$ in 2D.

The above results have a 3D understanding. Let us consider the situation where two loops, b and c, are threaded by string a (see Fig. 6). If the a-string is the type-B_{3D} string, then the b and c-strings must also be

$\otimes$	$\mathbf{1}$	A^1	A^2	B	B^1	C	C^1	C^2
$\mathbf{1}$	$\mathbf{1}$	A^1	A^2	B	B^1	C	C^1	C^2
A^1	A^1	$\mathbf{1}$	A^2	B^1	B	C	C^1	C^2
A^2	A^2	A^2	$\mathbf{1} \oplus A^1 \oplus A^2$	$B \oplus B^1$	$B \oplus B^1$	$C^1 \oplus C^2$	$C \oplus C^2$	$C \oplus C^1$
B	B	B^1	$B \oplus B^1$	$\mathbf{1} \oplus A^2 \oplus C \oplus C^1 \oplus C^2$	$A^1 \oplus A^2 \oplus C \oplus C^1 \oplus C^2$	$B \oplus B^1$	$B \oplus B^1$	$B \oplus B^1$
B^1	B^1	B	$B \oplus B^1$	$A^1 \oplus A^2 \oplus C \oplus C^1 \oplus C^2$	$\mathbf{1} \oplus A^2 \oplus C \oplus C^1 \oplus C^2$	$B \oplus B^1$	$B \oplus B^1$	$B \oplus B^1$
C	C	C	$C^1 \oplus C^2$	$B \oplus B^1$	$B \oplus B^1$	$\mathbf{1} \oplus A^1 \oplus C$	$C^2 \oplus A^2$	$C^1 \oplus A^2$
C^1	C^1	C^1	$C \oplus C^2$	$B \oplus B^1$	$B \oplus B^1$	$C^2 \oplus A^2$	$\mathbf{1} \oplus A^1 \oplus C^1$	$C \oplus A^2$
C^2	C^2	C^2	$C \oplus C^1$	$B \oplus B^1$	$B \oplus B^1$	$C^1 \oplus A^2$	$C \oplus A^2$	$\mathbf{1} \oplus A^1 \oplus C^2$

TABLE I. Fusion rules of two-dimensional $D(S_3)$ model. Here B and C correspond to pure flux excitations, A^1 and A^2 pure charge excitations, $\mathbf{1}$ the vacuum sector while B^1, C^1 and C^2 are charge-flux composites. If we add the subscript $3D$, the table becomes a list of the 3D particle/string excitations, and their fusion rules.

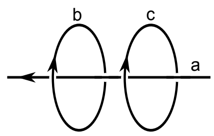


FIG. 6. Three string configuration, where two loops of type b and c are threaded by a string of type a.

the type-B_{3D} string. So the type B_{3D} string in the center forbids the 3D strings C_{3D}, C^1_{3D}, and C^2_{3D} to loop around it. This is just like the gauge flux B through the small compactified circle forbids the 3D string excitations C_{3D}, C^1_{3D}, and C^2_{3D} to wrap around the small compactified circle. So the type-B_{3D} string in the center corresponds to the gauge flux B through the small compactified circle.

The fusion and braiding of the 2D particle e is very simple: it is an boson with fusion $e \otimes e = \mathbf{1}$. This is consistent with the fact that the corresponding 3D particle A^1_{3D} is a boson with fusion $A^1_{3D} \otimes A^1_{3D} = \mathbf{1}_{3D}$. The fusion and braiding of the 2D particle m is also very simple, since it is also an boson $m \otimes m = \mathbf{1}$. This suggests that the 3D type-B_{3D} string excitations has a simple fusion and braiding property, *provided that those 3D string excitations are threaded by a type-B_{3D} string going through their center* (see Fig. 6). For example, from the 2D fusion rule $m \otimes m = \mathbf{1}$, we find that the fusion of two type-B_{3D} loops give rise to a trivial string

$$B_{3D} \otimes B_{3D} = \mathbf{1}_{3D}. \tag{16}$$

As suggested by the 2D braiding of two m particles, when a type-B_{3D} string going around another type-B_{3D} string, the induced phase is zero (*i.e.* the mutual braiding "statistics" is trivial).

Similarly, we can choose the gauge flux through the small compactified circle to be C. In this case $\mathcal{C}^{3D}_{S_3} \to \mathcal{C}^{2D}_{\mathbb{Z}_3}$, and $\mathcal{C}^{2D}_{\mathbb{Z}_3}$ is a $\mathbb{Z}_3$ topological order in 2D which has 9 particle types: $\mathbf{1}$, e_1, e_2, m_1, m_2, $e_i m_j|_{i,j=1,2}$. In this case, the gauge flux C through the small compactified circle forbids the 3D string excitations B_{3D} and B^1_{3D} to wrap around the small compactified circle. So there is no 2D excitations that correspond to the 3D string excitations B_{3D} and B^1_{3D}. The 3D string excitation C_{3D} wrapping around the small compactified circle gives rise to a composite $\mathbb{Z}_3$ vortex $m_1 \oplus m_2$ in 2D. (This is because there are two non-trivial group elements in S_3 that commute with a group element in the conjugacy class C). Also, from the $S_3 \to \mathbb{Z}_3$ symmetry breaking: $A^1 \to 1$ and $A^2 \to e_1 \oplus e_2$, we see that the 3D A^1_{3D} charge reduces to type-$\mathbf{1}$ particle in 2D, and the 3D A^2_{3D} charge reduce to a composite particle $e_1 \oplus e_2$ in 2D.

The fusion of the composite 2D particle $c = m_1 \oplus m_2$ is given by

$$c \otimes c = 2\mathbf{1} \oplus c. \tag{17}$$

This leads to the corresponding fusion rule for the 3D type-C_{3D} loops

$$C_{3D} \otimes C_{3D} = 2\mathbf{1}_{3D} \oplus C_{3D} \text{ or } \mathbf{1}_{3D} \oplus A^1_{3D} \oplus C_{3D}, \tag{18}$$

provided that those 3D loops are threaded by a type-C_{3D} string going through their center (see Fig. 6). (The ambiguity arises because the 3D charge A^1_{3D} reduces to $\mathbf{1}$ in 2D.)

Now, let us choose the gauge flux through the small compactified circle to be trivial. In this case $\mathcal{C}^{3D}_{S_3} \to \mathcal{C}^{2D}_{S_3}$, which has 8 particle types: $\mathbf{1}$, A^1, A^2, B, B^1, C, C^1, C^2. The 3D string excitation B_{3D} and C_{3D} wrapping around the small compactified circle gives rise to the 2D excitation B and C. The fusion of the 2D particle C is given by

$$C \otimes C = \mathbf{1} \oplus A^1 \oplus C. \tag{19}$$

This leads to the corresponding fusion rule for the 3D type-C_{3D} loops

$$C_{3D} \otimes C_{3D} = \mathbf{1}_{3D} \oplus A^1_{3D} \oplus C_{3D}, \tag{20}$$

provided that those 3D loops are not threaded by any non-trivial string. The above fusion rule implies that when we fusion two C_{3D} loops, we obtain three accidentally degenerate states: the first one is a non-topological excitation, the second one is a S_3 charge A^1_{3D}, and the third one is a S_3 string C_{3D}.

Similarly, the fusion of the 2D particle B is given by

$$B \otimes B = \mathbf{1} \oplus A^2 \oplus C \oplus C^1 \oplus C^2. \tag{21}$$

a	A	B	C
Symmetry Breaking	$S_3 \to S_3$	$S_3 \to \mathbb{Z}_2$	$S_3 \to \mathbb{Z}_3$
$\mathbf{1}_{3D} \to$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$A^1_{3D} \to$	A^1	e	$\mathbf{1}$
$A^2_{3D} \to$	A^2	$\mathbf{1} \oplus e$	$e_1 \oplus e_2$
$B_{3D} \to$	B	m	-
$B^1_{3D} \to$	B^1	em	-
$C_{3D} \to$	C	-	$m_1 \oplus m_2$
$C^1_{3D} \to$	C^1	-	$e_1 m_1 \oplus e_1 m_2$
$C^2_{3D} \to$	C^2	-	$e_2 m_1 \oplus e_2 m_2$

TABLE II. The situation of figure 6, where strings are wrapped around another string of type $a = A, B, C$. Depending on a, fusion algebra and braiding statistics of each string will be related to a particle of some 2D topological order, as computed from the branching rules (6). See the text for more details.

This leads to the corresponding fusion rule for the 3D type-B_{3D} loops

$$B_{3D} \otimes B_{3D} = \mathbf{1}_{3D} \oplus A^2_{3D} \oplus C_{3D} \oplus C^1_{3D} \oplus C^2_{3D}. \quad (22)$$

This way, we can obtain the fusion algebra between all the 3D excitations A^1_{3D}, A^2_{3D}, B_{3D}, B^1_{3D}, C_{3D}, C^1_{3D}, C^2_{3D} (see Table I).

On the other hand, since the above 3D string loops are not threaded by any non-trivial string, we can shrink a single loop into a point. So we should be able to compute the fusion of 3D loops by shrinking them into a points. Mathematically we will define shrinking operation $\mathcal{S}$, which describes the shrinking process of loops.

Let $\mathcal{E}$ denote the set of 3D particle and string excitations. We would like to make sure that the shrinking operation is consistent with the fusion rules, ie $\mathcal{S}(a \otimes b) = \mathcal{S}(a) \otimes \mathcal{S}(b)$ for $a, b \in \mathcal{E}$. One can indeed check that this is the case for the following shrinking operations

$$\mathcal{S}(C_{3D}) = \mathbf{1}_{3D} \oplus A^1_{3D}, \quad \mathcal{S}(C^1_{3D}) = A^2_{3D}, \quad \mathcal{S}(C^2_{3D}) = A^2_{3D},$$
$$\mathcal{S}(B_{3D}) = \mathbf{1}_{3D} \oplus A^2_{3D}, \qquad \mathcal{S}(B^1_{3D}) = A^1_{3D} \oplus A^2_{3D}.$$

So indeed, we can compute the fusion of 3D loops by shrinking them into points. In particular, we find that the topological degeneracy for N type-C_{3D} loops is $2^N/2$. The topological degeneracy for two type-B_{3D} loops is 2. The topological degeneracy for N type-B_{3D} loops is of order 3^N in large N limit.

The above example suggests the following. Given a topological order in 3D, $\mathcal{C}^{3D}$, one may want to consider the situation illustrated in figure 6 where two loops b and c are threaded with a string a, and ask about the three-string braiding statistics. One way to compute this is to put the system on a 3-torus and compute the quantities (1), which give rise to a $SL(3,\mathbb{Z})$ representation. Then by finding the branching rules of this representation wrt. to the subgroup $SL(2,\mathbb{Z}) \subset SL(3\,\mathbb{Z})$, one finds how the systems decomposes in the 2D limit $\mathcal{C}^{3D} = \bigoplus_i \mathcal{C}^{2D}_i$, where there will be a sector i for each string type. The three-string statistics with string a in the middle, will be related to the 2D topological order $\mathcal{C}^{2D}_a$. To summarize:

- The representation branching rule (4) for $SL(3,\mathbb{Z}) \to SL(2,\mathbb{Z})$ leads to the dimension reduction branching rule (3).
- The number of the $SL(2,\mathbb{Z})$ representations (or the number of induced 2D topological orders) is equal to the number of 3D string types in the 3D topological order $\mathcal{C}^{3D}$.
- The $SL(2,\mathbb{Z})$ representations also contains information about two-string/three-string fusion, as described by eqns. (16,18,20,22). The two-string/three-string braiding can be obtained directly from the correspond 2D braiding of the corresponding particles.

VI. SOME GENERAL CONSIDERATIONS

To calculate the braiding statistics of strings and particles, we first need to know the topological degeneracy D in the presence of strings and particles before they braid. This is because the unitary matrix that describe the braiding is D by D matrix. To compute the topological degeneracy D, we need to know the topological types of strings and the particles since the topological degeneracy D depends on those types.

We have seen that, from the branching rules of $SL(3,\mathbb{Z})$ representation under $SL(3,\mathbb{Z}) \to SL(2,\mathbb{Z})$ (see eqn. (4)) we can obtain the number of the string types. How to obtain the number of the particle types?

To compute the number of the particle types, we start with a 3D sphere S^3, and then remove two small balls from it. The remaining 3D sphere will have two S^2 surfaces. This two surfaces may surround a particle and anti-particle. So the number of the particle types can be obtained by calculating the ground state degeneracy. But there is one problem with this approach, the two surfaces may carry gapless boundary excitations or some irrelevant symmetry breaking states.

To fix this problem, we note that the 3D space $S^2 \times I$ also have have two S^2 surfaces, where I is the 1D segment: $I = [0, 1]$. We can glue the space $S^2 \times I$ onto the 3D sphere S^3 with two balls removed, along the two 2D spheres S^2. The resulting space is $S^2 \times S^1$. This way, we show that the topological degeneracy on $S^2 \times S^1$ is equal to the number of the particle types.

For the gauge theory of finite gauge group G, the topologically degenerate ground states on $S^2 \times S^1$ are labelled by the group elements $g \in G$ (which describe the monodromy along the non-contractible loop in $S^2 \times S^1$), but not in an one-to-one fashion. Two elements g and $g' = h^{-1} g h$ label the same ground state since g and g' are related by a gauge transformation. So the topological degeneracy on $S^2 \times S^1$ is equal to the number of conjugacy classes of G. The number of conjugacy classes is equal to the number of irreducible representations of G, which is also the number of the particle types, a well known result for gauge theory.

Once we know the types of particles and strings, the simple fusion and braiding of those excitations can be obtained from the dimensional reduction as described in this paper.

VII. CONCLUSION

In a recent work Ref. 19, we proposed that for a gapped d-dimensional theory on a manifold $\mathcal{M}$, the overlaps (1) give rise to a representation of $\mathtt{MCG}(\mathcal{M})$ and that these are robust against any local perturbation that do not close the energy gap. In this paper we studied a simple class of $\mathbb{Z}_N$ models on $\mathcal{M} = T^3$ and computed the corresponding representations of $\mathtt{MCG}(T^3) = SL(3,\mathbb{Z})$. We argued that, similar to in 2D, the $\tilde{T}$ generator contains information about particle and string excitations above the ground state, although computed from the ground states. In an independent work Ref. 21, the authors studied the matrices (1) using some Abelian models on T^3. They argued that the generator $\tilde{S}$ contains information about braiding processes involving three loops.

Furthermore we studied a dimensional reduction process in which the 3D topological order can be viewed as several 2D topological orders $\mathcal{C}^{3D} = \bigoplus_i \mathcal{C}_i^{2D}$. This decomposition can be computed from branching rules of a $SL(3,\mathbb{Z})$ representation into representations of a $SL(2,\mathbb{Z}) \subset SL(3,\mathbb{Z})$ subgroup. Interestingly, this reduction encodes all the information about three-string statistics discussed in Ref. 20 for Abelian groups. This approach, however, also provide information about fusion and braiding statistics of non-Abelian string excitations in 3D.

We also discussed how to obtain information about particles by putting the theory on $S^2 \times S^1$. All this lends support for our conjecture[19], that the overlaps (1) for different manifold topologies $\mathcal{M}$, completely characterize topological order with finite ground state degeneracy in any dimension.

This research is supported by NSF Grant No. DMR-1005541, NSFC 11074140, and NSFC 11274192. It is also supported by the John Templeton Foundation. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research.

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