Bosonic short-range entangled states beyond group cohomology classification

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Bosonic Short Range Entangled states Beyond Group Cohomology classification

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We explore and construct a class of *bosonic* short range entangled (BSRE) states in all $4k+2$ spatial dimensions, which are higher dimensional generalizations of the well-known Kitaev's E_8 state in $2d$[1,2]. These BSRE states share the following properties: (1) their bulk is fully gapped and nondegenerate; (2) their $(4k+1)d$ boundary is described by a "self-dual" rank-$2k$ antisymmetric tensor gauge field, and it is guaranteed to be gapless without assuming any symmetry; (3) their $(4k+1)d$ boundary has intrinsic gravitational anomaly once coupled to the gravitational field; (4) their bulk is described by an effective Chern-Simons field theory with rank-$(2k+1)$ antisymmetric tensor fields, whose K^{IJ} matrix is identical to that of the E_8 state in $2d$; (5) The existence of these BSRE states lead to various bosonic symmetry protected topological (BSPT) states as their descendants in other dimensions; (6) These BSRE states can be constructed by confining fermionic degrees of freedom from 8 copies of $(4k+2)d$ SRE states with fermionic $2k-$branes; (7) After compactifying the $(4k+2)d$ BSRE state on a closed $4k$ dimensional manifold, depending on the topology of the compact $4k$ manifold, the system could reduce to nontrivial $2d$ BSRE states.

I. 1. INTRODUCTION

In the last few years, successful classification of bosonic symmetry protected topological (BSPT) states[46] have significantly enriched our understanding of states of matter under strong quantum fluctuations[3,4]. The most significant property of a SPT state is the contrast between its bulk and boundary: while the bulk is completely gapped and nondegenerate, its boundary remains nontrivial (gapless or degenerate) as long as the entire system preserves certain symmetry. Meanwhile, for a BSPT state in d-dimensional space, its $(d-1)$-dimensional boundary cannot be realized (regularized) as a $(d-1)$-dimensional system itself. The original BSPT classification[3,4] was based purely on the symmetry and dimensionality of the system, and the symmetry group cohomology generates a list of BSPT states.

However, it is known that in both $2d$ and $3d$ spaces[47], there is one special bosonic state that is beyond the classification table given in Ref. 3,4. In $2d$ space, this state is usually called the "E_8" state which was first proposed by Kitaev[1,2]. The $1d$ boundary of this bosonic state has gapless boundary state which is described by a chiral conformal field theory with chiral central charge $c=8$. This $2d$ bosonic state should belong to a more general concept called bosonic short range entangled (BSRE) state[48] as it does not need any symmetry to protect its boundary states. This state is most conveniently described by a Chern-Simons (CS) theory in $(2+1)d$ space-time[5]:

$$\mathcal{S} = \int K^{IJ} \frac{i}{4\pi} a_I \wedge da_J \quad (1)$$

with an 8×8 K matrix, which is precisely the Cartan matrix of the E_8 group. In $3d$ space, a descendant of this state was discovered[6]. It is constructed as a BSPT state protected by the time-reversal symmetry $\mathcal{T}$, and naively it can be viewed as a state in $3d$ space with fluctuating two dimensional $\mathcal{T}$-breaking domain walls with a E_8 state confined in each domain wall. The effective field theory for this state is $\mathcal{S} = \int \frac{i}{8\pi} K^{IJ} da_I \wedge da_J$[6].

Both states mentioned above are beyond the group cohomology classification, and they are also beyond another description of BSPT states using semiclassical nonlinear sigma model field theory[7]. Now an obvious question is, do these BSRE states beyond group cohomology exist only in $2d$ and $3d$, or can they be generalized to higher spatial dimensions?

Let us look at the E_8 state in $2d$ space first. Unlike all the other $2d$ BSPT states, the boundary of the E_8 state is chiral, namely it only has right moving modes but no left moving modes. This is why it does not need any symmetry to protect its boundary states, because a chiral mode can never be back-scattered. All the other $2d$ BSPT states have nonchiral boundary states, then their boundary state is stable only because the left and right moving modes carry different symmetry quantum numbers, thus the back scattering is forbidden only with the presence of symmetry. Thus to generalize the E_8 state to higher dimensions, we need to look for higher dimensional BSRE states whose boundary is purely "chiral".

Another way of making the same statement is that, unlike all the other $2d$ BSPT states, the E_8 state boundary has intrinsic gravitational anomaly due to its chiral nature, namely after coupling to the gravitational field, the general coordinate transformation at the boundary space-time is no longer a symmetry, or in other words, the energy-momentum tensor is no longer conserved with a background gravitational field[8]. Thus to generalize the E_8 state to higher dimensions, we should look for higher dimensional bosonic states whose boundary has gravitational anomalies. Because according to the anomaly matching condition[9,10], if a gapless system has perturbative gauge anomalies after coupling to a gauge field, then it cannot be gapped out by any perturbation preserving the gauge symmetry.

In lower dimensions, gravitational anomaly is usually related to the thermal Hall effect (see for instance Ref. 11, 12). In higher dimensions we can also use gravitational anomaly as a tool to construct higher dimensional SRE states. The relation between SRE states and boundary gravitational anomalies was also discussed in Ref. 13,14.

Unlike bosonic states, free fermions without interaction can already form nontrivial SPT states. The free fermion SPT states have been well understood and classified in Ref. 15–17, and recent studies suggest that interaction may not lead to new SRE states, but it can reduce the classification of fermionic SRE states[18–26]. It remains an open question that whether interaction can lead to any new FSPT or FSRE state that is intrinsically different from any free fermion state.

In this paper we will demonstrate that in all $(4k+2)d$ space, there exists a BSRE state with stable gapless boundary without assuming any symmetry. The existence of this BSRE state in $(4k+2)d$ space generates a list of bosonic SPT states in higher dimensions as its descendants. We also argue that in all $(4k+2)d$ space, there exists a SRE state constructed with "fermionic" $2k-$branes, *e.g.* $2k$ dimensional objects with fermionic statistics under exchange.

II. 2. FIELD THEORY FOR SRE STATES IN $(4k+2)d$ SPACE

Ref. 8 already demonstrated that in $(4k+1)d$ space ($(4k+2)d$ space-time), a "self-dual" rank-$2k$ antisymmetric gauge field $\Theta_{\mu_1\cdots\mu_{2k}}$ has gravitational anomaly, namely the general coordinate transformation is no longer a symmetry of the system. The self-dual condition in the $(4k+2)d$ space-time is

$$F = \star F, \tag{2}$$

where $F = d\Theta$. When $k=0$, Θ becomes a scalar boson θ, and this self-dual condition reduces to the familiar chiral condition in $(1+1)d$ space-time: $\partial_t\theta = \partial_x\theta$.

The chiral boson in $(1+1)d$ space-time is the boundary of a CS field theory in $(2+1)d$ space-time. In fact, it is straightforward to prove that a self-dual boson field in $(4k+2)d$ space-time is the boundary of a CS field theory in $(4k+3)d$ space-time with the following action[27]:

$$\mathcal{S}_C = \int \frac{iK^{IJ}}{4\pi} C^I \wedge dC^J, \tag{3}$$

where C^I is a $(2k+1)$-form antisymmetric tensor field. At the boundary, the relation between C and Θ is $C^I = d\Theta^I$. Thus we propose that the field theory Eq. 3 with a proper choice of *symmetric* matrix K^{IJ} is the field theory that describes the desired BSRE states.

Field theory Eq. 3 appears in every $(4k+3)d$ space-time instead of every odd space-time dimension. This is because for $(4k+1)d$ space-time, although a similar CS theory can be written down, the K matrix must be antisymmetric instead of symmetric, and the analysis of those states will be very different from the ones discussed in this paper. Also, as was shown in Ref. 8, only for $(4k+2)d$ space-time, the self-dual boson field Θ can have perturbative gravitational anomaly.

The action Eq. 3 with $k=1$ ($6d$ space) and the simplest choice of the matrix $K=1$ were studied in Ref. 27,28, and the authors demonstrated that the boundary of the system must be nontrivial. In our paper we will argue that this action with $K=1$ corresponds to a $(4k+2)d$ SRE state constructed with fermionic branes; while with the same K matrix as the $2d$ E_8 state, Eq. 3 corresponds to a $(4k+2)d$ BSRE state that is beyond the group cohomology classification.

If Eq. 3 describes a SRE state, then there are constraints on K. With $k=0$ ($2d$ space), Eq. 3 usually has topological order and topological degeneracy. On a $2d$ torus, the ground state degeneracy (GSD) of Eq. 3 is $\mathrm{Det}[K]$. On a $(4k+2)d$ torus, the GSD of Eq. 3 is (please see appendix for derivation)

$$\mathrm{GSD} = (\mathrm{Det}[K])^{\frac{(4k+2)!}{2[(2k+1)!]^2}}. \tag{4}$$

Since the desired BSRE and FSRE states should have no bulk topological order or topological degeneracy, then it must have $\mathrm{Det}[K]=1$.

If Eq. 3 describes a nontrivial SRE state with nonzero boundary gravitational anomalies, then after diagonalizing the K matrix, the number of positive eigenvlues n_+ must be different from the number of negative eigenvalues n_-, *i.e.* $n_+ \neq n_-$. For the simplest case $k=0$, this implies that the $(1+1)d$ boundary has a chiral central charge; for larger k, this implies that the gravitational anomalies at the boundary do not cancel out.

If Eq. 3 describes a BSRE state, then the K matrix not only needs to satisfy the previous two conditions, it also needs to yield bosonic statistics between all excitations, which imposes further constraints on the K matrix.

III. 3. STATISTICS BETWEEN $2k$-BRANES

The "matter field" that directly couples to the $(2k+1)$-form gauge field is a $(2k+1)$-form current J, which corresponds to the motion of a $2k$-dimensional matter ($2k$-brane) in the space-time:

$$\mathcal{S} \ = \ \mathcal{S}_C + \int \sum_I m_I C^I \wedge \star J. \tag{5}$$

$m = (m_1, \cdots)^T$ is the charge vector carried by the $2k$-brane, whose components $m_I \in \mathbb{Z}$ are all integers. Thus the simplest microscopic construction for our SRE states is a construction based on membranes, rather than point particles.

Suppose we consider two closed current configurations $\mathcal{J}^A$ and $\mathcal{J}^B$ (both are closed $(2k+1)$-manifolds) in the space-time with charge vectors m_A and m_B respectively, then after integrating out the gauge field C, the following action will be generated:

$$\mathcal{S} = \int 2\pi i \left(m_A^T K^{-1} m_B\right) J^A \wedge M^B, \tag{6}$$

where M^B is a $(2k+2)$-form field in the space-time satisfying $\star d \star M^B = J^B$, and the current field configuration is given by $J^{A(B)} = \delta(x \in \mathcal{J}^{A(B)})\,\mathrm{vol}(\mathcal{J}^{A(B)})$ with $\mathrm{vol}(\mathcal{J})$ being the volume form of the manifold $\mathcal{J}$. The field M^B actually describes an open $(2k+2)$-manifold $\mathcal{M}^B$ bordered by $\mathcal{J}^B$, *i.e.* $\mathcal{J}^B = \partial\mathcal{M}^B$. The non-zero contribution to the action in Eq. (6) only comes from the intersection of $\mathcal{J}^A$ and $\mathcal{M}^B$, *i.e.* when $\mathcal{J}^A$ and $\mathcal{J}^B$ are linked in the space-time. The linking of $\mathcal{J}^A$ and $\mathcal{J}^B$ can be interpreted as the braiding of the $2k$-branes labeled by the charge vectors m_A and m_B. This kind of braiding is well-defined in arbitrary D-dimensional space, between two branes of D_1 and D_2 dimensions, as long as $D = D_1 + D_2 + 2$. For instance, in $2d$ space, we can discuss braiding between two $0d$ point particles; in $3d$ space, we can discuss braiding between a $1d$ loop and a $0d$ point particle; and here in $(4k+2)d$ space, Eq. (6) simply describes the braiding statistics between two $2k$-branes.

If we take the simplest K matrix, *i.e.* $K = 1$, then two currents J^A and J^B with charge $+1$ after a full braiding will acquire phase 2π, *i.e.* two identical $2k$-branes after exchange (half-braiding) would acquire phase π. Thus the simplest choice of $K = 1$ corresponds to a fermionic membrane theory.

For a bosonic state without bulk topological order or fractionalization, all the excitations must have bosonic statistics. Thus if we want to construct a BSRE state, we must demand all the $2k$-branes have bosonic exchange statistics. The simplest K matrix that satisfies this criterion, while simultaneously satisfying $\mathrm{Det}[K] = 1$ and $n_+ \neq n_-$, is the same K matrix as the E_8 state in $2d$ space:

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \quad (7)$$

A bosonic brane can be constructed with point bosons on a lattice by imposing local constraints on all the allowed configurations of bosons, just like string configurations can be constructed with ordinary local bosons and spins[29]. Thus a bosonic membrane model can be viewed as a local quantum boson model as well. However, it is less obvious how to construct fermionic branes based on local fermions. Thus the exact connection between the fermionic membranes discussed here and local point fermion particles requires further exploration.

The simplest fermionic SRE state in $6d$ is the integer quantum Hall (IQH) state, whose boundary is a $5d$ chiral fermion. The IQH state is also two copies of a $6d$ topological superconductor in the so-called D-class[15–17]. We believe that our fermionic membrane state with $K = 1$, though it is unclear if it can be constructed with point fermion particles, must be fundamentally different from these free fermion SRE states. The reason is that, as was shown in Ref. 30, at the boundary of $(4k+2)d$ space, the gravitational anomalies have multiple terms for $k > 0$. For example, for $k = 1$, under the general coordinate transformation $x^i \to x^i + \eta^i(x)$ at the $(5+1)d$ boundary space-time, the gravitational anomalies have two terms:

$$\eta^i D^j T_{ij} \sim D^a \eta^b \epsilon_{ijklmn} \times$$
$$(\alpha R_{abij} R_{cdkl} R_{cdmn} + \beta R_{bcij} R_{cdkl} R_{damn}). \quad (8)$$

The boundary of the $(4k+2)d$ IQH state is the $(4k+1)d$ chiral fermion, whose gravitational anomaly is *linearly independent* of that of the self-dual tensor field Θ in $(4k+2)d$ space-time except for the special case $k = 0$. This is because unlike the simplest case $k = 0$ where there is only one gravitational anomaly term at the $(1+1)d$ boundary, in all the higher dimensions the gravitational anomaly is always a linear combination of several different terms, like Eq. 8. This implies that for $d \geq 6$, the free fermion SRE states, and the fermionic brane state described by Eq. 3 with $K = 1$ cannot be connected to each other by deforming the boundary Hamiltonian, they must be separated by a bulk quantum phase transition.

IV. 4. DESCENDANT BSPT STATES IN OTHER DIMENSIONS

Once we identify a SRE state in $(4k+2)d$ space, we can construct a series of SPT states in other dimensions as its descendants. Unlike their parent SRE state in $(4k+2)d$ space, these SPT states do need certain symmetry to protect their boundary states.[49]

One such BSPT state in $3d$ space was constructed in Ref. 6 as a descendant of the $2d$ E_8 state. This $3d$ BSPT state is protected by the time-reversal symmetry $\mathcal{T}$. This BSPT state is constructed as follows: we first break $\mathcal{T}$, then the system will become a trivial state. Then $\mathcal{T}$ can be restored by proliferating the domain walls between $\mathcal{T}$-breaking domains. It is the domain walls that distinguish this $3d$ BSPT state from other trivial $3d$ states: there is a $2d$ E_8 state sandwiched in each $2d$ $\mathcal{T}$-breaking domain wall. Also, the ordinary $3d$ topological insulator can be viewed as a $3d$ state with proliferating $\mathcal{T}$-breaking domain walls with a $2d$ $\nu = 1$ IQH state sandwiched in each domain wall.

The same construction can be trivially generalized to any $(4k+3)d$ space: a $(4k+3)d$ BSPT state can be constructed by proliferating domain walls of $\mathcal{T}$-breaking domains, with the $(4k+2)d$ BSRE state sandwiched in each domain wall. The effective field theory for such $(4k+3)d$ state is

$$\mathcal{S} = \int \frac{i\theta}{2\pi} \frac{1}{4\pi} K^{IJ} dC^I \wedge dC^J. \quad (9)$$

Analogous to the simplest case with $k = 0$ discussed in Ref. 6, and similar to the ordinary $3d$ topological insulator[31], when $\theta = \pi$ this state is a nontrivial $(4k+3)d$

BSPT with time-reversal symmetry, while when $\theta = 0$ mod 2π, this state is trivial. Thus this $(4k+3)d$ BSPT has Z_2 classification.

The Z_2 classification can be understood as the follows. On a closed $(4k+4)d$ space-time manifold, the integral $\int \frac{1}{4\pi^2} dC \wedge dC$ is an even integer on any manifold whose $(2k+2)$th Wu-class vanishes, while it is an odd integer otherwise[28,32]. When $k = 0$, the second Wu-class is equivalent to the second Stiefel-Whitney class[32], thus the condition of vanishing Wu-class reduces to the well-known condition for a manifold to allow a spin structure. However, with the BSRE K matrix in Eq. 7, the integral $\int \frac{1}{4\pi^2} K^{IJ} dC^I \wedge dC^J$ is always an even integer on any closed $(4k+4)d$ space-time manifold. This implies that in Eq. 9 θ and $\theta + 2\pi$ are equivalent. And because time-reversal transformation takes θ to $-\theta$, if the system has time-reversal symmetry, it demands θ take only discrete values $\theta = \pi n$. While with $\theta = 2\pi$, if $\mathcal{T}$ is broken at the boundary, the $(4k+2)d$ boundary of Eq. 9 is precisely the $(4k+2)d$ theory described in Eq. 3, thus the boundary can be regularized as a $(4k+2)d$ system itself. Thus $\theta = 2\pi$ corresponds to a trivial state. When $\theta = \pi$, naively the boundary corresponds to "half" of the $(4k+2)d$ BSRE state in Eq. 3, thus it cannot be realized as a $(4k+2)d$ state itself. When $\theta = \pi$, if the boundary preserves $\mathcal{T}$, based on our experiences with $3d$ topological insulator, we expect the boundary to be either gapless, or have $(4k+2)d$ topological order that cannot be regularized on $(4k+2)d$ space itself.

Similar constructions can be generalized to BSPT states with other symmetries as well. For every $(4k+4)d$ space, we can construct a BSPT state with U(1) symmetry as follows: We start with the superfluid phase with spontaneous U(1) symmetry breaking, then the U(1) symmetry can be restored by proliferating U(1) vortices. In $(4k+4)d$ space, a vortex is a $(4k+2)$-brane. For ordinary systems, after proliferating the vortices the system will enter a trivial gapped disordered phase. However, if there is a $(4k+2)d$ BSRE state described by Eq. 3 confined in each vortex, then after vortex proliferation the system will enter the desired BSPT state with U(1) symmetry. Once we couple the U(1) symmetry to an external U(1) gauge field A_μ, this system is described by the following effective field theory:

$$\mathcal{S} = \int \frac{i}{8\pi^2} K^{IJ} C^I \wedge dC^J \wedge dA. \tag{10}$$

This state has Z classification, since for arbitrary copies of this system, it remains nontrivial.

At the boundary of the $(4k+4)d$ theory Eq. 10, after coupling to both gravitational field and U(1) gauge field, there is a mixed gauge-gravitational anomaly, *i.e.* the conservation of energy momentum tensor is violated in the flux of the U(1) gauge field. For example, for $k = 1$, at the seven dimensional boundary (eight dimensional boundary space-time), the mixed anomaly reads:

$$\eta^i D^j T_{ij} \sim D^a \eta^b \epsilon_{ijklmnpq} \times$$

$$(\alpha R_{abij} R_{cdkl} R_{cdmn} + \beta R_{bcij} R_{cdkl} R_{damn}) \partial_p A_q. \tag{11}$$

For every $(4k+2+2n)d$ space, we can construct a family of BSPT state with U(1) symmetry, based on the BSRE state in $(4k+2)d$. This state is constructed as follows: In $(4k+2+2n)d$ space, a U(1) vortex is a $(4k+2n)$ dimensional membrane. Then n vortices will intersect on a $(4k+2)$ dimensional membrane, where the $(4k+2)d$ BSRE state Eq. 3 can reside. Then this BSPT state discussed in this paragraph can be obtained by proliferating these vortices with BSRE state Eq. 3 confined in each $n-$vortex intersection. After coupling the system to an external U(1) gauge field, the effective field theory will be

$$\mathcal{S} = \int \frac{i}{4\pi (2\pi)^n n!} K^{IJ} C^I \wedge dC^J \wedge (dA)^n. \tag{12}$$

Again, at the $(4k+2n+1)d$ boundary of Eq. 12, after coupling to both gravitational field and U(1) gauge field, there is a mixed gauge-gravitational anomaly. All these BSPT states constructed in this section are beyond the group cohomology classification developed in Ref. 3,4.

V. 5. CONFINE FERMIONIC BRANES

In this section we will demonstrate that the BSRE state with $K = K_{E_8}$ in Eq. 7 can be constructed with fermionic brane theories with $K = 1$.

We start from 8 copies of $K = 1$ theory, which is still a CS field theory but with the K matrix being an 8×8 identity matrix, denoted as I_8. To make connection to the E_8 theory, we first attach a trivial bosonic state to the system, which is described by the K matrix σ^x. Thus the entire system has K matrix $I_8 \oplus \sigma^x$. σ^x has eigenvalues $+1$ and -1, thus if a system has $K = \sigma^x$, its boundary has both self-dual and anti self-dual fields, *i.e.* its boundary has no gravitational anomaly, hence it is a trivial bosonic state in $(4k+2)d$.

Just like $2d$ Abelian quantum Hall state, different K matrices can correspond to the same physical state, as long as these K matrices only differ from each other by a $\mathrm{GL}(N,\mathbb{Z})$ transformation W ($\det W = 1$)[33]. The physical meaning of the $\mathrm{GL}(N,\mathbb{Z})$ transformation is just to relabel all the excitations. It is straight forward to verify that there exists the following transformation W,

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -3 & 5 \end{pmatrix}, \tag{13}$$

such that

$$W^\intercal (I_8 \oplus \sigma^x) W = K_{E_8} \oplus \sigma^z. \tag{14}$$

Existence of this transformation implies that 8 copies of fermionic membrane theories plus a trivial boson state, is topologically equivalent to the desired $(4k+2)d$ BSRE state attached to a trivial fermionic brane state with $K=\sigma^z$ (the boundary of Eq. 3 with $K=\sigma^z$ is also nonchiral, hence it corresponds to a trivial state).

Since eventually the K matrix $K_{E_8}\oplus\sigma^z$ is block diagonal, fermionic and bosonic degrees of freedom are decoupled. This implies that we can safely tune the gap of fermionic branes to infinity without interfering with the bosonic sector. Thus 8 copies of fermionic brane theory with $K=1$ will automatically become the desired BSRE state with $K=K_{E_8}$ given by Eq. 7, after the fermionic branes are confined.

Here we also want to note that the relation between $2d$ $\nu=8$ integer quantum Hall state and the $2d$ E_8 state was previously also studied in Ref. 34,35.

VI. 6. REDUCTION TO $2d$ SPACE

A natural question to ask about the SRE states constructed by Eq. 3 is that, if we define the system on a $(4k+2)d$ space manifold that is $\mathbb{R}^2\otimes\mathcal{M}_{4k}$, where $\mathbb{R}^2$ is the infinite two dimensional plane, while $\mathcal{M}_{4k}$ is a closed compact $4k-$dimensional manifold, then does this state reduce to any nontrivial SRE state in $2d$?

Let us take $k=1$ ($6d$ space with $5d$ boundary) and $K=1$ as an example. At the $5d$ boundary, we can choose a gauge to make $\Theta_{0j}=0$. Then the self-dual condition reads

$$\partial_t\Theta_{ij}=\epsilon_{ijklm}\partial_k\Theta_{lm}. \tag{15}$$

The dynamical modes in the $(1+1)d$ space-time (t,x_1) are Θ_{ab} with $a,b=2\cdots5$:

$$\begin{aligned}\partial_t\Theta_{23} &= \partial_1\Theta_{45}+\partial_4\Theta_{51}+\partial_5\Theta_{14};\\ \partial_t\Theta_{45} &= \partial_1\Theta_{23}+\partial_2\Theta_{31}+\partial_3\Theta_{12};\\ \cdots &= \cdots\end{aligned}\tag{16}$$

Let us define a $1-$form vector field $\vec{A}=(\Theta_{12},\Theta_{13},\Theta_{14},\Theta_{15})$ on the compact $4-$manifold $\mathcal{M}_4$, then the chirality of the reduced field theory on the remaining $(1+1)d$ space-time, depends on whether the corresponding theory on $\mathcal{M}_4$ is self-dual or anti self-dual. For example, if $d_4A=\star d_4A$ (self-dual) on $\mathcal{M}_4$, then $\Theta_{23}+\Theta_{45}$, $\Theta_{34}+\Theta_{25}$, $\Theta_{35}+\Theta_{42}$ are three left moving modes; while if $d_4A=-\star d_4A$ (anti self-dual) on $\mathcal{M}_4$, then $\Theta_{23}-\Theta_{45}$, $\Theta_{34}-\Theta_{25}$, $\Theta_{35}-\Theta_{42}$ are three right moving modes. Each self-dual (anti self-dual) configuration $\vec{A}$ on $\mathcal{M}_4$ will produce three left (right) moving modes on the reduced $(1+1)d$ space-time.

Thus with $K=1$, the chiral central charge of the reduced $(1+1)d$ theory is

$$c=3(n_+-n_-), \tag{17}$$

where $n_\pm$ is the number of independent 2-form $\omega_\pm$ on $\mathcal{M}_4$ that satisfies

$$\begin{aligned}\mathbf{d}\omega_+=0, &\quad \omega_+=\star\omega_+,\\ \mathbf{d}\omega_-=0, &\quad \omega_-=-\star\omega_-,\end{aligned}\tag{18}$$

where $\mathbf{d}=d-\star d\star$. Mathematically ω_+ and ω_- are the kernel and cokernal of operator $\mathbf{d}$, and n_+-n_- is the index of operator $\mathbf{d}$, and according to the Atiyah-Singer theorem, the index is a topological invariant of $\mathcal{M}_4$. In this particular case n_+-n_- corresponds to the signature (also called the $L-$genus) of $\mathcal{M}_4$: $n_+-n_-=\mathrm{Sign}[\mathcal{M}_4]$. The signature of a manifold is determined by its topology only. The signature of some example $4-$manifolds is listed in the follows[36]:

$$\mathrm{Sign}[S^4]=0,\quad \mathrm{Sign}[T^4]=0,\quad \mathrm{Sign}[\mathrm{CP}^2]=1. \tag{19}$$

On S^4 and T^4, because their signature is zero, after compactification the reduced $(1+1)d$ theory is nonchiral, thus the reduced $2d$ bulk theory is a trivial disordered state. However because $\mathrm{Sign}[\mathrm{CP}^2]=1$, if we take $K=1$ in Eq. 3, then after compatifying Eq. 3 on CP^2, the reduced theory on $(1+1)d$ space-time has chiral central charge 3, thus the $2d$ bulk is presumably equivalent to $\nu=3$ integer quantum Hall state in $2d$. If we take $K=K_{E_8}$ in Eq. 7, then the reduced $2d$ bulk becomes a nontrivial BSRE state in $2d$, which presumably is equivalent to three copies of the $2d$ E_8 states.

VII. 7. SUMMARY AND DISCUSSION

In this work we construct BSRE states in $(4k+2)d$ space, along with its descendant BSPT states in other dimensions, using the fact that the boundary of Eq. 3 has intrinsic gravitational anomalies after coupling to the gravitational field. Due to the anomaly matching condition, the gravitational anomalies imply that the boundary of Eq. 3 cannot be gapped out, even though we assume no symmetry at all in Eq. 3.

So far the gravitational anomalies we used in our work were all perturbative gravitational anomalies. There is another type of global gravitational anomalies that we have not discussed yet. In Ref. 30, it was demonstrated that a single (or odd flavors of) Majorana fermion in $8k-1$ and $8k$ dimensional space has global gravitational anomalies, namely the partition function of the system changes sign under a large global coordinate transformation. These global gravitational anomalies have Z_2 classification, because even flavors of Majorana fermions in these dimensions have no global gravitational anomalies. These global anomalies precisely correspond to the Z_2 classification of topological superconductors in $8k$ and $8k+1$ dimensional space without any symmetry (the so called D class[15–17]), whose boundary is precisely massless Majorana fermion in $8k-1$ and $8k$ dimensional space.

Ref. 13,37–40, proposed that there exists a $4d$ BSRE state without any symmetry, and this state has Z_2 classification. This result implies that the $3d$ boundary of odd flavors of this BSRE state should have global gravitational anomalies. Ref. 41 proposed that the effective field theory description of this $4d$ BSRE state is a $(4+1)d$ version of Eq. 3, and the gauge field involved is a $2-$form field. Also Ref. 41 pointed out that this state corresponds to the $4d$ BSRE state discussed in Ref. 42 whose boundary is a $3d$ QED with fermionic gauge charge and monopole. More general connection between global gravitational anomaly and SRE states is desired, which we will leave to future studies.

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[46] Sometimes this kind of states are also called "symmetry protected trivial" states in literature, depending on the taste and level of terminological rigor of authors.
[47] In order to avoid possible confusion, in this paper the term "space" always refers to real physical space instead of space-time.
[48] Here we define short range entangled state as states with fully gapped and nondegenerate ground state, these states have no topological entanglement entropy, and hence we call them "short range entangled".
[49] In this section all these SPT states are constructed with the "decorated defect" picture. But we admit that we do not have an explicit lattice construction for these states, but we expect (without proof) our "decorated defect" picture can be realized in some properly designed lattice model. We will leave this to future study.

Appendix A: Quantization of K

The $(2k+1)$-form gauge fields C that we considered here are *compact* U(1) gauge fields. The compactness arise from the assumption that the matter of $2k$-brane only carries quantized (integer) gauge charge, *i.e.* any $2k$-brane is an integer multiple of the elementary $2k$-brane. Therefore all physical observables in the gauge theory are given by (functions of) the Wegner-Willson amplitudes on some closed $(2k+1)d$ current manifold $\mathcal{J}$ with the elementary gauge charge,

$$\mathcal{W}[\mathcal{J}] = e^{\mathrm{i}\int_{\mathcal{J}} C}. \tag{A1}$$

Any transformations of C that keep all the Wegner-Willson amplitudes $\mathcal{W}[\mathcal{J}]$ invariant are gauge transformations, which include the local gauge transformations $C \to C + \mathrm{d}\Theta$ (induced by arbitrary $2k$-form fields Θ) that trivially preserve $\mathcal{W}[\mathcal{J}]$, as well as the large gauge transformations which are allowed only in the compact gauge theory. In a $(4k+3)d$ space-time with periodicity L_i in the $\mathrm{d}x^i$ direction, the large gauge transformation takes the following form

$$C \to C + \delta C, \text{ with } \delta C = \frac{2\pi N_{i_1\cdots i_{2k+1}}}{L_{i_1}\cdots L_{i_{2k+1}}}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_{2k+1}}, \tag{A2}$$

with integers $N_{i_1\cdots i_{2k+1}} \in \mathbb{Z}$ for $i_1 < \cdots < i_{2k+1}$. For *contractable* current manifold $\mathcal{J} = \partial\mathcal{M}$, $\int_{\mathcal{J}} C = \int_{\partial\mathcal{M}} C = \int_{\mathcal{M}} \mathrm{d}C$ is simply invariant under the transformation in Eq. (A2) as $\mathrm{d}\delta C = 0$, so $\mathcal{W}[\mathcal{J}]$ is also invariant. For *non-contractable* current manifold $\mathcal{J}$, $\int_{\mathcal{J}} \delta C = 2\pi N$ will be an integer multiple of 2π, but $\mathcal{W}[\mathcal{J}] \to e^{\mathrm{i}2\pi N}\mathcal{W}[\mathcal{J}] = \mathcal{W}[\mathcal{J}]$ is still invariant. Hence it is verified that Eq. (A2) is indeed an allowed gauge transformation.

The invariance of the partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\mathcal{S}_C}$ with $\mathcal{S}_C = \int \frac{\mathrm{i}K^{IJ}}{4\pi} C^I \wedge \mathrm{d}C^J$ in Eq. (3) under the large gauge transformation Eq. (A2) necessarily requires the entries of the K matrix to be integers, *i.e.* $K^{IJ} \in \mathbb{Z}$. For simplicity, let us first take the case of single-component gauge field C, and consider the large gauge transformation $C_{0\cdots 2k} \to C_{0\cdots 2k} + \frac{2\pi}{L_0\cdots L_{2k}}$, the action will be changed by

$$\delta\mathcal{S}_C = \int \frac{\mathrm{i}K}{4\pi}\frac{2\times 2\pi}{L_0\cdots L_{2k}}\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{2k}\wedge\mathrm{d}C = \mathrm{i}K\int_{\Omega_{2k+2}}\mathrm{d}C, \tag{A3}$$

where Ω_{2k+2} is the closed $(2k+2)$-manifold parameterized by the remaining coordinates $(x^{2k+1},\cdots,x^{4k+2})$. It can be proved that $\int_{\Omega_{2k+2}}\mathrm{d}C$ must be an integer multiple of 2π on any closed $(2k+2)$-manifold Ω_{2k+2} by generalizing the argument in Ref. 43. First consider the manifold Ω_{2k+2} as patches glued together, such that on each patch the gauge field C is smooth and single-valued, then $\int_{\Omega_{2k+2}}\mathrm{d}C = \int_{\partial\Omega_{2k+2}} C$ with $\partial\Omega_{2k+2}$ being the boundary of the patches, on which C from both sides must be related by some gauge transformation $C \to C + \mathrm{d}\Theta$. So the integral will become $\int_{\Omega_{2k+1}}\mathrm{d}\Theta$ on one side of the patch boundary. Keep using the same trick on Ω_{2k+1}: consider Ω_{2k+1} as $(2k+1)$-dimensional patches glued along their $2k$-dimensional boundary, then the integral can be ascribed to the difference of Θ form both sides (denoted as $\Theta_\pm$) along the boundary manifold Ω_{2k} as $\int_{\Omega_{2k}}(\Theta_+ - \Theta_-)$. According to the compactification condition, $\int_{\Omega_{2k}}\Theta$ and $2\pi + \int_{\Omega_{2k}}\Theta$ are equivalent on any closed $2k$-manifold Ω_{2k}, because it corresponds to changing the phase of the matter field ($2k$ dimensional membranes) by 2π. So $\int_{\Omega_{2k}}\Theta_+$ and $\int_{\Omega_{2k}}\Theta_-$ can only differ by multiples of 2π, and hence proved $\int_{\Omega_{2k+2}}\mathrm{d}C = \int_{\Omega_{2k}}(\Theta_+ - \Theta_-) = 2\pi N$ with $N \in \mathbb{Z}$. Substituting this result back to Eq. (A3), we arrive at the conclusion that under large gauge transformation, the action is changed by $\delta\mathcal{S}_C = 2\pi\mathrm{i}KN$ with $N \in \mathbb{Z}$, so K must also be an integer to ensure the invariance of the partition function $\mathcal{Z} = \mathrm{Tr}\, e^{-\mathcal{S}_C}$ for any N. It is straight forward to generalize the above argument to multi-component C^I field, and the conclusion is that every element $K^{IJ} \in \mathbb{Z}$ in the K matrix must be an integer.

Appendix B: ground state degeneracy of Eq. 3

The topological ground state degeneracy of the Chern-Simon theory on the torus can be calculated following Ref. 44,45. We first fix the temporal gauge to $C_{0i_1\cdots i_{2k}} = 0$ (where $i_1,\cdots,i_{2k}$ are spacial indices) by the gauge transformation $C_{0i_1\cdots i_{2k}} \to C_{0i_1\cdots i_{2k}} + \partial_0\Theta_{i_1\cdots i_{2k}}$ with properly chosen $\Theta_{i_1\cdots i_{2k}}$ (there are totally $\binom{4k+2}{2k}$ of $C_{0i_1\cdots i_{2k}}$ to be fixed by the same number of $\Theta_{i_1\cdots i_{2k}}$). The equations of motion for $C_{0i_1\cdots i_{2k}}$ act as constraints requiring $\mathrm{d}C = 0$ in the $(4k+2)d$ space for the spatial components $C_{i_1\cdots i_{2k+1}}$. This implies that the gauge-inequivalent configurations are completely specified by the holonomies of the spacial gauge field C on the non-contractible $(2k+1)$-manifolds $\mathcal{J}$ of the torus, *i.e.* by $\int_{\mathcal{J}} C$ for every homology basis $\mathcal{J}$. This configuration space can be parameterized by an antisymmetric rank-$(2k+1)$ tensor X as

$$C_{i_1\cdots i_{2k+1}}(t,\boldsymbol{x}) = \frac{2\pi}{L_{i_1}\cdots L_{i_{2k+1}}} X_{i_1\cdots i_{2k+1}}(t). \tag{B1}$$

The large gauge transformation in Eq. (A2) takes $X_{i_1\cdots i_{2k+1}} \to X_{i_1\cdots i_{2k+1}} + N_{i_1\cdots i_{2k+1}}$ with $N_{i_1\cdots i_{2k+1}} \in \mathbb{Z}$, thus $X_{i_1\cdots i_{2k+1}} \sim X_{i_1\cdots i_{2k+1}} + 1$ labels the same quantum state, and the quantum mechanical wave function must respect this periodicity of the configuration space. With this parameterization, the Chern-Simons action reads

$$\mathcal{S}_C = \frac{2\pi\mathrm{i}K}{2}\int \mathrm{d}t\, X\wedge\dot{X}, \tag{B2}$$

so the momentum conjugate to X is $p_X = \delta\mathrm{i}\mathcal{S}_C/\delta\dot{X} = 2\pi K\star X$. There are all together $\frac{1}{2}\binom{4k+2}{2k+1} = \frac{(4k+2)!}{2[(2k+1)!]^2}$ pairs of canonical conjugate variables among the components of X. Each pair contributes K-fold ground state degeneracy (GSD), as the periodicity $X\sim X+1$ requires the momentum quantization $p_X = 2\pi K\star X = 2\pi n$ (with $n\in\mathbb{Z}$), then $\star X\sim\star X+1$ would imply $n\sim n+K$ so that $n=0,\cdots,K-1$ labels K degenerated states. So for $\frac{(4k+2)!}{2[(2k+1)!]^2}$ pairs of conjugate variables in X, the total GSD is

$$\mathrm{GSD} = K^{\frac{(4k+2)!}{2[(2k+1)!]^2}}. \tag{B3}$$

For multi-component gauge theory with a K matrix, the number K in the above formula is just replaced by $\det K$.