General procedure for determining braiding and statistics of anyons using entanglement interferometry

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General procedure for determining braiding and statistics of anyons using entanglement interferometry

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Recently, it was argued that the braiding and statistics of anyons in a two-dimensional topological phase can be extracted by studying the quantum entanglement of the degenerate ground-states on the torus. This construction either required a lattice symmetry (such as $\pi/2$ rotation) or tacitly assumed that the 'minimum entanglement states' (MESs) for two different bipartitions can be uniquely assigned quasiparticle labels. Here we describe a procedure to obtain the modular $\mathcal{S}$ matrix, which encodes the braiding statistics of anyons, which does not require making any of these assumptions. Our strategy is to coherently compare MESs of three independent entanglement bipartitions of the torus, which leads to a unique modular $\mathcal{S}$. This procedure also puts strong constraints on the modular $\mathcal{T}$ and $\mathcal{U}$ matrices without requiring any symmetries, and in certain special cases, completely determines it. Our method applies equally to Abelian and non-Abelian topological phases.

I. INTRODUCTION

Topological ordered phases in two dimensions such as fractional quantum Hall states and quantum spin-liquids are characterized by the presence of anyonic excitations which satisfy specific braid statistics rule when taken around each other[1–10]. Presence of anyons implies that a topological ordered phase possesses a degenerate set of ground states on a torus[11–15]. The intimate relation between the ground-state degeneracy and the presence of anyons suggests that the braid statistics rules must be encoded in the degenerate ground states themselves. In this paper, we generalize the discussion in Ref. 16 to obtain the braiding and statistics from the ground states.

Mathematically, the braiding statistics is encoded in the modular $\mathcal{S}$ and $\mathcal{U}$ matrices (or equivalently, $\mathcal{S}$ and $\mathcal{T}$ matrices where $\mathcal{U} = \mathcal{TST}$)[3,17–19]. The $\mathcal{S}$ matrix expresses the mutual statistics between anyons while the $\mathcal{U}$ matrix encodes the self-statistics. For chiral topological phases, there is an additional parameter, the central charge c for the edge states, which is determined modulo 8 by the $\mathcal{S}$ and $\mathcal{U}$ matrices[19]. Ref. 16 argued that the modular $\mathcal{S}$ and $\mathcal{U}$ matrices can be determined by calculating the quantum entanglement of degenerate ground states. This builds on the idea that the scaling of entanglement entropy with respect to the subsystem size often yields universal information about the corresponding phase of matter. For example, the von Neumann entanglement entropy S_{vN} of a gapped ground state for a contractible disk-shaped region in two dimensions with a boundary of size ℓ is given by $S_{vN} = \alpha\ell - \gamma + O(1/\ell)$, where γ is the so-called 'topological entanglement entropy'(TEE) and is given by $\gamma = \sqrt{\sum d_a^2}$ with d_a/γ being the first row of the modular $\mathcal{S}$ matrix[19–21].

This result motivates one to ask whether the full modular $\mathcal{S}$ and $\mathcal{U}$ matrices might also be extractable from the ground-state entanglement. Ref. 16 argued that the answer is indeed positive. The basic idea mainly consists of two steps: first, given a set of degenerate ground states $|\xi_a\rangle$, $a = 1, \ldots, N$, the TEE corresponding to a generic ground state $|\psi\rangle = \sum c_a|\xi_a\rangle$ for *non-contractible* subregions on a torus, e.g. partitioning the torus into two cylinders, generally differs from the value in trivial subregions, and is maximized by a special set of coefficients c_a, which can be identified using TEE as an indicator. Note that the TEE reduces the total entanglement entropy, such a state minimizes the total entanglement entropy and is therefore dubbed as the 'minimum entropy state' (MES). Given a non-contractible entanglement bipartition, the complete set of MESs forms a basis of the degenerate ground states, which correspond to the simultaneous eigenstates of the Hamiltonian as well as the operators that measure the quasiparticles through the boundary cycle of the entanglement bipartition. We hereafter denote the MESs basis for a bipartition α as $\left\{\left|\Xi_a^{(\alpha)}\right\rangle\right\} \equiv \left|\Xi^{(\alpha)}\right\rangle$. Then, the modular matrices can be related to the unitary transformations between two inequivalent sets of MESs defined for two different bipartitions (α's). For example, having identified the eigenstates of operators that measure quasiparticles, viz. the MES states, the elements of the modular $\mathcal{S}$ matrix are given by the overlap: $\mathcal{S}_{ab} = \left\langle \Xi_a^{(1)} | \Xi_b^{(2)} \right\rangle$ where the two entanglement bipartitions are along the $\hat{x}$ and $\hat{y}$ directions, respectively. This procedure can further simplified in the presence of certain rotation symmetry R, which relates the two sets MESs $\left|\Xi^{(2)}\right\rangle = R\left|\Xi^{(1)}\right\rangle$. For example, $R_{\pi/2} : \hat{x} \rightarrow \hat{y}$ gives $\mathcal{S} = \left\langle \Xi^{(1)} \right| R_{\pi/2} \left|\Xi^{(1)}\right\rangle$.

This method was successfully incorporated into matrix product state (MPS) and DMRG based techniques for finding ground states in quasi-2D systems[24–26], as well as new applications via variational Monte Carlo[26,27]. A slightly different method to obtain modular matrices was recently proposed in several papers[30]. It is also worth noting that for chiral topological phases, momentum polarization method[28,29] can also be used to obtain partial topological data viz. self-statistics of anyons and the chiral central charge. Finally, it was shown[31] that for a restricted class of Hamiltonians that can be written as sum of local commuting projectors, under certain reasonable assumptions one can obtain the modular matrix $\mathcal{S}$ using a single ground state.

An important detail in calculating an inner product such as $\left\langle \Xi_a^{(1)} | \Xi_b^{(2)} \right\rangle$, which was presumed in Ref. 16, is the relative ordering of the sets of MESs $\left\{ \left| \Xi_a^{(1)} \right\rangle \right\}$ and $\left\{ \left| \Xi_a^{(2)} \right\rangle \right\}$. In the presence of a consistent point group symmetry of the lattice, the relative ordering is automatically fixed[16]. This is however not true when such a symmetry is absent. This statement also holds true for extracting the modular $\mathcal{U}$ matrix where one utilizes overlap of MESs corresponding to bipartitions that differ by an angle of $2\pi/3$. An additional related concern is to identify the MES corresponding to the 'identity quasiparticle': conventionally, the first row and column of the $\mathcal{S}$ and $\mathcal{U}$ matrices is labeled by the identity quasiparticle. However, it is not obvious how the method described in Ref. 16 makes such an identification.

In this paper, we show that a *third* entanglement bipartition (in addition to the two bipartitions used in Ref. 16) helps to resolve this ambiguity. By considering the 'entanglement interferometry' that consists of a series of modular transformations that start and end with the same set of MESs, one can effectively cancel out the impact of the unknown details of the intermediate MESs. This procedure completely fixes the relative ordering of MESs for the disparate bipartitions. This also fixes the identification of quasiparticles vis-a-vis the MES states *upto an Abelian quasiparticle*. Our main result is that the modular $\mathcal{S}$ matrix is uniquely determined by considering overlap of MESs obtained from three entanglement bipartitions. As far as the modular $\mathcal{U}$ matrix is concerned, in the absence of any symmetries we are able to determine it only upto an additional phase factor for each quasiparticle, which is further constrained by the modular $\mathcal{S}$ matrix. In special cases, the constraints are strong enough to determine the $\mathcal{U}$ matrix fully, without requiring any symmetry.

The rest of the paper is organized as follows: In Sec. II, we briefly review the method in Ref. 16 and discuss its shortcoming in the absence of spatial symmetry; to resolve this issue, we propose in Sec. III a general algorithm with a third entanglement bipartition to extract the modular $\mathcal{S}$ matrix; in Sec. IV, we study its further application on the modular $\mathcal{U}$ matrix and quasiparticle spin of the topological ordered state; three illustrative examples are discussed in Sec. V.

II. THE MINIMUM ENTROPY STATES AND MODULAR MATRICES

For concreteness, let's denote $|\xi_a\rangle$, $a = 1, \ldots, N$ as the complete, orthonormal set of the degenerate ground states that we will use as our basis for the entire ground-state manifold.

For a given nontrivial entanglement bipartition, the MESs are by definition the ground states with minimum entanglement entropy (maximum TEE) and can be generated by $T_p^{(\alpha)}$: the insertion of the quasiparticles of the topological ordered state labeled by p through the non-contractible cycle enclosed by the α's entanglement bipartition boundary[16]. These MESs

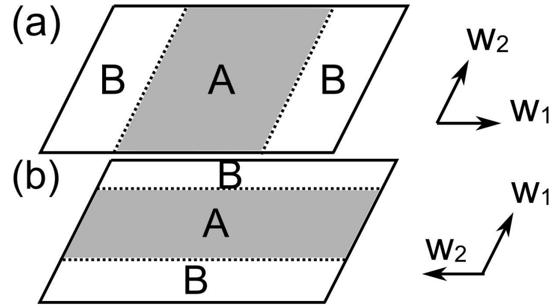

FIG. 1: Left panel: two nontrivial entanglement bipartitions shown as the dashed lines that separate the system into two cylindrical subsystems (periodic boundary condition is assumed through all boundaries); Right panel: two sets of primitive vectors for the same torus, we use $\vec{w}_2$ to label the boundary direction of each entanglement bipartition.

can be sequentially obtained by maximizing the TEE in the parameter space of the entire ground-state manifold $\sum_a c_a |\xi_a\rangle$. Let's denote the resulting MESs as

$$\left| \Xi_b^{(\alpha)} \right\rangle = e^{i\phi_b^{(\alpha)}} U_{ab}^{(\alpha)} |\xi_a\rangle \tag{1}$$

where the superscript α labels entanglement bipartitions with inequivalent non-contractible boundaries. $\phi_b^{(\alpha)}$ is an undetermined phase factor for each MES, which doesn't affect the resulting entanglement entropy.

The transformation between two MES bases is essentially a modular transformation $\mathcal{F}(\mathcal{S}, \mathcal{U})$[16]. Such a transformation can be also viewed as the transformation of the primitive vectors that define the torus and encoded in the $SL(2, \mathbb{Z})$ modular matrix $F(S, U)$, which can be expressed in terms of the two generators of $SL(2, \mathbb{Z})$

$$\begin{aligned} S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ U &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2}$$

For concreteness, let's consider a modular transformation from the primitive vectors $\vec{w}_{i=1,2}^{(1)}$ in Fig. 1(a) to $\vec{w}_{i=1,2}^{(2)}$ in Fig. 1(b), where we use $\vec{w}_2^{(\alpha)}$ to label the direction along the non-contractible boundary of the αth entanglement bipartition. The relation between the two sets of primitive vectors are

$$\begin{aligned} \vec{w}_1^{(2)} &= \vec{w}_2^{(1)} \\ \vec{w}_2^{(2)} &= -\vec{w}_1^{(1)} \end{aligned} \tag{3}$$

According to Eqn. 2, this implies $F(S, U) = S$, and therefore the transformation between the two MES bases $\left\langle \Xi^{(2)} | \Xi^{(1)} \right\rangle = \left(V^{(2)} \right)^{-1} \left(U^{(2)} \right)^{-1} U^{(1)} V^{(1)}$ is equivalent to the modular $\mathcal{S}$ matrix, where we denote the diagonal phase factors in Eqn. 1 $V^{(\alpha)} = \mathrm{diag}(e^{i\phi_b^{(\alpha)}})$. Since the undetermined $V^{(\alpha)}$ are diagonal phase factors, they are straightforward to determine with the knowledge that the elements of the first column and row of the modular $\mathcal{S}$ matrix are real and positive in accord with the definition of the identity particle[16]

$$\mathcal{S} = \left(V^{(2)}\right)^{-1}\left(U^{(2)}\right)^{-1}U^{(1)}V^{(1)} = R\left[\left(U^{(2)}\right)^{-1}U^{(1)}\right] \qquad (4)$$

where the function $R[X]$ corresponds to left and right matrix multiplication of the matrix X with certain diagonal phase factors, respectively, so as to make the elements of the first column and row of X real and positive.

However, there is an important complication that we have overlooked: in the derivation of Eqn. 4, we have assumed the same ordering for the two sets of MESs, while the maximum TEE requirement makes no distinction between the MESs with the same quantum dimension[16,22], in particular the MESs connected with the Abelian quasiparticles (including the identity particle) which have quantum dimension 1. Therefore, the ordering and particle content of the obtained MESs remain largely undetermined. In the absence of symmetry, the orderings of the MESs for different entanglement bipartitions are generally unrelated and may result in a scramble of the rows and the columns of the resulting modular matrices.

To be more specific, we now need to generalize Eqn. 1 by including possible permutations within each set of the MESs

$$\begin{aligned} U^{(\alpha)}_{ab} &= \bar{U}^{(\alpha)}_{ab'}P^{(\alpha)}_{b'b} \\ \left|\Xi^{(\alpha)}_b\right\rangle &= \bar{U}^{(\alpha)}_{ab'}V^{(\alpha)}_{b'b'}P^{(\alpha)}_{b'b}\left|\xi_a\right\rangle \end{aligned} \qquad (5)$$

where we have introduced a permutation matrix P_α that permutes the columns of U_α. As we have argued above, TEE calculations only determine $\bar{U}^{(\alpha)}$.

Following previous argument, we obtain instead of Eqn. 4

$$\begin{aligned} \mathcal{S} &= \left(P^{(2)}\right)^{-1}\left(V^{(2)}\right)^{-1}\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}V^{(1)}P^{(1)} \\ P^{(2)}\mathcal{S}\left(P^{(1)}\right)^{-1} &= \left(V^{(2)}\right)^{-1}\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}V^{(1)} \end{aligned} \qquad (6)$$

Due to the presence of the undetermined $P^{(1)}$ and $P^{(2)}$, we can no longer assume the elements of the first column and row of $\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}$ to be real and positive. In particular, the matrix $R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right]$ does not give the correct modular $\mathcal{S}$ matrix in general (symmetry of $\mathcal{S}$ matrix turns out to be insufficient to uniquely pick out the correct answer consistent with Eqn.6).

III. GENERAL ALGORITHM FOR THE MODULAR $\mathcal{S}$ MATRIX

To resolve the difficulty of properly ordering MESs, we now present a general algorithm to extract the modular $\mathcal{S}$ matrix for a generic topological ordered state with no implicit spatial symmetries. We find it fruitful to introduce an additional entanglement bipartition along the $\vec{w}^{(3)}_2 = \vec{w}^{(1)}_2 + \vec{w}^{(2)}_2$ direction and derive the corresponding MESs $\left|\Xi^{(3)}\right\rangle$, see Fig. 2 for illustration. Without loss of generality, we assume that for all three entanglement bipartitions, the first MES is associated with an Abelian quasiparticle, which in practice can be

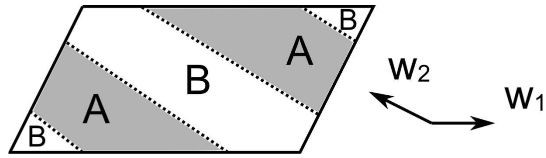

FIG. 2: Left panel: an additional and inequivalent entanglement bipartition scheme; Right panel: the corresponding primitive vectors, where $\vec{w}_2$ labeling the boundary direction of the entanglement bipartition is the sum of the previous two in Fig. 1.

verified by making sure that the corresponding quantum dimension, as obtained via TEE, is 1. Then, as we shall prove below, the modular $\mathcal{S}$ matrix is fully determined through

$$\mathcal{S} = \left\{R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right]\right\}^{-1} R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right]R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] \qquad (7)$$

this is the first main conclusion of this paper. Again, the function $R[X]$ corresponds to left and right matrix multiplication of the matrix X with certain diagonal phase factors, respectively, so as to make the elements of the first column and row of X real and positive.

To derive Eqn.7, we first note that the primitive vectors of the third entanglement bipartition may also be expanded as

$$\begin{aligned} \vec{w}^{(3)}_1 &= \vec{w}^{(1)}_1 \\ \vec{w}^{(3)}_2 &= -\vec{w}^{(1)}_1 + \vec{w}^{(1)}_2 \end{aligned} \qquad (8)$$

and equally

$$\begin{aligned} \vec{w}^{(2)}_1 &= \vec{w}^{(3)}_1 + \vec{w}^{(3)}_2 \\ \vec{w}^{(2)}_2 &= -\vec{w}^{(3)}_1 \end{aligned} \qquad (9)$$

In the Appendix, we further discuss the benefits of choosing $\vec{w}^{(3)}_2 = \vec{w}^{(1)}_2 + \vec{w}^{(2)}_2$ for the third entanglement bipartition.

With arguments similar to the last section, the modular matrices corresponding to the transformations from $\left|\Xi^{(1)}\right\rangle$ to $\left|\Xi^{(3)}\right\rangle$ and from $\left|\Xi^{(3)}\right\rangle$ to $\left|\Xi^{(2)}\right\rangle$ are $\mathcal{U}^{-1}\mathcal{S}\mathcal{U}^{-1}$ and $\mathcal{U}^{-1}\mathcal{S}$, respectively. Therefore, concerning the third entanglement bipartition we have

$$\begin{aligned} \mathcal{U}^{-1}\mathcal{S}\mathcal{U}^{-1} &= \left(P^{(3)}\right)^{-1}\left(V^{(3)}\right)^{-1}\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}V^{(1)}P^{(1)} \\ \mathcal{U}^{-1}\mathcal{S} &= \left(P^{(2)}\right)^{-1}\left(V^{(2)}\right)^{-1}\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}V^{(3)}P^{(3)} \end{aligned} \qquad (10)$$

Next, without loss of generality, it is straightforward to separate each permutation $P^{(\alpha)}$ into two distinct parts: $P^{(\alpha)} = \bar{P}^{(\alpha)}\tilde{P}^{(\alpha)}$, where $\bar{P}^{(\alpha)}$ is the permutation on columns other than the first one, while $\tilde{P}^{(\alpha)}$ maps each state to that with an *additional* Abelian quasiparticle, as is determined by the particle content of the first MES. By definition, any MES can be obtained from the MES associated with the identity quasiparticle through quasiparticle insertion

$$\mathcal{S}_{ab} = \left\langle \Xi^{(2)}_a | \Xi^{(1)}_b \right\rangle \equiv \left\langle 1^{(2)} \left| [T^{(2)}_a]^{-1} | T^{(1)}_b \right| 1^{(1)} \right\rangle \qquad (11)$$

where $\left|1^{(\alpha)}\right\rangle$ and $T_a^{(\alpha)}$ are the MES associated with the identity particle and insertion operator of an quasiparticle a along the $\vec{w}_2^{(\alpha)}$ direction, respectively. In comparison,

$$\begin{aligned}
\mathcal{S}_{ab}\left(\tilde{P}^{(1)}\right)^{-1} &= \left\langle \Xi_a^{(2)}\left|T_{p_1}^{(1)}\right|\Xi_b^{(1)}\right\rangle \\
&= \left\langle 1^{(2)}\left|[T_a^{(2)}]^{-1}|T_{p_1}^{(1)}T_b^{(1)}\right|1^{(1)}\right\rangle \\
&= \left\langle 1^{(2)}\left|T_{p_1}^{(1)}[T_a^{(2)}]^{-1}|T_b^{(1)}\right|1^{(1)}\right\rangle \theta_{p_1\times a}/\theta_{p_1}\theta_a \\
&= \left\langle 1^{(2)}\left|[T_a^{(2)}]^{-1}|T_b^{(1)}\right|1^{(1)}\right\rangle \theta_{p_1\times a}/\theta_{p_1}\theta_a \\
&= \left(\theta_{p_1\times a}/\theta_{p_1}\theta_a\right)\cdot\mathcal{S}_{ab} \qquad (12)
\end{aligned}$$

where $\times$ is the fusion product, p_1 is the added Abelian quasiparticle by $P^{(1)}$ with topological spin θ_{p_1}. $\theta_{p_1\times a}/\theta_{p_1}\theta_a = [T_a^{(2)}]^{-1}T_{p_1}^{(1)}T_a^{(2)}[T_{p_1}^{(1)}]^{-1}$ is the braiding between quasiparticles a and p_1. Physically, it is the induced phase for adiabatically moving a around p_1 and also encoded as the phase of the matrix element $\mathcal{S}_{p_1 a}$. We have also used the fact that for an Abelian quasiparticle p_1

$$\begin{aligned}
\left\langle 1^{(2)}\right|T_{p_1}^{(1)} &= \sum_b \mathcal{S}_{1b}\left\langle b^{(1)}\right|T_{p_1}^{(1)} = \sum_b \mathcal{S}_{1b}\left\langle (b\times p_1)^{(1)}\right| \\
&= \sum_b \mathcal{S}_{1b\times p_1}\left\langle (b\times p_1)^{(1)}\right| = \left\langle 1^{(2)}\right| \qquad (13)
\end{aligned}$$

More generally, denote the added Abelian quasiparticle by $P^{(2)}$ as p_2, one can show that

$$\begin{aligned}
\tilde{P}^{(2)}\mathcal{S}_{ab}\left(\tilde{P}^{(1)}\right)^{-1} &= \left\langle \Xi_a^{(2)}\left|[T_{p_2}^{(2)}]^{-1}T_{p_1}^{(1)}\right|\Xi_b^{(1)}\right\rangle \\
&= \left\langle 1^{(2)}\left|[T_a^{(2)}]^{-1}[T_{p_2}^{(2)}]^{-1}|T_{p_1}^{(1)}T_b^{(1)}\right|1^{(1)}\right\rangle \\
&= \left(\theta_{p_1\times p_2}/\theta_{p_1}\theta_{p_2}\right)\left(\theta_{p_1\times a}/\theta_{p_1}\theta_a\right)\cdot \\
&\quad \mathcal{S}_{ab}\cdot\left(\theta_{b\times p_2}/\theta_b\theta_{p_2}\right) \qquad (14)
\end{aligned}$$

Heuristically, the ij^{th} component of the modular $\mathcal{S}$ matrix encodes the braiding between the i^{th} and j^{th} quasiparticles. Eqn.14 formalizes this intuition so that the insertion of additional Abelian quasiparticles p_1 and p_2 leads to additional Abelian phases induced by the braiding between the i^{th} and p_1, p_2 and the j^{th} as well as p_1 and p_2 quasiparticles. To this end, the additional quasiparticles inserted by $\tilde{P}^{(\alpha)}$ and $\left(\tilde{P}^{(\beta)}\right)^{-1}$ effectively contribute additional phase factors to each rows and columns of the modular $\mathcal{S}$ matrix.

Now we note that (1) the undetermined components including $V^{(\alpha)}$, $\mathcal{U}$ and $\theta_{p_1\times a}/\theta_{p_1}\theta_a$, etc. are all diagonal phase factors, and (2) $\bar{P}^{(\alpha)}$ ($\left(\bar{P}^{(\beta)}\right)^{-1}$) only involves the rows (columns) other than the first one, thus the elements in the first line and column of $\bar{P}^{(\alpha)}\mathcal{S}\left(\bar{P}^{(\beta)}\right)^{-1}$ remain real and positive just as in $\mathcal{S}$. Therefore, Eqns. 6, 10 and 14 together imply

$$\begin{aligned}
\bar{P}^{(2)}\mathcal{S}\left(\bar{P}^{(1)}\right)^{-1} &= R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right] \\
\bar{P}^{(2)}\mathcal{S}\left(\bar{P}^{(3)}\right)^{-1} &= R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right] \\
\bar{P}^{(3)}\mathcal{S}\left(\bar{P}^{(1)}\right)^{-1} &= R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] \qquad (15)
\end{aligned}$$

with which we can obtain the modular $\mathcal{S}$ matrix

$$\begin{aligned}
\mathcal{S} &\sim \bar{P}^{(1)}\mathcal{S}\left(\bar{P}^{(1)}\right)^{-1} \\
&= \left[P^{(2)}\mathcal{S}\left(\bar{P}^{(1)}\right)^{-1}\right]^{-1}\left[P^{(2)}\mathcal{S}\left(\bar{P}^{(3)}\right)^{-1}\right]\left[P^{(3)}\mathcal{S}\left(\bar{P}^{(1)}\right)^{-1}\right] \\
&= \left\{R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right]\right\}^{-1}R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right]R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] \qquad (16)
\end{aligned}$$

upto a trivial permutation ($=\bar{P}_1$) of the quasiparticles' ordering sequence. A physical interpretation of Eqn. 7 is that through our entanglement interferometry that consists of a series of modular transformations between bases $\vec{w}^{(1)}\to\vec{w}^{(2)}\to\vec{w}^{(3)}\to\vec{w}^{(1)}$, one effectively cancels the impact of undetermined quantities such as phase factors and relative orderings.

In passing, we recall that with the help of the Verlinde's formula

$$N_{ab}^c = \sum_x \frac{\mathcal{S}_{ax}\mathcal{S}_{bx}\mathcal{S}_{\bar{c}x}}{\mathcal{S}_{1x}} \qquad (17)$$

one can now also construct the fusion rule coefficients: $a\times b = \sum_c N_{ab}^c$ from the obtained modular $\mathcal{S}$ matrix.

IV. GENERAL CONSTRAINTS ON THE MODULAR $\mathcal{U}$ MATRIX

In this section, we use our three-entanglement-bipartition construction to extract information on the modular $\mathcal{U}$ matrix, a diagonal matrix whose ath element encodes the topological spin and self-statistics of the a quasiparticle. Recall[16] that in the presence of the $2\pi/3$ rotation symmetry $R_{2\pi/3}$, the modular $\mathcal{U}$ matrix is fully determined without any ambiguity $\mathcal{U}\mathcal{S} = \left\langle \Xi^{(1)}\right|R_{2\pi/3}\left|\Xi^{(1)}\right\rangle$.

For simplicity, we first reorder the MESs of the second and third entanglement bipartitions with $\bar{P}^{(1)}\left(\bar{P}^{(2)}\right)^{-1}$ and $\bar{P}^{(1)}\left(\bar{P}^{(3)}\right)^{-1}$, respectively, so that all $\bar{P}^{(\alpha)}$ are consistent and the remaining $\tilde{P}^{(\alpha)}$ parts only contribute some undetermined diagonal phase factors. Eqn.15 is thus simplified as

$$\begin{aligned}
\mathcal{S} &= R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right] = \Lambda^{1L}\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\Lambda^{1R} \\
&= R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] = \Lambda^{2L}\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\Lambda^{2R} \\
&= R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right] = \Lambda^{3L}\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\Lambda^{3R} \qquad (18)
\end{aligned}$$

where the Λ matrices are the diagonal phase factors used in the function R to make the elements of the first column and row of the argument matrix real and positive. We find that the quasiparticle topological spin θ_a is obtainable through

$$\theta_a \propto \left[\Lambda_a^{1R}\right]^{-1}\Lambda_a^{2R}\left(\theta_a\theta_p/\theta_{p\times a}\right) = \left[\Lambda_a^{1R}\right]^{-1}\Lambda_a^{2R}\left|\mathcal{S}_{ap}\right|/\mathcal{S}_{ap} \qquad (19)$$

where $\theta_{p\times a}/\theta_a\theta_p$ is the braiding between a and some as yet undetermined Abelian quasiparticle p, which is encoded as the phase factors of the elements in the pth column of the modular $\mathcal{S}$ matrix.

To derive Eqn. 19, we first compare Eqn. 6, 10, 14 and 18 and realize that the contributions to the diagonal phase factors originate from the modular $\mathcal{U}$ matrix, the $V^{(\alpha)}$ conventions and the remaining relative orderings $\tilde{P}^{(\alpha)}$. More specifically, we have

$$\begin{aligned}
\Lambda_a^{1R} &\propto V_a^{(1)}\left(\theta_a\theta_{p_2}/\theta_{a\times p_2}\right) \\
\Lambda_a^{2R} &\propto V_a^{(1)}[\tilde{P}^{(1)}\mathcal{U}\left(\tilde{P}^{(1)}\right)^{-1}]_a\left(\theta_a\theta_{p_3}/\theta_{a\times p_3}\right) \\
&= V_a^{(1)}\left(\theta_{a\times p_1}\theta_a\theta_{p_3}/\theta_{a\times p_3}\right)
\end{aligned} \tag{20}$$

where p_1, p_2, p_3 are the Abelian quasiparticles added by $\tilde{P}^{(1)}$, $\tilde{P}^{(2)}$, $\tilde{P}^{(3)}$ and determined by the particle content of the first MES of each entanglement bipartition $\alpha = 1, 2, 3$, respectively. Then we can eliminate the unknown $V_a^{(1)}$ part

$$\left[\Lambda_a^{1R}\right]^{-1}\Lambda_a^{2R} \propto \theta_{a\times p_1}\theta_{a\times p_2}/\theta_{a\times p_3} \propto \theta_{a\times p} \tag{21}$$

upto some overall phases. Here $p = p_1 \times p_2 \times \bar{p}_3$ is also an Abelian quasiparticle where $\bar{p}_3$ is the anti-particle of p_3.

Since the particle content of p is undetermined, we need to consider all cases where p is Abelian, which gives the following potential solutions of θ_a

$$\theta_{p\times a} \propto \theta_a(\theta_{p\times a}/\theta_a\theta_p) = \theta_a\mathcal{S}_{ap}/\left|\mathcal{S}_{ap}\right| \tag{22}$$

together with Eqn. 21 we obtain our result in Eqn. 19. Similar expressions can be straightforwardly obtained with $\Lambda_a^{3L}\left[\Lambda_a^{1L}\right]^{-1}$ and $\Lambda_a^{2L}\Lambda_a^{3R}$ instead.

The overall phase of θ_a can be fixed by requiring $\theta_1 = 1$ for the identity particle. In addition, more information on p can be obtained by imposing certain consistency requirements. In particular, the self-braiding - the phase obtained when an Abelian quasiparticle a braids around another a should be twice as much as the self-statistics - the phase when two a quasiparticles exchange with each other, therefore $\theta_a^2 = \theta_{a\times a}/\theta_a\theta_a = \mathcal{S}_{aa}/|\mathcal{S}_{aa}|$ equals the phase factor of the ath diagonal element in the modular $\mathcal{S}$ matrix. Therefore, if θ_a is a consistent solution, another solution $\theta'_a = \theta_a\mathcal{S}_{ap}/\left|\mathcal{S}_{ap}\right| = \theta_{\alpha\times a}/\theta_p$ is also consistent if and only if $\left(\theta_{p\times a}/\theta_p\right)^2 = (\theta'_a)^2 = \mathcal{S}_{aa}/|\mathcal{S}_{aa}| = \theta_a^2$ for the choice of p. Correspondingly, $\theta_{p\times a}/\theta_p\theta_a = \mathcal{S}_{pa}/\left|\mathcal{S}_{pa}\right| = \pm 1$, the Abelian elements of the pth column in the modular $\mathcal{S}$ matrix need to be fully real. In particular, when the first column is the only column in the modular $\mathcal{S}$ matrix where all Abelian elements are real, our algorithm completely determines θ_a. We show later an example of the $\mathbb{Z}_3$ gauge theory where θ_a can be completely determined given the modular $\mathcal{S}$ matrix and $\theta_{a\times p}$.

In addition, when the topological ordered state is bosonic, the modular $\mathcal{U}$ matrix by definition $\mathcal{U}_a = \theta_a\exp(-i2\pi c/24)$ and the corresponding central charge c can be determined (modulo 8) by the requirement that $(\mathcal{U}\mathcal{S})^3 = 1$.

V. EXAMPLES: THE $\mathbb{Z}_2$ GAUGE THEORY (TORIC CODE MODEL), THE $SU(2)_3$ CHERN SIMONS THEORY AND THE $\mathbb{Z}_3$ GAUGE THEORY

A. Obtaining the Modular $\mathcal{S}$ matrix of the $\mathbb{Z}_2$ gauge theory

In this subsection, we use Kitaev's square lattice toric code model[23] as an example for our algorithm. The ground state is an equal superposition of all possible configurations of closed electric field loops on the lattice. On a torus, the four degenerate ground states $|\xi_{ab}\rangle$, $a, b = 0, 1$ are distinguished by the winding number parities a, b of the electric field loops around the two cycles of the torus and cannot be mixed by any local operator, constituting the $\mathbb{Z}_2$ gauge theory.

The nature of the MESs for the toric code model was studied in Ref. 16. For a nontrivial entanglement bipartition, the MESs are the simultaneous eigenstates of electric and magnetic fluxes threading the entanglement bipartition boundary. The MESs for entanglement bipartition along the $\vec{w}_2^{(1)} = \hat{y}$ direction are

$$\begin{aligned}
|\Xi_1\rangle &= \frac{e^{i\varphi_1}}{\sqrt{2}}(|\xi_{00}\rangle + |\xi_{01}\rangle) \\
|\Xi_2\rangle &= \frac{e^{i\varphi_2}}{\sqrt{2}}(|\xi_{00}\rangle - |\xi_{01}\rangle) \\
|\Xi_3\rangle &= \frac{e^{i\varphi_3}}{\sqrt{2}}(|\xi_{10}\rangle + |\xi_{11}\rangle) \\
|\Xi_4\rangle &= \frac{e^{i\varphi_4}}{\sqrt{2}}(|\xi_{10}\rangle - |\xi_{11}\rangle)
\end{aligned} \tag{23}$$

where φ_i are undetermined phases for each MES. The unitary matrix $\bar{U}_1$ connecting the $\vec{w}_2^{(1)}$ MESs and the electric field parity states $\{|\xi_{00}\rangle, |\xi_{01}\rangle, |\xi_{10}\rangle, |\xi_{11}\rangle\}$ is

$$\bar{U}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi_1} & e^{i\varphi_2} & & \\ e^{i\varphi_1} & -e^{i\varphi_2} & & \\ & & e^{i\varphi_3} & e^{i\varphi_4} \\ & & e^{i\varphi_3} & -e^{i\varphi_4} \end{pmatrix} \tag{24}$$

On the other hand, it is straightforward to verify that for entanglement bipartition boundary along the $\vec{w}_2^{(2)} = -\hat{x}$ direction the corresponding MESs are

$$\begin{aligned}
|\Xi'_1\rangle &= \frac{e^{i\varphi'_2}}{\sqrt{2}}(|\xi_{00}\rangle - |\xi_{10}\rangle) \\
|\Xi'_2\rangle &= \frac{e^{i\varphi'_3}}{\sqrt{2}}(|\xi_{01}\rangle + |\xi_{11}\rangle) \\
|\Xi'_3\rangle &= \frac{e^{i\varphi'_4}}{\sqrt{2}}(|\xi_{01}\rangle - |\xi_{11}\rangle) \\
|\Xi'_4\rangle &= \frac{e^{i\varphi'_1}}{\sqrt{2}}(|\xi_{00}\rangle + |\xi_{10}\rangle)
\end{aligned} \tag{25}$$

where again φ'_i are undetermined phases for each MES. We have purposefully scrambled the ordering of the MESs so that $|\Xi_i\rangle$ and $|\Xi'_i\rangle$ do not necessarily correspond to the same quasiparticle and the modular $\mathcal{S}$ matrix is not directly obtainable from only two sets of MESs. The unitary matrix $\bar{U}_2$ connecting the $\vec{w}_2^{(2)}$ MESs and the electric field parity states is

$$\bar{U}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi_2'} & & & e^{i\varphi_1'} \\ & e^{i\varphi_3'} & e^{i\varphi_4'} & \\ -e^{i\varphi_2'} & & & e^{i\varphi_1'} \\ & e^{i\varphi_3'} & -e^{i\varphi_4'} & \end{pmatrix} \tag{26}$$

Now we need to introduce just another entanglement bipartition. Let's consider taking the boundary along the $\vec{w}_2^{(3)} = -\hat{x}+\hat{y}$ direction so that $\vec{w}_2^{(3)} = \vec{w}_2^{(1)} + \vec{w}_2^{(2)}$. The corresponding MESs are

$$\begin{aligned} |\Xi_1''\rangle &= \frac{e^{i\varphi_3''}}{\sqrt{2}}(|\xi_{01}\rangle + |\xi_{10}\rangle) \\ |\Xi_2''\rangle &= \frac{e^{i\varphi_4''}}{\sqrt{2}}(|\xi_{01}\rangle - |\xi_{10}\rangle) \\ |\Xi_3''\rangle &= \frac{e^{i\varphi_2''}}{\sqrt{2}}(|\xi_{00}\rangle - |\xi_{11}\rangle) \\ |\Xi_4''\rangle &= \frac{e^{i\varphi_1''}}{\sqrt{2}}(|\xi_{00}\rangle + |\xi_{11}\rangle) \end{aligned} \tag{27}$$

where φ_i'' are undetermined phases and we have once again scrambled the ordering of the quasiparticles to make a difference from the previous two. The unitary matrix $\bar{U}_3$ connecting the $\vec{w}_2^{(3)}$ MESs and the electric field parity states is

$$\bar{U}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} & & e^{i\varphi_2''} & e^{i\varphi_1''} \\ e^{i\varphi_3''} & e^{i\varphi_4''} & & \\ e^{i\varphi_3''} & -e^{i\varphi_4''} & & \\ & & -e^{i\varphi_2''} & e^{i\varphi_1''} \end{pmatrix} \tag{28}$$

From Eqn. 24, 26 and 28, we can construct matrices $\bar{U}_2^{-1}\bar{U}_1$, $\bar{U}_3^{-1}\bar{U}_1$ and $\bar{U}_2^{-1}\bar{U}_3$. By setting the elements of the first rows and columns to be real and positive, we find:

$$\begin{aligned} \bar{P}_2^{-1}\mathcal{S}\bar{P}_1 = R(\bar{U}_2^{-1}\bar{U}_1) &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ \bar{P}_3^{-1}\mathcal{S}\bar{P}_1 = R(\bar{U}_3^{-1}\bar{U}_1) &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ \bar{P}_2^{-1}\mathcal{S}\bar{P}_3 = R(\bar{U}_2^{-1}\bar{U}_3) &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \end{aligned} \tag{29}$$

where $\bar{P}_\alpha$ are permutation matrices acting on the 2nd, 3rd and 4th columns and rows. According to Eqn. 16, we obtain the following solution consistent with the $\mathbb{Z}_2$ gauge theory:

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{30}$$

with $\bar{P}_1 = I$ and:

$$\bar{P}_2 = \bar{P}_2^{-1} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}, \bar{P}_3 = \bar{P}_3^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \tag{31}$$

In addition, we can obtain the diagonal Λ matrices by comparing the matrices $\bar{U}_2^{-1}\bar{U}_1$, $\bar{U}_3^{-1}\bar{U}_1$ and $\bar{U}_2^{-1}\bar{U}_3$ before and after the function R. In particular, we have $\Lambda^{1L} \propto \operatorname{diag}\left(e^{-i\varphi_1}, e^{-i\varphi_2}, -e^{-i\varphi_3}, -e^{-i\varphi_4}\right)$ and $\Lambda^{2L} \propto \operatorname{diag}\left(e^{-i\varphi_1}, -e^{-i\varphi_2}, e^{-i\varphi_3}, e^{-i\varphi_4}\right)$. According to Eqn. 21, this leads to:

$$\theta_{\alpha\times a} \propto \left[\Lambda^{1R}\right]^{-1}\Lambda^{2R} \propto \operatorname{diag}(1,-1,-1,-1) \tag{32}$$

together with the obtained modular $\mathcal{S}$ matrix and Eqn. 22, the possible solutions of the quasiparticle spins θ_a are:

$$\begin{aligned} \theta_a &= (1,1,1,-1), c=0 \\ \text{or } \theta_a &= (1,1,-1,1), c=0 \\ \text{or } \theta_a &= (1,-1,1,1), c=0 \\ \text{or } \theta_a &= (1,-1,-1,-1), c=4 \end{aligned} \tag{33}$$

where we have used $(\mathcal{US})^3 = 1$ and $\mathcal{U}_a = \theta_a \exp(-i2\pi c/24)$ to extract the value of c. Since all elements of the modular $\mathcal{S}$ matrix are real, we cannot refine these candidates further. In Sec. V C, we discuss another example, the $\mathbb{Z}_3$ gauge theory, where there is only one fully real column in the modular $\mathcal{S}$ matrix thus θ_a can be uniquely determined.

B. The non-Abelian $SU(2)_3$ Chern Simons theory

In this subsection, we provide an example of applying our algorithm to a non-Abelian state: the $SU(2)_3$ Chern Simons theory. The modular $\mathcal{S}$ matrix of $SU(2)_3$ topological ordered phase is:

$$\mathcal{S} = \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & 1 & -1 & -\sigma \\ \sigma & -1 & -1 & \sigma \\ 1 & -\sigma & \sigma & -1 \end{pmatrix} \tag{34}$$

where $\sigma = \left(1+\sqrt{5}\right)/2$ is the golden ratio. Corresponding to each column (row), the four quasiparticles have quantum dimensions $d = 1, \sigma, \sigma, 1$ respectively – clearly the second and third quasiparticles are non-Abelian. If we introduce only two entanglement bipartitions, the TEE can make a distinction between the MESs associated with the Abelian quasiparticles and those with the non-Abelian quasiparticles, yet it can not distinguish between the first and fourth MESs, both associated with Abelian quasiparticles, as well as between the second and third MESs, both associated with non-Abelian quasiparticle of equal quantum dimensions.

Following the algorithm in the main text, we introduce three entanglement bipartitions along the $\vec{w}^{(1)}$, $\vec{w}^{(2)}$ and $\vec{w}^{(3)} =$

$\vec{w}^{(1)} + \vec{w}^{(2)}$ directions, and shuffle their respective ordering of the MESs with the only requirement that the first MES is either $\left|\Xi_1^{(i)}\right\rangle$ or $\left|\Xi_4^{(i)}\right\rangle$, which are associated with Abelian quasiparticles. As a particular example: MESs along the $w^{(1)}$: $\left\{\left|\Xi_4^{(1)}\right\rangle, \left|\Xi_3^{(1)}\right\rangle, \left|\Xi_2^{(1)}\right\rangle, \left|\Xi_1^{(1)}\right\rangle\right\}$; MESs along the $w^{(2)}$: $\left\{\left|\Xi_4^{(2)}\right\rangle, \left|\Xi_2^{(2)}\right\rangle, \left|\Xi_3^{(2)}\right\rangle, \left|\Xi_1^{(2)}\right\rangle\right\}$; MESs along the $w^{(3)}$: $\left\{\left|\Xi_1^{(3)}\right\rangle, \left|\Xi_3^{(3)}\right\rangle, \left|\Xi_2^{(3)}\right\rangle, \left|\Xi_4^{(3)}\right\rangle\right\}$. After neutralizing all the diagonal phase factors in Eq. 6 and 10 from the modular $\mathcal{U}$ matrix and the $V^{(i)}$ conventions, we obtain the transformation between these MES bases:

$$\begin{aligned}
R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] &= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}R\left[\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & -1 & -1 & \sigma \\ -\sigma & -1 & 1 & \sigma \\ -1 & \sigma & -\sigma & 1 \end{pmatrix}\right] \\
&= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & -1 & -1 & \sigma \\ \sigma & 1 & -1 & -\sigma \\ 1 & -\sigma & \sigma & -1 \end{pmatrix} \\
R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right] &= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}R\left[\begin{pmatrix} 1 & \sigma & -\sigma & -1 \\ \sigma & -1 & 1 & -\sigma \\ \sigma & -1 & -1 & \sigma \\ 1 & \sigma & \sigma & 1 \end{pmatrix}\right] \\
&= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & -1 & -1 & \sigma \\ \sigma & -1 & 1 & -\sigma \\ 1 & \sigma & -\sigma & -1 \end{pmatrix} \\
R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right] &= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}R\left[\begin{pmatrix} -1 & \sigma & -\sigma & 1 \\ -\sigma & -1 & 1 & \sigma \\ \sigma & -1 & -1 & \sigma \\ 1 & \sigma & \sigma & 1 \end{pmatrix}\right] \\
&= \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & -1 & -1 & \sigma \\ \sigma & 1 & -1 & -\sigma \\ 1 & -\sigma & \sigma & -1 \end{pmatrix}
\end{aligned} \tag{35}$$

Then, it is straightforward to check Eq. 7 gives the consistent modular $\mathcal{S}$ matrix:

$$\left\{R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(1)}\right]\right\}^{-1} R\left[\left(\bar{U}^{(2)}\right)^{-1}\bar{U}^{(3)}\right] R\left[\left(\bar{U}^{(3)}\right)^{-1}\bar{U}^{(1)}\right] = \sqrt{\frac{2}{5}}\sin\frac{\pi}{5}\begin{pmatrix} 1 & \sigma & \sigma & 1 \\ \sigma & 1 & -1 & -\sigma \\ \sigma & -1 & -1 & \sigma \\ 1 & -\sigma & \sigma & -1 \end{pmatrix} \tag{36}$$

C. Obtaining the modular $\mathcal{U}$ matrix of the $\mathbb{Z}_3$ gauge theory

In this subsection, we briefly introduce another example where the modular $\mathcal{S}$ matrix together with $\theta_{\alpha\times\alpha}$ completely determines the values of θ_a even if the Abelian quasiparticle α is yet undetermined.

The modular $\mathcal{S}$ matrix of the Abelian $\mathbb{Z}_3$ gauge theory, which is fully obtainable by similar argument to the last section, is:

$$\mathcal{S} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & q & q & q & q^2 & q^2 & q^2 \\ 1 & 1 & 1 & q^2 & q^2 & q^2 & q & q & q \\ 1 & q & q^2 & 1 & q & q^2 & 1 & q & q^2 \\ 1 & q & q^2 & q & q^2 & 1 & q^2 & 1 & q \\ 1 & q & q^2 & q^2 & 1 & q & q & q^2 & 1 \\ 1 & q^2 & q & 1 & q^2 & q & 1 & q^2 & q \\ 1 & q^2 & q & q & 1 & q^2 & q^2 & q & 1 \\ 1 & q^2 & q & q^2 & q & 1 & q & 1 & q^2 \end{pmatrix} \tag{37}$$

where $q = e^{i2\pi/3}$. Its diagonal elements as the self-braiding should be consistent with the self-statistics of the quasiparticles with presumed ordering of $(1, e, e^2, m, em, e^2m, m^2, em^2, e^2m^2)$:

$$\theta_a^2 = (1, 1, 1, 1, q^2, q, 1, q, q^2) \tag{38}$$

On the other hand, given $\theta_{a\times p}$ without knowing the actual Abelian particle content of p, we can derive the following possibilities for θ_a according to Eqn. 22 by considering each column of the modular $\mathcal{S}$ matrix above:

$$\begin{aligned}
\theta_a &= (1, 1, 1, 1, q, q^2, 1, q^2, q) \\
\text{or } \theta_a &= (1, 1, 1, q, q^2, 1, q^2, q, 1) \\
\text{or } \theta_a &= (1, 1, 1, q^2, 1, q, q, 1, q^2) \\
\text{or } \theta_a &= (1, q, q^2, 1, q^2, q, 1, 1, 1) \\
\text{or } \theta_a &= (1, q, q^2, q, 1, q^2, q^2, q^2, q^2) \\
\text{or } \theta_a &= (1, q, q^2, q^2, q, 1, q, q, q) \\
\text{or } \theta_a &= (1, q^2, q, 1, 1, 1, 1, q, q^2) \\
\text{or } \theta_a &= (1, q^2, q, q, q, q, q^2, 1, q) \\
\text{or } \theta_a &= (1, q^2, q, q^2, q^2, q^2, q, q^2, 1)
\end{aligned} \tag{39}$$

however, only the first candidate is consistent with Eqn. 38 from the modular $\mathcal{S}$ matrix. Therefore the θ_a solution is unique, and the statistics of the quasiparticles can be uniquely determined. In addition, with $(\mathcal{US})^3 = 1$ we can obtain the modular $\mathcal{U}$ matrix: $\mathcal{U} = \text{diag}(\theta_a) = (1, 1, 1, 1, q, q^2, 1, q^2, q)$.

VI. CONCLUSION

In this paper, we extended the discussion in Ref.[16] to characterize a two-dimensional topological ordered phase with only its complete set of ground-state wavefunctions. Based on a closed sequence of modular transformations between three inequivalent entanglement bipartitions, our algorithm derives the modular $\mathcal{S}$ matrix and the corresponding quasiparticle braiding of topologically ordered phase without presuming any lattice symmetries. It also constrains the modular $\mathcal{U}$ matrix to a few discrete possibilities, and in certain cases determines it fully. Our algorithm is applicable to Abelian and non-Abelian phases alike.

In general, however, our algorithm still does not guarantee a definitive solution to the modular $\mathcal{U}$ matrix and thus, the quasiparticle self-statistics. For chiral phases, momentum polarization can determine the self-statistics and the chiral edge central charge[28]. Recently, another method to obtain the modular matrices based on universal wavefunction overlap[30] has been introduced, yet limited to small system sizes due to an exponentially small prefactor. A universally applicable method for the modular $\mathcal{U}$ matrix is still under study.

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Appendix A: Results for general entanglement bipartitions

In this Appendix, we discuss the generalization to our choices of entanglement bipartitions in Eqn. 3, 8 and 9 as well as that of Fig. 1 and 2. Without loss of generality, we can always define:

$$\begin{aligned} \vec{w}_1^{(3)} &= n_1\vec{w}_1^{(1)} + m_1\vec{w}_2^{(1)} \\ \vec{w}_2^{(3)} &= n_2\vec{w}_1^{(1)} + m_2\vec{w}_2^{(1)} \end{aligned} \tag{A1}$$

and:

$$\begin{aligned} \vec{w}_1^{(2)} &= n_3\vec{w}_1^{(3)} + m_3\vec{w}_2^{(3)} \\ \vec{w}_2^{(2)} &= n_4\vec{w}_1^{(3)} + m_4\vec{w}_2^{(3)} \end{aligned} \tag{A2}$$

with $n_1m_2 - m_1n_2 = 1$ and $n_3m_4 - m_3n_4 = 1$ by definition of the modular transformation.

First, in order to make our algorithm work, we would like all these transformations between different MES bases to be the form of the modular $\mathcal{S}$ matrix times some diagonal phase factors, which requires the cross products $\vec{w}_2^{(1)} \times \vec{w}_2^{(2)} = \vec{w}_2^{(1)} \times \vec{w}_2^{(3)} = \vec{w}_2^{(3)} \times \vec{w}_2^{(2)} = A$ where A is the (signed) surface area of the torus. For example, for the transformation in Eqn. A1 this requires $n_2 = -1$. Then the corresponding $SL(2,\mathbb{Z})$ matrix has the following expansion:

$$\begin{aligned} F = \begin{pmatrix} n_1 & 1-n_1m_2 \\ -1 & m_2 \end{pmatrix} &= \begin{pmatrix} 1 & -n_1 \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & -m_2 \\ & 1 \end{pmatrix} \\ &= U^{-n_1} S U^{-m_2} \end{aligned} \tag{A3}$$

thus the modular matrix is $\mathcal{F}_{13}(\mathcal{S},\mathcal{U}) = \mathcal{U}^{-n_1}\mathcal{S}\mathcal{U}^{-m_2}$.

Similarly we require $n_4 = -1$. It is also straightforward to derive the $SL(2,\mathbb{Z})$ matrix for the transformation from the $\vec{w}^{(1)}$ to $\vec{w}^{(2)}$ from the product of Eqn. A1 and Eqn. A2:

$$\begin{aligned} &\begin{pmatrix} n_3 & 1-n_3m_4 \\ -1 & m_4 \end{pmatrix}\begin{pmatrix} n_1 & 1-n_1m_2 \\ -1 & m_2 \end{pmatrix} \\ &= \begin{pmatrix} n_3n_1 + n_3m_4 - 1 & n_3 - n_1n_3m_2 + m_2 - n_3m_2m_4 \\ -n_1 - m_4 & m_2m_4 + n_1m_2 - 1 \end{pmatrix} \end{aligned} \tag{A4}$$

therefore we have another requirement $-n_1 - m_4 = -1$.

Under these constraints, the $SL(2,\mathbb{Z})$ matrices for the transformation from the $\vec{w}^{(3)}$ to $\vec{w}^{(2)}$ and $\vec{w}^{(1)}$ to $\vec{w}^{(2)}$ may be expanded in terms of generators in Eqn. 2 as:

$$\begin{pmatrix} n_3 & 1-n_3+n_1n_3 \\ -1 & 1-n_1 \end{pmatrix} = \begin{pmatrix} 1 & -n_3 \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & n_1-1 \\ & 1 \end{pmatrix} \tag{A5}$$

and:

$$\begin{pmatrix} n_3-1 & n_3+m_2-n_3m_2 \\ -1 & m_2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1-n_3 \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & 1-m_2 \\ & 1 \end{pmatrix} \tag{A6}$$

The corresponding modular matrices for the transformations between the sets of MESs are:

$$\mathcal{F}_{23}(\mathcal{S},\mathcal{U}) = \mathcal{U}^{-n_3}\mathcal{S}\mathcal{U}^{n_1-1} \tag{A7}$$

and

$$\mathcal{F}_{12}(\mathcal{S},\mathcal{U}) = \mathcal{U}^{1-n_3}\mathcal{S}\mathcal{U}^{1-m_2} \tag{A8}$$

respectively.

Especially, it is clear that $\vec{w}_2^{(3)} = -\vec{w}_1^{(1)} + m_2\vec{w}_2^{(1)}$ and $\vec{w}_2^{(2)} = -\vec{w}_1^{(1)} + (m_2-1)\vec{w}_2^{(1)}$, thus we have to have $\vec{w}_2^{(3)} = \vec{w}_2^{(1)} + \vec{w}_2^{(2)}$. This explains our choice of the third entanglement bipartition in the main text.

The remaining degrees of freedom n_1, m_2 and n_3 are concerned only with the choices of the $\vec{w}_1^{(\alpha)}$ directions and bring no essential change to our line of reasoning. The specific choice in the main text corresponds to $n_1 = m_2 = n_3 = 1$.

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