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Shaking Maxwell's Rule - Locally Resonant Bandgaps in Periodic Beam Lattices by Tuning Their Connectivity

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Lattice structures have long fascinated physicists and engineers not only because of their outstanding functionalities, but also for their ability to control the propagation of elastic waves. While the study of the relation between the connectivity of these systems and their static properties has a long history that goes back to Maxwell, rules that connect the dynamic response to the network topology have not been established. Here, we demonstrate that by tuning the average connectivity of a beam network (\bar{z}), locally resonant bandgaps can be generated in the structures without embedding additional resonating units. In particular, a critical threshold for \bar{z} is identified, far from which the bandgap size is purely dictated by the global lattice topology. By contrast, near this critical value, the detailed local geometry of the lattice also has strong effects. Moreover, in stark contrast to the static case, we find that the nature of the joints is irrelevant to the dynamic response of the lattices. Our results not only shed new light on the rich dynamic properties of periodic lattices, but also outline a new strategy to manipulate mechanical waves in elastic systems.

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The topology of structures comprising an interconnected network of elastic beams can be effectively described by the coordination number (\bar{z}), which is defined as the average number of connections at joints. From a static point of view, reducing \bar{z} makes the structure less rigid until a critical threshold is reached, below which deformation modes of zero energy emerge¹⁻⁵. A global stability criterion that purely depends on \bar{z} was first determined by Maxwell for pin-joined lattices comprising spring-like ligaments¹, and then modified to account for the nature (pin or welded) of the joints⁶, the bending stiffness of the struts^{7,8}, self-stresses⁹, dislocation defects¹⁰, collapse mechanisms¹¹ and boundary modes¹²⁻¹⁵. In recent years, the dynamic response of periodic lattices has also attracted considerable interest¹⁶⁻¹⁹ because of their ability to tailor the propagation of elastic waves through directional transmissions²⁰⁻²³ and bandgaps (frequency ranges of strong wave attenuation)²¹⁻²⁴. However, though several studies have shown that the wave propagation properties of periodic lattices are highly sensitive to the architecture of the network²⁰⁻²⁴, a global criterion connecting the frequency and size of bandgaps to the lattice topology is still not yet in place.

In this letter, we study the effects of both the global coordination number (\bar{z}) and the local geometric features of periodic networks made of slender beams with finite bending stiffness on the propagation of elastic waves. We consider two dimensional periodic lattices made of Euler-Bernoulli beams supporting both bending and axial deformations. Each beam is made of a linear elastic isotropic material, has length L , mass per unit length m , bending stiffness EI , axial stiffness EA and is characterized by a slenderness ratio $\lambda = L\sqrt{EA}/(EI) = 20$. However, it is important to note that all results presented in the paper are not affected by this specific choice of λ and that identical

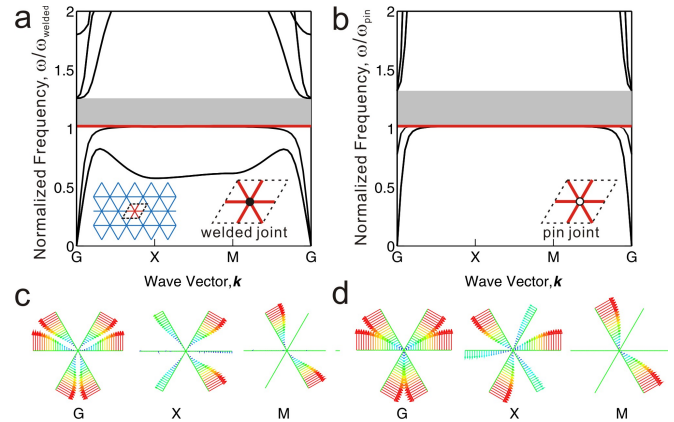


FIG. 1: Band structures of triangular periodic beam lattices. (a) Dispersion relation of the triangular lattice with welded joints. (b) Dispersion relation of the triangular lattice with pin joints. The lattice and unit cells with both welded and pin joints are shown as insets. Bandgaps are shown as grey-shaded areas and the flat bands at their lower edge are highlighted in red. The Bloch modes of the flat band (the third mode) at high symmetry points of the Brillouin zone (G, X and M)²⁵ are shown in (c) and (d) for the case of welded and pin joints, respectively. A plot presenting both band structures normalized by the same factor is included in the Supplementary Material²⁵

findings can be obtained for network of beams characterized by any value of $\lambda > 10$. To study the propagation of small amplitude elastic waves in such lattices, we perform frequency-domain wave propagation analysis^{26,27} within the finite element (FE) framework using the commercial package Abaqus/Standard and Bloch-type boundary conditions are applied to the edges of the unit cell. We then solve the frequency-domain wave equation for wave vectors in the Brillouin zone using a perturbation method²⁵.

We start by investigating the propagation of elastic waves in the triangular lattice (characterized by connectivity $\bar{z} = 6$) with both pin and welded joints. Figures 1a and 1b show the band structures in term of the normalized frequency ω/ω_{welded} and ω/ω_{pin} , where $\omega_{welded} = 22.4\sqrt{EI/(mL^4)}$ and $\omega_{pin} = \pi^2\sqrt{EI/(mL^4)}$ are the first natural frequency of the a single beam with both ends fixed (welded) and free to rotate (pin), respectively. As previously predicted²¹, the dispersion relations indicate that the structure is characterized by a bandgap. However, while such bandgap is believed to be due to Bragg scattering, our analysis surprisingly indicates that it is generated by localized resonance, a fact that has not been clearly pointed out before. This important observation is clearly supported by the fact that, regardless of the type of joints, the band at the lower edge of the bandgap is completely flat (see red lines in Figs. 1a and 1b) and located in correspondence of the first natural frequency of the beams (i.e. $\omega/\omega_{welded} = 1$ and $\omega/\omega_{pin} = 1$). Furthermore, the Bloch mode shapes of the flat band at the high-symmetry points G, X and M reported in Figs. 1c and 1d confirm that each beam vibrates independently according to its natural mode. As a result, the vibration energy is localized by the single-beam resonant mode, preventing the propagation of the elastic waves. In previous studies, the local resonance mechanism was typically realized in heterogeneous systems comprising two²⁷⁻²⁹, three³⁰⁻³² or four³³ different constituent materials. Remarkably, in periodic triangular lattices, even with a single material and building block, local resonances can be exploited to generate bandgaps, providing a foundation for the design of a new class of systems to manipulate the propagation of elastic waves. Furthermore, our results also demonstrate that, in order to attenuate the propagation of elastic waves through localized resonances, it is not necessary to embed additional resonating components^{23,24,34-42} within the beam lattices. Such single-building-block and single-material system with locally resonant bandgap has been previously realized in one-dimensional setup only⁴³.

Having demonstrated that the triangular beam lattice is characterized by a locally resonant bandgap regardless of the type of joints, we now investigate the dynamic response of the hexagonal lattice, which has the same lattice symmetries of the triangular lattice but a much smaller coordination number (i.e. $\bar{z} = 3$). As shown in Fig. 2a, the hexagonal lattice with welded joints also exhibits an almost flat band at the resonant frequency of the beams (i.e. $\omega = \omega_{welded}$). However, in this case the flat-band resonant mode does not open a bandgap regardless of the types of joints (identical behavior is found in the case of pin joints²⁵). By comparing to the triangular lattice, we note that the absence of the locally resonant bandgap in the hexagonal lattice could arise either because of the low connectivity number \bar{z} or the different angle α between connected beams. To further inspect the contribution of the lat-

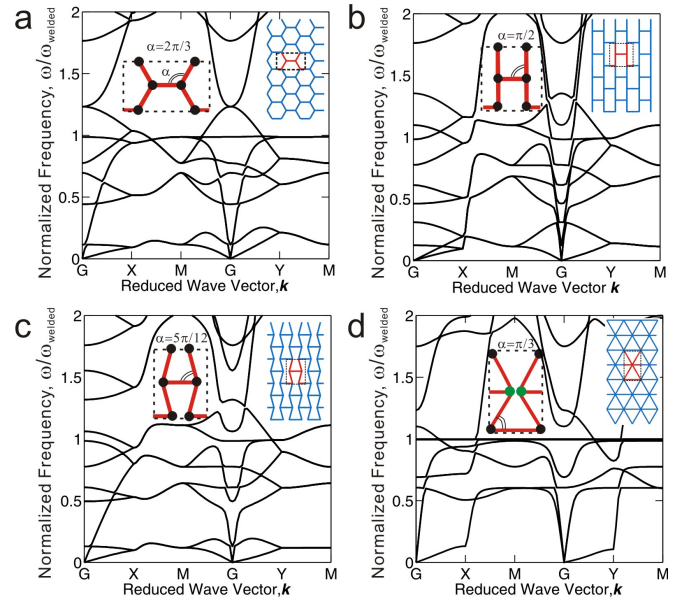


FIG. 2: Band structure of periodic lattices with $\bar{z} = 3$ and welded joints. (a) Hexagonal lattice ($\alpha = 2\pi/3$); (b) Topologically equivalent lattice with $\alpha = \pi/2$; (c) Topologically equivalent lattice with $\alpha = 5\pi/12$; and (d) Topologically equivalent lattice with $\alpha = \pi/3$. The lattice structures and unit cells used in the calculations are shown as insets. Note that for $\alpha = \pi/3$ the arrangement of the beams is the same as for the triangular lattice, but the connectivity is still $\bar{z} = 3$. In fact, although for clarity the green-colored joints are drawn separately in the unit cell of **d**, they are positioned at the same spatial location. No locally resonant bandgap is found for any of the configurations. Results for the same lattices with pin joints are provided in the Supplementary Material²⁵.

ter factor, by keeping $\bar{z} = 3$ and varying α from $2\pi/3$ to $\pi/3$, we construct periodic lattices that are topologically equivalent to the hexagonal lattice and study their dynamic response. As clearly shown in Figs. 2b-2d, though the phononic band structure evolves significantly as α changes, no bandgap is observed. It is especially important to note that for $\alpha = \pi/3$ the beams are arranged exactly as in the triangular lattice. However, the network topology is different, as only three, instead of six, beams connect to each other at each joint (see inset in Fig. 2d). Thus, these results conclusively show that the global topology of the network described by the coordination number plays a crucial role in determining the existence of locally resonant bandgaps. In addition, our analysis indicates that this conclusion is not affected by the nature of the joints (additional results for pin joints are included in the Supporting Information²⁵).

Next, to further demonstrate the role of the coordination number on the formation of locally resonant bandgaps, we investigate a number of periodic beam lattices with $3 < \bar{z} < 6$, as shown in Fig. 3a. These hybrid lattices are generated by considering enlarged unit cells of the triangular lattice and randomly remov-

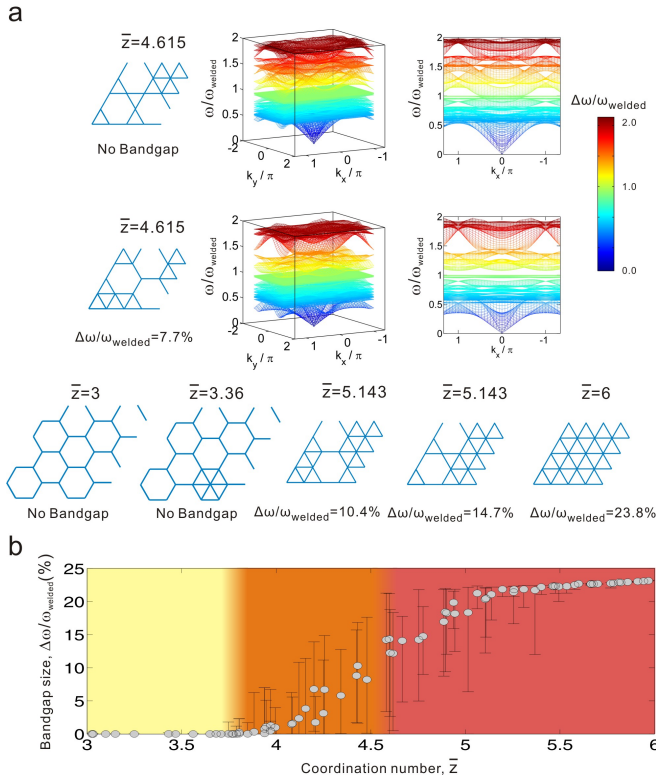


FIG. 3: Dynamic response of periodic lattices with $3 < \bar{z} < 6$. (a) Representative unit cells constructed either starting from the triangular lattice and removing a number of randomly chosen nodes or starting from the hexagonal lattice and filling a number of randomly chosen hexagons with six triangles. For two unit cells with $\bar{z} = 4.615$ also the band structures are shown. **b**, Relation between normalized average bandgap size $\Delta\omega/\omega_{welded}$ and coordination number \bar{z} . The bandgap size is computed as the difference between the upper and lower edges of the gap. The error bar at each data point indicates the bandgap range of frequencies spanned by all different configurations characterized by the same value of \bar{z} .

ing a number of joints and all beams attached to them. Alternatively, lattices with \bar{z} close to 3 are constructed starting with an enlarged unit cell of the hexagonal lattice and filling some randomly chosen hexagons with six triangles. For each value of \bar{z} multiple configurations are analyzed and, since these periodic lattices do not share the same spatial symmetry of the triangular and hexagonal lattices, additional Bloch vectors are considered to determine the presence of bandgaps²⁵. The results are summarized in Fig. 3b, where the width of the bandgap $\Delta\omega$ is reported for different hybrid lattices with $3 < \bar{z} < 6$. Each circular marker in the plot represents the average bandgap width of all configurations considered for that particular value of \bar{z} , while the corresponding error bar spans the range of observed $\Delta\omega$. Interestingly, for lattices with low connectivity (i.e. $3.0 < \bar{z} < 3.7$ - yellow region in Fig. 3b), no locally resonant bandgap is observed at $\omega = \omega_{welded}$.

In contrast, when $4.6 < \bar{z} < 6.0$ (red region in Fig.

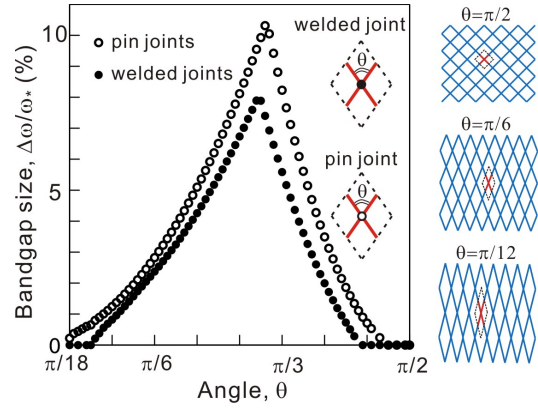


FIG. 4: Normalized bandgap width $\Delta\omega/\omega_*$ for rhombic lattices that are topologically equivalent to the square lattice. The lattice structures and joint types used in the calculations are shown as insets. The results for lattices with both welded and pin joints are reported. Note that $\omega_* = \omega_{welded}$ and $\omega_* = \omega_{pin}$ for lattices with welded and pin joints, respectively.

3b), all considered lattices possess a locally resonant bandgap. In particular, for structures with a very high average connectivity (i.e. $\bar{z} > 5.5$), the bandgap width $\Delta\omega$ is linearly correlated to the coordination number \bar{z} . On the other hand, if $4.6 < \bar{z} < 5.7$, although a locally resonant bandgap always exists, its size is not solely determined by \bar{z} , but also affected by the specific arrangement of the beams within the unit cell. Finally, when $3.7 < \bar{z} < 4.6$ (orange region in Fig. 3b), depending on the local geometric features, the lattice may either have or not have a locally resonant bandgap²⁵.

The results presented in Fig. 3 not only confirm that the global topology of the lattice is the leading factor in determining the existence of a locally resonant bandgap, but also indicate that for an intermediate range of \bar{z} the detailed geometry of the lattice plays an important role. To further understand the effect of the arrangement of the beams in lattices characterized by intermediate values of \bar{z} , we also study rhombic lattices with $\bar{z} = 4$ and investigate the effect of the angle θ between the beams at the joints. We start from the case of $\theta = \pi/2$, the well-known square lattice, which does not have a locally resonant bandgap²¹. However, as shown in Fig. 4, when the angle θ is progressively reduced, a locally resonant bandgap appears for rhombic lattices. Interestingly, we also find that a maximum bandgap width is achieved for $\theta = 56\pi/180$, regardless of the joint types. Therefore, our results indicate that, when extended to the dynamic response of the beam lattices, Maxwell's rule can be relaxed. In fact, while $\bar{z} = 4$ represents the critical threshold below which a lattice made of spring-like ligaments becomes unstable, the rhombic lattices at the threshold can still possess a locally resonant bandgap by carefully choosing the angle θ .

In summary, we have numerically investigated the dynamic response of periodic beam lattices. We have

found that, in highly connected lattices, the beams themselves act as mechanical resonators, enabling the generation of locally resonant bandgaps. Similar to the observation reported for the effective static properties of lattices^{1,5,6,8,13,14}, our results indicate that the presence and width of the locally resonant bandgap depends on the coordination number (i.e. average lattice connectivity, \bar{z}). A reduction of \bar{z} results in a decrease of the band width, until a critical threshold is reached, below which the bandgap completely closes. On the other hand, we have found that, different from the ground state static properties^{6,7,44}, the dynamic response of the system is not sensitive to the type of joints. Under dynamic loading, the lattices are in an excited state characterized by a finite level of energy, and the response is not qualitatively affected by the bending stiffness of the joints. Moreover, we have also shown that the average connectivity is not enough to predict the dynamic characteristics of a system when \bar{z} is near the critical threshold. In fact, we identified a transition region where the dynamic response of the lattices is sensitive to the detailed architecture of the network. Our work paves the way towards the design of a new class of systems made of identical elastic

beams that can effectively attenuate the propagation of elastic waves at low frequencies by exploiting local resonances. As an example, highly connected periodic lattices with beam length ~ 40 mm, thickness ~ 2.0 mm, which are made of an acrylic polymer that can be easily 3D-printed (Young's modulus ~ 1.14 GPa, density ~ 1050 kg/m³), exhibit a locally resonant bandgap in the audible frequency range (i.e. at around 590 Hz and 1340 Hz for the case of pin and welded joints, respectively).

Acknowledgments

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