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Phys. Rev. B **90**, 205101 — Published 4 November 2014

DOI: 10.1103/PhysRevB.90.205101

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(Dated: September 29, 2014)

A scheme is proposed to construct integer and fractional topological quantum states of fermions in two spatial dimensions. We devise models for such states by coupling wires of non-chiral Luttinger liquids of electrons, that are arranged in a periodic array. Which inter-wire couplings are allowed is dictated by symmetry and the compatibility criterion that they can simultaneously acquire a finite expectation value, opening a spectral gap between the ground state(s) and all excited states in the bulk. First, with these criteria at hand, we reproduce the tenfold classification table of integer topological insulators, where their stability against interactions becomes immediately transparent in the Luttinger liquid description. Second, we construct an example of a strongly interacting fermionic topological phase of matter with short-range entanglement that lies outside of the tenfold classification. Third, we expand the table to long-range entangled topological phases with intrinsic topological order and fractional excitations.

I. INTRODUCTION

The study of topological phases of matter is one of the most vibrant directions of research in contemporary condensed matter physics. One core accomplishment has been the theoretical modeling and experimental discovery of two-dimensional topological insulators.[1–4] The integer quantum Hall effect (IQHE) was an early example of how states could be classified into distinct topological classes using an integer, the Chern number, to express the quantized Hall conductivity.[5–7] In the IQHE, the number of delocalized edge channels is directly tied to the quantized Hall conductivity through the Chern number. More recently, it has been found that the symmetry under reversal of time acts as a protective symmetry for edge modes in (bulk) insulators with strong spin-orbit interactions in two and three dimensions,[1,8] and that these systems are characterized by a $\mathbb{Z}_2$ topological invariant.

The discovery of $\mathbb{Z}_2$ topological insulators has triggered a search for a classification of phases of fermionic matter that are distinct by some topological attribute. For non-interacting electrons, a complete classification, the tenfold way, has been accomplished in arbitrary dimensions.[9–11] In this scheme, three discrete symmetries that act locally in position space – time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral or sublattice symmetry (SLS) – play a central role when defining the quantum numbers that identify the topological insulating fermionic phases of matter within one of the ten symmetry classes (see columns 1-3 from Table I).

The tenfold way is believed to be robust to a perturbative treatment of short-ranged electron-electron interactions for the following reasons. First, the unperturbed ground state in the clean limit and in a closed geometry is non-degenerate and given by the filled bands of a band insulator. The band gap provides a small expansion parameter, namely the ratio of the characteristic interacting energy scale to the band gap. Second, the quantized topological invariant that characterizes the filled bands, provided its definition and topological character survives the presence of electron-electron interactions as is the case for the symmetry class A in two spatial dimensions, cannot change in a perturbative treatment of short-range electron-electron interactions.[12]

On the other hand, the fate of the tenfold way when electron-electron interactions are strong is rather subtle.[12–15] For example, short-range interactions can drive the system through a topological phase transition at which the energy gap closes,[16,17] or they may spontaneously break a defining symmetry of the topological phase. Even when short-range interactions neither spontaneously break the symmetries nor close the gap, it may be that two phases from the non-interacting tenfold way cease to be distinguishable in the presence of interactions. In fact, it was shown for the symmetry class BDI in one dimension by Fidkowski and Kitaev that the non-interacting $\mathbb{Z}$ classification was too fine in that it must be replaced by a $\mathbb{Z}_8$ classification when generic short-range interactions are allowed. How to construct a counterpart to the tenfold way for interacting fermion (and boson) systems has thus attracted a lot of interest.[18–28]

The fractional quantum Hall effect (FQHE) is the paradigm for a situation by which interactions select topologically ordered ground states of a very different kind than the non-degenerate ground states from the tenfold way. On a closed two-dimensional manifold of genus g, interactions can stabilize incompressible many-body ground states with a g-dependent degeneracy. Excited states in the bulk must then carry fractional quantum numbers (see Ref. 29 and references therein). Such phases of matter, that follow the FQHE paradigm, appear in the literature under different names: fractional topological insulators, long-range entangled phases, topologically ordered phases, or symmetry enriched topological phases. In this paper we use the terminology long-range entangled (LRE) phase for all phases with nontrivial g-dependent ground state degeneracy. All other phases, i.e., those that follow the IQHE paradigm, are called short-range entangled (SRE) phases. (In doing so, we follow the terminology of Ref. 26, that differs slightly from the one used in Ref. 21. The latter counts all chiral phases irrespective of their ground state degeneracy as LRE.)

While there are nontrivial SRE and LRE phases in the absence of any symmetry constraint, many SRE and LRE phases are defined by some protecting symmetry they obey. If this protecting symmetry is broken, the topological attribute of the phase is not well defined any more. However, there is a sense in which LRE phases are more robust than SRE phases against a weak breaking of the defining symmetry. The topological attributes of LRE phases are not confined to the boundary in space between two distinct topological realizations of these phases, as they are for SRE phases. They also characterize intrinsic bulk properties such as the existence of gapped deconfined fractionalized excitations. Hence, whereas gapless edge states are gapped by any breaking of the defining symmetry, topological bulk properties are robust to a weak breaking of the defining symmetry as long as the characteristic energy scale for this symmetry breaking is small compared to the bulk gap in the LRE phase, for a small breaking of the protecting symmetry does not wipe out the gapped deconfined fractionalized bulk excitations.

The purpose of this paper is to implement a classification scheme for interacting electronic systems in two spatial dimensions that treats SRE and LRE phases on equal footing. To this end, we use a coupled wire construction for each of the symmetry classes from the tenfold way. This approach has been pioneered in Refs. 30 and 31 for the IQHE and in Refs. 32 and 33 for the FQHE (see also related work in Refs. 34–39).

To begin with, non-chiral Luttinger liquids are placed in a periodic array of coupled wires. In doing so, forward-scattering two-body interactions are naturally accounted for within each wire. We then assume that the back-scattering (i.e., tunneling) within a given wire or between neighboring wires are the dominant energy scales. Imposing symmetries constrains these allowed tunnelings. Whether a given arrangement of tunnelings truly gaps out all bulk modes, except for some ungapped edge states on the first and last wire, is verified with the help of a condition that applies to the limit of strong tunneling. We name this condition the Haldane criterion, as it was introduced by Haldane in his study of the stability of non-maximally chiral edge states in the quantum Hall effect.[40] We show that, for a proper choice of the tunnelings, all bulk modes are gapped. Moreover, in five out of the ten symmetry classes of the tenfold way, there remain gapless edge states in agreement with the tenfold way. It is the character of the tunnelings that determines if this wire construction selects a SRE or a LRE phase. Hence, this construction, predicated as it is on the strong tunneling limit, generalizes the tenfold way for SRE phases to LRE phases. It thereby delivers LRE phases that have not yet appeared in the literature before. Evidently, this edge-centered classification scheme does not distinguish between LRE phases of matter that do not carry protected gapless edge modes at their interfaces. For example, some fractional, time-reversal-symmetric, incompressible and topological phases of matter can have fractionalized excitations in the bulk, while not supporting protected gapless modes at their boundaries.[41–43]

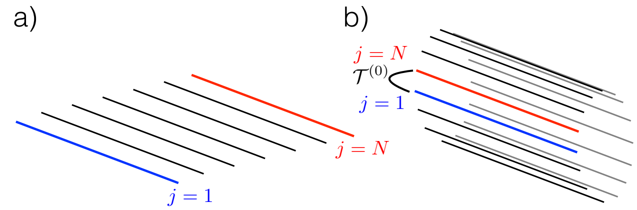


FIG. 1. (Color online) The boundary conditions determine whether a topological phase has protected gapless modes or not. (a) With open boundary conditions, gapless modes exist near the wires $j=1$ and $j=N$, the scattering between them is forbidden by imposing locality in the limit $N\to\infty$. (b) Periodic boundary conditions allow the scattering vector $\mathcal{T}^{(0)}$ that gaps modes which were protected by locality before.

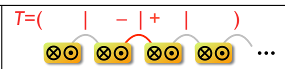
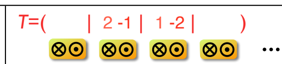
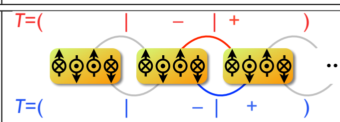
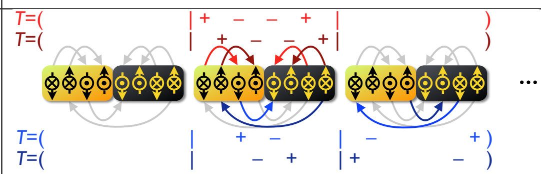
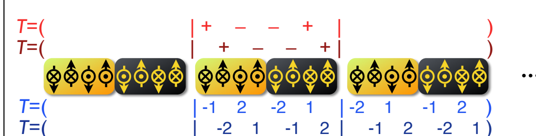
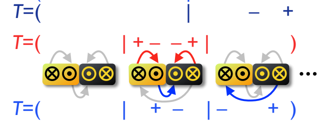
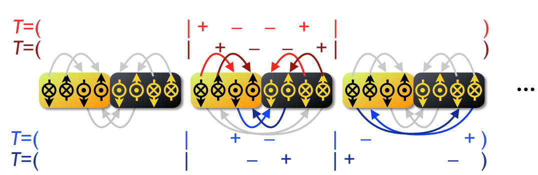
Stated in a slightly more constructive way, we can think of our approach as (1) fixing, in a first step, a given desired edge theory at the boundary, and (2) continue, in a second step, by asking whether such an edge can be consistently defined with a set of symmetry-allowed periodic tunneling terms between wires which manage to gap out all other modes. Alluding to a related strategy in philosophy, this is what we call wire deconstructionism of topological phases.

The paper is organized as follows. We define the array of Luttinger liquids in Sec. II. The Haldane criterion, which plays an essential role for the stability analysis of the edge theory, is reviewed in Sec. III C. All five SRE entries of Table I are derived in Sec. IV, while all five LRE entries of Table I are derived in Sec. V. We conclude with Sec. VI, where we allude to the generalization of our approach to additional symmetries, bosonic systems, and higher spatial dimensions.

II. DEFINITIONS

We consider an array of N parallel wires that stretch along the x direction of the two-dimensional embedding Euclidean space (see Fig. 1). We label a wire by the Latin letter $i=1,\cdots,N$. Each wire supports fermions that carry an even integer number M of internal degrees of freedom that discriminate between left- and right-movers, the projection along the spin-1/2 quantization axis, and particle-hole quantum numbers, among others (e.g., flavors). We label these internal degrees of freedom by the Greek letter $\gamma=1,\cdots,M$. We combine those

TABLE I. (Color online) Realization of a two-dimensional array of quantum wires in each symmetry class of the tenfold way. For each of the symmetry classes A, AII, D, DIII, and C, the ground state supports propagating gapless edge modes localized on the first and last wire that are immune to local and symmetry-preserving perturbations. The first column labels the symmetry classes according to the Cartan classification of symmetric spaces. The second column dictates if the operations for reversal of time ($\widehat{\Theta}$ with the single-particle representation Θ), exchange of particles and holes ($\widehat{\Pi}$ with the single-particle representation Π), and reversal of chirality ($\widehat{C}$ with the single-particle representation C) are the generators of symmetries with their single-particle representations squaring to $+1$, -1, or are not present in which case the entry 0 is used. (See the footnote 44 for a definition of $\widehat{C}$.) The third column is the set to which the topological index from the tenfold way, defined as it is in the non-interacting limit, belongs to. The fourth column is a pictorial representation of the interactions (a set of tunnelings vectors T) for the two-dimensional array of quantum wires that delivers short-range entangled (SRE) gapless edge states. A wire is represented by a colored box with the minimum number of channels compatible with the symmetry class. Each channel in a wire is either a right mover ($\otimes$) or a left mover ($\odot$) that may or may not carry a spin quantum number ($\uparrow, \downarrow$) or a particle (yellow color) or hole (black color) attribute. The lines describe tunneling processes within a wire or between consecutive wires in the array that are of one-body type when they do not carry an arrow or of strictly many-body type when they carry an arrow. Arrows point toward the sites on which creation operators act and away from the sites on which annihilation operators act. For example in the symmetry class A, the single line connecting two consecutive wires in the SRE column represents a one-body backward scattering by which left and right movers belonging to consecutive wires are coupled. The lines have been omitted for the fifth (LRE) column, only the tunneling vectors are specified.

	Θ^2	Π^2	C^2		Short-range entangled (SRE) topological phase	Long-range entangled (LRE) topological phase
A	0	0	0	$\mathbb{Z}$		
AIII	0	0	+		NONE	
AII	−	0	0	$\mathbb{Z}_2$		
DIII	−	+	+	$\mathbb{Z}_2$		
D	0	+	0	$\mathbb{Z}$		
BDI	+	+	+		NONE	
AI	+	0	0		NONE	
CI	+	−	+		NONE	
C	0	−	0	$\mathbb{Z}$		
CII	−	−	+		NONE	

two indices in a collective index $\mathsf{a} \equiv (i,\gamma)$. Correspondingly, we introduce the $M \times N$ pairs of creation $\hat{\psi}^{\dagger}_{\mathsf{a}}(x)$ and annihilation $\hat{\psi}_{\mathsf{a}}(x)$ field operators obeying the fermionic equal-time algebra

$$\left\{\hat{\psi}_{\mathsf{a}}(x), \hat{\psi}^{\dagger}_{\mathsf{a}'}(x')\right\} = \delta_{\mathsf{a},\mathsf{a}'}\, \delta(x-x') \tag{2.1a}$$

with all other anticommutators vanishing and the collective labels $\mathsf{a},\mathsf{a}' = 1,\cdots,M\times N$. The notation

$$\widehat{\Psi}^{\dagger}(x) \equiv \begin{pmatrix} \hat{\psi}^{\dagger}_{1}(x) & \cdots & \hat{\psi}^{\dagger}_{MN}(x) \end{pmatrix}, \quad \widehat{\Psi}(x) \equiv \begin{pmatrix} \hat{\psi}_{1}(x) \\ \vdots \\ \hat{\psi}_{MN}(x) \end{pmatrix}, \tag{2.1b}$$

is used for the operator-valued row ($\widehat{\Psi}^{\dagger}$) and column ($\widehat{\Psi}$) vector fields. We assume that the many-body quantum dynamics of the fermions supported by this array of wires is governed by the Hamiltonian $\hat{H}$, whereby interactions within each wire are dominant over interactions between wires so that we may represent $\hat{H}$ as N coupled Luttinger liquids, each one of which is composed of M interacting fermionic channels.

By assumption, we may thus bosonize the $M \times N$ fermionic channels making up the array. To this end, we follow Ref. 45. Within Abelian bosonization, this is done by postulating first the $MN \times MN$ matrix

$$\mathcal{K} \equiv (\mathcal{K}_{\mathsf{aa}'}) \tag{2.2a}$$

to be symmetric with integer-valued entries. Because we are after an array of identical wires, each of which having its quantum dynamics governed by that of a Luttinger liquid, it is natural to assume that $\mathcal{K}$ is reducible,

$$\mathcal{K}_{\mathsf{aa}'} = \delta_{ii'}\, K_{\gamma\gamma'}, \quad i,i' = 1,\cdots,N, \quad \gamma,\gamma' = 1,\cdots,M. \tag{2.2b}$$

A second $MN \times MN$ matrix is then defined by

$$\mathcal{L} \equiv (\mathcal{L}_{\mathsf{aa}'}) \tag{2.3a}$$

where

$$\mathcal{L}_{\mathsf{aa}'} := \mathrm{sgn}(\mathsf{a}-\mathsf{a}')\,(\mathcal{K}_{\mathsf{aa}'}+1)\,. \tag{2.3b}$$

Third, one verifies that, for any pair $\mathsf{a},\mathsf{a}' = 1,\cdots,MN$, the Hermitian fields $\hat{\phi}_{\mathsf{a}}$ and $\hat{\phi}_{\mathsf{a}'}$, defined by the Mandelstam formula

$$\hat{\psi}_{\mathsf{a}}(x) \equiv\ :\exp\left(+\mathrm{i}\mathcal{K}_{\mathsf{aa}'}\,\hat{\phi}_{\mathsf{a}'}(x)\right): \tag{2.4a}$$

as they are, obey the bosonic equal-time algebra

$$\left[\hat{\phi}_{\mathsf{a}}(x), \hat{\phi}_{\mathsf{a}'}(x')\right] = -\mathrm{i}\pi\left(\mathcal{K}^{-1}_{\mathsf{aa}'}\,\mathrm{sgn}(x-x') + \mathcal{K}^{-1}_{\mathsf{ab}}\,\mathcal{L}_{\mathsf{bc}}\,\mathcal{K}^{-1}_{\mathsf{ca}'}\right). \tag{2.4b}$$

Here, the notation $:(\cdots):$ stands for normal ordering of the argument $(\cdots)$ and the summation convention over repeated indices is implied. In line with Eq. (2.1b), we use the notation

$$\widehat{\Phi}^{\mathsf{T}}(x) \equiv \begin{pmatrix} \hat{\phi}_{1}(x) & \cdots & \hat{\phi}_{MN}(x) \end{pmatrix}, \quad \widehat{\Phi}(x) \equiv \begin{pmatrix} \hat{\phi}_{1}(x) \\ \vdots \\ \hat{\phi}_{MN}(x) \end{pmatrix}, \tag{2.4c}$$

for the operator-valued row ($\widehat{\Phi}^{\mathsf{T}}$) and column ($\widehat{\Phi}$) vector fields. Periodic boundary conditions along the x direction parallel to the wires are imposed by demanding that

$$\mathcal{K}\,\widehat{\Phi}(x+L) = \mathcal{K}\,\widehat{\Phi}(x) + 2\pi\,\mathcal{N}, \qquad \mathcal{N} \in \mathbb{Z}^{MN}. \tag{2.4d}$$

Equipped with Eqs. (2.2)–(2.4), we decompose additively the many-body Hamiltonian $\hat{H}$ for the MN interacting fermions propagating on the array of wires into

$$\hat{H} = \hat{H}_{\mathcal{V}} + \hat{H}_{\{\mathcal{T}\}}. \tag{2.5a}$$

Hamiltonian

$$\hat{H}_{\mathcal{V}} := \int \mathrm{d}x\, \left(\partial_x \widehat{\Phi}^{\mathsf{T}}\right)(x)\; \mathcal{V}\; \left(\partial_x \widehat{\Phi}\right)(x), \tag{2.5b}$$

even though quadratic in the bosonic field, encodes both local one-body terms as well as contact many-body interactions between the M fermionic channels in any given wire from the array through the block-diagonal, real-valued, and symmetric $MN \times MN$ matrix

$$\mathcal{V} := (\mathcal{V}_{\mathsf{aa}'}) \equiv \left(\mathcal{V}_{(i,\gamma)(i',\gamma')}\right) = \mathbb{1}_N \otimes (V_{\gamma\gamma'}). \tag{2.5c}$$

Hamiltonian

$$\begin{aligned} \hat{H}_{\{\mathcal{T}\}} &:= \int \mathrm{d}x\, \sum_{\mathcal{T}} \frac{h_{\mathcal{T}}(x)}{2} \left(e^{+\mathrm{i}\alpha_{\mathcal{T}}(x)} \prod_{\mathsf{a}=1}^{MN} \hat{\psi}^{\mathcal{T}_{\mathsf{a}}}_{\mathsf{a}}(x) + \mathrm{H.c.}\right) \\ &= \int \mathrm{d}x\, \sum_{\mathcal{T}} h_{\mathcal{T}}(x)\, \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi}(x) + \alpha_{\mathcal{T}}(x)\right) \end{aligned} \tag{2.5d}$$

is not quadratic in the bosonic fields. With the understanding that the operator-multiplication of identical fermion fields at the same point x along the wire requires point splitting, and with the short-hand notation $\hat{\psi}^{-1}_{\mathsf{a}}(x) \equiv \hat{\psi}^{\dagger}_{\mathsf{a}}(x)$, we interpret $\hat{H}_{\{\mathcal{T}\}}$ as (possibly many-body) tunnelings between the fermionic channels. Here, we introduced the set $\{\mathcal{T}\}$ comprised of *all* integer-valued tunneling vectors

$$\mathcal{T} \equiv (\mathcal{T}_{\mathsf{a}}) \tag{2.5e}$$

obeying the condition

$$\sum_{\mathsf{a}=1}^{MN} \mathcal{T}_{\mathsf{a}} = \begin{cases} 0 \bmod 2, & \text{for D, DIII, C, and CI,} \\ \\ 0, & \text{otherwise,} \end{cases} \tag{2.5f}$$

and we assigned to each $\mathcal{T}$ from the set $\{\mathcal{T}\}$ the real-valued functions

$$h_{\mathcal{T}}(x)=h^{*}_{\mathcal{T}}(x)\geq 0 \tag{2.5g}$$

and

$$\alpha_{\mathcal{T}}(x)=\alpha^{*}_{\mathcal{T}}(x). \tag{2.5h}$$

The condition (2.5f) ensures that these tunneling events preserve the parity of the total fermion number for the superconducting symmetry classes (symmetry classes D, DIII, C, and CI in Table I), while they preserve the total fermion number for the non-superconducting symmetry classes (symmetry classes A, AIII, AI, AII, BDI, and CII in Table I). We emphasize that the integer

$$q:=\sum_{\mathsf{a}=1}^{MN}\frac{|\mathcal{T}_{\mathsf{a}}|}{2} \tag{2.6}$$

dictates that $\mathcal{T}$ encodes a q-body interaction in the fermion representation.

III. STRATEGY FOR CONSTRUCTING TOPOLOGICAL PHASES

Our strategy consists in choosing the many-body Hamiltonian $\hat{H}=\hat{H}_{\mathcal{V}}+\hat{H}_{\{\mathcal{T}\}}$ defined in Eq. (2.5) so that (i) it belongs to any one of the ten symmetry classes from the tenfold way (with the action of symmetries defined in Sec. III A) and (ii) all excitations in the bulk are gapped by a *specific* choice of the tunneling vectors $\{\mathcal{T}\}$ entering $\hat{H}_{\{\mathcal{T}\}}$ (with the condition for a spectral gap given in Sec. III C). The energy scales in $\hat{H}_{\{\mathcal{T}\}}$ are assumed sufficiently large compared to those in $\hat{H}_{\mathcal{V}}$ so that it is $\hat{H}_{\mathcal{V}}$ that may be thought of as a perturbation of $\hat{H}_{\{\mathcal{T}\}}$ and not the converse.

We anticipate that for five of the ten symmetry classes there can be protected gapless edge states because of locality and symmetry. Step (ii) for each of the five symmetry classes supporting gapless edge states is represented pictorially as is shown in the fourth and fifth columns of Table I. In each symmetry class, topologically trivial states that do not support protected gapless edge states in the tenfold classification can be constructed by gapping all states in each individual wire from the array.

A. Representation of symmetries

The classification is based on the presence or the absence of the TRS and the PHS that are represented by the antiunitary many-body operator $\widehat{\Theta}$ and the unitary many-body operator $\widehat{\Pi}$, respectively. Each of $\widehat{\Theta}$ and $\widehat{\Pi}$ can exist in two varieties such that their single-particle representations Θ and Π square to the identity operator up to the multiplicative factor ±1,

$$\Theta^{2}=\pm1,\qquad \Pi^{2}=\pm1, \tag{3.1}$$

respectively. By assumption, the set of all degrees of freedom in each given wire is invariant under the actions of $\widehat{\Theta}$ and $\widehat{\Pi}$. If so, we can represent the actions of $\widehat{\Theta}$ and $\widehat{\Pi}$ on the fermionic fields in two steps. First, we introduce two $M\times M$-dimensional matrix representations P_{Θ} and P_{Π} of the permutation group of M elements, which we combine into the block-diagonal $MN\times MN$ real-valued and orthogonal matrices

$$\mathcal{P}_{\Theta}:=\mathbb{1}_{N}\otimes P_{\Theta},\qquad \mathcal{P}_{\Pi}:=\mathbb{1}_{N}\otimes P_{\Pi}, \tag{3.2a}$$

where $\mathbb{1}_{N}$ is the $N\times N$ unit matrix and we make sure that P_{Θ} and P_{Π} represent products of transpositions so that

$$P_{\Theta}=P^{-1}_{\Theta}=P^{\mathsf{T}}_{\Theta},\qquad P_{\Pi}=P^{-1}_{\Pi}=P^{\mathsf{T}}_{\Pi}. \tag{3.2b}$$

Second, we introduce two column vectors $I_{\Theta}\in\mathbb{Z}^{M}$ and $I_{\Pi}\in\mathbb{Z}^{M}$, which we combine into the two column vectors

$$\mathcal{I}_{\Theta}:=\begin{pmatrix}I_{\Theta}\\ \vdots\\ I_{\Theta}\end{pmatrix},\qquad \mathcal{I}_{\Pi}:=\begin{pmatrix}I_{\Pi}\\ \vdots\\ I_{\Pi}\end{pmatrix}, \tag{3.2c}$$

and the $MN\times MN$ diagonal matrices

$$\mathcal{D}_{\Theta}:=\mathrm{diag}\,(\mathcal{I}_{\Theta}),\qquad \mathcal{D}_{\Pi}:=\mathrm{diag}\,(\mathcal{I}_{\Pi}), \tag{3.2d}$$

with the components of the vectors $\mathcal{I}_{\Theta}$ and $\mathcal{I}_{\Pi}$ as diagonal matrix elements. The vectors I_{Θ} and I_{Π} are not chosen arbitrarily. We demand that the vectors $(1+\mathcal{P}_{\Theta})\,\mathcal{I}_{\Theta}$ and $(1+\mathcal{P}_{\Pi})\,\mathcal{I}_{\Pi}$ are made of even [for the $+1$ in Eq. (3.1)] and odd [for the -1 in Eq. (3.1)] integer entries only, while

$$e^{+\mathrm{i}\pi\,\mathcal{D}_{\Theta}}\,\mathcal{P}_{\Theta}=\pm\mathcal{P}_{\Theta}\,e^{+\mathrm{i}\pi\,\mathcal{D}_{\Theta}} \tag{3.2e}$$

and

$$e^{+\mathrm{i}\pi\,\mathcal{D}_{\Pi}}\,\mathcal{P}_{\Pi}=\pm\mathcal{P}_{\Pi}\,e^{+\mathrm{i}\pi\,\mathcal{D}_{\Pi}}, \tag{3.2f}$$

in order to meet $\Theta^{2}=\pm1$ and $\Pi^{2}=\pm1$, respectively. The operations of reversal of time and interchanges of particles and holes are then represented by

$$\widehat{\Theta}\,\widehat{\Psi}\,\widehat{\Theta}^{-1}=e^{+\mathrm{i}\,\pi\,\mathcal{D}_{\Theta}}\,\mathcal{P}_{\Theta}\,\widehat{\Psi}, \tag{3.2g}$$

$$\widehat{\Pi}\,\widehat{\Psi}\,\widehat{\Pi}^{-1}=e^{+\mathrm{i}\pi\,\mathcal{D}_{\Pi}}\,\mathcal{P}_{\Pi}\,\widehat{\Psi}, \tag{3.2h}$$

for the fermions and

$$\widehat{\Theta}\,\widehat{\Phi}\,\widehat{\Theta}^{-1}=\mathcal{P}_{\Theta}\,\widehat{\Phi}+\pi\,\mathcal{K}^{-1}\,\mathcal{I}_{\Theta}, \tag{3.2i}$$

$$\widehat{\Pi}\,\widehat{\Phi}\,\widehat{\Pi}^{-1}=\mathcal{P}_{\Pi}\,\widehat{\Phi}+\pi\,\mathcal{K}^{-1}\,\mathcal{I}_{\Pi}, \tag{3.2j}$$

for the bosons. One verifies that Eq. (3.1) is fulfilled.

Hamiltonian (2.5) is TRS if

$$\widehat{\Theta}\,\hat{H}\,\widehat{\Theta}^{-1}=+\hat{H}. \tag{3.3a}$$

This condition is met if

$$P_\Theta\, V\, P_\Theta^{-1} = +V, \tag{3.3b}$$

$$P_\Theta\, K\, P_\Theta^{-1} = -K, \tag{3.3c}$$

$$h_{\mathcal{T}}(x) = h_{-\mathcal{P}_\Theta \mathcal{T}}(x), \tag{3.3d}$$

$$\alpha_{\mathcal{T}}(x) = \alpha_{-\mathcal{P}_\Theta \mathcal{T}}(x) - \pi\, \mathcal{T}^{\mathsf{T}}\, \mathcal{P}_\Theta\, \mathcal{I}_\Theta. \tag{3.3e}$$

The Hamiltonian (2.5) is PHS if

$$\widehat{\Pi}\, \hat{H}\, \widehat{\Pi}^{-1} = +\hat{H}. \tag{3.4a}$$

This condition is met if (see Appendix A)

$$P_\Pi\, V\, P_\Pi^{-1} = +V, \tag{3.4b}$$

$$P_\Pi\, K\, P_\Pi^{-1} = +K, \tag{3.4c}$$

$$h_{\mathcal{T}}(x) = h_{+\mathcal{P}_\Pi \mathcal{T}}(x), \tag{3.4d}$$

$$\alpha_{\mathcal{T}}(x) = \alpha_{\mathcal{P}_\Pi \mathcal{T}}(x) + \pi\, \mathcal{T}^{\mathsf{T}}\, \mathcal{P}_\Pi\, \mathcal{I}_\Pi. \tag{3.4e}$$

B. Particle-hole symmetry in interacting superconductors

The total number of fermions is a good quantum number in any metallic or insulating phase of fermionic matter. This is not true anymore in the mean-field treatment of superconductivity. In a superconductor, within a mean-field approximation, charge is conserved modulo two as Cooper pairs can be created and annihilated. The existence of superconductors and the phenomenological success of the mean-field approximation suggest that the conservation of the total fermion number operator should be relaxed down to its parity in a superconducting phase of matter. If we only demand that the parity of the total fermion number is conserved, we may then decompose any fermionic creation operator in the position basis into its real and imaginary parts, thereby obtaining two Hermitean operators called Majorana operators. Any Hermitean Hamiltonian that is build out of even powers of Majorana operators necessarily preserves the parity of the total fermion number operator, but it might break the conservation of the total fermion number. By definition, any such Hamiltonian belongs to the symmetry class D.

The tool of Abelian bosonization allows to represent a fermion operator as a single exponential of a Bose field. In Abelian bosonization, a Majorana operator is the sum of two exponentials, and this fact makes it cumbersome to apply Abelian bosonization for Majorana operators. It is possible to circumvent this difficulty by representing any Hamiltonian from the symmetry class D in terms of the components of Nambu-Gorkov spinors obeying a reality condition. Indeed, we may double the dimensionality of the single-particle Hilbert space by introducing Nambu-Gorkov spinors with the understanding that (i) a reality condition on the Nambu-Gorkov spinors must hold within the physical subspace of the enlarged single-particle Hilbert space and (ii) the dynamics dictated by the many-body Hamiltonian must be compatible with this reality condition. The reality condition keeps track of the fact that there are many ways to express an even polynomial of Majorana operators in terms of the components of a Nambu-Gorkov spinor. The complication brought about by this redundency is compensated by the fact that it is straightforward to implement Abelian bosonization in the Nambu-Gorkov representation.

We implement this particle-hole doubling by assigning to every pair of fermionic operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ (whose indices we have been omitted for simplicity) related to each other by the reality condition

$$\widehat{\Pi}\, \hat{\psi}\, \widehat{\Pi}^\dagger = \hat{\psi}^\dagger, \tag{3.5a}$$

the pair of bosonic field operators $\hat{\phi}$ and $\hat{\phi}'$ related by the reality condition

$$\widehat{\Pi}\, \hat{\phi}\, \widehat{\Pi}^\dagger = -\hat{\phi}'. \tag{3.5b}$$

Invariance under this transformation has to be imposed on the (interacting) Hamiltonian in the doubled (Nambu-Gorkov) representation. In addition to the PHS, we also demand, when describing the superconducting symmetry classes, that the parity of the total fermion number is conserved. This discrete global symmetry, the symmetry of the Hamiltonian under the reversal of sign of all fermion operators, becomes a continuous $U(1)$ global symmetry that is responsible for the conservation of the electric charge in all non-superconducting symmetry classes. In this way, all 9 symmetry classes from the tenfold way descend from the symmetry class D by imposing a composition of TRS, $U(1)$ charge conservation, and the chiral (sublattice) symmetry.

We close this discussion of the particle-hole symmetry by documenting its past use in connection to this work. Altland and Zirnbauer introduced four random matrix ensembles (denoted by the Cartan labels for the symmetric spaces D, DIII, C, and CI) to describe the level statistics and transport properties of non-interacting disordered superconducting quantum dots.[46] They also extended the threefold way for random matrices introduced by Dyson in Ref. 47 to the tenfold way. The extension of the tenfold way for random matrices to the theory of Anderson localization in d-dimensional space ($d > 0$) delivered the periodic table for the topological insulators in the hands of Schnyder, Ryu, Furusaki, and Ludwig in 2008 and 2010.[9,11] In the approach by Schnyder et al., the particle-hole symmetry plays a crucial role when deriving the non-linear-sigma models describing disordered Bogoliubov-deGennes superconductors and when identifying for which symmetry class and for which dimensions these non-linear-sigma models can be augmented by a topological or a Wess-Zumino-Witten term that allows for topologically protected boundary states. The combined effects of disorder and interactions in superconductors was studied in Refs. 48–51 starting from the Nambu-Gorkov formalism to derive a non-linear-sigma model for the Goldstone modes relevant to the interplay between the physics of Anderson localization and that of interactions. The stability of Majorana zero modes to interactions preserving the particle-hole symmetry was studied in Ref. 52.

C. Conditions for a spectral gap

Hamiltonian $\hat{H}_{\mathcal{V}}$ in the decomposition (2.5) has MN gapless modes. However, $\hat{H}_{\mathcal{V}}$ does not commute with $\hat{H}_{\{\mathcal{T}\}}$ and the competition between $\hat{H}_{\mathcal{V}}$ and $\hat{H}_{\{\mathcal{T}\}}$ can gap some, if not all, the gapless modes of $\hat{H}_{\mathcal{V}}$. For example, a tunneling amplitude that scatters the right mover into the left mover of each flavor in each wire will gap out the spectrum of $\hat{H}_{\mathcal{V}}$.

A term in $\hat{H}_{\{\mathcal{T}\}}$ has the potential to gap out a gapless mode of $\hat{H}_{\mathcal{V}}$ if the condition (in the Heisenberg representation)[45,53]

$$\partial_x \left[\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi}(t,x)+\alpha_{\mathcal{T}}(x)\right]=C_{\mathcal{T}}(x) \tag{3.6}$$

holds for some time-independent real-valued functions $C_{\mathcal{T}}(x)$ on the canonical momentum $(4\pi)^{-1}\,\mathcal{K}\,(\partial_x\widehat{\Phi})(t,x)$ that is conjugate to $\widehat{\Phi}(t,x)$, when applied to the ground state. The locking condition (3.6) removes a pair of chiral bosonic modes with opposite chiralities from the gapless degrees of freedom of the theory. However, not all scattering vectors $\mathcal{T}$ can simultaneously lead to such a locking due to quantum fluctuations. The set of linear combinations $\{\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi}(t,x)\}$ that can satisfy the locking condition (3.6) simultaneously is labeled by the subset $\{\mathcal{T}\}_{\text{locking}}$ of all tunneling matrices $\{\mathcal{T}\}$ defined by Eqs. (2.5e) and (2.5f) obeying the Haldane criterion (3.7)[45,53]

$$\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\mathcal{T}=0 \tag{3.7a}$$

for any $\mathcal{T}\in\{\mathcal{T}\}_{\text{locking}}$ and

$$\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\mathcal{T}'=0 \tag{3.7b}$$

pairwise for any $\mathcal{T}\neq\mathcal{T}'\in\{\mathcal{T}\}_{\text{locking}}$.

IV. REPRODUCING THE TENFOLD WAY

Our first goal is to apply the wire construction in order to reproduce the classification of non-interacting topological insulators (symmetry classes A, AIII, AI, AII, BDI, and CII in Table I) and superconductors (symmetry classes D, DIII, C,and CI in Table I) in $(2+1)$ dimensions (see Table I).[9–11] In this section, we will carry out the classification scheme within the bosonized description of quantum wires. Here, we will restrict the classification to one-body tunneling terms, i.e., $q=1$ in Eq. (2.6), for the non-superconducting symmetry classes, and to two-body tunneling terms, i.e., $q=2$ in Eq. (2.6), for the superconducting symmetry classes. In Sec. V, we generalize this construction to the cases $q>1$ and $q>2$ of multi-particle tunnelings in the non-superconducting and superconducting symmetry classes, respectively. The topological stability of edge modes will be an immediate consequence of the observation that no symmetry-respecting local terms can be added to the models that we are going to construct.

Within the classification of non-interacting Hamiltonians, superconductors are nothing but fermionic bilinears with a particle-hole symmetry. The physical interpretation of the degrees of freedom as Bogoliubov quasiparticles is of no consequence to the analysis. In particular, they still carry an effective conserved $U(1)$ charge in the non-interacting description.

A. Symmetry class A

1. SRE phases in the tenfold way

Topological insulators in symmetry class A can be realized without any symmetry aside from the $U(1)$ charge conservation. The wire construction starts from wires supporting spinless fermions, so that the minimal choice $M=2$ only counts left- and right-moving degrees of freedom. The K-matrix reads

$$K:=\operatorname{diag}(+1,-1). \tag{4.1a}$$

The entry $+1$ of the K-matrix corresponds to a right mover. It is depicted by the symbol $\otimes$ in the first line of Table I. The entry -1 of the K-matrix corresponds to a left mover. It is depicted by the symbol $\odot$ in the first line of Table I. The operation for reversal of time in any one of the N wires is represented by [one verifies that Eq. (3.2e) holds]

$$P_{\Theta}:=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad I_{\Theta}:=\begin{pmatrix}0\\0\end{pmatrix}. \tag{4.1b}$$

We define $\hat{H}_{\{\mathcal{T}\}}$ by choosing $(N-1)$ scattering vectors, whereby, for any $j=1,\cdots,(N-1)$,

$$\mathcal{T}^{(j)}_{(i,\gamma)}:=\delta_{i,j}\,\delta_{\gamma,2}-\delta_{i-1,j}\,\delta_{\gamma,1} \tag{4.2a}$$

with $i=1,\cdots,N$ and $\gamma=1,2$. In other words,

$$\mathcal{T}^{(j)}:=(0,0|\cdots|0,+1|-1,0|\cdots|0,0)^{\mathsf{T}} \tag{4.2b}$$

for $j=1,\cdots,N-1$. Intent on helping with the interpretation of the tunneling vectors, we use the $|$'s in Eq. (4.2b) to compartmentalize the elements within a given wire. Henceforth, there are $M=2$ vector components within each pair of $|$'s that encode the $M=2$ degrees of freedom within a given wire. The jth scattering vector (4.2b) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f) and breaks TRS, since the scattering vector $(0,+1)^{\mathsf{T}}$ is mapped into the scattering vector

$(+1,0)^{\mathsf{T}}$ by the permutation P_{Θ} that represents reversal of time in a wire by exchanging right- with left-movers. For any $j=1,\cdots,(N-1)$, we also introduce the amplitude

$$h_{\mathcal{T}^{(j)}}(x)\geq 0 \tag{4.2c}$$

and the phase

$$\alpha_{\mathcal{T}^{(j)}}(x)\in\mathbb{R} \tag{4.2d}$$

according to Eqs. (3.3d) and (3.3e), respectively. The choices for the amplitude (4.2c) and the phase (4.2d) are arbitrary. In particular the amplitude (4.2c) can be chosen to be sufficiently large so that it is $\hat{H}_{\mathcal{V}}$ that may be thought of as a perturbation of $\hat{H}_{\{\mathcal{T}\}}$ and not the converse.

One verifies that all $(N-1)$ scattering vectors (4.2a) satisfy the Haldane criterion (3.7), i.e.,

$$\mathcal{T}^{(i)\mathsf{T}}\,\mathcal{K}\,\mathcal{T}^{(j)}=0,\qquad i,j=1,\cdots,N-1. \tag{4.3}$$

Correspondingly, the term $\hat{H}_{\{\mathcal{T}\}}$ gaps out $2(N-1)$ of the $2N$ gapless modes of $\hat{H}_{\mathcal{V}}$. Two modes of opposite chirality that propagate along the first and last wire, respectively, remain in the low energy sector of the theory. These edge states are localized on wire $i=1$ and $i=N$, respectively, for their overlaps with the gapped states from the bulk decay exponentially fast as a function of the distance away from the first and end wires. The energy splitting between the edge state localized on wire $i=1$ and the one localized on wire $i=N$ that is brought about by the bulk states vanishes exponentially fast with increasing N. Two gapless edge states with opposite chiralities emerge in the two-dimensional limit $N\to\infty$.

At energies much lower than the bulk gap, the effective $\mathcal{K}$-matrix for the edge modes is

$$\mathcal{K}_{\text{eff}}:=\operatorname{diag}(+1,0|0,0|\cdots|0,0|0,-1). \tag{4.4}$$

Here, $\mathcal{K}_{\text{eff}}$ follows from replacing the entries in the $2N\times 2N$ $\mathcal{K}$ matrix for all gapped modes by 0. The pictorial representation of the topological phase in the symmetry class A with one chiral edge state per end wire through the wire construction is shown on the first row and fourth column of Table I. The generalization to an arbitrary number n of gapless edge states sharing a given chirality on the first wire that is opposite to that of the last wire is the following. We enlarge $M=2$ to $M=2n$ by making n identical copies of the model depicted in the first row and fourth column of Table I. The stability of the n chiral gapless edge states in wire 1 and wire N is guaranteed because back-scattering among these gapless edges state is not allowed kinematically within wire 1 or within wire N, while back-scattering across the bulk is exponentially suppressed for N large by locality and the gap in the bulk. The number of robust gapless edge states of a given chirality is thus integer. This is the reason why $\mathbb{Z}$ is found in the third column on the first line of Table I.

2. *SRE phases beyond the tenfold way*

It is imperative to ask whether the phases that we constructed so far exhaust all possible SRE phases in the symmetry class A. By demanding that one-body interactions are dominant over many-body interactions, we have constructed all phases from the (exhaustive) classification for non-interacting fermions in class A and only those. In these phases, the same topological invariant controls the Hall and the thermal conductivities. However, it was observed that interacting fermion systems can host additional SRE phases in the symmetry class A where this connection is lost.[26] These phases are characterized by an edge that includes charge-neutral chiral modes. While such modes contribute to the quantized energy transport (i.e., the thermal Hall conductivity), they do not contribute to the quantized charge transport (i.e., the charge Hall conductivity). By considering the thermal and charge Hall conductivity as two independent quantized topological responses, this enlarges the classification of SPT phases in the symmetry class A to $\mathbb{Z}\times\mathbb{Z}$.

Starting from identical fermions of charge e, we now construct an explicit wire model that stabilizes a SRE phase of matter in the symmetry class A carrying a non-vanishing Hall conductivity but a vanishing thermal Hall conductivity. In order to build a wire-construction of such a strongly interacting SRE phase in the symmetry class A, we group three spinless electronic wires into one unit cell, i.e.,

$$K:=\operatorname{diag}(+1,-1,+1,-1,+1,-1). \tag{4.5a}$$

It will be useful to arrange the charges $Q_{\gamma}=1$ measured in units of the electron charge e for each of the modes $\hat{\phi}_{\gamma}$, $\gamma=1,\cdots,M$, into a vector

$$Q=(1,1,1,1,1,1)^{\mathsf{T}}. \tag{4.5b}$$

The physical meaning of the tunneling vectors (interactions) that we define below is most transparent, if we employ the following linear transformation on the bosonic field variables

$$\widehat{\Phi}(x)=:\mathcal{W}\,\widetilde{\Phi}(x), \tag{4.6a}$$

$$\mathcal{T}=:\mathcal{W}\,\widetilde{\mathcal{T}}, \tag{4.6b}$$

$$K=:W\,\widetilde{K}\,W^{\mathsf{T}}, \tag{4.6c}$$

where $\mathcal{W}$ is a $MN\times MN$ block-diagonal matrix with the block W having integer entries and unit determinant. The transformation W that we employ is given by

$$W:=\begin{pmatrix}0&+1&-1&0&0&0\\+1&-1&+1&0&0&0\\-1&0&-1&0&0&0\\0&0&0&-1&0&-1\\0&0&0&+1&-1&+1\\0&0&0&-1&+1&0\end{pmatrix}. \tag{4.7}$$

It brings K to the form

$$\widetilde{K} := \begin{pmatrix} 0 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.8}$$

As we can read off from Eq. (2.4b), the parity of $K_{\gamma\gamma}$ determines the self-statistics of particles of type $\gamma = 1, \cdots, N$. As Eq. (2.4b) is form invariant under the transformation (4.6), we conclude that, with the choice (4.7), the transformed modes $\gamma = 1, 2$ as well as the modes $\gamma = 5, 6$ are pairs of bosonic degrees of freedom, while the third and fourth mode remain fermonic. Furthermore, the charges transported by the transformed modes $\widetilde{\phi}_\gamma$ are given by

$$\widetilde{Q} = W^{-1} Q = (+2, -2, -3, -3, -2, +2)^{\mathsf{T}}. \tag{4.9}$$

We may then define the charge-conserving tunneling vectors

$$\begin{aligned} \widetilde{\mathcal{T}}_1^{(j)} :=& (0,0,0,0,0,0|\cdots|0,0,+1,-1,0,0|\cdots|0,0,0,0,0,0)^{\mathsf{T}}, \qquad j = 1, \cdots, N, \\ \widetilde{\mathcal{T}}_2^{(j)} :=& (0,0,0,0,0,0|\cdots|0,0,0,0,+1,0|0,-1,0,0,0,0|\cdots|0,0,0,0,0,0)^{\mathsf{T}}, \qquad j = 1, \cdots, N-1, \\ \widetilde{\mathcal{T}}_3^{(j)} :=& (0,0,0,0,0,0|\cdots|0,0,0,0,0,+1|-1,0,0,0,0,0|\cdots|0,0,0,0,0,0)^{\mathsf{T}}, \qquad j = 1, \cdots, N-1. \end{aligned} \tag{4.10}$$

Using Eq. (4.6b), these tunneling vectors can readily be rewritten in the original electronic degrees of freedom.

These tunneling vectors gap all modes in the bulk and the remaining gapless edge modes on the left edge are

$$\widetilde{K}_{\mathrm{eff,left}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{Q}_{\mathrm{eff,left}} = \begin{pmatrix} +2 \\ -2 \end{pmatrix}. \tag{4.11}$$

The only charge-conserving tunneling vector that could gap out this effective edge theory, $\widetilde{T} = (1,1)^{\mathsf{T}}$, is not compatible with Haldane's criterion (3.7). We conclude that the edge theory (4.11) is stable against charge conserving perturbations. The Hall conductivity supported by this edge theory is given by

$$\widetilde{Q}^{\mathsf{T}}_{\mathrm{eff,left}} \, \widetilde{K}^{-1}_{\mathrm{eff,left}} \, \widetilde{Q}_{\mathrm{eff,left}} = -8 \tag{4.12}$$

in units of e^2/h. This is the minimal Hall conductivity of a SRE phase of bosons, if each boson is interpreted as a pair of electrons carrying the electronic charge $2e$.[26] On the other hand, the edge theory (4.11) supports two modes with opposite chiralities, for the symmetric matrix $\widetilde{K}_{\mathrm{eff,left}}$ has the pair of eigenvalues ± 1. Thus, the net energy transported along the left edge, and with it the thermal Hall conductivity, vanishes.

B. Symmetry class AII

Topological insulators in symmetry class AII can be realized by demanding that $U(1)$ charge conservation holds and that TRS with $\Theta^2 = -1$ holds. The wire construction starts from wires supporting spin-1/2 fermions because $\Theta^2 = -1$, so that the minimal choice $M = 4$ counts two pairs of Kramers degenerate left- and right-moving degrees of freedom carrying opposite spin projections on the spin quantization axis, i.e., two pairs of Kramers degenerate helical modes. The K-matrix reads

$$K := \mathrm{diag}\,(+1, -1, -1, +1). \tag{4.13a}$$

The entries in the K-matrix represent, from left to right, a right-moving particle with spin up, a left-moving particle with spin down, a left-moving particle with spin up, and a right-moving particle with spin down. The operation for reversal of time in any one of the N wires is represented by [one verifies that Eq. (3.2e) holds]

$$P_\Theta := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad I_\Theta := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{4.13b}$$

We define $\hat{H}_{\mathcal{V}}$ by choosing any symmetric 4×4 matrix V that obeys

$$V = P_\Theta \, V \, P_\Theta^{-1}. \tag{4.13c}$$

We define $\hat{H}_{\{\mathcal{T}_{\mathrm{SO}}\}}$ by choosing $2(N-1)$ scattering vectors as follows. For any $j = 1, \cdots, (N-1)$, we introduce the pair of scattering vectors

$$\mathcal{T}_{\mathrm{SO}}^{(j)} := (0,0,0,0|\cdots|0,0,+1,0|-1,0,0,0|\cdots|0,0,0,0)^{\mathsf{T}} \tag{4.14a}$$

and

$$\overline{\mathcal{T}}_{\mathrm{SO}}^{(j)} := -\mathcal{P}_\Theta \, \mathcal{T}_{\mathrm{SO}}^{(j)}. \tag{4.14b}$$

The scattering vector (4.14a) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f). It scatters a left mover with spin up from wire j into a right mover with spin up in wire $j+1$. For any $j = 1, \cdots, (N-1)$, we also introduce the pair of amplitudes

$$h_{\mathcal{T}_{\mathrm{SO}}^{(j)}}(x) = h_{\overline{\mathcal{T}}_{\mathrm{SO}}^{(j)}}(x) \geq 0 \tag{4.14c}$$

and the pair of phases

$$\alpha_{\mathcal{T}^{(j)}_{\mathrm{SO}}}(x) = \alpha_{\overline{\mathcal{T}}^{(j)}_{\mathrm{SO}}}(x) \in \mathbb{R} \tag{4.14d}$$

according to Eqs. (3.3d) and (3.3e), respectively. The choices for the amplitude (4.14c) and the phase (4.14d) are arbitrary. The subscript SO refers to the intrinsic spin-orbit coupling. The rational for using it shall be shortly explained.

One verifies that all $2(N-1)$ scattering vectors (4.13c) and (4.14a) satisfy the Haldane criterion (3.7), i.e.,

$$\mathcal{T}^{(i)\mathsf{T}}_{\mathrm{SO}}\,\mathcal{K}\,\mathcal{T}^{(j)}_{\mathrm{SO}} = \overline{\mathcal{T}}^{(i)\mathsf{T}}_{\mathrm{SO}}\,\mathcal{K}\,\overline{\mathcal{T}}^{(j)}_{\mathrm{SO}} = \mathcal{T}^{(i)\mathsf{T}}_{\mathrm{SO}}\,\mathcal{K}\,\overline{\mathcal{T}}^{(j)}_{\mathrm{SO}} = 0, \tag{4.15}$$

for $i,j = 1,\cdots,N-1$. Correspondingly, the term $\hat{H}_{\{\mathcal{T}_{\mathrm{SO}}\}}$ gaps out $4(N-1)$ of the $4N$ gapless modes of $\hat{H}_{\mathcal{V}}$. Two pairs of Kramers degenerate helical edge states that propagate along the first and last wire, respectively, remain in the low energy sector of the theory. These edge states are localized on wire $i = 1$ and $i = N$, respectively, for their overlaps with the gapped states from the bulk decay exponentially fast as a function of the distance away from the first and end wires. The energy splitting between the edge state localized on wire $i = 1$ and wire $i = N$ brought about by the bulk states vanishes exponentially fast with increasing N. Two pairs of gapless Kramers degenerate helical edge states emerge in the two-dimensional limit $N \to \infty$.

At energies much lower than the bulk gap, the effective $\mathcal{K}$-matrix for the two pairs of helical edge modes is

$$\begin{aligned}\mathcal{K}_{\mathrm{eff}} := \mathrm{diag}(&+1,-1,0,0|0,0,0,0|\\ &\cdots|0,0,0,0|0,0,-1,+1).\end{aligned} \tag{4.16}$$

Here, $\mathcal{K}_{\mathrm{eff}}$ follows from replacing the entries in the $4N \times 4N$ $\mathcal{K}$ matrix for all gapped modes by 0. We are going to show that the effective scattering vector

$$\mathcal{T}_{\mathrm{eff}} := (+1,-1,0,0|0,0,0,0|\cdots)^{\mathsf{T}}, \tag{4.17}$$

with the potential to gap out the pair of Kramers degenerate helical edge modes on wire $i = 1$ since it fulfills the Haldane criterion (3.7), is not allowed by TRS.[54] On the one hand, $\mathcal{T}_{\mathrm{eff}}$ maps to itself under reversal of time,

$$\mathcal{T}_{\mathrm{eff}} = -\mathcal{P}_{\Theta}\,\mathcal{T}_{\mathrm{eff}}. \tag{4.18}$$

On the other hand,

$$\mathcal{T}^{\mathsf{T}}_{\mathrm{eff}}\,\mathcal{P}_{\Theta}\,\mathcal{I}_{\Theta} = -1. \tag{4.19}$$

Therefore, the condition (3.3e) for $\mathcal{T}_{\mathrm{eff}}$ to be a TRS perturbation is not met, for the phase $\alpha_{\mathcal{T}_{\mathrm{eff}}}(x)$ associated to $\mathcal{T}_{\mathrm{eff}}$ then obeys

$$\alpha_{\mathcal{T}_{\mathrm{eff}}}(x) = \alpha_{\mathcal{T}_{\mathrm{eff}}}(x) - \pi, \tag{4.20}$$

a condition that cannot be satisfied.

Had we imposed a TRS with $\Theta = +1$ instead of $\Theta = -1$ as is suited for the symmetry class AI that describes spinless fermions with TRS, we would only need to replace I_{Θ} in Eq. (4.13b) by the null vector. If so, the scattering vector (4.17) is compatible with TRS since the condition (3.3e) for TRS then becomes

$$\alpha_{\mathcal{T}_{\mathrm{eff}}}(x) = \alpha_{\mathcal{T}_{\mathrm{eff}}}(x) \tag{4.21}$$

instead of Eq. (4.20). This is the reason why symmetry class AI is always topologically trivial in two-dimensional space from the point of view of the wire construction.

Note also that if we would not insist on the condition of charge neutrality (2.5f), the tunneling vector

$$\mathcal{T}'_{\mathrm{eff}} := (+1,+1,0,0|0,0,0,0|\cdots)^{\mathsf{T}}, \tag{4.22}$$

that satisfies the Haldane criterion and is compatible with TRS could gap out the Kramers degenerate pair of helical edge states.

To address the question of what happens if we change $M = 4$ to $M = 4n$ with n any strictly positive integer in each wire from the array, we consider, without loss of generality as we shall see, the case of $n = 2$. To this end, it suffices to repeat all the steps that lead to Eq. (4.17), except for the change

$$\mathcal{K}_{\mathrm{eff}} := \mathrm{diag}\,(+1,-1,0,0;+1,-1,0,0|0,0,0,0;0,0,0,0|\cdots|0,0,0,0;0,0,0,0|0,0,-1,+1;0,0,-1,+1). \tag{4.23}$$

One verifies that the scattering vectors

$$\mathcal{T}'_{\mathrm{eff}} := (+1,0,0,0;0,-1,0,0|0,0,0,0;0,0,0,0|\cdots)^{\mathsf{T}} \tag{4.24}$$

and

$$\mathcal{T}''_{\mathrm{eff}} := (0,-1,0,0;+1,0,0,0|0,0,0,0;0,0,0,0|\cdots)^{\mathsf{T}} \tag{4.25}$$

are compatible with the condition that TRS holds in that the pair is a closed set under reversal of time,

$$\mathcal{T}'_{\mathrm{eff}} = -\mathcal{P}_{\Theta}\,\mathcal{T}''_{\mathrm{eff}}. \tag{4.26}$$

One verifies that these scattering vectors fulfill the Haldane criterion (3.7). Consequently, inclusion in $\hat{H}_{\{\mathcal{T}_{\mathrm{SO}}\}}$ of the two cosine potentials with $\mathcal{T}'_{\mathrm{eff}}$ and $\mathcal{T}''_{\mathrm{eff}}$ entering in their arguments, respectively, gaps out the pair of Kramers degenerate helical modes on wire $i=1$. The same treatment of the wire $i=N$ leads to the conclusion that TRS does not protect the gapless pairs of Kramers degenerate edge states from perturbations when $n=2$. The generalization to $M=4n$ channels is that it is only when n is odd that a pair of Kramers degenerate helical edge modes is robust to the most generic $\hat{H}_{\{\mathcal{T}_{\mathrm{SO}}\}}$ of the form depicted in the fourth column on line 3 of Table I. Since it is the parity of n in the number $M=4n$ of channels per wire that matters for the stability of the Kramers degenerate helical edge states, we use the group of two integers $\mathbb{Z}_2$ under addition modulo 2 in the third column on line 3 of Table I.

If we were to impose conservation of the projection of the spin-1/2 quantum number on the quantization axis, we must then preclude from all scattering vectors processes by which a spin is flipped. In particular, the scattering vectors (4.24) and (4.25) are not admissible anymore. By imposing the $U(1)$ residual symmetry of the full $SU(2)$ symmetry group for a spin-1/2 degree of freedom, we recover the group of integers $\mathbb{Z}$ under the addition that encodes the topological stability in the quantum spin Hall effect (QSHE).

We close the discussion of the symmetry class AII by justifying the interpretation of the index SO as an abbreviation for the intrinsic spin-orbit coupling. To this end, we introduce a set of $(N-1)$ pairs of scattering vectors

$$\mathcal{T}^{(j)}_{\mathrm{R}} := (0,0,0,0|\cdots|0,+1,0,0|-1,0,0,0|\cdots|0,0,0,0)^{\mathsf{T}} \tag{4.27a}$$

and

$$\overline{\mathcal{T}}^{(j)}_{\mathrm{R}} := -\mathcal{P}_{\Theta}\,\mathcal{T}^{(j)}_{\mathrm{R}} \tag{4.27b}$$

for $j=1,\cdots,N-1$. The scattering vector (4.27a) labels a one-body interaction in the fermion representation that fulfills Eq. (2.5f). The index R is an acronym for Rashba as it describes a backward scattering process by which a left mover with spin down from wire j is scattered into a right mover with spin up on wire $j+1$ and conversely. For any $j=1,\cdots,(N-1)$, we also introduce the pair of amplitudes

$$h_{\mathcal{T}^{(j)}_{\mathrm{R}}}(x) = h_{\overline{\mathcal{T}}^{(j)}_{\mathrm{R}}}(x) \geq 0 \tag{4.27c}$$

and the pair of phases

$$\alpha_{\mathcal{T}^{(j)}_{\mathrm{R}}}(x) = \alpha_{\overline{\mathcal{T}}^{(j)}_{\mathrm{R}}}(x) + \pi \in \mathbb{R} \tag{4.27d}$$

according to Eqs. (3.3d) and (3.3e), respectively. In contrast to the intrinsic spin-orbit scattering vectors, the Rashba scattering vectors (4.27a) fail to meet the Haldane criterion (3.7) as

$$\mathcal{T}^{(j)\mathsf{T}}_{\mathrm{R}}\,\mathcal{K}\,\overline{\mathcal{T}}^{(j+1)}_{\mathrm{R}} = -1, \qquad j=1,\cdots,N-1. \tag{4.28}$$

Hence, the Rashba scattering processes fail to open a gap in the bulk, as is expected of a Rashba coupling in a two-dimensional electron gas. On the other hand, the intrinsic spin-orbit coupling can lead to a phase with a gap in the bulk that supports the spin quantum Hall effect in a two-dimensional electron gas.

C. Symmetry class D

The simplest example among the topological superconductors can be found in the symmetry class D that is defined by the presence of a PHS with $\Pi^2=+1$ and the absence of TRS.

With the understanding of PHS as discussed in Sec. III B, we construct a representative phase in class D from identical wires supporting right- and left-moving spinless fermions each of which carry a particle or a hole label, i.e., $M=4$. The K-matrix reads

$$K := \mathrm{diag}(+1,-1,-1,+1). \tag{4.29a}$$

The entries in the K-matrix represent, from left to right, a right-moving particle, a left-moving particle, a left-moving hole, and a right-moving hole. The operation for the exchange of particles and holes in any one of the N wires is represented by [one verifies that Eq. (3.2f) holds]

$$P_{\Pi} := \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}, \qquad I_{\Pi} := \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}. \tag{4.29b}$$

We define $\hat{H}_{\mathcal{V}}$ by choosing any symmetric 4×4 matrix V that obeys

$$V = +P_{\Pi}\,V\,P^{-1}_{\Pi}. \tag{4.29c}$$

We define $\hat{H}_{\{\mathcal{T}\}}$ by choosing $2N-1$ scattering vectors as follows. For any wire $j=1,\cdots,N$, we introduce the scattering vector

$$\mathcal{T}^{(j)} := (0,0,0,0|\cdots|+1,-1,-1,+1|\cdots|0,0,0,0)^{\mathsf{T}}. \tag{4.30a}$$

Between any pair of neighboring wires we introduce the scattering vector

$$\overline{\mathcal{T}}^{(j)} := (0,0,0,0|\cdots|0,+1,-1,0|-1,0,0,+1|\cdots|0,0,0,0)^{\mathsf{T}}, \tag{4.30b}$$

for $j=1,\cdots,(N-1)$. We observe that both $\mathcal{T}^{(j)}$ and $\overline{\mathcal{T}}^{(j)}$ are eigenvectors of the particle-hole transformation in that

$$\mathcal{P}_{\Pi}\,\mathcal{T}^{(j)} = +\mathcal{T}^{(j)}, \qquad \mathcal{P}_{\Pi}\,\overline{\mathcal{T}}^{(j)} = -\overline{\mathcal{T}}^{(j)}. \tag{4.30c}$$

Thus, to comply with PHS, we have to demand that the phases

$$\alpha_{\overline{\mathcal{T}}^{(j)}}(x) = 0, \tag{4.30d}$$

while $\alpha_{\mathcal{T}^{(j)}}(x)$ are unrestricted. Similarly, the amplitudes $h_{\mathcal{T}^{(j)}}(x)$ and $h_{\overline{\mathcal{T}}^{(j)}}(x)$ can take arbitrary real values.

One verifies that the set of scattering vectors defined by Eqs. (4.30a) and (4.30b) satisfies the Haldane criterion. Correspondingly, the term $\hat{H}_{\{\mathcal{T}\}}$ gaps out $(4N-2)$ of the $4N$ gapless modes of $\hat{H}_{\mathcal{V}}$. Furthermore, one identifies with

$$\overline{\mathcal{T}}^{(0)} = (-1,0,0,+1|0,0,0,0|\cdots|0,0,0,0|0,+1,-1,0)^{\mathsf{T}} \tag{4.31}$$

a unique (up to an integer multiplicative factor) scattering vector that satisfies the Haldane criterion with all existing scattering vectors Eqs. (4.30a) and (4.30b) and could thus potentially gap out the remaining pair of modes. However, the tunneling $\overline{\mathcal{T}}^{(0)}$ is non-local for it connects the two edges of the system when open boundary conditions are chosen. We thus conclude that the two remaining modes are exponentially localized near wire $i=1$ and wire $i=N$, respectively, and propagate with opposite chirality.

To give a physical interpretation of the resulting topological (edge) theory in this wire construction, one has to keep in mind that the degrees of freedom were artificially doubled. We found, in this doubled theory, a single chiral boson (with chiral central charge $c=1$). To interpret it as the edge of a chiral $(p_x+\mathrm{i}p_y)$ superconductor, we impose the reality condition to obtain a single chiral Majorana mode with chiral central charge $c=1/2$.

The pictorial representation of the topological phase in the symmetry class D through the wire construction is shown on the fifth row of Table I. The generalization to an arbitrary number n of gapless chiral edge modes is analogous to the case discussed in symmetry class A. The number of robust gapless chiral edge states of a given chirality is thus integer. This is the reason why the group of integers $\mathbb{Z}$ is found in the third column on the fifth line of Table I.

D. Symmetry classes DIII and C

The remaining two topological nontrivial superconducting classes DIII (TRS with $\Theta^2=-1$ and PHS with $\Pi^2=+1$) and C (PHS with $\Pi^2=-1$) involve spin-1/2 fermions. Each wire thus features no less than $M=8$ internal degrees of freedom corresponding to the spin-1/2, chirality, and particle/hole indices. The construction is very similar to the cases we already presented. We relegate details to the appendices B and C.

The scattering vectors that are needed to gap out the bulk for each class of class DIII and C are represented pictorially in the fourth column on lines 4 and 9 of Table I.

E. Summary

We have provided an explicit construction by way of an array of wires supporting fermions that realizes all five insulating and superconducting topological phases of matter with a nondegenerate ground state in two-dimensional space according to the tenfold classification of band insulators and superconductors. The topological protection of edge modes in the bosonic formulation follows from imposing the Haldane criterion (3.7) along with the appropriate symmetry constraints. In the next section we shall extend the wire construction to allow many-body tunneling processes that delivers fractionalized phases with degenerate ground states.

V. FRACTIONALIZED PHASES

The power of the wire construction goes much beyond what we have used in Sec. IV to reproduce the classification of the SRE phases. In this section we describe how to construct models for interacting phases of matter with intrinsic topological order and fractionalized excitations by relaxing the condition on the tunnelings between wires that they be of the one-body type. While these phases are more complex, the principles for constructing the models and proving the stability of edge modes remain the same: All allowed tunneling vectors have to obey the Haldane criterion (3.7) and the respective symmetries.

A. Symmetry class A: Fractional quantum Hall states

First, we review the models of quantum wires that are topologically equivalent to the Laughlin state in the FQHE,[55] following the construction in Ref. 32 for Abelian fractional quantum Hall states. Here, we want to emphasize that the choice of scattering vectors is determined by the Haldane criterion (3.7) and at the same time prepare the grounds for the construction of fractional topological insulators with TRS in Sec. V B.

We want to construct the fermionic Laughlin series of states indexed by the positive odd integer m.[55] (By the same method, other fractional quantum Hall phases from the Abelian hierarchy could be constructed.[32]) The elementary degrees of freedom in each wire are spinless right- and left-moving fermions with the K-matrix

$$K = \mathrm{diag}\,(+1,-1), \tag{5.1a}$$

as is done in Eq. (4.1a). Reversal of time is defined through P_Θ and I_Θ given in Eq (4.1b). Instead of Eq (4.2), the scattering vectors that describe the interactions between the wires are now defined by

$$\mathcal{T}^{(j)} := \left(0,0\,\middle|\cdots\middle|\,m_+,-m_-\,\middle|\,m_-,-m_+\,\middle|\cdots\middle|\,0,0\right)^{\mathsf{T}}, \tag{5.1b}$$

for any $j = 1, \cdots, N-1$, where $m_{\pm} = (m \pm 1)/2$ [see Table I for an illustration of the scattering process].

For any $j = 1, \cdots, N-1$, the scattering (tunneling) vectors (5.1b) preserve the conservation of the total fermion number in that they obey Eq. (2.5f), and they encode a tunneling interaction of order $q = m$, with q defined in Eq. (2.6). As a set, all tunneling interactions satisfy the Haldane criterion (3.7), for

$$\mathcal{T}^{(i)\mathsf{T}}\,\mathcal{K}\,\mathcal{T}^{(j)} = 0, \quad i,j = 1, \cdots, N-1. \tag{5.2}$$

We note that the choice of tunneling vector in Eq. (5.1b) is unique (up to an integer multiplicative factor) if one insists on charge conservation, compliance with the Haldane criterion (3.7), and only includes scattering between neighboring wires.

The bare counting of tunneling vectors shows that the wire model gaps out all but two modes. However, we still have to convince ourself that the remaining two modes (i) live on the edge, (ii) cannot be gapped out by other (local) scattering vectors and (iii) are made out of fractionalized quasiparticles.

To address (i) and (ii), we note that the remaining two modes can be gapped out by a unique (up to an integer multiplicative factor) charge-conserving scattering vector that satisfies the Haldane criterion (3.7) with all existing scatterings, namely

$$\mathcal{T}^{(0)} := \big(\, m_{-}, -m_{+} \big|\, 0, 0 \big| \cdots \big| 0, 0\, \big| m_{+}, -m_{-}\, \big)^{\mathsf{T}}. \tag{5.3}$$

Connecting the opposite ends of the array of wires through the tunneling $\mathcal{T}^{(0)}$ is not an admissible perturbation, for it violates locality in the two-dimensional thermodynamic limit $N \to \infty$. Had we chosen periodic boundary conditions corresponding to a cylinder geometry (i.e., a tube as in Fig. 1) by which the first and last wire are nearest neighbors, $\mathcal{T}^{(0)}$ would be admissible. Hence, the gapless nature of the remaining modes when open boundary conditions are chosen depends on the boundary conditions. These gapless modes have support near the boundary only and are topologically protected.

Applying the transformation (4.6) with

$$W := \begin{pmatrix} -m_{-} & m_{+} \\ m_{+} & -m_{-} \end{pmatrix}, \tag{5.4}$$

where

$$\det W = -m, \tag{5.5}$$

transforms the K-matrix into

$$\widetilde{K} = \begin{pmatrix} -m & 0 \\ 0 & +m \end{pmatrix}. \tag{5.6}$$

As its determinant is not unity, the linear transformation (5.4) changes the compactification radius of the new field $\widetilde{\Phi}(x)$ relative to the compactification radius of the old field $\widehat{\Phi}(x)$ accordingly. Finally, the transformed tunneling vectors are given by

$$\widetilde{\mathcal{T}}^{(j)} = (0,0|\cdots|0,0|0,+1|-1,0|0,0|\cdots|0,0)^{\mathsf{T}}, \tag{5.7}$$

where $\mathcal{W} := \mathbb{1}_N \otimes W$ and $j = 1, \cdots, N-1$.

In view of Eqs. (4.6c) and (5.7), the remaining effective edge theory is described by

$$\widetilde{\mathcal{K}}_{\text{eff}} = \operatorname{diag}(-m, 0|0,0|\cdots|0,0|0,+m). \tag{5.8}$$

This is a chiral theory at each edge that cannot be gapped by local perturbations. Equation (5.8) is precisely the edge theory for anyons with statistical angle $1/m$ and charge e/m,[29] where e is the charge of the original fermions.

B. Symmetry Class AII: Fractional topological insulators

Having understood how fractionalized quasiparticles emerge out of a wire construction, it is imperative to ask what other phases can be obtained when symmetries are imposed on the topologically ordered phase. Such symmetry enriched topological phases have been classified by methods of group cohomology.[27] Here, we shall exemplify for the case of TRS with $\Theta^2 = -1$ how the wire construction can be used to build up an intuition for these phases and to study the stability of their edge theory.

The elementary degrees of freedom in each wire are spin-1/2 right- and left-moving fermions with the K-matrix

$$K := \operatorname{diag}(+1, -1, -1, +1), \tag{5.9a}$$

as is done in Eq. (4.13a). Reversal of time is defined through P_Θ and I_Θ given in Eq (4.13b). Instead of Eq (4.14a), the scattering vectors that describe the interactions between the wires are now defined by

$$\mathcal{T}^{(j)} := \big(0,0,0,0\,\big|\cdots\big|-m_{-},0,+m_{+},0\,\big|-m_{+},\ 0,+m_{-},0\big|\cdots\big|\,0,0,0,0\big)^{\mathsf{T}} \tag{5.9b}$$

and

$$\overline{\mathcal{T}}^{(j)} := -\mathcal{P}_\Theta\,\mathcal{T}^{(j)}, \tag{5.9c}$$

for any $j = 1, \cdots, N-1$, m a positive odd integer, and $m_\pm = (m \pm 1)/2$.

For any $j = 1, \cdots, N-1$, the scattering (tunneling) vectors (5.9b) preserve conservation of the total fermion number in that they obey Eq. (2.5f), and they encode a tunneling interaction of order $q = m$ with q defined in Eq. (2.6). They also satisfy the Haldane criterion (3.7) as a set [see Table I for an illustration of the scattering process].

Applying the transformation (4.6) with

$$W := \begin{pmatrix} -m_- & 0 & m_+ & 0 \\ 0 & -m_- & 0 & m_+ \\ m_+ & 0 & -m_- & 0 \\ 0 & m_+ & 0 & -m_- \end{pmatrix}, \tag{5.10}$$

to the bosonic fields, leaves the representation of time-reversal invariant

$$W^{-1}\, P_\Theta\, W = P_\Theta, \tag{5.11}$$

while casting the theory in a new form with the transformed $\widetilde{K}$-matrix given by

$$\widetilde{K} = \operatorname{diag}(-m, +m, +m, -m), \tag{5.12}$$

and, for any $j = 1, \cdots, N-1$, with the transformed pair of scattering vectors $(\widetilde{\mathcal{T}}^j, \widetilde{\overline{\mathcal{T}}}^j)$ given by

$$\widetilde{\mathcal{T}}^{(j)} = (0,0,0,0|\cdots|+1,0,0,0|0,0,-1,0|\cdots|0,0,0,0)^{\mathsf{T}} \tag{5.13}$$

and

$$\widetilde{\overline{\mathcal{T}}}^{(j)} = (0,0,0,0|\cdots|0,-1,0,0|0,0,0,+1|\cdots|0,0,0,0)^{\mathsf{T}}. \tag{5.14}$$

When these scattering vectors have gapped out all modes in the bulk, the effective edge theory is described by

$$\widetilde{\mathcal{K}}_{\text{eff}} = \operatorname{diag}(0,0,+m,-m|0,0,0,0|\cdots|0,0,0,0|-m,+m,0,0). \tag{5.15}$$

This effective K-matrix describes a single Kramers degenerate pair of $1/m$ anyons propagating along the first wire and another single Kramers degenerate pair of $1/m$ anyons propagating along the last wire. Their robustness to local perturbations is guaranteed by TRS.

Unlike in the tenfold way, the correspondence between the bulk topological phase and the edge theories of LRE phases is not one-to-one. For example, while a bulk topological LRE phase supports fractionalized topological excitations in the bulk, its edge modes may be gapped out by symmetry-allowed perturbations. For the phases discussed in this section, namely the Abelian and TRS fractional topological insulators, it was shown in Refs. 45 and 56 that the edge, consisting of Kramers degenerate pairs of edge modes, supports at most one stable Kramers degenerate pair of delocalized quasiparticles that are stable against disorder. (Note that this does not preclude the richer edge physics of non-Abelian TRS fractional topological insulators.[57])

We will now argue that the wire constructions with edge modes given by Eq. (5.15) exhaust all stable edge theories of Abelian topological phases which are protected by TRS with $\Theta^2 = -1$ alone.

Let the single protected Kramers degenerate pair be characterized by the linear combination of bosonic fields

$$\hat{\varphi}(x) := \mathcal{T}^{\mathsf{T}}\, \mathcal{K}'\, \widehat{\Phi}(x) \tag{5.16}$$

and its time-reversed partner

$$\hat{\bar{\varphi}}(x) := \overline{\mathcal{T}}^{\mathsf{T}}\, \mathcal{K}'\, \widehat{\Phi}(x), \tag{5.17}$$

where the tunneling vector $\mathcal{T}$ was constructed from the microscopic information from the theory in Ref. 45 and $\mathcal{K}'$ is the K-matrix of a TRS bulk Chern-Simons theory from the theory in Ref. 45. [In other words, the theory encoded by $\mathcal{K}'$ has nothing to do a priori with the array of quantum wires defined by Eq. (5.9).] The Kramers degenerate pair of modes $(\hat{\varphi}, \hat{\bar{\varphi}})$ is stable against TRS perturbations supported on a single edge if and only if

$$\frac{1}{2}|\mathcal{T}^{\mathsf{T}}\, \mathcal{Q}| \tag{5.18}$$

is an odd number. Here, $\mathcal{Q}$ is the charge vector with integer entries that determines the coupling of the different modes to the electromagnetic field. Provided $(\hat{\varphi}, \hat{\bar{\varphi}})$ is stable, its equal-time commutation relations follow from Eq. (2.4b) as

$$[\hat{\varphi}(x), \hat{\varphi}(x')] = -\,\mathrm{i}\pi\, \left(\mathcal{T}^{\mathsf{T}}\, \mathcal{K}'\, \mathcal{T}\, \operatorname{sgn}(x-x') + \mathcal{T}^{\mathsf{T}}\, \mathcal{L}\, \mathcal{T}\right), \tag{5.19a}$$

$$\left[\hat{\bar{\varphi}}(x), \hat{\bar{\varphi}}(x')\right] = -\,\mathrm{i}\pi\, \left(-\mathcal{T}^{\mathsf{T}}\, \mathcal{K}'\, \mathcal{T}\, \operatorname{sgn}(x-x') + \overline{\mathcal{T}}^{\mathsf{T}}\, \mathcal{L}\, \overline{\mathcal{T}}\right), \tag{5.19b}$$

where we used that $\mathcal{K}'$ anticommutes with $\mathcal{P}_\Theta$ according to Eq. (3.3c). By the same token, we can show that the fields $\hat{\varphi}$ and $\hat{\bar{\varphi}}$ commute, for

$$\mathcal{T}^{\mathsf{T}}\, \mathcal{K}'\, \overline{\mathcal{T}} = \mathcal{T}^{\mathsf{T}}\, \mathcal{P}_\Theta\, \mathcal{K}'\, \mathcal{T} = -\overline{\mathcal{T}}^{\mathsf{T}}\, \mathcal{K}'\, \mathcal{T} = 0. \tag{5.20}$$

We conclude that the effective edge theory for *any* Abelian TRS fractional topological insulator build from fermions has the effective form of one Kramers degenerate pairs

$$\mathcal{K}_{\text{eff}} = \begin{pmatrix} \mathcal{T}^{\mathsf{T}}\mathcal{K}'\mathcal{T} & 0 \\ 0 & -\mathcal{T}^{\mathsf{T}}\mathcal{K}'\mathcal{T} \end{pmatrix}, \tag{5.21}$$

and is thus entirely defined by the single integer

$$m := \mathcal{T}^{\mathsf{T}}\mathcal{K}'\mathcal{T}. \tag{5.22}$$

With the scattering vectors (5.9c) we have given an explicit wire construction for each of these cases, thus exhausting all possible stable edge theories for Abelian fractional topological insulators.

For each positive odd integer m, we can thus say that the fractionalized mode has a $\mathbb{Z}_2$ character: It can have either one or none stable Kramers degenerate pair of m quasiparticles.

C. Symmetry Class D: Fractional superconductors

In Sec. V B we have imposed TRS on the wire construction of fractional quantum Hall states and obtained the fractional topological insulator in symmetry class AII. In complete analogy, we can impose PHS with $\Pi^2 = +1$ on the wire construction of a fractional quantum Hall state, thereby promoting it to symmetry class D. Physically, there follows a model for a superconductor with "fractionalized" Majorana fermions or Bogoliubov quasiparticles.

Lately, interest in this direction has been revived by the investigation of exotic quantum dimensions of twist defects embedded in an Abelian fractional quantum Hall liquid,[58–60] along with heterostructures of superconductors combined with fractional quantum Hall effect,[61–63] or fractional topological insulators.[64] Furthermore, the Kitaev quantum wire has been generalized to $\mathbb{Z}_n$ clock models hosting parafermionic edge modes,[65,66] along with efforts to transcend the Read-Rezayi quantum Hall state[67] to spin liquids[68,69] and superconductors,[37] all of which exhibit parafermionic quasiparticles.

As in the classification of non-interacting insulators, we treat the Bogoliubov quasiparticles under bosonization as if they were Dirac fermions. The fractional phase is driven by interactions among the Bogoliubov quasiparticles.

The elementary degrees of freedom in each wire are spinless right- and left-moving fermions and holes as was defined for symmetry class D in Eqs. (4.29a)-(4.29c). We construct the fractional topological insulator using the set of PHS scattering vectors $\mathcal{T}^{(j)}$, for $j = 1, \cdots, N$ with $\mathcal{T}^{(j)}$ as defined in Eq. (4.30a) in each wire and the PHS as defined in Eq. (4.29b). We complement them with the set of PHS scattering vectors $\overline{\mathcal{T}}^{(j)}$, for $j = 1, \cdots, N-1$ defined by

$$\overline{\mathcal{T}}^{(j)} = \big(0,0,0,0\,\big|\cdots\big|\,{-m_-},m_+,-m_+,m_-\big|-m_+,m_-,-m_-,m_+\big|\cdots\big|\,0,0,0,0\big)^{\mathsf{T}}, \qquad m_\pm = (m\pm 1)/2, \tag{5.23}$$

with m an odd positive integer. Notice that $\overline{\mathcal{T}}^{(j)} := -\mathcal{P}_{\Pi}\,\overline{\mathcal{T}}^{(j)}$ so that we have to demand that $\alpha_{\overline{\mathcal{T}}^{(j)}} = 0$ has to comply with PHS. Thus, together the $\mathcal{T}^{(j)}$ and $\overline{\mathcal{T}}^{(j)}$ gap out $(4N-2)$ of the $4N$ chiral modes in the wire. We can identify a unique (up to an integer multiplicative factor) scattering vector

$$\overline{\mathcal{T}}^{(0)} = \big(\,{-m_+},m_-,-m_-,m_+\big|\,0,0,0,0\big|\cdots\big|0,0,0,0\,\big|{-m_-},m_+,-m_+,m_-\,\big)^{\mathsf{T}}, \qquad m_\pm = (m\pm 1)/2, \tag{5.24}$$

with m the ame odd positive integer as in Eq. (5.23) that satisfies the Haldane criterion with all $\mathcal{T}^{(j)}$ and $\overline{\mathcal{T}}^{(j)}$ and thus can potentially gap out the 2 remaining modes. However, it is physically forbidden for it represents a non-local scattering from one edge to the other. We conclude that each boundary supports a single remaining chiral mode that is an eigenstate of PHS.

To understand the nature of the single remaining chiral mode on each boundary, we use the local linear transformation W of the bosonic fields

$$W = \begin{pmatrix} -m_- & +m_+ & 0 & 0 \\ +m_+ & -m_- & 0 & 0 \\ 0 & 0 & -m_- & +m_+ \\ 0 & 0 & +m_+ & -m_- \end{pmatrix}, \quad m_\pm = \frac{m\pm 1}{2}, \tag{5.25}$$

with determinant $\det W = m^4$. When applied to the non-local scattering vector $\overline{\mathcal{T}}^{(0)}$ that connects the two remaining chiral edge modes,

$$\begin{aligned} \widetilde{\overline{\mathcal{T}}}^{(0)} &= \mathcal{W}^{-1}\,\overline{\mathcal{T}}^{(0)} \\ &= (0,-1,+1,0|0,0,0,0|\cdots|0,0,0,0|+1,0,0,-1), \end{aligned} \tag{5.26}$$

while the K matrix changes under this transformation to

$$\widetilde{K} = \operatorname{diag}(-m,m,m,-m). \tag{5.27}$$

Noting that the representation of PHS is unchanged

$$W^{-1}\,P_{\Pi}\,W = P_{\Pi}, \tag{5.28}$$

we can interpret the remaining chiral edge mode as a PHS superposition of a Laughlin quasiparticle and a Laughlin quasihole. It thus describes a fractional chiral edge mode on either side of the two-dimensional array of quantum wires. The definite chirality is an important difference to the case of the fractional $\mathbb{Z}_2$ topological insulator discussed in Sec. V B. It guarantees that any integer number $n \in \mathbb{Z}$ layers of this theory is stable, for no tunneling vector that acts locally on one edge can satisfy the Haldane criterion (3.7). For each m, we can thus say that the parafermion mode has a $\mathbb{Z}$ character, as does the SRE phase in symmetry class D.

D. Symmetry classes DIII and C: More fractional superconductors

Needed are the many-body tunneling matrices for class DIII and C. We refer the reader to the appendices B and C for their definitions. For class DIII, the edge excitations (and bulk quasiparticles) of the phase are TRS fractionalized Bogoliubov quasiparticles that have also been discussed in one-dimensional realizations. (In the latter context, these TRS fractionalized Bogoliubov quasiparticles are rather susceptible to perturbations.[70,71])

VI. DISCUSSION

In this work, we have developed a wire construction to build models of short-range entangled and long-range entangled topological phases in two spatial dimensions, so as to yield immediate information about the topological stability of their edge modes. As such, we have promoted the periodic table of integer topological phases to its fractional counterpart. The following paradigms were applied.

(1) Each Luttinger liquid wire describes (spinfull or spinless) electrons. We rely on a bosonized description.

(2) Back-scattering and short-range interactions within and between wires are added. Modes are gapped out if these terms acquire a finite expectation value.

(3) A mutual compatibility condition, the Haldane criterion, is imposed among the terms that acquire an expectation value. It is an incarnation of the statement that the operators have to commute if they are to be replaced simultaneously by their expectation values.

(4) A set of discrete and local symmetries are imposed on all terms in the Hamiltonian. When modes become massive, they may not break these symmetries.

(5) We do not study the renormalization group flow of the interaction and back-scattering terms, but analyze the model in a strong-coupling limit.

It has become fashionable to write papers in condensed matter physics that take Majorana fermions as the building blocks of lattice models. Elegant mathematical results have been obtained in this way, some of which having the added merit for bringing conceptual clarity. However, the elementary building blocks of condensed matter are ions and electrons whose interactions are governed by quantum electrodynamics. Majorana fermions in condensed matter physics can only emerge in a nonperturbative way through (i) the interactions between the electrons from the valence bands of a material, or (ii) as the low-energy excitations of exotic quantum magnets. For Majorana fermions to be observable in condensed matter physics, a deconfining transition must have taken place, a notoriously non-perturbative phonomenon. One of the challenges that we have undertaken in this paper is to find interacting models for itinerant electrons with local interactions that support Majorana fermions at low energies and long wave lengths. We achieved this goal, starting from non-interacting itinerant electrons, by constructing local many-body interactions that conserve the electron charge and that stabilize two-dimensional bulk superconductors supporting gapless Majorana fermions along their two-dimensional boundaries. This is why strictly many-body interactions are needed in the symmetry classes D, DIII, and C to realize SRE topological phases in the fourth column of Table I.

Using this strategy, the following directions present themselves for future work.

First, for symmetry class A, we have shown that sufficiently strong interactions among identical electrons can turn any topological phase with the same topological number controlling both the Hall and thermal conductivities into a SRE topological phase with independent quantized values of the Hall and thermal conductivities. [We only need to make the interaction encoded by Eq. (4.10) dominant.] Hence, it is natural to seek a putative breakdown of the topological counterpart to the Wiedemann-Franz law for metals in the symmetry class AII and for the LRE phases in the symmetry classes A and AII.

Second, we can impose on our wire construction additional, albeit less generic, symmetries such as a non-local inversion symmetry or such as a residual $U(1)$ spin symmetry.

Third, our construction can be extended to topological phases of systems that have bosons as their elementary degrees of freedom. For bosons, no analogue of the tenfold way exists to provide guidance. However, several works are dedicated to the classification of SRE and LRE phases of bosons, which might provide a helpful starting point.[26]

Fourth, extensions to higher dimensions could be considered.[72,73] This would, however, entail leaving the comfort zone of one-dimensional bosonization, with a necessary generalization of the Haldane criterion in a layer construction.

ACKNOWLEDGMENTS

We are grateful to Franz Wegner for a stimulating exchange that lead to the discovery of the transformation matrix (4.7). This work was supported by the European Research Council through the grant TOPOLECTRICS, ERC-StG-Thomale-336012, by DARPA SPAWARSYSCEN Pacific N66001-11-1-4110 (T.N.), and by DOE Grant DEF-06ER46316 (C.C.). CM and TN thank the Condensed Matter Theory Visitors Program at Boston University for support.

Appendix A: Conditions for particle-hole and time-reversal symmetry

The conditions (3.3) and (3.4) for TRS and PHS can be derived by adapting the derivations

$$\begin{aligned}\widehat{\Theta}\,\hat{H}_{\{\mathcal{T}\}}\,\widehat{\Theta}^{-1} &= \int \mathrm{d}x \sum_{\mathcal{T}} h_{\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}(\mathcal{P}_{\Theta}\,\widehat{\Phi} + \pi\,\mathcal{K}^{-1}\,\mathcal{I}_{\Theta}) + \alpha_{\mathcal{T}}\right)\\ &= \int \mathrm{d}x \sum_{\mathcal{T}} h_{-\mathcal{P}_{\Theta}\,\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi} - \pi\,\mathcal{T}^{\mathsf{T}}\,\mathcal{P}_{\Theta}\,\mathcal{I}_{\Theta} + \alpha_{-\mathcal{P}_{\Theta}\,\mathcal{T}}\right)\\ &\stackrel{!}{=} \int \mathrm{d}x \sum_{\mathcal{T}} h_{\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi} + \alpha_{\mathcal{T}}\right)\end{aligned} \tag{A1a}$$

and

$$\begin{aligned}\widehat{\Pi}\,\hat{H}_{\{\mathcal{T}\}}\,\widehat{\Pi}^{-1} &= \int \mathrm{d}x \sum_{\mathcal{T}} h_{\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}(\mathcal{P}_{\Pi}\,\widehat{\Phi} + \pi\,\mathcal{K}^{-1}\,\mathcal{I}_{\Pi}) + \alpha_{\mathcal{T}}\right)\\ &= \int \mathrm{d}x \sum_{\mathcal{T}} h_{\mathcal{P}_{\Pi}\,\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi} + \pi\,\mathcal{T}^{\mathsf{T}}\,\mathcal{P}_{\Pi}\,\mathcal{I}_{\Pi} + \alpha_{\mathcal{P}_{\Pi}\,\mathcal{T}}\right)\\ &\stackrel{!}{=} \int \mathrm{d}x \sum_{\mathcal{T}} h_{\mathcal{T}} \cos\left(\mathcal{T}^{\mathsf{T}}\,\mathcal{K}\,\widehat{\Phi} + \alpha_{\mathcal{T}}\right)\end{aligned} \tag{A1b}$$

of Eqs. (3.3d) and (3.3e), respectively.

Appendix B: Symmetry class C

Class C is defined on line 9 of Table I by the operator $\widehat{\Pi}$ for the PHS obeying $\Pi^2 = -1$ with neither TRS nor chiral symmetry present (as is implied by the entries 0 for Θ^2 and C^2 in Table I). In physical terms, class C describes a generic superconductor for which full spin $SU(2)$ symmetry is retained but TRS is broken. The only difference to the case of class D considered in the main text is that the number of degrees of freedom is doubled. We postulate that under PHS the following transformation rules hold

$$b^{\dagger}_{\uparrow,\mathrm{R}} \stackrel{\widehat{\Pi}}{\to} -b_{\downarrow,\mathrm{R}}, \qquad b^{\dagger}_{\downarrow,\mathrm{R}} \stackrel{\widehat{\Pi}}{\to} +b_{\uparrow,\mathrm{R}}, \tag{B1}$$

for the creation operators of Bogoliubov-deGennes quasiparticles that are right (R) movers at the Fermi energy and carry the spin quantum numbers $\uparrow,\downarrow$. We apply the same transformation law to the creation operators of Bogoliubov-deGennes quasiparticles that are left (L) movers at the Fermi energy and carry the spin quantum numbers $\uparrow,\downarrow$.

We consider identical wires with quasiparticles of type 1 and 2, "particles" and "holes", as well as left- and right-moving degrees of freedom. For any given wire with the basis ($b^{\dagger}_{\uparrow,\mathrm{L}}$, $b^{\dagger}_{\downarrow,\mathrm{L}}$, $b^{\dagger}_{\uparrow,\mathrm{R}}$, $b^{\dagger}_{\downarrow,\mathrm{R}}$, $b_{\uparrow,\mathrm{R}}$, $b_{\downarrow,\mathrm{R}}$, $b_{\uparrow,\mathrm{L}}$, $b_{\downarrow,\mathrm{L}}$), the K-matrix reads

$$K := \mathrm{diag}\,(+1,+1,-1,-1,-1,-1,+1,+1), \tag{B2a}$$

where PHS has the representation

$$P_{\Pi} := \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}, \qquad I_{\Pi} := \begin{pmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{pmatrix}. \tag{B2b}$$

1. SRE phase

To complete the definition of an array of quantum wires realizing a SRE phase in the symmetry class C, we specify the $(4N-2)$ scattering vectors

$$\mathcal{T}^{(j)}_{1,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|+1,0,-1,0,-1,0,+1,0|\cdots|0,0,0,0,0,0,0,0), \tag{B3a}$$

$$\mathcal{T}^{(j)}_{2,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,+1,0,-1,0,-1,0,+1|\cdots|0,0,0,0,0,0,0,0), \tag{B3b}$$

$$\mathcal{T}^{(l)}_{3,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,0,+1,0,-1,0,0,0|0,-1,0,0,0,0,0,+1|\cdots|0,0,0,0,0,0,0,0), \tag{B3c}$$

$$\mathcal{T}^{(l)}_{4,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,0,0,-1,0,+1,0,0|+1,0,0,0,0,0,-1,0|\cdots|0,0,0,0,0,0,0,0), \tag{B3d}$$

for $j=1,\cdots,N$, and $l=1,\cdots,N-1$. These scattering vectors gap out all modes in the bulk and comply both with PHS and with the Haldane criterion (3.7). Of the remaining four modes, two are localized at either edge of the system. The remaining two modes on either edge share the same chirality, for they could be gapped out by the non-local scattering vectors

$$\mathcal{T}^{(0)}_{3,\mathrm{SRE}} := (0,-1,0,0,0,0,0,+1|\cdots|0,0,+1,0,-1,0,0,0), \tag{B4a}$$

$$\mathcal{T}^{(0)}_{4,\mathrm{SRE}} := (+1,0,0,0,0,0,-1,0|\cdots|0,0,0,-1,0,+1,0,0), \tag{B4b}$$

which act on modes with $+$ chirality on the left and and of $-$ chirality on the right edge only. We conclude that the pair of chiral modes on either edge is protected from backscattering. Extending this construction to any integer number of layers yields the $\mathbb{Z}$ classification of class C.

2. LRE phase

To complete the definition of an array of quantum wires realizing a LRE phase in the symmetry class C, we use the $2N$ scattering vectors $\mathcal{T}^{(j)}_{1,\mathrm{SRE}}$ and $\mathcal{T}^{(j)}_{2,\mathrm{SRE}}$, $j=1,\cdots,N$ defined in Eq. (B3) and supplement them with the $2(N-1)$ scattering vectors

$$\mathcal{T}^{(j)}_{3,\mathrm{LRE}} := (0,0,0,0,0,0,0,0|\cdots|0,-m_-,m_+,0,-m_+,0,0,m_-|0,-m_+,m_-,0,-m_-,0,0,m_+|\cdots|0,0,0,0,0,0,0,0), \tag{B5a}$$

$$\mathcal{T}^{(j)}_{4,\mathrm{LRE}} := (0,0,0,0,0,0,0,0|\cdots|m_-,0,0,-m_+,0,m_+,-m_-,0|m_+,0,0,-m_-,0,m_-,-m_+,0|\cdots|0,0,0,0,0,0,0,0), \tag{B5b}$$

for $j=1,\cdots,N-1$, and $m_\pm=(m\pm1)/2$ as well as m an odd positive integer. These tunneling vectors gap out all modes in the bulk and comply both with PHS and with the Haldane criterion (3.7). One verifies that there exists a linear transformation with integer entires W and $|\det W|=m^8$ such that

$$\mathcal{T}^{(j)}_{l,\mathrm{SRE}} = \mathcal{W}^{-1}\,\mathcal{T}^{(j)}_{l,\mathrm{LRE}}, \qquad j=1,\cdots,N-1, \qquad l=3,4. \tag{B6}$$

The K-matrix transforms according to Eq. (4.6c), leaving the effective effective edge theory with two chiral modes of PHS symmetric superpositions of Laughlin quasiparticles with Laughlin quasiholes on either edge of the system. As with the SRE phase of symmetry class C, this is a completely chiral theory and no back-scattering mechanism can gap out modes by the Haldane criterion (3.7). Extending this construction to any integer number of layers yields a $\mathbb{Z}$ classification of the LRE phase in symmetry class C for every positive odd integer m.

Appendix C: Symmetry class DIII

Class DIII is defined on line 4 of Table I by the operator $\widehat{\Pi}$ for the PHS obeying $\Pi^2=+1$ and with the TRS $\Theta^2=-1$. In physical terms, class DIII describes a generic superconductor for which full spin $SU(2)$ symmetry is broken but TRS is retained. We use the same basis and K-matrix as specified for class C in Eq. (B2a), namely

$$K := \mathrm{diag}\,(+1,+1,-1,-1,-1,-1,+1,+1). \tag{C1a}$$

The PHS now has the representation

$$P_{\Pi} := \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \qquad I_{\Pi} := \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \tag{C1b}$$

while TRS is defined by

$$P_{\Theta} := \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}, \qquad I_{\Theta} := \begin{pmatrix} 0\\1\\0\\1\\0\\1\\0\\1 \end{pmatrix}. \tag{C1c}$$

1. SRE phase

To complete the definition of an array of quantum wires realizing a SRE phase in the symmetry class DIII, we specify the $(4N-2)$ tunneling vectors

$$\mathcal{T}^{(j)}_{1,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|+1,0,-1,0,-1,0,+1,0|\cdots|0,0,0,0,0,0,0,0), \tag{C2a}$$

$$\mathcal{T}^{(j)}_{2,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,+1,0,-1,0,-1,0,+1|\cdots|0,0,0,0,0,0,0,0), \tag{C2b}$$

$$\mathcal{T}^{(l)}_{3,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,0,+1,0,-1,0,0,0|-1,0,0,0,0,0,+1,0|\cdots|0,0,0,0,0,0,0,0), \tag{C2c}$$

$$\mathcal{T}^{(l)}_{4,\mathrm{SRE}} := (0,0,0,0,0,0,0,0|\cdots|0,-1,0,0,0,0,0,+1|0,0,0,+1,0,-1,0,0|\cdots|0,0,0,0,0,0,0,0), \tag{C2d}$$

for $j=1,\cdots,N$, and $l=1,\cdots,N-1$. These tunnelings gap out all the bulk modes. Here, $\mathcal{T}^{(j)}_{1,\mathrm{SRE}}$ and $\mathcal{T}^{(j)}_{2,\mathrm{SRE}}$ as well as $\mathcal{T}^{(j)}_{3,\mathrm{SRE}}$ and $\mathcal{T}^{(j)}_{4,\mathrm{SRE}}$ are pairwise related by TRS, while each of the tunneling vectors is in itself PHS. The phases of the corresponding cosine terms in Hamiltonian (2.5d) comply with both TRS and PHS as if

$$\alpha_{\mathcal{T}^{(j)}_{1,\mathrm{SRE}}} = \alpha_{\mathcal{T}^{(j)}_{2,\mathrm{SRE}}}, \qquad \alpha_{\mathcal{T}^{(j)}_{3,\mathrm{SRE}}} = \alpha_{\mathcal{T}^{(j)}_{4,\mathrm{SRE}}} = 0. \tag{C3}$$

On both wire $j=1$ and wire $j=N$, there remains a single Kramers degenerate pair of propagating modes. Now, the tunneling vector

$$\mathcal{T}^{\mathrm{L}} = (-1,0,0,+1,0,-1,+1,0|\cdots|0,0,0,0,0,0,0,0) \tag{C4}$$

acts locally on the left edge, satisfies the Haldane criterion with all existing scattering vectors, and is unique up to an integer multiplicative factor. It might thus be concluded that the left pair of Kramers degenerate edge states can be gapped by the tunneling $\mathcal{T}^{\mathrm{L}}$. This is not so however. Indeed, while $\mathcal{T}^{\mathrm{L}}$ itself is both compliant with PHS and TRS, its contribution to $\hat{H}_{\{\mathcal{T}\}}$ induces another expectation value that breaks TRS spontaneously. To see this, we note that, by the particle-hole redundancy, the bosonic fields $\hat{\phi}_1(x)$ and $-\hat{\phi}_7(x)$ as well as $\hat{\phi}_4(x)$ and $-\hat{\phi}_6(x)$ have to be identified. Thus, it is $\cos\left(T^{\mathrm{L}\,\mathsf{T}}\,K\,\hat{\Phi}(x)\right) \sim \cos\left(2\hat{\phi}_1(x)-2\hat{\phi}_4(x)\right)$ that acquires an expectation value. Now, the term $\cos\left(\hat{\phi}_1(x)-\hat{\phi}_4(x)\right)$ is more relevant from the point of view of the renormalization group than $\cos\left(2\hat{\phi}_1(x)-2\hat{\phi}_4(x)\right)$. If $\cos\left(2\hat{\phi}_1(x)-2\hat{\phi}_4(x)\right)$ acquires an expectation value, so does $\cos\left(\hat{\phi}_1(x)-\hat{\phi}_4(x)\right)$. However, $\cos\left(\hat{\phi}_1(x)-\hat{\phi}_4(x)\right)$ corresponds to

$$\bar{\mathcal{T}}^{\mathrm{L}} = (-1,0,0,+1,0,0,0,0|\cdots|0,0,0,0,0,0,0,0) \tag{C5}$$

(and scattering vectors related by PHS) must then break TRS spontaneously, for the resulting condition $\alpha_{\bar{\mathcal{T}}^{\mathrm{L}}} = \alpha_{-\mathcal{P}_{\Theta}\bar{\mathcal{T}}^{\mathrm{L}}} + \pi = \alpha_{\bar{\mathcal{T}}^{\mathrm{L}}} + \pi$ on the phase of its cosine term cannot be met. If we rule out the spontaneous breaking of TRS on the left edge, we must rule out the tunnelings $n\,\mathcal{T}^{\mathrm{L}}$ for any integer n. Under this assumption, there remains a single gapless left pair of Kramers degenerate edge states.

We conclude that there is no possibility to localize the remaining edge modes with perturbations that comply with both TRS and PHS. Had we considered two layers of this wire model, edge modes in both layers can be gapped out pairwise, similar to the case of class AII that we discussed in the main text. We conclude that the SRE phase of symmetry class DIII features a $\mathbb{Z}_2$ topological classification.

2. LRE phase

To complete the definition of an array of quantum wires realizing a LRE phase in the symmetry class D III, we use the $2N$ scattering vectors $\mathcal{T}^{(j)}_{1,\mathrm{SRE}}$ and $\mathcal{T}^{(j)}_{2,\mathrm{SRE}}$, $j = 1,\cdots,N$ defined in Eq. (C2) and supplement them with the $2(N-1)$ scattering vectors

$$\mathcal{T}^{(j)}_{3,\mathrm{LRE}} := (0,0,0,0,0,0,0,0|\cdots|-m_-,0,m_+,0,-m_+,0,m_-,0|-m_+,0,m_-,0,-m_-,0,m_+,0|\cdots|0,0,0,0,0,0,0,0), \tag{C6a}$$

$$\mathcal{T}^{(j)}_{4,\mathrm{LRE}} := (0,0,0,0,0,0,0,0|\cdots|0,-m_+,0,m_-,0,m_-,0,m_+|0,-m_-,0,m_+,0,-m_+,0,m_-|\cdots|0,0,0,0,0,0,0,0), \tag{C6b}$$

for $j = 1,\cdots,N-1$, and $m_{\pm} = (m \pm 1)/2$ as well as m an odd positive integer. These tunneling vectors gap out all modes in the bulk and comply both with PHS and with the Haldane criterion (3.7). One verifies that there exists a linear transformation with integer entires W and $|\det W| = m^8$ such that

$$\mathcal{T}^{(j)}_{l,\mathrm{SRE}} = \mathcal{W}^{-1}\,\mathcal{T}^{(j)}_{l,\mathrm{LRE}}, \qquad j = 1,\cdots,N-1, \qquad l = 3,4. \tag{C7}$$

The K-matrix transforms according to Eq. (4.6c), leaving the effective effective edge theory with one Kramers degenerate pair of PHS symmetric superpositions of Laughlin quasiparticles with Laughlin quasi-holes on either edge of the system. As with the SRE phase of symmetry class DIII, this edge theory is protected by PHS and TRS. Two copies of it, however, can be fully gapped out while preserving PHS and TRS. This yields a $\mathbb{Z}_2$ classification of the LRE phase in symmetry class DIII for every positive odd integer m.

[1] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).

[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).

[3] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).

[4] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science **318**, 766 (2007).

[5] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[6] R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).

[7] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[8] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).

[9] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).

[10] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).

[11] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics **12**, 065010 (2010).

[12] V. Gurarie, Phys. Rev. B **83**, 085426 (2011).

[13] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010).

[14] S. R. Manmana, A. M. Essin, R. M. Noack, and V. Gurarie, Phys. Rev. B **86**, 205119 (2012).

[15] Z. Wang and S.-C. Zhang, Phys. Rev. X **2**, 031008 (2012).

[16] S. Raghu, X.-L. Qi, C. Honerkamp, and S.-C. Zhang, Phys. Rev. Lett. **100**, 156401 (2008).

[17] J. C. Budich, R. Thomale, G. Li, M. Laubach, and S.-C. Zhang, Phys. Rev. B **86**, 201407 (2012).

[18] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).

[19] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).

[20] N. Schuch, D. Perez-Garcia, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).

[21] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).

[22] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **84**, 235128 (2011).

[23] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011).

[24] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011).
[25] Z.-C. Gu and X.-G. Wen, ArXiv e-prints (2012).
[26] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[27] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[28] Y.-M. Lu and A. Vishwanath, ArXiv e-prints (2013).
[29] X.-G. Wen, Int. J. Mod. Phys. **B5**, 1641 (1991).
[30] V. M. Yakovenko, Phys. Rev. B **43**, 11353 (1991).
[31] D.-H. Lee, Phys. Rev. B **50**, 10788 (1994).
[32] C. L. Kane, R. Mukhopadhyay, and T. C. Lubensky, Phys. Rev. Lett. **88**, 036401 (2002).
[33] J. C. Y. Teo and C. L. Kane, Phys. Rev. B **89**, 085101 (2014).
[34] S. L. Sondhi and K. Yang, Phys. Rev. B **63**, 054430 (2001).
[35] J. Klinovaja and D. Loss, Phys. Rev. Lett. **111**, 196401 (2013).
[36] J. Klinovaja and D. Loss, arxiv 1305.1569 .
[37] R. S. K. Mong *et al.*, Phys. Rev. X **4**, 011036 (2014).
[38] I. Seroussi, E. Berg, and Y. Oreg, (2014).
[39] A. Vaezi, Phys. Rev. X **4**, 031009 (2014).
[40] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).
[41] X. G. Wen, Phys. Rev. B **44**, 2664 (1991).
[42] S. Sachdev and N. Read, Int. J. Mod. Phys. B **05**, 219 (1991).
[43] C. Mudry and E. Fradkin, Phys. Rev. B **49**, 5200 (1994).
[44] A chiral symmetry is preseent if there exists a chiral operator $\widehat{C}$ that is antiunitary and commutes with the Hamiltonian. The single-particle representation C of $\widehat{C}$ is a unitary operator that anticommutes with the single-particle Hamiltonian. In a basis in which C is strictly block off diagonal, C reverses the chirality. This chirality is unrelated to the direction of propagation of left and right movers which is also called chirality in this paper.
[45] T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165107 (2011).
[46] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[47] F. J. Dyson, Journal of Mathematical Physics **3**, (1962).
[48] M. Jeng, A. W. W. Ludwig, T. Senthil, and C. Chamon, eprint arXiv:cond-mat/0112044 (2001).
[49] M. S. Foster and A. W. W. Ludwig, Phys. Rev. B **73**, 155104 (2006).
[50] M. S. Foster and A. W. W. Ludwig, Phys. Rev. B **77**, 165108 (2008).
[51] M. S. Foster, H.-Y. Xie, and Y.-Z. Chou, Phys. Rev. B **89**, 155140 (2014).
[52] G. Goldstein and C. Chamon, Phys. Rev. B **86**, 115122 (2012).
[53] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[54] Even integer multiples of $\mathcal{T}_{\text{eff}}$ would gap the edge states, but they must also be discarded as explained in Ref. 45.
[55] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[56] M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009).
[57] B. Scharfenberger, R. Thomale, and M. Greiter, Phys. Rev. B **84**, 140404 (2011).
[58] M. Barkeshli and X.-L. Qi, Phys. Rev. X **2**, 031013 (2012).
[59] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **87**, 045130 (2013).
[60] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 235103 (2013).
[61] A. Vaezi, Phys. Rev. B **87**, 035132 (2013).
[62] N. H. Lindner, E. Berg, G. Refael, and A. Stern, Phys. Rev. X **2**, 041002 (2012).
[63] D. J. Clarke, J. Alicea, and K. Shtengel, Nature Comm. **4**, 1348 (2013).
[64] M. Cheng, Phys. Rev. B **86**, 195126 (2012).
[65] A. Y. Kitaev, Phys.-Usp. **44**, 131 (2001).
[66] P. Fendley, J. Stat. Mech. P11020 (2012).
[67] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[68] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).
[69] M. Greiter, D. F. Schroeter, and R. Thomale, Phys. Rev. B **89**, 165125 (2014).
[70] Y. Oreg, E. Sela, and A. Stern, Phys. Rev. B **89**, 115402 (2014).
[71] J. Klinovaja and D. Loss, ArXiv e-prints (2013).
[72] C. Wang, A. C. Potter, and T. Senthil, Science **343**, 629 (2014).
[73] C. Wang and T. Senthil, Phys. Rev. B **89**, 195124 (2014).