Effective field theory for the bulk-edge correspondence in a two-dimensional Z_{2} topological insulator with Rashba interactions

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Effective field theory for the bulk-edge correspondence in a two-dimensional $\mathbb{Z}_2$ topological insulator with Rashba interactions

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We determine the effective field theory in $(2+1)$-dimensional space and time that it captures the long-wave-length and low-energy limit of fermions hopping on a honeycomb lattice at half-filling when both a dominant intrinsic and subdominant Rashba spin-orbit couplings are present. This effective field theory for a $\mathbb{Z}_2$ topological insulator (the Kane-Mele model at vanishing uniform and staggered chemical potentials) is a perturbation around a double Chern-Simons theory, with the $U(1)$ gauge invariance associated to spin conservation explicitly broken due to the Rashba spin orbit coupling. Nonetheless, we find that the effective field theory has a BRST symmetry that allows us to construct the bulk-edge correspondence.

I. INTRODUCTION

There has been a great amount of interest in the field of research opened up by the discovery of materials known as topological insulators.[1–3] Topological insulators represent a new quantum state of matter that is characterized by bulk properties like those of ordinary band insulators, but supporting protected conducting boundary states on their edges or surfaces. These states are possible due to a combination of spin-orbit interactions and time-reversal symmetry.

The Kane-Mele model introduced in Refs. 4 and 5 is an example of a band insulator in two-dimensional space for which time-reversal symmetry guarantees the stability of gapless edge states that are perfectly conducting along any boundary. The Kane-Mele model is a tight-binding representation for electrons in graphene in the presence of an intrinsic spin-orbit coupling and of a Rashba spin-orbit coupling. Even though the magnitudes of the spin-orbit couplings in graphene are too small to lead to observable effects with the present experimental resolution in energy and temperature, the Kane-Mele model aroused considerable interest and led to the predictions and discoveries of $\mathbb{Z}_2$ topological insulators both in two- and three-dimensional space (see Refs. 1, 2, and 3 for reviews).

The Kane-Mele model at vanishing uniform and staggered chemical potentials and in the absence of the Rashba spin-orbit coupling simplifies to a reducible massive Dirac Hamiltonian with Dirac matrices of rank 8 at long wave lengths and low energies. In turn, each irreducible block realizes a massive Dirac Hamiltonian with Dirac matrices of rank 4. There are thus two Dirac masses that enter with opposite signs so that time-reversal symmetry holds. Both the electronic charge and the projection of the electronic spin quantum number along the quantization axis in spin space are conserved when the Rashba terms are switched off in the Kane-Mele Hamiltonian. Integration of the electrons in the Kane-Mele Hamiltonian at vanishing uniform and staggered chemical potentials, without Rashba terms, but coupled to two $U(1)$ external gauge fields, one that couples to the conserved $U(1)$ charge and one that couples to the $U(1)$ spin current, delivers a double Chern-Simons (CS) theory.[6] As there is a bulk-edge correspondence associated to each of the CS terms, there follows the existence and stability of an integer number of pairs of helical edge states in any geometry with boundaries. Correspondingly, the Kane-Mele model in the absence of the Rashba terms supports the quantum-spin Hall effect.[7] The quantization of the spin Hall response is lost for any Rashba spin-orbit coupling. The insight of Kane and Mele was to recognize that, as long as time-reversal symmetry holds, a single pair of helical edge states persists in the form of a perfectly conducting channel, provided there was an odd number of pairs of helical edge states prior to switching on the (not too large) Rashba spin-orbit coupling.

The goal of this work is to derive the effective quantum field theory in $(2+1)$-dimensional space and time that encodes at long wave lengths and low energies the Kane-Mele model at vanishing uniform and staggered chemical potentials together with an intrinsic spin-orbit coupling that dominates over a Rashba spin-orbit coupling, and understand how the gapless edge dynamics arises from this bulk action. A brief summary with the main results of the paper follows.

Starting with a Dirac Hamiltonian coupled to the pair $A^{(+)}_\mu$ and $A^{(-)}_\mu$ of gauge fields and after integrating out the massive Dirac fermions, we obtain the one-loop effective action

$$I[A^{(+)},A^{(-)}] = -\frac{1}{4\pi}\int \mathrm{d}^3x \left[\epsilon^{\mu\alpha\nu}\,\left(A^{(+)}_\mu\partial_\alpha A^{(-)}_\nu + A^{(-)}_\mu\partial_\alpha A^{(+)}_\nu\right) - \frac{\lambda_{\mathrm{R}}^2}{|\eta|}\left(A^{(-)}_0\right)^2\right], \tag{1.1a}$$

where the real-valued parameters η and λ_{R} are the spin-orbit and Rashba couplings, respectively. This action is invariant under gauge transformations of the field $A^{(+)}_\mu$ as charge is conserved. It is not invariant under gauge transformations of the field $A^{(-)}_\mu$ as the spin-1/2 symmetry is completely broken by the Rashba spin-orbit coupling. As it is known for the quantum Hall effect,[8–10] gauge invariance is sufficient to show the existence of the propagating chiral states along the edge. The question we are thus after is how to construct the bulk-edge correspondence without the complete $U(1)\times U(1)$ gauge invariance of the effective action when $\lambda_{\mathrm{R}}=0$. The important point is that we can interpret the correction $(A^{(-)}_0)^2$ as a gauge fixing term. By using the Faddeev-Popov procedure,[11] we can introduce the ghost action

$$S_{\mathrm{ghost}} := -\frac{1}{4\pi}\int \mathrm{d}^3x\,\bar{C}\,\partial_t C, \tag{1.1b}$$

where C and $\bar{C}$ are fermionic ghosts fields, such that the complete action

$$S := I + S_{\mathrm{ghost}} \tag{1.1c}$$

changes by a total derivative under the combination of the usual gauge transformation for the $A^{(+)}_\mu$ field,

$$A^{(+)}_\mu \to A^{(+)}_\mu + \partial_\mu\Lambda^{(+)}, \tag{1.2a}$$

with the BRST transformations[12–15] for $A^{(-)}_\mu$, C, and $\bar{C}$,

$$\begin{aligned} &A^{(-)}_\mu \to A^{(-)}_\mu + \theta\,\partial_\mu C,\\ &C \to C,\\ &\bar{C} \to \bar{C} + 2\,\frac{\lambda_{\mathrm{R}}^2}{|\eta|}\,\theta\,A^{(-)}_0, \end{aligned} \tag{1.2b}$$

where θ is a constant Grassmann-valued parameter. Notice that when $\lambda_{\mathrm{R}}\to 0$, the ghosts do not change and the transformation of the gauge field $A^{(-)}_\mu$ reduces to a usual gauge transformation with parameter $\Lambda^{(-)}\equiv\theta\,C$. For any manifold with boundaries, imposing the symmetry under this $U(1)\times$ BRST is sufficient to derive the bulk-edge correspondence, as will be shown later.

The paper is organized as follows. In Sec. II, we introduce the model and formulate the problem in a field theory form. In Sec. III, we perform the one-loop calculation of the gauge effective action. Section IV is dedicated to the study of the edge theory. A summary and additional comments are presented in the Sec. V Three appendices contain further details of some calculations.

II. THE MODEL

A. Hamiltonian

In this work, we consider the single-particle Kane-Mele Hamiltonian in the Dirac approximation. In other words, we start from the tight-binding Hamiltonian for graphene perturbed by an intrinsic spin-orbit coupling and a Rashba spin-orbit coupling. At half-filling, the dispersion of graphene, to linear order in a gradient expansion in the deviations about the Fermi momenta, is that of an 8-dimensional representation of the massless Dirac Hamiltonian in two-dimensional space. The intrinsic spin-orbit coupling is represented by a mass term in the Dirac approximation. Unlike the spin-orbit coupling, the Rashba spin-orbit coupling is represented by an element of the Clifford algebra that does not anticommute with the kinetic energy. The resulting second-quantized Hamiltonian

$$H := H_0 + H_{\mathrm{gauge}} + H_{\mathrm{SO}} + H_{\mathrm{R}} \tag{2.1a}$$

comprises four quadratic terms in the creation and annihilation operators obeying the fermion algebra. There is the kinetic energy

$$H_0 := \psi^\dagger \begin{pmatrix} -\mathrm{i}\alpha_i\,\partial_i & 0\\ 0 & -\mathrm{i}\alpha_i\,\partial_i \end{pmatrix}\psi, \tag{2.1b}$$

where the Latin index $i=1,2$ is reserved for the space coordinates and the summation convention over repeated indices is assumed. There is the coupling (the coupling e is real valued)

$$H_{\mathrm{gauge}} := e\,\psi^\dagger \begin{pmatrix} \alpha_i\,A^u_i & 0\\ 0 & \alpha_i\,A^d_i \end{pmatrix}\psi \tag{2.1c}$$

to the independent pair of classical vector gauge fields A^u_i and A^d_i. There is the intrinsic spin-orbit coupling (the coupling η is real valued)

$$H_{\mathrm{SO}} := \mathrm{i}\eta\,\psi^\dagger\,\alpha_1\,\alpha_2\otimes s_3\,\psi \tag{2.1d}$$

that anticommutes with H_0 and H_{gauge}. There is the Rashba spin-orbit coupling (the coupling λ_{R} is real valued)

$$H_{\mathrm{R}} := \lambda_{\mathrm{R}}\,\psi^\dagger\,(\alpha_1\otimes s_2 - \alpha_2\otimes s_1)\,\psi. \tag{2.1e}$$

In these expressions, ψ denotes the 8-component operator-valued spinor

$$\psi := \begin{pmatrix}\psi^u\\ \psi^d\end{pmatrix}, \qquad \psi^{u,d} := \begin{pmatrix}\psi^{u,d}_{+A}\\ \psi^{u,d}_{+B}\\ \psi^{u,d}_{-B}\\ \psi^{u,d}_{-A}\end{pmatrix}, \tag{2.1f}$$

where the index u (d) refers to the spin up (down) projection along the spin-1/2 quantization axis of the electrons in graphene selected by the intrinsic spin-orbit coupling, the indices A and B represent the two sublattices of the honeycomb lattice of graphene, and the indices + and − refer to the two Dirac points of graphene at half-filling. Finally, the Dirac matrices α_i with $i = 1, 2$ are chosen to be

$$\alpha_i \coloneqq \sigma_3 \otimes \tau_i, \qquad \alpha_3 \coloneqq \sigma_3 \otimes \tau_3, \tag{2.1g}$$

where $\sigma_\mu \equiv (\sigma_0, \boldsymbol{\sigma})$, $\tau_\mu \equiv (\tau_0, \boldsymbol{\tau})$, and $s_\mu \equiv (s_0, \boldsymbol{s})$ each represent three independent sets of the Pauli matrices augmented by the unit 2×2 matrices.

Alternatively, we may choose to quantize the theory with a path integral over the independent Grassmann-valued spinors $\bar{\psi}$ and ψ weighted by a Boltzmann weight with the Lagrangian density

$$\begin{aligned}\mathcal{L} \coloneqq\, & \bar{\psi}^u \left(\mathrm{i}\partial\!\!\!/ - e A\!\!\!/^u - \eta\gamma_5\,\gamma^3\right) \psi^u \\ & + \bar{\psi}^d \left(\mathrm{i}\partial\!\!\!/ - e A\!\!\!/^d + \eta\,\gamma_5\,\gamma^3\right) \psi^d \\ & + \lambda_{\mathrm{R}}\, \bar{\psi}^u \left(-\mathrm{i}\gamma^1 - \gamma^2\right) \psi^d + \lambda_{\mathrm{R}}\, \bar{\psi}^d \left(\mathrm{i}\gamma^1 - \gamma^2\right) \psi^u,\end{aligned} \tag{2.2a}$$

where $\bar{\psi}^{u,d} \equiv (\psi^{u,d})^\dagger \gamma^0$, $A\!\!\!/ \equiv \gamma^\mu A_\mu$ with the summation convention implied over the repeated index $\mu = 0, 1, 2$, and the Dirac matrices γ^μ are defined by

$$\begin{aligned} & \gamma^0 \equiv \beta \coloneqq \sigma_1 \otimes \tau_0, \\ & \gamma^1 \coloneqq \beta\,\alpha_1, \qquad \gamma^2 \coloneqq \beta\,\alpha_2, \qquad \gamma^3 \coloneqq \beta\,\alpha_3, \\ & \gamma_5 \equiv \gamma^5 \equiv -\mathrm{i}\alpha_1\,\alpha_2\,\alpha_3 = \mathrm{i}\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \sigma_3 \otimes \tau_0.\end{aligned} \tag{2.2b}$$

By using the fact that $\bar{\psi}$ and ψ are independent Grassmann-valued spinors, this Lagrangian density is brought to a more convenient form by introducing the spinors $\bar{\chi}^{u,d}$ and $\chi^{u,d}$ through

$$\bar{\psi}^{u,d} \eqqcolon \bar{\chi}^{u,d}\,\gamma_5\,\gamma^3, \qquad \psi^{u,d} \eqqcolon \chi^{u,d}, \tag{2.3a}$$

and in terms of which

$$\begin{aligned}\mathcal{L} =\, & \bar{\chi}^u \left(\mathrm{i}\partial\!\!\!/ - e A\!\!\!/^u - \eta\right) \chi^u \\ & + \bar{\chi}^d \left(\mathrm{i}\partial\!\!\!/ - e A\!\!\!/^d + \eta\right) \chi^d \\ & + \lambda_{\mathrm{R}}\, \bar{\chi}^u \left(-\mathrm{i}\Gamma^1 - \Gamma^2\right) \chi^d + \lambda_{\mathrm{R}}\, \bar{\chi}^d \left(\mathrm{i}\Gamma^1 - \Gamma^2\right) \chi^u,\end{aligned} \tag{2.3b}$$

$A\!\!\!/ \equiv \Gamma^\mu A_\mu$, and

$$\Gamma^\mu \equiv \gamma_5\,\gamma^3\,\gamma^\mu, \qquad \{\Gamma^\mu, \Gamma^\nu\} = 2\,g^{\mu\nu}, \tag{2.3c}$$

for $\mu, \nu = 0, 1, 2$. The signature of the Minkowski metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$. Some useful properties of Dirac matrices are presented in Appendix A.

The Lagrangian density (2.3b) is a special case of

$$\begin{aligned}\mathcal{L} =\, & \bar{\chi}^u \left(\mathrm{i}\partial\!\!\!/ - \eta\right) \chi^u + \bar{\chi}^d \left(\mathrm{i}\partial\!\!\!/ + \eta\right) \chi^d - e\,\bar{\chi}^u A\!\!\!/^u\,\chi^u \\ & - e\,\bar{\chi}^d A\!\!\!/^d\,\chi^d + \lambda_{\mathrm{R}}\,\bar{\chi}^u\,V\!\!\!/\,\chi^d + \lambda_{\mathrm{R}}\,\bar{\chi}^d W\!\!\!\!/\,\chi^u,\end{aligned} \tag{2.4a}$$

with the choice

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} \coloneqq \begin{pmatrix} 0 \\ -\mathrm{i} \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix} \coloneqq \begin{pmatrix} 0 \\ +\mathrm{i} \\ -1 \end{pmatrix}. \tag{2.4b}$$

We will leave the vectors V_μ and W_μ arbitrary throughout the perturbative calculations to come. Notice that the Hermiticity condition for the Lagrangian only demands that $W^*_\mu = V_\mu$.

One fundamental property of the Lagrangian (2.4a) is its invariance under reversal of time. The transformation law of the Dirac fields under reversal of time is

$$\begin{aligned} & \chi^u \to \mathrm{i}\sigma_1 \otimes \tau_1\,\chi^d, \qquad \bar{\chi}^u \to -\mathrm{i}\bar{\chi}^d\,\sigma_1 \otimes \tau_1, \\ & \chi^d \to -\mathrm{i}\sigma_1 \otimes \tau_1 \chi^u, \qquad \bar{\chi}^d \to +\mathrm{i}\bar{\chi}^u\,\sigma_1 \otimes \tau_1,\end{aligned} \tag{2.5}$$

while the transformation law of the gauge fields is

$$A^{u,d}_0 \to +A^{d,u}_0, \qquad A^{u,d}_i \to -A^{d,u}_i. \tag{2.6}$$

Notice here the interchange between the flavors up and down. The invariance of the Lagrangian density (2.4a) under reversal of time is achived if and only if

$$V_0 = W_0 = 0. \tag{2.7}$$

Reversal of time does not restrict the spatial components V_i and $W_i (= V^*_i)$ for $i = 1, 2$.

Needed is the effective action generated for the fields A^u_μ and A^d_μ in the background V_μ and W_μ from integrating out the massive Dirac fermions. We are going to show that

$$
\begin{aligned}
I[A^u, A^d] = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Bigg[& A^u_\mu(-k) \left(I^{(2)}_{uu}\right)^{\mu\nu}(k)\, A^u_\nu(k) + A^u_\mu(-k) \left(I^{(2)}_{ud}\right)^{\mu\nu}(k)\, A^d_\nu(k) \\
& + A^d_\mu(-k) \left(I^{(2)}_{du}\right)^{\mu\nu}(k)\, A^u_\nu(k) + A^d_\mu(-k) \left(I^{(2)}_{dd}\right)^{\mu\nu}(k)\, A^d_\nu(k) + \cdots \Bigg].
\end{aligned} \tag{2.8}
$$

The dots include terms of higher order than quadratic in the gauge fields. In this expression, $I^{(2)}_{IJ}$ represents the one-particle irreducible (1PI) two-point vertex functions for the pair of gauge fields labeled by $I = u, d$ and $J = u, d$.

B. Propagators

Our first task is to choose the propagators associated to the Lagrangian (2.4a). To obtain the exact free propagators, we rewrite the Lagrangian density (2.4a) as

$$\mathcal{L}^0 \coloneqq \bar{\chi}^I\, M^{IJ}\, \chi^J, \tag{2.9}$$

with $I, J = u, d$. The matrix M in momentum space is

$$M(p) = \begin{pmatrix} \not{p} - \eta & \lambda_{\mathrm{R}} \not{V} \\ \lambda_{\mathrm{R}} \not{W} & \not{p} + \eta \end{pmatrix}. \tag{2.10}$$

The propagator S^{IJ} is defined to be

$$S(p) \coloneqq \mathrm{i}\, M^{-1}(p). \tag{2.11}$$

The multiplicative factor i is chosen by convention.

Observe that, aside from the propagators S^{uu} and S^{dd}, there are the mixed propagators S^{ud} and S^{du}. We define the inverse matrix

$$M^{-1}(p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.12}$$

where $A, B, C,$ and D are matrices to be determined. Imposing the condition $MM^{-1} = 1$, we obtain the set of equations

$$(\not{p} - \eta)\, A + \lambda_{\mathrm{R}} \not{V}\, C = 1, \tag{2.13a}$$

$$(\not{p} - \eta)\, B + \lambda_{\mathrm{R}} \not{V}\, D = 0, \tag{2.13b}$$

$$\lambda_{\mathrm{R}} \not{W}\, A + (\not{p} + \eta)\, C = 0, \tag{2.13c}$$

and

$$\lambda_{\mathrm{R}} \not{W}\, B + (\not{p} + \eta)\, D = 1. \tag{2.13d}$$

The formal solutions to these equations are

$$S^{uu}(p) = \frac{\mathrm{i}}{\not{p} - \eta - \frac{\lambda_{\mathrm{R}}^2}{p^2 - \eta^2} \not{V}\, (\not{p} - \eta) \not{W}}, \tag{2.14a}$$

$$S^{du}(p) = -\lambda_{\mathrm{R}}\, \frac{1}{\not{p} + \eta} \not{W}\, \frac{\mathrm{i}}{\not{p} - \eta - \frac{\lambda_{\mathrm{R}}^2}{p^2 - \eta^2} \not{V}\, (\not{p} - \eta) \not{W}}, \tag{2.14b}$$

$$S^{dd}(p) = \frac{\mathrm{i}}{\not{p} + \eta - \frac{\lambda_{\mathrm{R}}^2}{p^2 - \eta^2} \not{W}\, (\not{p} + \eta) \not{V}}, \tag{2.14c}$$

and

$$S^{ud}(p) = -\lambda_{\mathrm{R}}\, \frac{1}{\not{p} - \eta} \not{V}\, \frac{\mathrm{i}}{\not{p} + \eta - \frac{\lambda_{\mathrm{R}}^2}{p^2 - \eta^2} \not{W}\, (\not{p} + \eta) \not{V}}. \tag{2.14d}$$

These propagators display an intricate matrix structure, making diagrammatic calculations impracticable. Hence, we will consider the limit $|\lambda_{\mathrm{R}}| \ll |\eta|$ and perform an expansion in powers of $|\lambda_{\mathrm{R}}|/|\eta|$.

III. PERTURBATIVE EXPANSION IN THE RASHBA SPIN-ORBIT COUPLING

A. Feynman rules

In order to develop perturbation theory, we use the following Feynman rules associated to the Lagrangian density (2.4a). The fermion free propagators are defined by setting $\lambda_{\mathrm{R}} = 0$ in Eqs. (2.14a-2.14d). The non-vanishing propagators are

$$S^u(p) \coloneqq \frac{\mathrm{i}}{\not{p} - \eta + \mathrm{i}\epsilon}, \qquad S^d(p) \coloneqq \frac{\mathrm{i}}{\not{p} + \eta + \mathrm{i}\epsilon}, \tag{3.1}$$

where we have introduced the iϵ prescription to regulate poles. These propagators are represented by the lines shown in Fig. 1. We have four types of vertices, as shown in Fig. 2.

B. Double Chern-Simons contributions

We start by calculating the contributions of order e^2 and $(e^2\, \lambda_{\mathrm{R}}^2)^0$ to the 2-point 1PI vertex functions of the fields A^u_μ and A^d_μ, namely the ground-state expectation values $\langle A^u_\mu\, A^u_\nu \rangle$ and $\langle A^d_\mu\, A^d_\nu \rangle$. These contributions are responsible for generating the doubled Chern-Simons action and correspond to the diagrams shown in Figs. 3 and 4. Up to this order, we do not have contributions to

FIG. 1: Fermionic propagators.

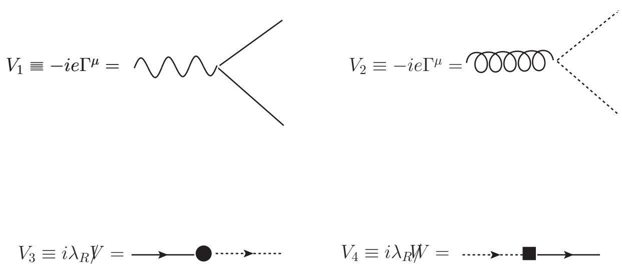

FIG. 2: Vertices representing the coupling between the fermions and the gauge fields. The wavy line in the vertex V_1 refers to the gauge field A^u_μ, whereas the curly line in V_2 refers to A^d_μ.

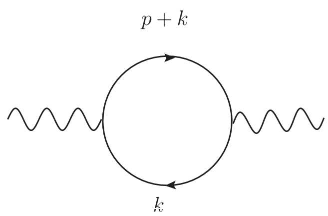

FIG. 3: Diagram contributing to the Chern-Simons term of A^u_μ.

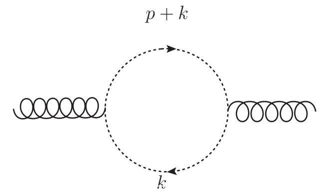

FIG. 4: Diagram contributing to the Chern-Simons term of A^d_μ.

the mixed ground-state expectation values $\langle A^u_\mu \, A^d_\nu \rangle$ and $\langle A^d_\mu \, A^u_\nu \rangle$.

According to the Feynman rules, the expression corresponding to the Feynman diagram 3 is

$$\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p)\Big|_{e^2} = e^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{Tr}\left[\Gamma^\mu \, S^u(p+k) \, \Gamma^\nu \, S^u(k)\right]. \tag{3.2}$$

This integral is linearly divergent but the Chern-Simons contribution turns out to be finite. Selecting only the Chern-Simons contribution, we find

$$\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p)\Big|_{e^2} = -4\mathrm{i}e^2 \, \eta \, p_\alpha \, \epsilon^{\mu\alpha\nu} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{[(p+k)^2 - \eta^2 + \mathrm{i}\epsilon][k^2 - \eta^2 + \mathrm{i}\epsilon]} + \cdots . \tag{3.3}$$

For simplicity, from now on we will omit the iϵ prescription. The identity

$$\frac{1}{a\,b} = \int_0^1 \mathrm{d}x \frac{1}{[a\,x + b\,(1-x)]^2}, \tag{3.4}$$

allows to perform the integration in the loop momentum. One finds

$$\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p)\Big|_{e^2} = \frac{e^2}{2\pi} \eta \, p_\alpha \, \epsilon^{\mu\alpha\nu} \int_0^1 \mathrm{d}x \, \frac{1}{[\eta^2 - x\,(1-x)\,p^2]^{1/2}} + \cdots . \tag{3.5}$$

For small momentum, i.e., $p^2/\eta^2 \ll 1$, we perform an expansion of the integrand in powers of p^2/η^2. The result is the Chern-Simons kernel

$$\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p)\Big|_{e^2} = +\frac{e^2}{2\pi} \frac{\eta}{|\eta|} \, p_\alpha \, \epsilon^{\mu\alpha\nu} + \cdots . \tag{3.6}$$

Higher order corrections in p^2/η^2 generate terms quadratic in the gauge fields of higher order in the derivatives. For example, there is the term $\epsilon^{\mu\nu\rho} \, A_\mu \Box \partial_\nu \, A_\rho$, with the metric-dependent d'Alembert operator $\Box \equiv \partial_\mu \partial^\mu$. This term is infra-red irrelevant by power counting.

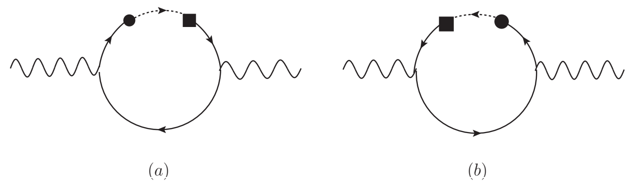

FIG. 5: Rashba contributions of order $e^2\,\lambda_{\rm R}^2$ to $\langle A^u_\mu\, A^u_\nu\rangle$.

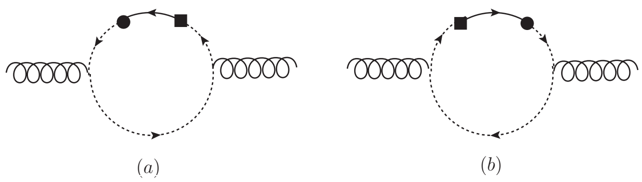

FIG. 6: Rashba contributions of order $e^2\,\lambda_{\rm R}^2$ to $\langle A^d_\mu\, A^d_\nu\rangle$.

Similarly, the contribution of the Feynman diagram depicted in Fig. 4 to the 2-point 1PI vertex function of the gauge field A^d_μ is

$$\mathrm{i}(I^{(2)}_{dd})^{\mu\nu}(p)\Big|_{e^2} = e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\, S^d(p+k)\,\Gamma^\nu\, S^d(k)\right]. \tag{3.7}$$

The only difference between the right-hand side of Eqs. (3.7) and (3.6) is the sign with which the mass η enters the free propagators. Hence, we find

$$\mathrm{i}(I^{(2)}_{dd})^{\mu\nu}(p)\Big|_{e^2} = -\frac{e^2}{2\pi}\,\frac{\eta}{|\eta|}\,p_\alpha\,\epsilon^{\mu\alpha\nu} + \cdots. \tag{3.8}$$

Collecting Eqs. (3.6) and (3.8), we find for the effective field theory the double Chern-Simons theory

$$\mathcal{L}_{\rm eff}\Big|_{e^2} = \frac{e^2}{2\pi}\,\frac{\eta}{|\eta|}\,\epsilon^{\mu\alpha\nu}\left(A^u_\mu\partial_\alpha A^u_\nu - A^d_\mu\partial_\alpha A^d_\nu\right) + \cdots \tag{3.9}$$

up to order e^2 and $(e^2\,\lambda_{\rm R}^2)^0$ in the couplings and to the first non-vanishing order in a gradient expansion.

C. Rashba corrections

We are after the corrections of order $e^2\,\lambda_{\rm R}^2$ to the 2-point functions of A^u_μ and A^d_μ.

1. *The 1PI vertex function* $\langle A^u_\mu A^u_\nu\rangle$

Starting with the function $\langle A^u_\mu\, A^u_\nu\rangle$, we have the contributions of the two diagrams of Fig. 5, with the corresponding expressions

$$\begin{aligned}\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p)\,\Big|_{\lambda_{\rm R}^2\,e^2} =& -\lambda_{\rm R}^2\,e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\, S^u(p+k)\,\not{V}\, S^d(p+k)\not{W}\, S^u(p+k)\,\Gamma^\nu\, S^u(k)\right]\\ &-\lambda_{\rm R}^2\,e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\, S^u(p+k)\,\Gamma^\nu\, S^u(k)\,\not{V}\, S^d(k)\,\not{W}\, S^u(k)\right].\end{aligned} \tag{3.10}$$

These integrals are finite. As the leading terms do not depend on the external momentum, we can simplify the calculation by setting $p=0$,

$$\begin{aligned}\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p=0)\,\Big|_{\lambda_{\rm R}^2\,e^2} =& -\lambda_{\rm R}^2\,e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\, S^u(k)\,\not{V}\, S^d(k)\,\not{W}\, S^u(k)\,\Gamma^\nu\, S^u(k)\right]\\ &-\lambda_{\rm R}^2\,e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\nu\, S^u(k)\,\not{V}\, S^d(k)\,\not{W}\, S^u(k)\,\Gamma^\mu\, S^u(k)\right],\end{aligned} \tag{3.11}$$

where we used the cyclicity of the trace to rewrite the second term in such way that it differs from the first one only by the change $\mu\leftrightarrow\nu$. The calculation of this expression is simplified by the introduction of the rank-4 tensor

$$\begin{aligned}A^{\mu\nu\alpha\beta} :=& \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\Big[\Gamma^\mu\, S^u(k)\,\Gamma^\alpha\, S^d(k)\\ &\times\Gamma^\beta\, S^u(k)\,\Gamma^\nu\, S^u(k)\Big].\end{aligned} \tag{3.12}$$

A detailed evaluation of the trace as well as the loop integral is done in Appendix B, where it is shown that

$$A^{\mu\nu\alpha\beta} = \frac{\mathrm{i}}{12\,\pi\,|\eta|}\,\left(g^{\mu\nu}\,g^{\alpha\beta} - g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha}\right). \tag{3.13}$$

Notice that $A^{\mu\nu\alpha\beta}$ is symmetric in the exchange of the pairs of indices (μ,ν) and (α,β). Equation (3.11) becomes

$$\mathrm{i}(I^{(2)}_{uu})^{\mu\nu}(p=0)\Big|_{\lambda_{\rm R}^2\,e^2} = -\frac{\mathrm{i}\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,[g^{\mu\nu}\,(V\cdot W) - V^\mu\,W^\nu - V^\nu\,W^\mu]\,. \tag{3.14}$$

The effective Lagrangian in the coordinate space reads

$$\mathcal{L}^{uu}_{\rm eff}\Big|_{\lambda_{\rm R}^2\,e^2} = -\frac{\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,[(A^u\cdot A^u)\,(V\cdot W) - 2\,(A^u\cdot V)\,(A^u\cdot W)]\,. \tag{3.15}$$

2. The 1PI vertex function $\langle A^d_\mu\, A^d_\nu\rangle$

We turn our attention to the 1PI vertex function $\langle A^d_\mu\, A^d_\nu\rangle$. The one-loop contribution is given by the Feynman diagrams of Fig. 6,

$$\begin{aligned}\mathrm{i}(I^{(2)}_{dd})^{\mu\nu}(p)\Big|_{\lambda_{\rm R}^2\,e^2} =& -\lambda_{\rm R}^2\,e^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\,S^d(p+k)\,\Gamma^\nu\,S^d(k)\,\not{W}\,S^u(k)\,\not{V}\,S^u(k)\right]\\ & -\lambda_{\rm R}^2\,e^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\,S^d(p+k)\,\not{W}\,S^u(p+k)\,\not{V}\,S^d(p+k)\,\Gamma^\nu\,S^d(k)\right].\end{aligned} \tag{3.16}$$

Equation (3.10) is mapped into (3.16) by changing the sign of the mass, $\eta\to-\eta$, and interchanging $\not{V}$ and $\not{W}$. Hence,

$$\mathrm{i}(I^{(2)}_{dd})^{\mu\nu}(p=0)\Big|_{\lambda_{\rm R}^2\,e^2} = -\frac{\mathrm{i}\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,[g^{\mu\nu}\,(V\cdot W) - V^\mu\,W^\nu - V^\nu\,W^\mu] \tag{3.17}$$

and the corresponding effective Lagrangian density in the coordinate space

$$\mathcal{L}^{dd}_{\rm eff}\Big|_{\lambda_{\rm R}^2\,e^2} = -\frac{\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,\left[(A^d\cdot A^d)\,(V\cdot W) - 2\,(A^d\cdot V)\,(A^d\cdot W)\right]. \tag{3.18}$$

3. The 1PI vertex function $\langle A^u_\mu\, A^d_\nu\rangle$

The 1PI vertex function $\langle A^u_\mu\, A^d_\nu\rangle$ has the one-loop contribution shown in the Feynman diagram of Fig. 7,

$$\mathrm{i}(I^{(2)}_{ud})^{\mu\nu}(p)\Big|_{\lambda_{\rm R}^2\,e^2} = -\lambda_{\rm R}^2\,e^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\,S^u(p+k)\,\not{V}\,S^d(p+k)\,\Gamma^\nu\,S^d(k)\,\not{W}\,S^u(k)\right]. \tag{3.19}$$

The trace and loop integral are performed in Appendix B. The result for $p=0$ is

$$\mathrm{i}(I^{(2)}_{ud})^{\mu\nu}(p)\Big|_{\lambda_{\rm R}^2\,e^2} = \frac{\mathrm{i}\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,[g^{\mu\nu}\,(V\cdot W) - V^\mu\,W^\nu - V^\nu\,W^\mu]\,. \tag{3.20}$$

By turning this equation to the coordinate space, it follows that

$$\mathcal{L}^{ud}_{\rm eff}\Big|_{\lambda_{\rm R}^2\,e^2} = \frac{\lambda_{\rm R}^2\,e^2}{6\,\pi\,|\eta|}\,\left[(A^u\cdot A^d)\,(V\cdot W) - (A^u\cdot V)\,(A^d\cdot W) - (A^u\cdot W)\,(A^d\cdot V)\right]. \tag{3.21}$$

4. The 1PI vertex function $\langle A^d_\mu\, A^u_\nu\rangle$

The 1PI vertex function $\langle A^d_\mu\, A^u_\nu\rangle$ has the one-loop contribution shown in the Feynman diagram of Fig. 8,

$$\mathrm{i}(I^{(2)}_{du})^{\mu\nu}(p)\Big|_{\lambda^2_{\mathrm{R}}\,e^2} = -\lambda^2_{\mathrm{R}}\,e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{Tr}\left[\Gamma^\mu\, S^d(p+k)\, \not{W}\, S^u(p+k)\,\Gamma^\nu\, S^u(k)\,\not{V}\, S^d(k)\right]. \tag{3.22}$$

Equation (3.19) maps into Eq. (3.22) by doing the substitutions $V \leftrightarrow W$ and $\eta \to -\eta$. Under these substitutions, Eq. (3.20) is turned into

$$\mathrm{i}(I^{(2)}_{du})^{\mu\nu}(p)\Big|_{\lambda^2_{\mathrm{R}}\,e^2} = \frac{\mathrm{i}\lambda^2_{\mathrm{R}}\,e^2}{6\,\pi\,|\eta|}\,\left[g^{\mu\nu}\,(V\cdot W) - V^\mu\, W^\nu - V^\nu\, W^\mu\right]. \tag{3.23}$$

The effective Lagrangian density reads

$$\mathcal{L}^{du}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} = \frac{\lambda^2_{\mathrm{R}}\,e^2}{6\,\pi\,|\eta|}\,\left[(A^u\cdot A^d)\,(V\cdot W) - (A^u\cdot V)\,(A^d\cdot W) - (A^u\cdot W)\,(A^d\cdot V)\right]. \tag{3.24}$$

5. Summary

Adding Eqs. (3.15), (3.18), (3.21), and (3.24), we obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} &= \mathcal{L}^{uu}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} + \mathcal{L}^{dd}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} + \mathcal{L}^{ud}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} + \mathcal{L}^{du}_{\mathrm{eff}}\Big|_{\lambda^2_{\mathrm{R}}\,e^2} \\ &= -\frac{e^2\lambda^2_{\mathrm{R}}}{3\,\pi\,|\eta|}\,\left[\frac{1}{2}\,(A^{(-)})^2\,(V\cdot W) - [(A^{(-)}\cdot V]\,[A^{(-)}\cdot W]\right],\end{aligned} \tag{3.25a}$$

where we defined the gauge fields

$$A^{(\pm)}_\mu \equiv A^u_\mu \pm A^d_\mu. \tag{3.25b}$$

Thus, with the condition (2.7), the time-reversal-symmetric one-loop effective action is

$$\begin{aligned}I[A^{(+)},A^{(-)}] = \frac{e^2}{4\pi}\int \mathrm{d}^3x\,\Bigg\{&\frac{\eta}{|\eta|}\,\epsilon^{\mu\alpha\nu}\,\left(A^{(+)}_\mu\partial_\alpha A^{(-)}_\nu + A^{(-)}_\mu\partial_\alpha A^{(+)}_\nu\right) \\ &+ \frac{2\,\lambda^2_{\mathrm{R}}}{3\,|\eta|}\,\left[(A^{(-)}_0)^2\,(\vec{V}\cdot\vec{W}) + A^{(-)}_i\,A^{(-)}_j\,\left(\delta_{ij}\,\vec{V}\cdot\vec{W} - V_i\,W_j - V_j\,W_i\right)\right]\Bigg\},\end{aligned} \tag{3.26a}$$

where

$$\vec{V}\cdot\vec{W} \equiv V_i\,W_i. \tag{3.26b}$$

For the particular case of the Rashba spin-orbit coupling, $V^\top = (0,-\mathrm{i},-1)$ and $W^\top = (0,+\mathrm{i},-1)$ and this effective action reduces to

$$I[A^{(+)},A^{(-)}] = \frac{e^2}{4\pi}\int \mathrm{d}^3x\,\left[\frac{\eta}{|\eta|}\,\epsilon^{\mu\alpha\nu}\,\left(A^{(+)}_\mu\partial_\alpha A^{(-)}_\nu + A^{(-)}_\mu\partial_\alpha A^{(+)}_\nu\right) + \frac{4\,\lambda^2_{\mathrm{R}}}{3\,|\eta|}\,\left(A^{(-)}_0\right)^2\right]. \tag{3.27}$$

The coupling constant e^2 multiplies the integrand in this effective action. Hence, it can be absorbed by the rescaling

$$|e|\,A^{(+)}_\mu \to A^{(+)}_\mu, \qquad |e|\,A^{(-)}_\mu \to A^{(-)}_\mu, \tag{3.28}$$

of the gauge fields. For convenience, we also do the redefinition

$$\frac{4}{3}\,\lambda^2_{\mathrm{R}} \to \lambda^2_{\mathrm{R}} \tag{3.29}$$

of the Rashba spin-orbit coupling. Finally, we choose without loss of generality the sign

$$\mathrm{sgn}(\eta) = - \tag{3.30}$$

for the intrinsic spin-orbit coupling. In this way, we arrive at Eq. (1.1a).

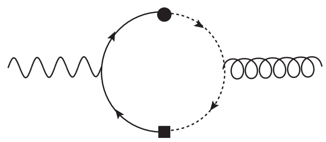

FIG. 7: Rashba contributions of order $e^2\,\lambda_{\mathrm{R}}^2$ to $\langle A^u_\mu\, A^d_\nu\rangle$.

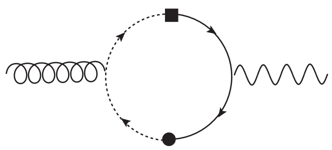

FIG. 8: Rashba contributions of order $\lambda_{\mathrm{R}}^2\, e^2$ to $\langle A^d_\mu\, A^u_\nu\rangle$.

IV. EDGE THEORY

This section is devoted to deriving the bulk-edge correspondence when the effective action (1.1a) is defined on a manifold with boundaries. To this end, we need to extract from the effective action (1.1a) the effective Lagrangian density

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4\pi}\left[\epsilon^{\mu\alpha\nu}\left(A^{(+)}_\mu\,\partial_\alpha A^{(-)}_\nu + A^{(-)}_\mu\,\partial_\alpha A^{(+)}_\nu\right) - \frac{\lambda_{\mathrm{R}}^2}{|\eta|}\left(A^{(-)}_0\right)^2\right]. \tag{4.1}$$

A. Pure Chern-Simons Theory

As a warm up we first study the bulk-edge correspondence in the absence of Rashba spin-orbit coupling. In doing so, we shall emphasize the ingredients that will be useful for the extension to the case with Rashba spin-orbit coupling.

The effective action (1.1a) reduces to the double Chern-Simons action

$$S_{\mathrm{CS}} = -\frac{1}{4\pi}\int \mathrm{d}^3x\,\epsilon^{\mu\nu\lambda}\left(A^{(+)}_\mu\partial_\nu A^{(-)}_\lambda + A^{(-)}_\mu\partial_\nu A^{(+)}_\lambda\right) \tag{4.2}$$

when $\lambda_{\mathrm{R}} = 0$. The variation of the Lagrangian density (4.1) with $\lambda_{\mathrm{R}} = 0$ under the gauge transformations

$$A^{(+)}_\mu \to A^{(+)}_\mu + \partial_\mu\Lambda^{(+)}, \qquad A^{(-)}_\mu \to A^{(-)}_\mu + \partial_\mu\Lambda^{(-)}, \tag{4.3}$$

is a total derivative. Thus, if the manifold has no boundary, the theory is gauge invariant. In this case, we have the freedom to fix any one of the components of $A^{(+)}_\mu$ and any one of the components of $A^{(-)}_\mu$. Correspondingly, the equations of motion

$$0 = \frac{\delta S_{\mathrm{CS}}}{\delta A^{(+)}_\mu} = \epsilon^{\mu\nu\lambda}\,\partial_\nu A^{(-)}_\lambda \tag{4.4}$$

and

$$0 = \frac{\delta S_{\mathrm{CS}}}{\delta A^{(-)}_\mu} = \epsilon^{\mu\nu\lambda}\,\partial_\nu A^{(+)}_\lambda \tag{4.5}$$

dictate that

$$F^{(-)}_{\mu\nu} := \partial_\mu A^{(-)}_\nu - \partial_\nu A^{(-)}_\mu = 0, \tag{4.6}$$

and

$$F^{(+)}_{\mu\nu} := \partial_\mu A^{(+)}_\nu - \partial_\nu A^{(+)}_\mu = 0, \tag{4.7}$$

respectively. Hence, the doubled Chern-Simons action does not support gapless excitations when two-dimensional space has no boundary.

This freedom to fix all the components of the gauge fields is lost if the space manifold has a boundary. To appreciate this point, we choose a manifold Ω in two-dimensional position space with an edge running along the x axis at $y = 0$, i.e.,

$$\Omega := \{(x,y)|x\in\mathbb{R}, y\leq 0\}, \tag{4.8}$$

in addition to the time coordinate defined by $-\infty < t < +\infty$. The 3-dimensional manifold over which the doubled Chern-Simons Lagrangian density

$$\mathcal{L}_{\mathrm{CS}} = -\frac{1}{4\pi}\,\epsilon^{\mu\nu\lambda}\left(A^{(+)}_\mu\partial_\nu A^{(-)}_\lambda + A^{(-)}_\mu\partial_\nu A^{(+)}_\lambda\right) \tag{4.9}$$

is to be integrated is thus

$$\Omega\times\mathbb{R}. \tag{4.10}$$

Under the gauge transformations (4.3), the variation of the doubled Chern-Simons action (4.2) is the edge action

$$\begin{aligned}\delta S_{\mathrm{CS}} =& -\frac{1}{4\pi}\int \mathrm{d}x\mathrm{d}t\,\Lambda^{(+)}\left(\partial_t A^{(-)}_1 - \partial_x A^{(-)}_0\right)\Big|_{y=0}\\ & -\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\Lambda^{(-)}\left(\partial_t A^{(+)}_1 - \partial_x A^{(+)}_0\right)\Big|_{y=0}.\end{aligned} \tag{4.11}$$

Evidently, gauge invariance is lost for arbitrary $\Lambda^{(+)}$ and $\Lambda^{(-)}$ and so is the freedom to fix all the components of the gauge fields.

1. *Restoring gauge symmetry by restricting the allowed gauge transformations*

One way to preserve the gauge invariance on the space manifold (4.8) and the space-time manifold (4.10) is to restrict the gauge transformation in Eq. (4.3) by imposing the conditions

$$\Lambda^{(+)}\Big|_{y=0} = \Lambda^{(-)}\Big|_{y=0} = 0 \tag{4.12}$$

for any coordinate x along the edge and any time t. Restricting the allowed functions $\Lambda^{(\pm)}$ by imposing the constraint (4.12) on the edge restores gauge invariance. However, this gauge symmetry, restricted as it is on the edge, allows for gapless degrees of freedom to be supported on the boundary, as we demonstrate now.

We fix the gauge fields $A_0^{(+)}$ and $A_0^{(-)}$ by demanding that they be proportional to the gauge fields $A_1^{(-)}$ and $A_1^{(+)}$, respectively,

$$A_0^{(+)} = v\, A_1^{(-)}, \qquad A_0^{(-)} = v\, A_1^{(+)}. \tag{4.13}$$

The proportionality constant v is arbitrary and carries the dimension of velocity. It will shortly be identified with the characteristic velocity of the edge states. The arbitrariness in choosing v reflects the fact that the value of v is fixed by the contributions to the effective action of higher order in the derivative expansion than the leading terms that have been kept, i.e., v is independent of the microscopic physics encoded by the double Chern-Simons action.

As the components $A_0^{(+)}$ and $A_0^{(-)}$ are not independent dynamical degrees of freedom anymore, their equations of motion, Eqs. (4.5) and (4.4) with $\mu = 0$, respectively, become the constraints

$$F_{12}^{(-)} := \partial_x A_2^{(-)} - \partial_y A_1^{(-)} = 0 \tag{4.14}$$

and

$$F_{12}^{(+)} := \partial_x A_2^{(+)} - \partial_y A_1^{(+)} = 0, \tag{4.15}$$

respectively. Both constraints are met by

$$A_i^{(-)} = \partial_i \varphi^{(-)}, \qquad A_i^{(+)} = \partial_i \varphi^{(+)}, \tag{4.16}$$

for $i = x, y$ if the scalar fields $\varphi^{(-)}$ and $\varphi^{(+)}$ are smooth.

One verifies that the action (4.2) becomes

$$S_{\mathrm{edge}} = \frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \left(\partial_t \varphi^{(-)} \partial_x \varphi^{(+)} + \partial_t \varphi^{(+)} \partial_x \varphi^{(-)} - v\, \partial_x \varphi^{(+)} \partial_x \varphi^{(+)} - v\, \partial_x \varphi^{(-)} \partial_x \varphi^{(-)} \right) \tag{4.17}$$

if we make use of Eqs. (4.13) and (4.16). This is the action for a pair of massless relativistic counter propagating chiral bosonic modes in (1+1)-dimensional space and time. As promised, gapless excitations are supported by the edge even though the theory in the bulk is massive.

2. *Restoring gauge symmetry by adding dynamical degrees of freedom on the edges*

An alternative strategy to restore the gauge invariance on the space manifold (4.8) and the space-time manifold (4.10) is to add to the action (4.2) an action that cancels the anomalous term (4.11) acquired under the gauge transformation (4.3). In other words, the action

$$S := S_{\mathrm{CS}} + S_{\mathrm{edge}}, \tag{4.18a}$$

where

$$\begin{aligned} S_{\mathrm{edge}} := &\frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \varphi^{(+)} \left(\partial_t A_1^{(-)} - \partial_x A_0^{(-)} \right) \\ &+ \frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \varphi^{(-)} \left(\partial_t A_1^{(+)} - \partial_x A_0^{(+)} \right), \end{aligned} \tag{4.18b}$$

is invariant under the gauge transformations defined by Eqs. (4.3) and

$$\varphi^{(+)} \to \varphi^{(+)} + \Lambda^{(+)}, \qquad \varphi^{(-)} \to \varphi^{(-)} + \Lambda^{(-)}. \tag{4.18c}$$

The violation of the gauge symmetry in the bulk is exactly compensated by the violation of the gauge symmetry at the edge. This is the celebrated bulk-edge correspondence.[8–10]

We now proceed to identifying the physical degrees of freedom at the edge, by eliminating redundant degrees of freedom using the symmetries of the edge action.

The path integral that defines the quantized theory along the edge is to be performed over the 6 fields

$$\left\{ \varphi^{(-)}, \varphi^{(+)}, A_0^{(-)}, A_0^{(+)}, A_1^{(-)}, A_1^{(+)} \right\}. \tag{4.19}$$

The symmetries of the action on the edge follow from the gauge transformations

$$A_0^{(-)} \to A_0^{(-)} + \partial_t \chi^{(-)}, \qquad A_1^{(-)} \to A_1^{(-)} + \partial_x \chi^{(-)} \tag{4.20}$$

and

$$A_0^{(+)} \to A_0^{(+)} + \partial_t \chi^{(+)}, \qquad A_1^{(+)} \to A_1^{(+)} + \partial_x \chi^{(+)}. \tag{4.21}$$

The symmetry under these transformations allows to fix 2 degrees of freedom, say by demanding that

$$A_0^{(-)} = v\, A_1^{(+)}, \qquad A_0^{(+)} = v\, A_1^{(-)}, \tag{4.22}$$

where the proportionality constant v is an arbitrary real-valued number carrying the dimension of velocity. Insertion of the gauge-fixing conditions (4.22) into the edge action (4.18b) gives

$$\begin{aligned} S_{\mathrm{edge}} = &\frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \varphi^{(+)} \left(\partial_t A_1^{(-)} - v\, \partial_x A_1^{(+)} \right) \\ &+ \frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \varphi^{(-)} \left(\partial_t A_1^{(+)} - v\, \partial_x A_1^{(-)} \right). \end{aligned} \tag{4.23}$$

The path integral that defines the quantized theory along the edge is now to be performed over the 4 fields

$$\left\{ \varphi^{(-)}, \varphi^{(+)}, A_1^{(-)}, A_1^{(+)} \right\}. \tag{4.24}$$

The action on the edge is symmetric under the residual gauge symmetry defined by

$$A_1^{(-)} \to A_1^{(-)} + v\, \partial_x \zeta, \qquad A_1^{(+)} \to A_1^{(+)} + \partial_t \zeta \tag{4.25}$$

and

$$\varphi^{(+)} \to \varphi^{(+)} + v\, \partial_x \xi, \qquad \varphi^{(-)} \to \varphi^{(-)} + \partial_t \xi, \tag{4.26}$$

provided ζ and ξ satisfy the Klein-Gordon equations

$$(\partial_t^2 - v^2\, \partial_x^2)\, \zeta = 0, \qquad (\partial_t^2 - v^2\, \partial_x^2)\, \xi = 0, \tag{4.27}$$

respectively. The functions ζ and ξ that parametrize the residual gauge symmetry obey the Klein-Gordon equation and as such can be decomposed into a linear superposition of ingoing and outgoing waves,

$$\zeta(x,t) = \zeta^{(+)}(x + v\, t) + \zeta^{(-)}(x - v\, t), \tag{4.28a}$$

$$\xi(x,t) = \xi^{(+)}(x + v\, t) + \xi^{(-)}(x - v\, t). \tag{4.28b}$$

The components $\zeta^{(+)}, \xi^{(+)}$ and $\zeta^{(-)}, \xi^{(-)}$ when $v > 0$ are also known as left- and right- moving waves or as chiral and anti-chiral waves, respectively.

The functions ζ and ξ are not the only ones obeying the Klein-Gordon equation. So do the dynamical fields $A_1^{(\pm)}$ and $\varphi^{(\pm)}$,

$$(\partial_t^2 - v^2\, \partial_x^2)\, A_1^{(\pm)} = 0, \qquad (\partial_t^2 - v^2\, \partial_x^2)\, \varphi^{(\pm)} = 0, \tag{4.29}$$

as follows from the equations of motion derived from the action on the edge (4.23). Correspondingly, the fields $A_1^{(\pm)}$ and $\varphi^{(\pm)}$ also obey an additive decomposition into chiral and anti-chiral components. This observation allows to impose the gauge-fixing condition

$$A_1^{(-)} = -\partial_x\, \varphi^{(-)}, \qquad A_1^{(+)} = -\partial_x\, \varphi^{(+)}. \tag{4.30}$$

Implementing the condition (4.30) in the action on the edge (4.23) delivers

$$S_{\mathrm{edge}} = \frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t\, \left(\partial_t \varphi^{(-)} \partial_x \varphi^{(+)} + \partial_t \varphi^{(+)} \partial_x \varphi^{(-)} - v\, \partial_x \varphi^{(+)} \partial_x \varphi^{(+)} - v\, \partial_x \varphi^{(-)} \partial_x \varphi^{(-)} \right) \tag{4.31}$$

in agrement with Eq. (4.17). The derivation of Eq. (4.31) is the one that we will extend to the case when Rashba spin-orbit coupling is present.

B. Including Rashba Terms - BRST Approach

The effective action in the presence of Rashba spin-orbit coupling is

$$I[A^{(+)}, A^{(-)}] = -\frac{1}{4\pi} \int \mathrm{d}^3 x\, \left[\epsilon^{\mu\nu\lambda} \left(A_\mu^{(+)}\, \partial_\nu A_\lambda^{(-)} + A_\mu^{(-)}\, \partial_\nu A_\lambda^{(+)} \right) - \frac{\lambda_{\mathrm{R}}^2}{|\eta|}\, (A_0^{(-)})^2 \right]. \tag{4.32}$$

Owing to the term $(A_0^{(-)})^2$, we no longer have the full gauge symmetry (4.3) (nor the Lorentz symmetry), that was used to establish the bulk-edge correspondence in the case of the doubled Chern-Simons theory. On the other hand, we are in the situation where $|\lambda_{\rm R}| \ll |\eta|$. Hence, the existence of the gap in the bulk is not affected by switching on adiabatically the Rashba spin-orbit coupling. The topological stability of the parity in the number of pairs of gapless helical edge states implies that at least one pair remains gapless when $|\lambda_{\rm R}| \ll |\eta|$. Our task is now to understand how to get the edge states from the effective field theory (4.32). It is natural to expect that some weaker symmetry replaces the gauge symmetry of Sec. IV A.

We start with a manifold without boundaries. The central point of our construction is that the Rashba term can be interpreted as a gauge-fixing term for the field $A^{(-)}_{\mu}$. If so, the action (4.32) is to be thought of as a doubled Chern-Simons action augmented by a gauge-fixing term, which we may implement through the Faddeev-Popov procedure[11], as we now show. To this end, we define the ghost action

$$S_{\rm ghost} := -\frac{1}{4\pi}\int \mathrm{d}^3x\, \bar{C}\,\partial_t C, \tag{4.33a}$$

where $\bar{C}$ and C enter as Grassmann-valued ghosts fields in the partition function. The augmented action is

$$S := I + S_{\rm ghost}. \tag{4.33b}$$

We may then write

$$\begin{aligned} Z &:= \int \mathcal{D}A\; e^{+\mathrm{i}I} \\ &\propto \int \mathcal{D}A \int \mathcal{D}C\,\mathcal{D}\bar{C}\; e^{\mathrm{i}(I+S_{\rm ghost})}, \end{aligned} \tag{4.33c}$$

for there is no coupling between the gauge and ghost fields and the integration over the ghosts just produces a constant multiplicative factor that can be absorbed in the integration measure,

$$\int \mathcal{D}C\,\mathcal{D}\bar{C}\; e^{\mathrm{i}S_{\rm ghost}} = \text{constant}. \tag{4.34}$$

The partition function (4.33c) is independently invariant (as the action changes by a a total derivative) under the gauge transformation

$$A^{(+)}_{\mu} \to A^{(+)}_{\mu} + \partial_{\mu}\Lambda^{(+)} \tag{4.35}$$

for the gauge field $A^{(+)}_{\mu}$ and the BRST transformations[12–15]

$$\begin{aligned} A^{(-)}_{\mu} &\to A^{(-)}_{\mu} + \theta\,\partial_{\mu}C, \\ \bar{C} &\to \bar{C} + 2\,\frac{\lambda^2_{\rm R}}{|\eta|}\,\theta\,A^{(-)}_{0}, \\ C &\to C, \end{aligned} \tag{4.36}$$

for the gauge field $A^{(-)}_{\mu}$ and for the pair $\bar{C}$ and C of ghost fields. Here, θ is a global Grassmannian parameter of the BRST transformation. Observe that the transformation of the $A^{(-)}_{\mu}$ is essentially a gauge transformation with parameter $\theta\, C$. The form of the BRST transformation shows that when the Rashba coupling constant $\lambda_{\rm R} \to 0$, the ghosts fields no longer transform and we recover the transformations (4.3) with the identification $\Lambda^{(-)} \equiv \theta\, C$. That is the reason for which we do not need to invoke the ghosts fields in the doubled Chern-Simons theory. The BRST approach in the up-down basis is discussed in the appendix C.

On the one hand, the inclusion of the ghost action in (4.32) in our effective field theory is innocuous, for the ghost fields can be thought of as being hidden, i.e., integrated out, and it is a mere matter of convenience to make them explicit. On the other hand, the inclusion of the ghost action is important to understand how the bulk effective theory (4.32) delivers gapless edge states.

In the presence of the space manifold (4.8) and the space-time manifold (4.10), the action (4.33b) is no longer invariant under the gauge and BRST transformations (4.35) and (4.36), respectively. The action (4.33b changes under the transformations (4.35) and (4.36) by the boundary action

$$\begin{aligned} \delta S = &-\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\Lambda^{(+)}\,\left(\partial_t A^{(-)}_{1} - \partial_x A^{(-)}_{0}\right)\Big|_{y=0} \\ &-\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\theta\, C\,\left(\partial_t A^{(+)}_{1} - \partial_x A^{(+)}_{0}\right)\Big|_{y=0}. \end{aligned} \tag{4.37}$$

Invariance under the transformations (4.35) and (4.36) is achieved by the partition function with the action

$$S_{\rm tot} := I + S_{\rm ghost} + S_{\rm edge}, \tag{4.38a}$$

where

$$\begin{aligned} S_{\rm edge} := &\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\varphi^{(+)}\,\left(\partial_t A^{(-)}_{1} - \partial_x A^{(-)}_{0}\right) \\ &+\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\varphi^{(-)}\,\left(\partial_t A^{(+)}_{1} - \partial_x A^{(+)}_{0}\right), \end{aligned} \tag{4.38b}$$

and the edge fields $\varphi^{(\pm)}$ transform according to the law

$$\varphi^{(+)} \to \varphi^{(+)} + \Lambda^{(+)}, \qquad \varphi^{(-)} \to \varphi^{(-)} + \theta\, C. \tag{4.38c}$$

The action on the edge (4.38b) is none but the action (4.18b). The gauge fixing from Sec. IV A 2 is applicable and delivers

$$S_{\rm edge} = \frac{1}{4\pi} \int \mathrm{d}x\, \mathrm{d}t \, \left(\partial_t \varphi^{(-)} \, \partial_x \varphi^{(+)} + \partial_t \varphi^{(+)} \, \partial_x \varphi^{(-)} - v \, \partial_x \varphi^{(+)} \, \partial_x \varphi^{(+)} - v \, \partial_x \, \varphi^{(-)} \, \partial_x \varphi^{(-)} \right), \tag{4.39}$$

in agrement with Eqs. (4.17) and (4.31). Hence, we have shown that the existence of a single pair of gapless helical edge states in the quantum-spin Hall effect is robust to the adiabatic switching of a Rashba spin-orbit coupling $|\lambda_{\rm R}| \ll |\eta|$.

V. DISCUSSION

In this work, we have obtained the low-energy and long-wave length effective field theory that encodes the Kane-Mele model with a dominant intrinsic spin-orbit coupling and a subdominant Rashba spin-orbit coupling at vanishing uniform and staggered chemical potentials[16]. Without Rashba spin-orbit coupling the fermionic Lagrangian density (2.4a) has the Lorentz, $U(1) \times U(1)$ gauge, and time-reversal symmetries. All these symmetries are encoded by the doubled Chern-Simons effective Lagrangian density (4.9) that follows from integrating out the fermions to lowest order in a gradient expansion.

The effect of the Rashba coupling is the additive correction $(A^{(-)}_0)^2$ to the double Chern-Simons Lagrangian that breaks the gauge invariance of $A^{(-)}_\mu$ as well as the Lorentz symmetry of the theory, while preserving time-reversal symmetry. On the other hand, the gauge invariance of $A^{(+)}_\mu$ is preserved due to the conservation of electric charge.

The requirement of gauge invariance when the physics in the bulk and at the boundaries are treated on equal footing is the ingredient sufficient to establish the bulk-edge correspondence for the doubled Chern-Simons with the Lagrangian density (4.9). However, the correction due to the Rashba coupling partially breaks the $U(1) \times U(1)$ gauge invariance down to $U(1)$. Nevertheless, topological arguments constructed from the Bloch states associated with the band electrons guarantee the existence of an odd number of pairs of gapless helical edge states whenever $|\lambda_{\rm R}|$ is small compared to the spin-orbit coupling $|\eta|$. Thus, the question is how to determine the bulk-edge correspondence, in this case without the $U(1) \times U(1)$ gauge invariance. Our strategy was to interpret the correction $\sim (A^{(-)}_0)^2$ as a gauge fixing term for the $A^{(-)}_\mu$ field. In this way, in replacement of the $U(1)$ gauge (residual spin) symmetry, we find a BRST symmetry after the appropriate ghost action is accounted for. For a manifold with a boundary, the BRST symmetry delivers the bulk-edge correspondence leading to a pair of gapless helical edge states. As there is no interaction between ghost and gauge fields and as the BRST transformation reduces to the usual gauge transformations in the limit $\lambda_{\rm R} \to 0$, the $U(1) \times U(1)$ gauge symmetry of the doubled Chern-Simons action is recovered in the $\lambda_{\rm R} \to 0$ limit.

Having succeeded in establishing the bulk-edge correspondence in the presence of the Rashba spin-orbit coupling using the BRST symmetry, we now turn the discussion to open problems. The approach we proposed needs to be extended to more general situations in which time-reversal symmetry is preserved. This is the case when we consider arbitrary vectors V_i and W_i ($W_i = V^*_i$ and $V_0 = W_0 = 0$). From our one-loop calculations, we infer that the low energy effective Lagrangian density is

$$\begin{aligned} \mathcal{L} = {}& \epsilon^{\mu\nu\lambda} \, \left(A^{(+)}_\mu \, \partial_\nu A^{(-)}_\lambda + A^{(-)}_\mu \, \partial_\nu A^{(+)}_\lambda \right) \\ & - \frac{\lambda^2_{\rm R}}{|\eta|} \, f^{\mu\nu} \, A^{(-)}_\mu \, A^{(-)}_\nu , \end{aligned} \tag{5.1}$$

where $f^{\mu\nu}$ is an arbitrary real-valued symmetric matrix with $f^{0i} = 0$ that can be read from Eq. (3.25a), i.e.,

$$f^{\mu\nu} \propto \left[\frac{1}{2} \, g^{\mu\nu} \, (V \cdot W) - \frac{1}{2} \, (V^\mu \, W^\nu + V^\nu W^\mu) \right]. \tag{5.2}$$

One verifies that the BRST procedure cannot be directly applied to this more general situation. This is so because the correction $f^{\mu\nu} \, A^{(-)}_\mu \, A^{(-)}_\nu$ does not correspond to a gauge fixing term. In the sense of the gauge fixing, it fixes more components than allowed by gauge invariance. Thus, a remaining problem is how to determine the bulk-edge correspondence in this situation.

It is encouraging to view the problem from the following perspective. We do know that the bulk, described by Eqs. (5.1) and (5.2), does have gapless edge modes, because it realizes a $\mathbb{Z}_2$ topological insulator. We succeeded in uncovering a weaker symmetry than the $U(1)$ gauge symmetry to establish the bulk-edge correspondence when a small Rashba spin-orbit coupling is present. The BRST symmetry is perhaps sufficient, yet not necessary, to establish the bulk-edge correspondence, in which case a weaker condition than BRST would be the guarantor for the bulk-edge correspondence.

VI. ACKNOWLEDGMENTS

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Appendix A: Some useful properties of Dirac matrices

In this appendix, we recall some properties of Dirac matrices useful in the calculation of the Feynman diagrams contributing to the effective action. The first property is the product of two Dirac matrices, that can be decomposed as

$$\begin{aligned}\Gamma^\mu \Gamma^\nu &= \frac{1}{2}\left\{\Gamma^\mu, \Gamma^\nu\right\} + \frac{1}{2}\left[\Gamma^\mu, \Gamma^\nu\right] \\ &= g^{\mu\nu} - \mathrm{i}\epsilon^{\mu\nu\rho}\,\Gamma_\rho,\end{aligned} \tag{A1}$$

where we used $[\Gamma^\mu, \Gamma^\nu] = -2\mathrm{i}\epsilon^{\mu\nu\rho}\,\Gamma_\rho$, with the convention $\epsilon^{012} \equiv 1$. This property enable us to reduce the number of Dirac matrices in products with several matrices. From Eq. (A1), we may easily obtain the trace of products of Dirac matrices

$$\mathrm{Tr}\,(\Gamma^\mu\,\Gamma^\nu) = 4g^{\mu\nu}, \tag{A2}$$

$$\mathrm{Tr}\,\left(\Gamma^\mu\,\Gamma^\nu\,\Gamma^\rho\right] = -4\mathrm{i}\epsilon^{\mu\nu\rho}, \tag{A3}$$

and

$$\mathrm{Tr}\,[\Gamma^\mu\,\Gamma^\nu\,\Gamma^\rho\,\Gamma^\sigma] = 4(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma} + g^{\mu\sigma}\,g^{\rho\nu}). \tag{A4}$$

A helpful property involving the Levi-Civita tensor is

$$\epsilon^{\mu\nu\sigma}\,\epsilon^{\alpha\beta}{}_\sigma = g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha}. \tag{A5}$$

Appendix B: Calculation of Diagrams

This appendix is dedicated to the calculation of Feynman diagrams involved in the determination of the low energy effective field theory underlying the Kane-Mele Hamiltonian.

We will discuss a procedure to obtain the expression for the tensor $A^{\mu\nu\alpha\beta}$ given in Eq. (3.13). At first, we need to deal with the following trace of Dirac matrices

$$\mathrm{Tr}\,\left[\Gamma^\mu\,(\not{k}+\eta)\,\Gamma^\alpha\,(\not{k}-\eta)\,\Gamma^\beta\,(\not{k}+\eta)\Gamma^\nu\,(\not{k}+\eta)\right]. \tag{B1}$$

Note that we can reduce the number of Dirac matrices in this product by using the algebra of Dirac matrices (2.3c) and the commutator $[\Gamma^\mu, \Gamma^\nu] = -2\mathrm{i}\epsilon^{\mu\nu\rho}\,\Gamma_\rho$. So we have

$$(\not{k}+\eta)\,\Gamma^\mu\,(\not{k}+\eta) = 2k^\mu\,(\not{k}+\eta) - (k^2-\eta^2)\,\Gamma^\mu \tag{B2}$$

and

$$(\not{k}+\eta)\,\Gamma^\mu\,(\not{k}-\eta) = (k^2-\eta^2)\,\Gamma^\mu + 2\mathrm{i}(\not{k}+\eta)\epsilon^{\rho\mu\sigma}\,k_\rho\,\Gamma_\sigma. \tag{B3}$$

With this, the trace in (B1) becomes

$$\begin{aligned}\mathrm{Tr}\Big[&2(k^2-\eta^2)\,k^\nu\,k_\rho\,\Gamma^\mu\,\Gamma^\alpha\,\Gamma^\beta\,\Gamma^\rho - (k^2-\eta^2)^2\,\Gamma^\mu\,\Gamma^\alpha\,\Gamma^\beta\,\Gamma^\nu + 4\mathrm{i}k_\lambda\,k_\delta\,k^\nu\,k_\rho\,\epsilon^{\rho\alpha}{}_\sigma\Gamma^\mu\,\Gamma^\lambda\,\Gamma^\sigma\,\Gamma^\beta\,\Gamma^\delta \\ &+ 4\mathrm{i}\eta^2\,k^\nu\,k_\rho\,\epsilon^{\rho\alpha}{}_\sigma\,\Gamma^\mu\,\Gamma^\sigma\,\Gamma^\beta - 2\mathrm{i}(k^2-\eta^2)\,k_\lambda\,k_\rho\,\epsilon^{\rho\alpha}{}_\sigma\,\Gamma^\mu\,\Gamma^\lambda\,\Gamma^\sigma\,\Gamma^\beta\,\Gamma^\nu\Big],\end{aligned} \tag{B4}$$

where we discarded terms with an odd number of loop momentum that vanish when integrated. For the terms involving a product of five Dirac matrices, it is convenient to use the decomposition $\Gamma^\sigma\,\Gamma^\beta = g^{\sigma\beta} - \mathrm{i}\epsilon^{\sigma\beta\eta}\,\Gamma_\eta$ in order to reduce the number of matrices. After that, eliminating the terms involving two Levi-Civita with one contracted index by means of the relation (A5), we obtain the following result for the trace

$$\begin{aligned}&8(k^2-\eta^2)\,(g^{\mu\alpha}\,k^\nu\,k^\beta - g^{\mu\beta}\,k^\nu\,k^\alpha + g^{\alpha\beta}\,k^\mu\,k^\nu) - 4(k^2-\eta^2)^2\,(g^{\mu\alpha}\,g^{\beta\mu} - g^{\mu\beta}\,g^{\alpha\nu} + g^{\mu\nu}\,g^{\alpha\beta}) \\ &+ 32\,k^\mu\,k^\nu\,k^\alpha\,k^\beta - 16\,k^2\,(g^{\mu\alpha}\,k^\beta\,k^\nu + g^{\alpha\beta}\,k^\mu\,k^\nu) - 16\,\eta^2(g^{\alpha\beta}\,k^\mu\,k^\nu - g^{\mu\alpha}\,k^\beta\,k^\nu) + 8\,(k^2-\eta^2)\,\epsilon^{\rho\alpha\beta}\,\epsilon^{\sigma\mu\nu}\,k_\rho\,k_\sigma \\ &- 8\,(k^2-\eta^2)\,(g^{\nu\alpha}\,k^\mu\,k^\beta - g^{\mu\alpha}\,k^\nu\,k^\beta + g^{\mu\nu}\,k^\alpha\,k^\beta) + 8\,k^2\,(k^2-\eta^2)\,g^{\alpha\beta}\,g^{\mu\nu}.\end{aligned} \tag{B5}$$

We can take advantage of the Lorentz invariance to do the following replacements

$$k^\mu\,k^\nu \to \frac{1}{D}\,g^{\mu\nu}\,k^2 \tag{B6}$$

and

$$k^\mu\,k^\nu\,k^\alpha\,k^\beta \to \frac{1}{D\,(D+2)}\,(g^{\mu\nu}\,g^{\alpha\beta} + g^{\mu\alpha}\,g^{\nu\beta} + g^{\mu\beta}\,g^{\nu\alpha})\,(k^2)^2, \tag{B7}$$

that are valid under the momentum integration. For our case $D = 3$. The result is

$$\begin{aligned} A^{\mu\nu\alpha\beta} = &\frac{8}{3}\,(g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha} + g^{\alpha\beta}\,g^{\mu\nu})\,J_3^1 - 4\,(g^{\mu\alpha}\,g^{\beta\mu} - g^{\mu\beta}\,g^{\alpha\nu} + g^{\mu\nu}\,g^{\alpha\beta})\,J_2^0 \\ &+ \frac{32}{15}\,(g^{\mu\nu}\,g^{\alpha\beta} + g^{\mu\alpha}\,g^{\nu\beta} + g^{\mu\beta}\,g^{\nu\alpha})\,J_4^2 - \frac{16}{3}\,(g^{\mu\alpha}\,g^{\beta\nu} + g^{\alpha\beta}\,g^{\mu\nu})\,J_4^2 \\ &- 16\,\eta^2\,(g^{\alpha\beta}\,g^{\mu\nu} - g^{\mu\alpha}\,g^{\beta\nu})\,J_4^1 + \frac{8}{3}\,(g^{\alpha\mu}\,g^{\beta\nu} - g^{\alpha\nu}\,g^{\beta\mu})J_3^1 \\ &- \frac{8}{3}\,(g^{\nu\alpha}\,g^{\mu\beta} - g^{\mu\alpha}\,g^{\nu\beta} + g^{\mu\nu}\,g^{\alpha\beta})\,J_3^1 + 8\,g^{\alpha\beta}\,g^{\mu\nu}\,J_3^1, \end{aligned} \tag{B8}$$

where we defined the integral $J_{P_1}^{P_2}$ to be

$$\begin{aligned} J_{P_1}^{P_2} &\equiv \int \frac{\mathrm{d}^D k}{(2\pi)^3}\,\frac{(k^2)^{P_1}}{(k^2 - \eta^2 + \mathrm{i}\epsilon)^{P_2}} \\ &= \mathrm{i}(-1)^{P_1 - P_2}\,\frac{\Omega_D}{(2\pi)^d}\,\frac{1}{\Gamma(P_2)}\,\Gamma\left(\frac{2P_1 + D}{2}\right)\,\Gamma\left(\frac{2P_2 - 2P_1 - D}{2}\right)\,\frac{1}{|\eta|^{2P_2 - 2P_1 - D}}, \end{aligned} \tag{B9}$$

with $\Omega_D \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}$. Using this result in (B8), we obtain (3.13),

$$A^{\mu\nu\alpha\beta} = \frac{\mathrm{i}}{12\,\pi\,|\eta|}\,\left(g^{\mu\nu}\,g^{\alpha\beta} - g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha}\right). \tag{B10}$$

A second type of 4-index tensor useful to deal with the diagrams in Figs. 7 and 8 is

$$B^{\mu\nu\alpha\beta} \equiv \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\,\mathrm{Tr}\,\left(\Gamma^\mu\,S^u(k)\,\Gamma^\alpha\,S^d(k)\,\Gamma^\nu\,S^d(k)\,\Gamma^\beta\,S^u(k)\right). \tag{B11}$$

The trace we need to consider is

$$\mathrm{Tr}\,\left[(\not{k} + \eta)\,\Gamma^\mu\,(\not{k} + \eta)\,\Gamma^\alpha\,(\not{k} - \eta)\,\Gamma^\nu\,(\not{k} - \eta)\,\Gamma^\beta\right]. \tag{B12}$$

Observe that this expression is more symmetric than Eq. (B1). In this case, we can use

$$(\not{k} + \eta)\,\Gamma^\mu\,(\not{k} + \eta) = 2\,k^\mu\,(\not{k} + \eta) - (k^2 - \eta^2)\,\Gamma^\mu \tag{B13}$$

and

$$(\not{k} - \eta)\,\Gamma^\nu(\not{k} - \eta) = 2\,k^\nu\,(\not{k} - \eta) - (k^2 - \eta^2)\,\Gamma^\nu. \tag{B14}$$

After that, by following essentially the same steps that yielded Eq. (B9), we obtain

$$B^{\mu\nu\alpha\beta} = -\frac{\mathrm{i}}{6\pi\,|\eta|}\,\left(g^{\mu\nu}\,g^{\alpha\beta} - g^{\mu\alpha}\,g^{\nu\beta} - g^{\mu\beta}\,g^{\nu\alpha}\right). \tag{B15}$$

Appendix C: BRST approach in the up-down basis

We will discuss the BRST approach with the gauge fields in the up-down basis. The action (4.32) written in terms of A^u_μ and A^d_μ fields is

$$S_{\mathrm{eff}} = -\frac{1}{2\pi}\int \mathrm{d}^3 x\left[\epsilon^{\mu\nu\lambda}\,\left(A^u_\mu\,\partial_\nu A^u_\lambda - A^d_\mu\,\partial_\nu A^d_\lambda\right) - \frac{\lambda_{\mathrm{R}}^2}{2|\eta|}\,(A^u_0 - A^d_0)^2\right]. \tag{C1}$$

The Rashba term breaks the gauge invariance of both A^u_μ and A^d_μ fields. So it is natural to expect the existence of two types of gauge fields (C^u, $\bar{C}^u$) and (C^d, $\bar{C}^d$). The ghost action is

$$S_{\mathrm{ghost}} = -\frac{1}{2\pi}\int \mathrm{d}^3 x\,\left(\bar{C}^u\,\partial_0 C^u + \bar{C}^d \partial_0 C^d\right). \tag{C2}$$

The BRST transformations are

$$\begin{aligned} &A^u_\mu \to A^u_\mu + \theta\, \partial_\mu C^u, \\ &C^u \to C^u, \\ &\bar{C}^u \to \bar{C}^u + \frac{\lambda_{\mathrm{R}}^2}{|\eta|}\, \theta\, (A^u_0 - A^d_0), \end{aligned} \tag{C3}$$

and

$$\begin{aligned} &A^d_\mu \to A^d_\mu + \theta\, \partial_\mu C^d, \\ &C^d \to C^d, \\ &\bar{C}^d \to \bar{C}^d - \frac{\lambda_{\mathrm{R}}^2}{|\eta|}\, \theta\, (A^u_0 - A^d_0). \end{aligned} \tag{C4}$$

Under these transformations the variation of the action (C1) is the surface term

$$\delta S_{\mathrm{eff}} = -\frac{1}{2\pi} \int \mathrm{d}^3x\, \partial_\mu \left[\epsilon^{\mu\nu\lambda} \left(\theta\, C^u\, \partial_\nu A^u_\lambda - \theta\, C^d\, \partial_\nu A^d_\lambda\right)\right]. \tag{C5}$$

If we choose a manifold with a boundary at $y = 0$, as before, we obtain the edge contribution

$$\begin{aligned} \delta S_{\mathrm{edge}} = &-\frac{1}{2\pi} \int \mathrm{d}x\, \mathrm{d}t\, \left[\theta\, C^u\, (\partial_t A^u_1 - \partial_x A^u_0)\right. \\ &\left. - \theta\, C^d\, (\partial_t A^d_1 - \partial_x A^d_0)\right]. \end{aligned} \tag{C6}$$

By analyzing the symmetries of the edge we can find the edge states.

We can connect the above construction with the discussion in the text by passing to the $\pm$ basis. We introduce the gauge fields $A^{(\pm)}_\mu \equiv A^u_\mu \pm A^d_\mu$ and similar definitions for the ghost fields $C^{(\pm)} \equiv C^u \pm C^d$ and $\bar{C}^{(\pm)} \equiv \bar{C}^u \pm \bar{C}^d$. With this, the gauge action is given by Eq. (4.32) whereas the ghost action (C2) becomes

$$S_{\mathrm{ghost}} \propto \int \mathrm{d}^3x\, \left(\bar{C}^{(+)}\, \partial_0 C^{(+)} + \bar{C}^{(-)}\, \partial_0 C^{(-)}\right). \tag{C7}$$

This is not the action we constructed in Eq. (4.33a). We have the presence of additional ghosts degrees of freedom. However, according to Eqs. (C3) and (C4), we see that the transformations of the ghosts fields $C^{(\pm)}$ and $\bar{C}^{(\pm)}$ are

$$\delta\bar{C}^{(+)} = 0, \qquad \delta C^{(+)} = 0 \tag{C8}$$

and

$$\delta\bar{C}^{(-)} = \frac{2\lambda_{\mathrm{R}}^2}{|\eta|}\, \theta\, (A^u_0 - A^d_0), \qquad \delta C^{(-)} = 0, \tag{C9}$$

besides the transformation of the gauge fields $\delta A^{(\pm)}_\mu = \theta\, \partial_\mu C^{(\pm)}$. The ghosts fields $\bar{C}^{(+)}$ and $C^{(+)}$ do not transform and hence the contribution $\bar{C}^{(+)}\partial_0 C^{(+)}$ can be discarded from the action (C7), yielding the desired result with the identifications $\bar{C}^{(-)} \equiv \bar{C}$ and $C^{(-)} \equiv C$. The transformation of the gauge field $A^{(+)}_\mu$ becomes the usual gauge transformation with parameter $\theta\, C^{(+)} \equiv \Lambda^{(+)}$.

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