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Landau Level Mixing and the Fractional Quantum Hall Effect

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We derive effective Hamiltonians for the fractional quantum Hall effect in $n = 0$ and $n = 1$ Landau levels that account perturbatively for Landau level mixing by electron-electron interactions. To second order in the ratio of electron-electron interaction to cyclotron energy, Landau level mixing is accounted for by constructing effective interaction Hamiltonians that include two-body and three-body contributions characterized by Haldane pseudopotentials. Our study builds upon previous treatments, using as a stepping stone the observation that the effective Hamiltonian is fully determined by the few-body problem with $N = 2$ and $N = 3$ electrons in the partially filled Landau level. For the $n = 0$ case we use a first quantization approach to provide a compact and transparent derivation of the effective Hamiltonian which captures a class of virtual processes omitted in earlier derivations of Landau-level-mixing corrected Haldane pseudopotentials.

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I. INTRODUCTION

In a two-dimensional electron gas external magnetic fields form macroscopically degenerate groups of single-particle kinetic energy eigenstates known as Landau levels (LLs). Many-electron state degeneracies grow exponentially with system size when these Landau levels are fractionally occupied and electron-electron interactions and disorder are ignored. In this limit the zero temperature chemical potential μ is constant over integer width intervals of the Landau level filling factor ν, jumping between single-particle eigenenergies at integer values of ν. ($\nu \equiv N/N_{LL}$ where N is the number of electrons in the gas and N_{LL} is the number of single-particle states in each Landau level.) The fractional quantum Hall effect[1] is a consequence of jumps in μ at non-integer values of ν, and therefore can occur only in interacting electron systems.

Because of the exponential degeneracy of the many-body ground state when interactions are neglected, it is not possible to understand the fractional quantum Hall effect by treating interactions as a weak perturbation. Instead, the problem of interactions in systems with partially filled Landau levels has traditionally been simplified by allowing occupation numbers to fluctuate only within the partially filled level. This projection of the interaction Hamiltonian onto a single Landau has a long history in theories of two-dimensional electron systems, and was first employed[2,3] even prior to the fractional quantum Hall effect's discovery.[1] It is strictly justified as a low energy theory, only when the interaction energy per particle is small compared to the energetic separation between the Landau levels. In this article we derive effective Hamiltonians which account for corrections to the projected Hamiltonian that are valid to leading order in interaction strength. These corrections account for quantum fluctuations in otherwise empty and full Landau levels, and are therefore normally referred to as Landau level mixing corrections.

Because of its non-perturbative character, the problem of interactions in a system with a partially filled Landau level has been a rich source of unique correlated electron phenomena, including fractional and non-Abelian quasiparticle statistics,[4] and electron-hole pair superfluidity.[5] We limit our attention in this paper to the case of a two dimensional electron system with parabolic bands. The semiconductor quantum well systems in which the fractional quantum Hall effect has most often been observed are well described by such a model. The small parameter on which our analysis is based is the ratio of characteristic interaction and kinetic energy parameters in the strong magnetic field limit of a two-dimensional parabolic band:

$$\kappa \equiv \frac{e^2}{\hbar\omega_c \epsilon l_0}, \tag{1}$$

where $\omega_c = eB/m^*c$ is the cyclotron frequency, $l_0 = \sqrt{\hbar c/eB}$ is the magnetic length, m^* is the parabolic band effective mass, and ϵ is the low-frequency dielectric constant of the environment hosting the two dimensional electron system. (Hereafter $\hbar = 1$.) Note that κ varies as $1/\sqrt{B}$, reaching its smallest values at the largest fields. The fact that the projected Hamiltonian is able to provide an adequate description of most properties of systems with fractionally filled Landau levels is, at first sight, somewhat surprising. In electron-doped gallium arsenide (GaAs), for example, $\kappa \sim 0.4$ even at the highest achievable steady magnetic fields, $B \sim 40T$. In hole-doped GaAs,[6] AlAs,[7] and in the recently studied ZnO heterostructures[8] effective masses are larger, reducing the cyclotron energy and increasing κ further, as summarized in Table I. Landau level mixing is also expected to be substantial in graphene[9] and in silicon quantum wells.[10] Additionally, some of the most interesting fractional quantum Hall states occur in higher levels, and therefore at weaker magnetic fields and hence larger κ values. Our goals in this paper are to shed light on why the influence of LL mixing on the fractional quantum Hall effect is often modest, and to make progress in understanding its role when it is essential.

There are many specific motivations for the study of interaction induced LL mixing. One is to attempt to bring theory and experiment into closer quantitative agreement in cases where the qualitative picture is already understood. It is widely recognized, for example, that theoretically predicted FQHE gaps are invariably larger than the experimentally measured ones.[11] Although part of the discrepancy can be attributed to disorder and to finite quantum well widths,[12,13] LL mixing is also expected to play an important role.[12,14–18]

Another motivation is to study the emergence of phases which would otherwise be unstable. Of particular interest is the potential role of LL mixing on the stabilization of the Moore-Read Pfaffian state,[19] generally believed to describe the incompressible state observed at filling fraction $\nu = 5/2$.[20,21] Indeed the Pfaffian and its particle-hole conjugate, the anti-Pfaffian, are distinct phases,[22,23] which are energetically degenerate in the absence of LL mixing. Which of these two states appears experimentally is completely determined by the particle-hole symmetry breaking terms that Landau level mixing generates. This state selection property for half-filled LLs applies to any incompressible state that is not particle-hole invariant. Many numerical studies support the view that a state of the Moore-Read type is favored at $\nu = 5/2$,[24–34] but the detailed form of the LL mixing would determine which of the two particle-hole mirror states is preferred. It is still unsettled which one of these two states is selected by the LL mixing induced by pure Coulomb interactions. One study has found the Pfaffian state to be favored,[35] while another has found the anti-Pfaffian to be favored,[36] although their detailed account for the LL mixing was different. Depending on the details of the confinement, the intersubband LL mixing could also play an important role,[37] specially in wide wells, where experimental studies have highlighted its influence of on the stability of the $\nu = 5/2$ state.[38]

Additionally, differences in gap sizes between filling factor $n+\nu$ and $n+1-\nu$ (or $2n+\nu$ and $2n+2-\nu$ when spin is an active degree-of-freedom), observed conspicuously in GaAs[39] and in large κ systems like AlAs[7,40] and hole-doped GaAs,[6] require particle-hole symmetry breaking and hence LL mixing. Finally, particle-hole asymmetries in critical densities for the emergence of charge density wave states at low particle or hole densities[41] also reflect LL mixing.

In this article we will construct an effective Hamiltonian which accounts for LL mixing to leading perturbative order in κ. This many-body Hamiltonian when solved exactly will be able to predict energies, in units of ω_c, correctly to order κ^2, and the projected many-body wavefunctions into the Landau level of interest to order κ. It is interesting to note that the new energy scale appearing in our analysis, $\omega_c\kappa^2$, is independent of magnetic field strength (unlike the dominant interaction scale of the FQHE, $\omega_c\kappa$, which grows as $\sqrt{B}$), and it is twice the effective Rydberg of the parabolic band system:

$$\omega_c\kappa^2 = \frac{m^* e^4}{\epsilon^2}. \tag{2}$$

TABLE I: Relative strength of Coulomb and cyclotron energies in GaAs, AlAs[7] and ZnO[8] (B is measured in Tesla).

	m^*/m_0	ϵ	κ
electron GaAs	0.069	13	$2.6/\sqrt{B}$
hole GaAs	0.39	13	$14.6/\sqrt{B}$
electron ZnO	0.29	8.5	$16.7/\sqrt{B}$
electron AlAs	0.46	10	$22.5/\sqrt{B}$

One of the earliest studies to account numerically for LL mixing in the second Landau level was performed by Rezayi and Haldane.[42] Analytical perturbative studies of LL mixing due to interactions were first carried out in the lowest Landau level by Murthy and Shankar.[18] These studies were subsequently extended to the second Landau level by Bishara and Nayak,[43] employing an analysis similar to the renormalization group (RG) for fermions. The present work complements these earlier papers by presenting new methods of derivation, adding some new results, and correcting some previous results. Our work has been developed essentially in parallel with two recent studies by Peterson and Nayak,[9] and Simon and Rezayi,[44] and our findings are largely in agreement with these two studies. A key observation in our approach is that the effective many body Hamiltonian can be constructed by solving $N = 2$ and $N = 3$ few-body problems, which we use to simplify some derivations. We compute two-body[45,46] and three-body[43,47,48] generalized Haldane pseudopotentials for these interactions, which can be incorporated into many-body numerical diagonalization studies. Although the values we list here for these pseudopotentials are specialized to the case of 2D Coulomb interactions, we have derived analytic and semianalytic formulae for all the pseudpotentials valid for any rotationally invariant interaction. Pseudopotential parameter values for more realistic interaction models which account for finite quantum well widths can be conveniently computed from these expressions.

Our paper is organized as follows. Section II provides a compact derivation of the effective Hamiltonian in the lowest Landau level ($n = 0$ LL) that is based on a first quantization formalism, and is valid for both bosons and fermions. In Sec. III we construct the effective Hamiltonian for a partially filled first excited Landau level ($n = 1$ LL), and compute its two-body and three-body generalized Haldane pseudopotentials. In Sec. IV we derive an effective Hamiltonian valid for $1 < \nu < 2$ that is valid for the special case of maximally polarized electronic states in which the majority spin state is full and the minority spin state is partially occupied. We also compute the two and three-body generalized Haldane pseudopotentials appropriate for these effective Halmitonians. In Sec. V we summarize our findings and present conclusions. We have relegated discussions of some effective interaction properties and calculation details to a series of appendices.

II. LOWEST LANDAU LEVEL

A. Many body Hamiltonian to order κ^2

We consider a two dimensional electron system subjected to a perpendicular magnetic field $\mathbf{B} = -B\mathbf{e_z}$. Measuring all energies in units of the cyclotron frequency and all lengths in units of the magnetic length, the single-particle spectrum of a non-interacting disorder-free spinless system consists of discrete Landau levels with energies, $\varepsilon_n = n + 1/2$.[49,50] The non-interacting Hamiltonian (including the Zeeman energy contribution) and the full Hamiltonian including interactions are given respectively by,

$$\mathcal{H}_0 = \sum_i \left(\hat{n}_i + 1/2 - g\sigma_i^z\right), \tag{3}$$

and

$$\mathcal{H} = \mathcal{H}_0 + \kappa \sum_{i<j} v_{ij}, \tag{4}$$

where $\hat{n}_i + 1/2$ is the kinetic energy operator of particle i, $g = g_s m^*/2m_e$ with m_e the mass of the electron in vacuum and g_s the effective g-factor of the host material, and $v_{ij} = 1/|r_i - r_j|$ is the dimensionless Coulomb potential.

Unless the filling fraction ν is an integer, the non-interacting many-body eigenstates are degenerate. From degenerate state perturbation theory, the eigen-energies can be determined to order κ by projecting the interaction term onto the degenerate manifold of non-interacting eigenstates with energy E_0,

$$\mathcal{H}_1 = E_0 + \kappa \sum_{i<j} \mathcal{P} v_{ij} \mathcal{P}, \tag{5}$$

$\mathcal{H}_1$ is the Hamiltonian commonly employed to study the fractional quantum Hall effect. Employing conventional degenerate second order perturbation theory, the correction to the next order, κ^2 in energies and κ in the projected wavefunctions, can be obtained from the effective Hamiltonian,

$$\mathcal{H}_2 = \mathcal{H}_1 - \kappa^2 \sum_{\substack{i<j \\ k<l}} \mathcal{P} v_{ij} \mathcal{P}_\perp \frac{1}{\mathcal{H}_0 - E_0} \mathcal{P}_\perp v_{kl} \mathcal{P}, \tag{6}$$

where $\mathcal{P}$ is the projector into the degenerate non-interacting ground state manifold of energy E_0, and $\mathcal{P}_\perp = 1 - \mathcal{P}$, is the projector onto its orthogonal complement. In the absence of Zeeman energy, $\mathcal{H}_2$ is the first quantization version of the Hamiltonian considered in Ref. 18.

In the case of the $n = 0$ LL, the degenerate manifold would be a subspace of the lowest kinetic energy eigenspace with a definite projection of the total spin along the z-axis, S_z, and Zeeman energy $E_0 = -2gS_z$. Nevertheless, the energy denominator in Eq. (6) only includes the kinetic energy difference between the virtual excited states and the states in the degenerate manifold. This is a consequence of the conservation of S_z, because the virtual excitations produced by the interactions v_{ij} and v_{kl} do not change it, and therefore, the Zeeman energy disappears from this energy denominator. Consequently, for any S_z, we can write the second order correction to the effective Hamiltonian when the lowest-energy $n = 0$ LL is partially filled, as,

$$\mathcal{H}_2 = \mathcal{H}_1 - \kappa^2 \sum_{\substack{i<j \\ k<l}} \mathcal{P} v_{ij} \mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{kl} \mathcal{P}, \tag{7}$$

with $\hat{n} = \sum_i \hat{n}_i$. There are three possibilities for the two pairs of indices (i, j) and (k, l) appearing in the sum in Eq. (7); they can share both particle indices (*i.e.* $i = k$ and $j = l$), they can share only one particle index while the other two are distinct, or they can share no indices. The last possibility does not contribute to $\mathcal{H}_2$ because the projection operators force each virtually excited particle to decay back into the lowest LL after both interactions act. The other two possibilities are non-vanishing and yield respectively two- and three-body effective interactions.[18,43] Below we address the two-body effective interactions first.

B. Two-body interactions

The effective interactions implied by Eq. (7) are independent of the state of the many body system in the $n = 0$ LL. We can therefore, without loss of generality, determine the effective Hamiltonian by considering only the few body $N = 2$ and $N = 3$ cases. The two body interactions can be written as,

$$\mathcal{V}^{2b} = \kappa \mathcal{P} v_{12} \mathcal{P} - \kappa^2 \mathcal{P} v_{12} \mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{12} \mathcal{P}. \tag{8}$$

We have explicitly verified that this interaction is identical to the the two body interaction in Ref. 18. In particular, after translating $\mathcal{V}^{2b}$ from Eq. (8) into its equivalent second quantized version, the piece for which both particles are virtually excited into higher Landau levels corresponds to the interaction δH_{00}^2 in Eq. (26) of Ref. 18, and the piece of Eq. (8) for which only one particle is excited into a higher Landau level corresponds, after normal ordering is performed, to the two body part obtained from δH_{00}^1 in Eq. (30) of Ref. 18.

The translational and rotational invariance of Eq. (8) permits its decomposition into Haldane pseudopotentials. The virtual excitations are most easily analyzed by decomposing the degrees of freedom into center of mass, R, and relative, r, coordinates. The total kinetic energy is then the sum of relative and center of mass kinetic energies, $\hat{n} = \hat{n}_1 + \hat{n}_2 = \hat{n}_r + \hat{n}_R$. Because the interaction acts only on the relative coordinate, only $\hat{n}_r$ enters the energy denominator. We write the two-body Haldane pseudopotentials as $V^{2b}_m = \kappa V_1(m) + \kappa^2 V_2(m)$ where $V_1(m)$ are the standard Haldane pseudopotentials[45,46,49,50] given by

$$V_1(m) = {}_r\langle 0, m|v_{12}|0, m\rangle_r = \frac{\Gamma(m + 1/2)}{2m!}. \quad (9)$$

Here $|n, m\rangle_r$ is an state with the relative guiding center and kinetic energy quantum numbers m and n respectively. Note that the form of the relative motion eigenstates differ from the familiar single-particle eigenstates only because of the difference between relative motion and single-particle motion magnetic lengths $\ell_r = \sqrt{2}\ell_0$. For $V_2(m)$ we obtain:

$$V_2(m) = -\sum_{n=1}^{\infty} \frac{|{}_r\langle n, m+n|v_{12}|0, m\rangle_r|^2}{n} = - \frac{[V_1(m)]^2}{4(m+1)} \, {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{3}{2}\\ 2, 2, m+2\end{matrix}; 1\right], \quad (10)$$

with ${}_4F_3$ the generalized hypergeometric function. Values of these pseudopotentials are presented in Table II, together with the usual Haldane pseudopotentials for comparison. These pseudopotentials agree with those computed in Refs. 9 and 44. Additionally very closely related numbers have been computed previously in a strong magnetic-field expansion of the spectrum of hydrogenic atoms.[51] $V_2(m)$ equals $\pi\alpha_{0,m}/4$ where $\alpha_{0,m}$ values are specified by Eq. (15) and Table II of Ref 51.

Our results for the leading order Haldane pseudopotential corrections differ from those presented in Table I of Ref. 43. The reason for this discrepancy is that the set of virtual processes in which only one particle is excited into higher Landau levels was omitted in Ref. 43. This discrepancy has been recently solved in Ref. 9. It was shown there that, in the RG language, these additional processes arise from keeping track of the correct normal ordering of the three-body interactions which give rise to additional contributions to the two-body interactions.

Since we have made no explicit reference to the statistics of the particles involved, our results apply equally well to fermions and bosons. The difference between the two cases is only in the constraint imposed by quantum statistics on the allowed states, which implies that odd (even) m pseudopotentials are associated with spin triplet wavefunction for fermions (bosons) and spin singlets for bosons (fermions). The case of a partially filled $n > 0$ LL, that we will discuss later in Sec. III, is relevant only for fermions since partially filled higher Landau level states are not non-interacting ground states in the bosonic case.

C. Three-body interactions

In order to derive the full Hamiltonian including the three body terms, it is sufficient to consider a $N = 3$ few particle problem. Three-body interactions follow from the terms in Eq. (7) where only one index is shared between the pairs (i, j) and (k, l) while the other two indices are distinct, *e.g.* $i = k$ but $j \neq l$. This contribution can be written as

$$\mathcal{V}^{3b} = -\kappa^2 \sum_{s \in S_3} \Pi_s \mathcal{P} v_{13} \mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{12} \mathcal{P} \Pi_s, \quad (11)$$

where the sum is over the six permutations of three objects and Π_s is the associated unitary permutation operator. Since bosonic and fermionic states are both eigenstates of the permutation operator $\Pi_s|\Psi\rangle = \pm|\Psi\rangle$, it follows that

$$\langle\Phi|\mathcal{V}^{3b}|\Psi\rangle = -6\kappa^2 \langle\Phi|\mathcal{P} v_{13} \mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{12} \mathcal{P}|\Psi\rangle, \quad (12)$$

for arbitrary states $\{|\Psi\rangle, |\Phi\rangle\}$ with the same parity under permutations. Generalized Haldane pseudopotentials for N-body interactions have been thoroughly discussed for spinless particles by Simon, Rezayi, and Cooper in Ref. 47, and for spinful particles by Davenport and Simon in Ref. 48. We specialize hereafter in the case of fermions. There are two possible values for the total spin of three particles, namely $S = 3/2$ and $S = 1/2$. The spatial wavefunction for $S = 3/2$ must be fully antisymmetric, while the one for $S = 1/2$ has mixed symmetry. For $S = 3/2$ we employ the fully antisymmetric wavefunctions for three particles in the lowest LL constructed by Laughlin in Ref. 52, whose polynomial part is

$$\Psi^{3/2}_{kl} = \frac{1}{Z^{3/2}_{kl}} (z_a^2 + z_b^2)^k \left[\frac{(z_a + iz_b)^{3l} - (z_a - iz_b)^{3l}}{2i}\right], \quad (13)$$

with $Z^{3/2}_{kl} = 2^{3l+2k+1}[\pi^3(3l + k)!k!]^{1/2}$, $k \geq 0$, $l \geq 1$, and

$$z_a = \sqrt{\frac{2}{3}} \left(\frac{z_1 + z_2}{2} - z_3\right), \quad z_b = \frac{z_1 - z_2}{\sqrt{2}}. \quad (14)$$

These wavefunctions form a complete orthonormal basis for the relative internal states of three particles, $\langle\Psi^{3/2}_{k'l'}|\Psi^{3/2}_{kl}\rangle = \delta_{k',k}\delta_{l',l}$. The relative orbital angular momentum of the states is $\hat{m}|\Psi^{3/2}_{kl}\rangle = (2k + 3l)|\Psi^{3/2}_{kl}\rangle$. Therefore, it follows from the rotational invariance of interactions that the generalized Haldane pseudopotential matrix satisfies,

TABLE II: Conventional two-body Haldane pseudopotentials in the $n = 0$ LL (V_1 from Eq. (9)), and the coefficients of their leading pertubative corrections (V_2 from Eq.(10)).

m	0	1	2	3	4	5	6	7	8	9
V_1	0.8862	0.4431	0.3323	0.2769	0.2423	0.2181	0.1999	0.1856	0.1740	0.1644
V_2	−0.3457	−0.0328	−0.0112	−0.0055	−0.0033	−0.0022	−0.0015	−0.0012	−0.0009	−0.0007

TABLE III: Coefficients of the leading $S = 3/2$ ($V_{3/2}$ from Eq. (15)) and $S = 1/2$ ($V_{1/2}$ from Eq. (21)) three-body Haldane pseudopotentials in lowest LL. For $S = 3/2$ there are two states with total angular momentum $m = 9$ so the pseudopotential is a 2×2 matrix. For $S = 1/2$ there are two states for $4 \leq m \leq 6$ and Haldane pseudopotentials are matrices, in these cases the listed pairs have been orthonormalized by rotating only the state with $\sigma = 2$, as described in Appendix A.

m	3	5	6	7	8	9
(k,l)	$(0,1)$	$(1,1)$	$(0,2)$	$(2,1)$	$(1,2)$	$(0,3)$ $(3,1)$
$V_{3/2}(k'l',kl)$	−0.0181	0.0033	−0.0107	0.0059	−0.0048	$\begin{pmatrix} -0.0049 & -0.0007 \\ -0.0007 & 0.0052 \end{pmatrix}$

m	1	2	3	4	5	6
(σ,k,l)	$(1,0,0)$	$(2,0,0)$	$(1,1,0)$	$(1,0,1)$ $(2,1,0)$	$(1,2,0)$ $(2,0,1)$	$(1,1,1)$ $(2,2,0)$
$V_{1/2}(\sigma'k'l',\sigma kl)$	−0.0345	−0.0540	0.0425	$\begin{pmatrix} -0.0343 & -0.0025 \\ -0.0025 & 0.0075 \end{pmatrix}$	$\begin{pmatrix} 0.0277 & 0.0067 \\ 0.0067 & -0.0176 \end{pmatrix}$	$\begin{pmatrix} -0.0119 & -0.0050 \\ -0.0050 & 0.0102 \end{pmatrix}$

$$\langle \Psi^{3/2}_{k'l'} | \mathcal{V}^{3b} | \Psi^{3/2}_{kl} \rangle = \kappa^2 \delta_{2k'+3l',2k+3l} V_{3/2}(k'l',kl). \tag{15}$$

In this way we find the $S = 3/2$ pseudopotentials listed in Table III. These pseudopotentials agree with those derived in Refs. 9 and 43. They are also in close agreement with the pseudopotential differences obtained in Ref. 36. Further details on the properties of these states and the derivation of these pseudopotential values can be found in Appendix A.

To construct the $S = 1/2$ states we follow the approach of Davenport and Simon.[48] We start with two primitive polynomials which are antisymmetric only under permutations of variables 1 and 2, and thus have the symmetry of the Young tableau[53]

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \; . \tag{16}$$

These primitive polynomials are,

$$\beta_1 = z_b, \;\; \beta_2 = z_a z_b. \tag{17}$$

The most general polynomials with the symmetry of this tableau are then obtained by multiplying the primitive polynomials by the most general fully symmetric translationally invariant polynomial. We choose a different basis for the fully symmetric polynomials from that employed in Ref. 48, which makes calculations simpler. Our basis for the symmetric polynomilas is the bosonic analog of the fermionic wavefunctions of Ref. 52. By multiplying this bosonic wavefunction by the primitive polynomials, β_σ, we obtain the polynomial part of the $S = 1/2$ fermionic wavefunctions,

$$\Psi^{1/2}_{\sigma kl} = \frac{\beta_\sigma}{Z^{1/2}_{\sigma kl}} (z_a^2 + z_b^2)^k \left[\frac{(z_a + i z_b)^{3l} + (z_a - i z_b)^{3l}}{2} \right], \tag{18}$$

with $\sigma = \{1, 2\}$, $k \geq 0$, $l \geq 0$. The normalization factor for $\sigma = 1$ states is given by $Z^{1/2}_{1kl} = Z^{3/2}_{kl} \sqrt{(1 + \delta_{l,0})(2k + 2 + 3l)}$, and for $\sigma = 2$ it is $Z^{1/2}_{2kl} = Z^{3/2}_{kl} \sqrt{(1 + \delta_{l,0})[(k + 1 + 3l)(k + 2 + 3l) + (k + 1)(k + 2)]}$, where $Z^{3/2}_{kl}$ is the normalization constant of the $\Psi^{3/2}_{kl}$ states appearing in Eq. (13).

The spin part of the $S = 1/2$ wavefunction has the symmetry of the Young tableau conjugate to (16), thus the fully antisymmetric wavefunction is,[48]

$$|\Psi^{1/2}_{\sigma kl}\rangle = \mathcal{A}\{\Psi^{1/2}_{\sigma kl} \otimes |\uparrow\uparrow\downarrow\rangle\}, \tag{19}$$

where $\mathcal{A} = (1 - \Pi_{23} + \Pi_{12}\Pi_{23})/\sqrt{3}$, is the partial antisymmetrization operator between $\uparrow$ and $\downarrow$ particles, and Π_{ij} is the unitary operator corresponding to elementary permutations between i and j. The states $|\Psi^{1/2}_{\sigma kl}\rangle$ are linearly indepenent and complete but not orthogonal:

$$\begin{aligned}\langle\Psi^{1/2}_{\sigma k'l'}|\Psi^{1/2}_{\sigma kl}\rangle &= \delta_{k',k}\delta_{l',l}\\ \langle\Psi^{1/2}_{1k'l'}|\Psi^{1/2}_{2kl}\rangle &= -\pi^3\frac{2^{4k'+6l'+1}}{Z^{1/2}_{1k'l'}Z^{1/2}_{2kl}}[k!(k'+3l')!\delta_{l',l+1}\delta_{k,k'+1}(1+\delta_{l,0})+k'!(k+3l)!\delta_{l,l'+1}\delta_{k',k+2}(1+\delta_{l',0})].\end{aligned} \tag{20}$$

The internal motion orbital angular momentum of these states is $\hat{m}|\Psi^{1/2}_{\sigma kl}\rangle = (\sigma+2k+3l)|\Psi^{1/2}_{\sigma kl}\rangle$. Thus the generalized Haldane pseudopotentials satisfy

$$\langle\Psi^{1/2}_{\sigma'k'l'}|\mathcal{V}^{3b}|\Psi^{1/2}_{\sigma kl}\rangle = \kappa^2\delta_{\sigma'+2k'+3l',\sigma+2k+3l}V_{1/2}(\sigma'k'l',\sigma kl), \tag{21}$$

and have the leading values listed in Table III. These pseudopotentials are also in agreement with those derived in Refs. 9 and 43 where the two leading three-body pseudopotentials for $S=1/2$ were computed. Further details on these states and the $S=1/2$ three-body pseudopotentials can be found in Appendix A.

III. $n=1$ LANDAU LEVEL

A. Many body Hamiltonian to order κ^2

The first quantization analysis presented in section II for a partially filled lowest LL is cumbersome in the case of a partially filled $n=1$ Landau level because of the need to account for virtual excitations of the full $n=0$ LL. In this section we therefore use a second quantization approach. By following a path entirely analogous to that of section II A, we arrive at an effective Hamiltonian to order κ^2 which acts in the $n=1$ LL.

The first order term is as usual simply the projection of the Hamiltonian onto the partially filled level,

$$\mathcal{H}_1 = E_0 + \frac{\kappa}{2}\sum_{\mu\nu m} v_{12,34}\; c^\dagger_{1m_1\nu}c^\dagger_{1m_2\mu}c_{1m_3\mu}c_{1m_4\nu}, \tag{22}$$

where $c^\dagger_{nm\mu}$ is a fermion creation operator in LL n with guiding center quantum number m, μ runs over all the fermion flavors, and E_0 is the non-interacting energy including the Zeeman term. For generality we assume f-flavors (*i.e.* for spin s, $f=2s+1$). To order κ^2 the effective Hamiltonian is

$$\begin{aligned}\mathcal{H}_2 = \mathcal{H}_1 - \frac{\kappa^2}{4}\sum_{\substack{1\cdots 8\\ \mu\nu\lambda\sigma}} v_{12,34}v_{56,78}\times\cdots\\ P_0 c^\dagger_{5\nu}c^\dagger_{6\mu}c_{7\mu}c_{8\nu}P_\perp\frac{1}{\hat{n}-N_1}P_\perp c^\dagger_{1\lambda}c^\dagger_{2\sigma}c_{3\sigma}c_{4\lambda}P_0,\end{aligned} \tag{23}$$

where the integers abbreviate single-particle kinetic and guiding center quantum numbers (*e.g.* $\{1\Leftrightarrow n_1m_1\}$), N_1 is the number of particles in the partially filled $n=1$ LL, P_0 is the projector into the many-body eigenspace with $\hat{n}=N_1$, and $P_\perp$ is the projector into its orthogonal complement.[54]

The classification of all the interaction terms arising from this Hamiltonian is a lengthy bookkeeping exercise that we describe in Appendix B. There are no four body terms, and the one body terms, which we do not compute, account only for exchange interactions with the full Landau level. These produce only a well known overall constant shift of the single-particle energies, that is equivalent to a change in the chemical potential.[42]

The non-vanishing two-body terms satisfy a kinetic energy balance condition, $n_5+n_6=n_3+n_4$, and can take values $n_3+n_4=\{2,1,0\}$ in Eq. (23). We employ the three possible values of this *incoming* kinetic energy as labels for the three allowed terms, labeled as $\{\mathcal{V}^{2b}(2),\mathcal{V}^{2b}(1),\mathcal{V}^{2b}(0)\}$, and find that:

$$\begin{aligned}\mathcal{V}^{2b}(2) &= -\frac{\kappa^2}{2}\sum_{\substack{1\cdots 6\\ \mu\nu}} v_{12,65}v_{56,34}\frac{\theta(n_5,n_6)}{n_5+n_6-2}c^\dagger_{1\nu}c^\dagger_{2\mu}c_{3\mu}c_{4\nu},\\ \mathcal{V}^{2b}(0) &= -\frac{\kappa^2}{2}\sum_{\substack{1\cdots 6\\ \mu\nu}} v_{12,65}v_{56,34}\frac{\delta(n_5,n_6)}{2}c^\dagger_{1\nu}c^\dagger_{2\mu}c_{3\mu}c_{4\nu},\end{aligned} \tag{24}$$

where the indices of the creation/annihilation operators are understood to be on the $n=1$ LL, $\theta(n_5,n_6)$ is a function that restricts $n_5+n_6\geq 3$ and $n_5\geq 1, n_6\geq 1$, and $\delta(n_5,n_6)$ restricts $n_5=n_6=0$. The term $\mathcal{V}^{2b}(1)$ can be split into three terms,

$$\begin{aligned}\mathcal{V}^{2b}(1) &= \mathcal{V}^{2b}_a(1)+\mathcal{V}^{2b}_b(1)+\mathcal{V}^{2b}_c(1),\\ \mathcal{V}^{2b}_a(1) &= -f\kappa^2\sum_{\substack{1\cdots 6\\ \mu\nu}} v_{15,64}v_{26,53}\frac{\tau(n_5,n_6)}{n_6}c^\dagger_{1\nu}c^\dagger_{2\mu}c_{3\mu}c_{4\nu},\\ \mathcal{V}^{2b}_b(1) &= \kappa^2\sum_{\substack{1\cdots 6\\ \mu\nu}} v_{15,36}v_{62,54}\frac{\tau(n_5,n_6)}{n_6}c^\dagger_{1\nu}c^\dagger_{2\mu}c_{3\mu}c_{4\nu},\\ \mathcal{V}^{2b}_c(1) &= \kappa^2\sum_{\substack{1\cdots 6\\ \mu\nu}}(v_{15,46}v_{62,35}+v_{51,46}v_{26,35})\times\cdots\\ &\qquad\frac{\tau(n_5,n_6)}{n_6}c^\dagger_{1\nu}c^\dagger_{2\mu}c_{3\mu}c_{4\nu},\end{aligned} \tag{25}$$

where $\tau(n_5,n_6)$ restricts $n_5=0$ and $n_6\geq 1$. Each of these two-body interactions has a simple diagrammatic representation. In particular $\mathcal{V}^{2b}(2)$ and $\mathcal{V}^{2b}(0)$ correspond to particle-particle ladder diagrams, and $\mathcal{V}^{2b}_a(1)$ is a screening diagram, hence the flavor multiplicity factor. $\mathcal{V}^{2b}_b(1)$ is a particle-hole ladder diagram, and $\mathcal{V}^{2b}_c(1)$ is a vertex correction diagram as summarized in Fig. 1. To

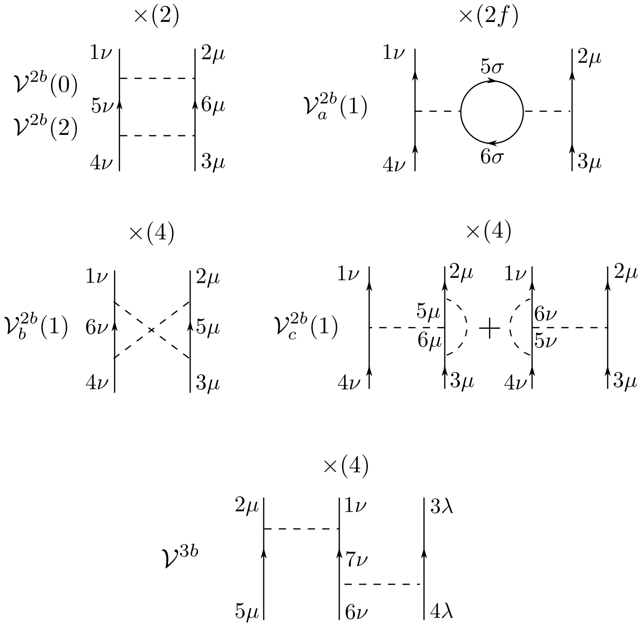

FIG. 1: Diagrams representing two- and three-body interactions at order κ^2 in the $n = 1$ LL. The symbol $\times(k)$ appearing above each diagram represents its multiplicity, *i.e.* there are k different contractions of the operators in Eq. (23) which give rise to the same diagram after relabeling dummy indices. The indices of the fermion lines are consistent with those appearing in Eqs. (24), (25) and (26).

get a feeling for the significant additional complication of the $n = 1$ LL effective interaction calculation, note that from all of these terms only the particle-particle ladder type diagram is present in the $n = 0$ LL. When our first quantization formulation approach was used for the two body interactions in the $n = 0$ LL case, the problem was reduced to the simplicity of a one-body second-order perturbation theory calculation.

By comparison, the three-body interactions are simpler. They can be combined into a single term,

$$\mathcal{V}^{3b} = -\kappa^2 \sum_{\substack{1\cdots 7\\ \lambda\mu\nu}} v_{12,57} v_{73,46} \frac{\omega(n_7)}{n_7 - 1} c^\dagger_{1\nu} c^\dagger_{2\mu} c^\dagger_{3\lambda} c_{4\lambda} c_{5\mu} c_{6\nu}, \tag{26}$$

where again the Landau level indices of the operators are understood to have $n = 1$ and $\omega(n_7)$ restricts $n_7 \neq 1$. The three-body interaction has a simple diagramatic representation depicted in Fig. 1. There is a compact first quantized version of Eq. (26), which for three particles in the $n = 1$ LL is

$$\mathcal{V}^{3b} = -\kappa^2 \sum_{s \in S_3} \Pi_s \mathcal{P} v_{13} \mathcal{P}_\perp \frac{1}{\hat{n} - 3} \mathcal{P}_\perp v_{12} \mathcal{P} \Pi_s, \tag{27}$$

where the notation is the same as that of Eq. (11), except that $\mathcal{P}$ is the projector into the eigenspace with $\hat{n} = \hat{n}_1 + \hat{n}_2 + \hat{n}_3 = 3$, and $\mathcal{P}_\perp$ into its orthogonal complement. This first quantized version significantly simplifies the evaluation of three-body Haldane pseudopotentials which we will discuss later on.

$\mathcal{V}^{2b}(2)$, $\mathcal{V}_a^{2b}(1)$, $\mathcal{V}_b^{2b}(1)$, $\mathcal{V}_c^{2b}(1)$, $\mathcal{V}^{2b}(0)$, and $\mathcal{V}^{3b}$, are separately rotationally and translationally invariant, as discussed in Appendix C. This property guarantees that the interactions can be represented by Haldane pseudopotentials as in the $n = 0$ LL case. To compute the two-body Haldane pseudopotentials, we consider the $N = fN_{LL} + 2$-body problem in which the lowest Landau level is completely full with fN_{LL} particles and only two particles are in the $n = 1$ LL. The state describing the orbital part of the relative motion of two particles in the $n = 1$ LL, $|m\rangle_1$, can be written as,[46]

$$|m\rangle_1 = a_1^\dagger a_2^\dagger |0, m\rangle_r, \tag{28}$$

where $|m, 0\rangle_r$ is the state of relative angular momentum m, of two particles in the $n = 0$ LL discussed next to Eq. (9), and $a_i^\dagger$ is the kinetic energy raising operator for particle i. Note that $|m\rangle_1$ is an entagled state of relative and center of mass quantum numbers as discussed in Appendix C. The usual Haldane pseudopotentials for the $n = 1$ LL are[25,46]

$$\begin{aligned} V_1(m) &= {}_1\langle m|v_{12}|m\rangle_1 = \\ &\qquad \frac{\Gamma(m+1/2)}{2m!} \frac{(m-3/8)(m-11/8)}{(m-1/2)(m-3/2)}. \end{aligned} \tag{29}$$

The Haldane pseudopotentials to order κ^2 are listed in Table IV. Our pseudopotentials are in agreement with those obtained in Refs. 9, 42, and 44, with small discrepancies pressumably arising from numerical error. We believe our pseudopotentials are essentially free from numerical errors because we have converted the effective interaction expressions to first quantization, as dicussed in Appendix C, which allows for very efficient calculations. We have explicitly verified that our two body interactions are equivalent to those of Ref. 9. More specifically, the sum of our ladder type interactions from Eq. (24), namely $\mathcal{V}^{2b}(2) + \mathcal{V}^{2b}(0)$, is the same as the sum of the BCS interaction of Eq. (19), with the two-body interaction appearing in the last line of Eq. (26) in Ref. 9. And the sum of the interactions appearing in Eq. (25), namely $\mathcal{V}_a^{2b}(1) + \mathcal{V}_b^{2b}(1) + \mathcal{V}_c^{2b}(1)$, is identical to the sum of the ZS and ZS' interactions appearing in Eqs. (17) and (18), with the remaining two-body interactions appearing in the second and third line of Eq. (26) in Ref. 9.

It is interesting to note from Table IV that the leading two-body pseudopotentials in the $n = 1$ LL, namely those with $m \leq 3$, are dominated by the screening interaction $\mathcal{V}_a^{2b}(1)$. The contributions from the remaining interactions to these pseudopotentials nearly cancel. In Appendix C we discuss further properties of $\mathcal{V}_a^{2b}(1)$, and explicitly show that it is equivalent to the κ^2 term in the RPA approximation for the statically screened potential in the presence of a completelly filled $n = 0$ LL.

The three-body states of the $N = fN_{LL}+3$ body problem in the $n = 1$ LL are mapped from those in the $n = 0$ LL by raising the kinetic energy of the three particles,

$$\begin{aligned} |\Psi^{3/2}_{kl}\rangle_1 &= a_1^\dagger a_2^\dagger a_3^\dagger |\Psi^{3/2}_{kl}\rangle, \\ |\Psi^{1/2}_{\sigma kl}\rangle_1 &= a_1^\dagger a_2^\dagger a_3^\dagger |\Psi^{1/2}_{\sigma kl}\rangle. \end{aligned} \tag{30}$$

With this construction the orthonormality of the $S = 3/2$ states in $n = 0$ LL and Eqs. (15),(20), and (21) are immediately extended to the $n = 1$ LL three body states. The three body Haldane pseudopotentials we obtain are listed in Table V. Further properties of these states and of the corresponding pseudopotentials are given in Appendix A. The three-body Haldane pseudopotentials in the $n = 1$ LL are in agreement with those computed in Refs. 9 and 43. There is a discrepancy with those obtained numerically in Ref. 44 which is likely to arise from the errors associated with the finite size effects and Landau level trunctaion present in such study.

IV. PARTIALLY FILLED SPIN POLARIZED LLL

Another instance for which a useful effective Hamiltonian can be obtained using the line of reasoning presented in the previous sections, is the case of a Landau level for which spin is a good quantum number, and f_o spin states are completely filled ($f_o < f$) while the remaining $f - f_o$ spin states are partially empty. This approach is useful for electrons, for example, in addressing those states at filling factors in the interval $1 < \nu < 2$ for which all $\uparrow$ $n = 0$ states are occupied, *i.e.* for maximally spin-polarized states. Although we could use the effective Hamiltonian discussed in Sec. II in this filling factor range, it is useful to derive an effective Hamiltonian which acts only on $\downarrow$ degrees of freedom, since the conservation of S_z prevents the $\uparrow$ spins from participating in the low energy dynamics. We emphasize that the ground state in the filling factor range $1 < \nu < 2$ is *not* always maximally spin-polarized, so this approach cannot always be used to describe the ground state.

We can construct an effective Hamiltonian for the partially empty flavors in which the interaction is exactly the same as the one discussed in Sec. II except that there is an additional two-body interaction of the screening type, in which occupied flavor electrons are virtually excited to higher Landau levels in a completely analogous manner to $\mathcal{V}_a^{2b}(1)$ from Eq. (25), with an analogous diagramatic representation as that appearing in Fig. 1, except that its multiplicity will be given only by the f_0 occupied flavors. This additional contribution to the effective interaction is:

$$\mathcal{V}_o^{2b} = -f_o\kappa^2 \sum_{\substack{1\cdots 6\\ \mu\nu}} v_{15,64} v_{26,53} \frac{\tau(n_5,n_6)}{n_6} c^\dagger_{1\nu} c^\dagger_{2\mu} c_{3\mu} c_{4\nu}, \tag{31}$$

where the orbital indices of the operators are understood to be in the $n = 0$ LL, and the spin indices run over the partially empty $f - f_o$ flavors only, *i.e.* all electrons have minority spins when only spin provides a flavor label. The additional Haldane pseudopotentials and the total Haldane pseudopotentials are listed in Table VI for the most common case of spin-1/2 fermions. For the case of spin-1/2 fermions, only odd m pseudopotentials are relevant for the Hilbert space where this effective Hamiltonian acts. For a more general case of f_o filled flavors, the pseudopotials $V_2(m, f_o)$ are,

$$V_2(m, f_o) = V_2(m) + f_o V_o^{2b}(m), \tag{32}$$

where the coefficients $V_2(m)$ are given by Eq. (10) and listed in Table II. The coefficients $V_o^{2b}(m)$, which are the pseudopotentials associated with the interaction of Eq. (31), are listed in Table VI. It is interesting to note the significant difference between the corrections to the pseudopotentials for $0 < \nu < 1$, listed in Table II, and those for spin polarized states with $1 < \nu < 2$, listed in Table VI. For example, the correction to the $m = 1$ pseudopotential, which is crucial in determining the gap of the Laughlin type states, is negative and about four times larger in magnitude compared to the $0 < \nu < 1$, indicating a higher reduction of this pseudopotential in the filling factor range $1 < \nu < 2$.

The three-body interactions between partially full flavor electrons remain unchanged and given by Eq. (11), thus, in the spin-1/2 case the three-body pseudopotentials would be those listed in Table III for the $S = 3/2$ states of Eq. (13), constructed for the partially filled flavor with spin $\downarrow$.

V. SUMMARY AND DISCUSSION

We have derived effective Hamiltonians which account for quantum fluctuations in full and empty Landau levels (Landau level mixing) to leading order in perturbation theory for the cases of a partially filled $n = 0$ Landau level and a partially filled $n = 1$ Landau level. These effective Hamiltonians describe fractional quantum Hall physics in the filling factor ranges $0 < \nu < 2$ and $2 < \nu < 4$ respectively in the case of spin-1/2 fermions. In both cases the effective Hamiltonians are a sum of two- and three-body terms. The three-body terms are responsible for particle-hole symmetry breaking within the Landau level. There has been considerable interest in these quantum fluctuation corrections because they are likely to play a decisive role in several outstanding problems in the fractional quantum Hall regime.

TABLE IV: Conventional two-body Haldane pseudopotentials in the $n = 1$ LL (V_1 from Eq. 29) and coefficients of their leading perturbative corrections for a completely filled spin-1/2 $n = 0$ LL (from Eqs. (24) and (25) with $f = 2$), which describe the physics for filling factors ν in the interval $2 < \nu < 4$. The last row is the sum of all the perturbative corrections, *i.e.* $V^{2b} = V^{2b}(2) + V_a^{2b}(1) + V_b^{2b}(1) + V_c^{2b}(1) + V^{2b}(0)$.

m	0	1	2	3	4	5	6	7	8	9
V_1	0.6093	0.4154	0.4500	0.3150	0.2635	0.2322	0.2101	0.1935	0.1803	0.1696
$V^{2b}(2)$	−0.0903	−0.0347	−0.1235	−0.0241	−0.0110	−0.0064	−0.0042	−0.0030	−0.0022	−0.0017
$V_a^{2b}(1)$	−0.3930	−0.2038	−0.1981	−0.1119	−0.0535	−0.0235	−0.0098	−0.0039	−0.0015	−0.0006
$V_b^{2b}(1)$	0.0247	0.0706	0.0803	0.0186	−0.0031	0.0091	−0.0004	0.0029	0.0007	0.0011
$V_c^{2b}(1)$	0.0750	−0.0475	0.0870	0.0141	−0.0128	−0.0167	−0.0139	−0.0104	−0.0077	−0.0058
$V^{2b}(0)$	0	0	−0.0276	−0.0023	−0.0006	−0.0003	−0.0001	-7×10^{-5}	-5×10^{-5}	-3×10^{-5}
V^{2b}	−0.3836	−0.2155	−0.1818	−0.1056	−0.0810	−0.0377	−0.0285	−0.0146	−0.0108	−0.0070

TABLE V: Coefficients of the second order correction to the $S = 3/2$ and $S = 1/2$ three-body Haldane pseudopotentials in the $n = 1$ LL. For $S = 1/2$ and $4 \leq m \leq 6$ the pairs listed have been orthonormalized by rotating only the state with $\sigma = 2$, as described in Appendix A.

m	3	5	6	7	8	9
(k, l)	$(0,1)$	$(1,1)$	$(0,2)$	$(2,1)$	$(1,2)$	$(0,3)$ $(3,1)$
$V_{3/2}(k'l', kl)$	−0.0147	−0.0054	−0.0099	0.0005	−0.0009	$\begin{pmatrix} -0.0088 & 0.0007 \\ 0.0007 & 0.0033 \end{pmatrix}$
m	1	2	3	4	5	6
(σ, k, l)	$(1,0,0)$	$(2,0,0)$	$(1,1,0)$	$(1,0,1)$ $(2,1,0)$	$(1,2,0)$ $(2,0,1)$	$(1,1,1)$ $(2,2,0)$
$V_{1/2}(\sigma' k' l', \sigma k l)$	−0.0319	−0.0305	−0.0131	$\begin{pmatrix} -0.0009 & -0.0004 \\ -0.0004 & -0.0100 \end{pmatrix}$	$\begin{pmatrix} -5\times 10^{-5} & -0.0056 \\ -0.0056 & 0.0229 \end{pmatrix}$	$\begin{pmatrix} 0.0067 & 0.0017 \\ 0.0017 & -0.0010 \end{pmatrix}$

Our work has been developed essentially in parallel with an analytic study by Peterson and Nayak,[9] and a numerical study by Simon and Rezayi.[44] The three works toghether provide a comprehensive view of the leading perturbative Landau level mixing corrections to the effective Hamiltonian in the lowest and second Landau levels, and they complement earlier works by Rezayi and Haldane,[42] Murthy and Shankar,[18] and Bishara and Nayak.[43] The source of discrepancy on the two-body pseudopotentials with the work of Bishara and Nayak[43] is now understood. In Ref. 9, this discrepancy was shown to arise from the normal ordering that must be kept in the three body interactions. When this normal ordering is dealt with properly, our two- and three-body interactions are exactly the same as those derived in Ref. 9.

Our two-body pseudopotentials in the lowest and second excited Landau levels are in agreement with Refs. 9 and 44. In the lowest Landau level the two-body pseudopotentials are also in agreement with the results of an earlier study of the perturbative expansion of the spectrum of hydrogenic atoms in a strong magnetic-field.[51] Our three-body pseudopotentials are in agreement with those of Refs. 9 and 43, and with those of Ref. 44 in the lowest Landau level. There is an appreciable difference with the three-body pseudopotential differences in the second Landau level reported in Ref. 44, which is likely to arise from finite size effects in this numerical study.

Now we would like to comment on the range of validity of the perturbative approach. It is impossible to know the range of validity of any perturbative expansion without a sense for the relative size of the higher order corrections. We do not believe it is feasible to carry out the same type of analysis we have discussed in this paper to higher orders in κ for the many-body problem in an analytic fashion. The reason is that it not possible to construct effective Hamiltonians to order κ^3 or higher, projected onto the Landau level of interest, without explicitly computing the energies to order κ. In other words it is necessary to solve the many-body problem exactly to order κ to be able to construct an effective Hamiltonian to order κ^3 projected into the degenerate manifold.

In spite of this seemingly insurmountable task, it is possible to get a sense, at least heuristically, for the size of the higher order corrections. One way this can be done is by studying the two-body problem to higher order in κ. This has been done, indirectly, for the lowest Landau level in the context of the problem hydrogenic atoms in a strong magnetic field in Ref. 51. More specifically from the coefficients $\alpha_{N,M}^{(i)}$ listed in Table II of Ref. 51, one obtains the following expression for the energies of two particles with relative angular momentum m in the lowest Landau level to order κ^4 in units of ω_c,

$$V_m = \sum_{p=1}^{4} \alpha_{0,m}^{(p)} \left(-\frac{\sqrt{\pi}}{2} \kappa \right)^p + \mathcal{O}(\kappa^5). \tag{33}$$

By reading the values of the coefficients $\alpha_{0,m}^{(p)}$ from Table II in Ref. 51, one learns that $\alpha_{0,m}^{(p)}$ decreases by about

TABLE VI: Coefficients of the second order corrections to the two-body Haldane pseudopotentials for spin-1/2 fermions in a state with completely filled majority spins and partially empty minority spins in the $n = 0$ LL.

m	0	1	2	3	4	5
$V_o^{2b}(m)$	-0.3662	-0.0959	-0.0268	-0.0078	-0.0023	-0.0007
$V_2(m,1)$	-0.7119	-0.1287	-0.0380	-0.0133	-0.0056	-0.0029

an order of magnitude at every order[55]. This indicates that higher order corrections remain parametrically small even at $\kappa \sim 1$. This observation suggests that higher order corrections might remain small even at values of $\kappa \sim 1$ in the Lowest landau level, and specially so in the dilute limit where the energies of the two-body problem are expected to dominate.

It is not possible to directly extract the energies of two fermions in the second landau level togheter with the completely filled lowest Landau level from the results of the bare two-body problem of Ref. 51. This is because the bare two-body problem fails to account for basic many-body effects like Pauli blocking that already arise at order κ in the energies in the second Landau level. It is thus hard to anticipate at this point the relative size of the κ^3 contributions in the second Landau level.

Most fractional quantum Hall samples have similar electron densities. For this reason the external magnetic field strength tends to be smaller, and κ correspondingly larger, for the experiments in the second Landau level than experiments in the lowest Landau level. Additionally by comparing Tables II and IV it is evident that even at a fixed field, the two-body pseudopotential corrections are larger for the second Landau level than for the lowest Landau level. Quantum fluctuations in Landau level occupations are therefore more likely to be important in the second Landau level case in which the fractional quantum Hall effect can be enriched by the appearance of striped states and even-denominator incompressible states.

Finally we would like to connect our study to the problem of the nature of the incompressible state observed at filling fraction $\nu = 5/2$ in GaAs. For the spin polarized case, the Moore-Read Pfaffian is known to be the unique highest density zero-energy state of a repulsive three-body Hamiltonian[25,56–58] for which only the lowest angular momentum three-body state is energetically penalized, namely $V_{3/2}(m = 3) > 0$ and the remainder pseudopotentials vanish.[47] Conversely the anti-Pfafffian is expected to be the ground for the particle-hole conjugated Hamiltonian,[31] which has $V_{3/2}(m = 3) < 0$. We have found in agreement with Refs. 9 and 43 that the leading value of this pseudopotential is negative. Nevertheless the two-particle pseudopotentials, which we have corrected for in this work, have been found to play decisive role in this competition, and they could drive the system into a compressible phase.[24–37] A reliable assessment of the influence of LL mixing on fractional quantum Hall states at even denominator fractions thus awaits the application of our pseudopotentials in many-body exact diagonalization studies.

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Appendix A: Three-body states and pseudopotentials

In this appendix we outline some properties of the three body states and their associated generalized Haldane pseudopotentials. We begin with the $n = 0$ LL. We are interested only in translationally invariant polynomials describing the internal state of relative motion of the three particles, and thus omit polynomial factors in the center of mass coordinates $(z_1 + z_2 + z_3)/3$. Translationally invariant polynomials would depend only on two independent translationally invariant coordinates, which can be chosen as

$$z_a = \sqrt{\frac{2}{3}}\left(\frac{z_1 + z_2}{2} - z_3\right), \quad z_b = \frac{z_1 - z_2}{\sqrt{2}}. \tag{A1}$$

The permutation operators on these coordinates act as a reflection, and as a composition of a reflection and rotations by $\pm 2\pi/3$,[52]

$$\Pi_{12} \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \Pi_{23} \doteq \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \ \Pi_{13} \doteq \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \tag{A2}$$

where the matrices are understood to act in a colum vector of the form $\left(\begin{smallmatrix} z_a \\ z_b \end{smallmatrix}\right)$.

The fully spin polarized $S = 3/2$ states, whose polynomial part is fully antisymmetric, were constructed in Ref. 52, and read as

$$\Psi^{3/2}_{kl} = \frac{1}{Z^{3/2}_{kl}}(z_a^2 + z_b^2)^k \left[\frac{(z_a + iz_b)^{3l} - (z_a - iz_b)^{3l}}{2i}\right], \tag{A3}$$

with $Z^{3/2}_{kl} = 2^{3l+2k+1}[\pi^3(3l+k)!k!]^{1/2}$, $k \geq 0$, $l \geq 1$. $\Psi^{3/2}_{kl}$ can be expanded as,

$$\Psi^{3/2}_{kl} = \frac{1}{Z^{3/2}_{kl}} \sum_{j=0}^{m} f^{3/2}_{jkl}\ z_b^j z_a^{m-j} \tag{A4}$$

where $m = 2k + 3l$ is the orbital angular momentum of $\Psi^{3/2}_{kl}$, and

$$f^{3/2}_{jkl} = \sum_{p=0}^{k}\sum_{q=0}^{3l} \binom{k}{p}\binom{3l}{q} \sin(q\pi/2)\delta_{j,q+2p}. \tag{A5}$$

In order to evaluate the pseudopotentials from Eq. (12) it is convenient to decompose the state $\Psi^{3/2}_{kl}$ into products of states with well defined relative numbers for the pair of particles 1 and 2, and states with well defined numbers for particle 3, as follows:

$$\Psi^{3/2}_{kl} = \sum_{j=0}^{m}\sum_{j'=0}^{m-j} C_{kl,jj'}|m-j-j',j\rangle_{12}|j'\rangle_3, \tag{A6}$$

where $|m, m'\rangle_{ij}$ abbreviates for the state with center of mass angular momentum m and relative angular momentum m' for particles i and j in the $n = 0$ LL, and $|l\rangle_i$ for the state of particle i with angular momentum l in the $n = 0$ LL. $C_{kl,jj'}$ can be found to be:

$$C_{kl,jj'} = (2\pi)^{3/2}\frac{f^{3/2}_{jkl}}{Z^{3/2}_{kl}}\binom{m-j}{j'}(-1)^{j'} \times \cdots \sqrt{\frac{2^{m+j'}}{3^{m-j}}(m-j-j')!j!j'!}. \tag{A7}$$

Consider now two states, $\Psi^{3/2}_{kl}$ and $\Psi^{3/2}_{k'l'}$, with the same angular momentum, $m = 2k + 3 = 2k' + 3l'$. Their associated generalized Haldane pseudopotential matrix elements, computed from Eqs. (12) and (15), is

$$V_{3/2}(k'l', kl) = 6\sum_{i=0}^{m}\sum_{i'=0}^{m-i} C_{k'l',ii'} \sum_{j=0}^{m}\sum_{j'=0}^{m-j} C_{kl,jj'} \sum_{n=1}^{\infty} \frac{{}_r\langle 0, i|v_{12}|n, i+n\rangle_{rr}\langle n, j+n|v_{12}|0, j\rangle_r}{2^n n} R^{m+n-i'}_{i+n,j'} R^{m+n-j'}_{j+n,i'}, \tag{A8}$$

where $R^L_{m,m'}$ are given by Eq. (C4), and the Coulomb matrix elements by Eq. (C6). Equation (A20) can be used to obtain the $S = 3/2$ pseudopotentials listed in Table III.

Let us now discuss the $S = 1/2$ three-body states. We begin by considering the bosonic analogue of $\Psi^{3/2}_{kl}$, namely

$$\Phi_{kl} = (z_a^2 + z_b^2)^k \left[\frac{(z_a + iz_b)^{3l} + (z_a - iz_b)^{3l}}{2}\right], \tag{A9}$$

with $k \geq 0$, $l \geq 0$, and we have not normalized these states yet. These states form a complete orthogonal basis for the fully symmetric translationally invariant polynomials. They can be expanded as

$$\Phi_{kl} = \sum_{j=0}^{2k+3l} f^{1/2}_{jkl}\ z_b^j z_a^{2k+3l-j}, \tag{A10}$$

with

$$f^{1/2}_{jkl} = \sum_{p=0}^{k}\sum_{q=0}^{3l} \binom{k}{p}\binom{3l}{q} \cos(q\pi/2)\delta_{j,q+2p}. \tag{A11}$$

The spatial part of the $S = 1/2$ states can the be written as[48]

$$\Psi^{1/2}_{\sigma kl} = \frac{\beta_\sigma}{Z^{1/2}_{\sigma kl}}\Phi_{kl}, \tag{A12}$$

with $\sigma = \{1, 2\}$, $k \geq 0$, $l \geq 0$, with the normalization constants $Z^{1/2}_{\sigma kl}$ given in the text below Eq. (18). Where the polynomials β_σ are[59]

$$\beta_1 = z_b,\ \beta_2 = z_a z_b. \tag{A13}$$

This basis is complete but not orthogonal. Fortunately, the inner products can be computed analytically and are listed in Eq. (20), thus othonormalization becomes trivial.

The fully antisymmetric wavefunction including the spin part is[48]

$$|\Psi^{1/2}_{\sigma kl}\rangle = \mathcal{A}\{\Psi^{1/2}_{\sigma kl} \otimes |\uparrow\uparrow\downarrow\rangle\}, \tag{A14}$$

where $\mathcal{A} = (1-\Pi_{23}+\Pi_{12}\Pi_{23})/\sqrt{3}$ is the antisymmetrization operator between $\uparrow$ and $\downarrow$ particles, and Π_{ij} is the unitary operator corresponding to elementary permutations between i and j.

The three-body interaction in Eq. (11) is rotationally invariant, hence, its associated three-body pseudopotentials are diagonal in the relative orbital angular momentum of $|\Psi^{1/2}_{\sigma kl}\rangle$, $m = \sigma + 2k + 3l$, therefore, for any two states with the same angular momentum we can write

$$\begin{aligned} V_{1/2}(\sigma' k' l', \sigma kl) &= -6\langle\Psi^{1/2}_{\sigma' k' l'}|v_{13}\mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{12}|\Psi^{1/2}_{\sigma kl}\rangle \\ &= 2\bar{\Psi}^{1/2}_{\sigma' k' l'}(h_0 + h_0^\dagger + h_1)\Psi^{1/2}_{\sigma kl}, \end{aligned} \tag{A15}$$

where $\hat{n} = \hat{n}_1 + \hat{n}_2 + \hat{n}_3$, $\mathcal{P}_\perp$ is the projector into the othogonal complement to the $\hat{n} = 0$ eigenspace. In the second line of Eq. (A15) it is understood that only the orbital part of the wavefunctions is involved, and it follows from Eq. (A14) using the fact that interactions are spin independent. The computation of Haldane pseudopotentials reduces to the computation of the matrix elements of h_0 and h_1, which stand for

$$h_0 = -v_{13}\mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{12}, \quad h_1 = -v_{13}\mathcal{P}_\perp \frac{1}{\hat{n}} \mathcal{P}_\perp v_{23}. \tag{A16}$$

In analogy with Eq. (A6), we can decompose $\Psi^{1/2}_{\sigma kl}$ as

$$\Psi^{1/2}_{\sigma kl} = \sum_{j=0}^{m} \sum_{j'=0}^{m-j} C^{\mu\nu}_{\sigma kl, jj'} |m-j-j', j\rangle_{\mu\nu} |j'\rangle_\gamma, \tag{A17}$$

where $m = \sigma + 2k + 3l$, and the indices $\{\mu\nu\gamma\}$ stand for any permutation of $\{123\}$. The decomposition is generally dependent on the particle ordering since the orbital part alone of the $S = 1/2$ states, $\Psi^{1/2}_{\sigma kl}$, is not an eigenstate of all permutations. $C^{\mu\nu}_{\sigma kl, jj'}$ can be found to be

$$\begin{aligned} C^{\mu\nu}_{\sigma kl, jj'} = (2\pi)^{3/2} \frac{g^{\mu\nu}_{\sigma jkl}}{Z^{1/2}_{\sigma kl}} \binom{m-j}{j'} (-1)^{j'} \times \cdots \\ \sqrt{\frac{2^{m+j'}}{3^{m-j}} (m-j-j')! j! j'!}, \end{aligned} \tag{A18}$$

with

$$\begin{aligned} g^{12}_{\sigma jkl} &= f^{1/2}_{j-1,kl}, \\ g^{13}_{1jkl} &= \frac{f^{1/2}_{j-1,kl} + \sqrt{3} f^{1/2}_{j,kl}}{2}, \\ g^{32}_{1jkl} &= \frac{f^{1/2}_{j-1,kl} - \sqrt{3} f^{1/2}_{j,kl}}{2}, \\ g^{13}_{2jkl} &= \frac{2f^{1/2}_{j-1,kl} + \sqrt{3}(f^{1/2}_{j-2,kl} - f^{1/2}_{j,kl})}{4}, \\ g^{32}_{2jkl} &= \frac{2f^{1/2}_{j-1,kl} - \sqrt{3}(f^{1/2}_{j-2,kl} - f^{1/2}_{j,kl})}{4}. \end{aligned} \tag{A19}$$

These expressions need not be derived independently, but can be obtained by deriving only the decomposition corresponding to $\{\mu\nu\} = \{12\}$, and then applying suitable permutation operators as represented in Eq. (A2). With this the evaluation of Eq. (A15) leads to,

$$\begin{aligned} \bar{\Psi}^{1/2}_{\sigma' k' l'} h_0 \Psi^{1/2}_{\sigma kl} &= -\sum_{i=0}^{m} \sum_{i'=0}^{m-i} C^{13}_{\sigma' k' l', ii'} \sum_{j=0}^{m} \sum_{j'=0}^{m-j} C^{12}_{\sigma kl, jj'} \sum_{n=1}^{\infty} \frac{{}_r\langle 0, i|v_{12}|n, i+n\rangle_{rr}\langle n, j+n|v_{12}|0, j\rangle_r}{2^n n} R^{m+n-i'}_{i+n,j'} R^{m+n-j'}_{j+n,i'}, \\ \bar{\Psi}^{1/2}_{\sigma' k' l'} h_1 \Psi^{1/2}_{\sigma kl} &= -\sum_{i=0}^{m} \sum_{i'=0}^{m-i} (-1)^i C^{13}_{\sigma' k' l', ii'} \sum_{j=0}^{m} \sum_{j'=0}^{m-j} C^{32}_{\sigma kl, jj'} \sum_{n=1}^{\infty} \frac{{}_r\langle 0, i|v_{12}|n, i+n\rangle_{rr}\langle n, j+n|v_{12}|0, j\rangle_r}{2^n n} R^{m+n-i'}_{i+n,j'} R^{m+n-j'}_{j+n,i'}. \end{aligned} \tag{A20}$$

Notice that all the matrix elements are purely real. Combining these expressions with Eqs. (C6), (C4) and (A15), one obtains the values listed in Table III.

The derivation in the $n = 1$ LL goes through in a completely analogous fashion. By raising the kinetic energy of the three particles, with the operator $a_1^\dagger a_2^\dagger a_3^\dagger$, we easily obtain the required representation of the states in one-to-one correspondence with the $n = 0$ LL.

To obtain the $S = 3/2$ three-body pseudopotentials in the $n = 1$ LL we apply $a_1^\dagger a_2^\dagger a_3^\dagger$ to Eq. (A6). Using the first quantized version of the three-body interactions appearing in Eq. (27), we arrive at the following expression for the pseudopotentials in the $n = 1$ LL,

$$V_{3/2}(k'l',kl) = 6\sum_{i=0}^{m}\sum_{i'=0}^{m-i} C_{k'l',ii'} \sum_{j=0}^{m}\sum_{j'=0}^{m-j} C_{kl,jj'} \Bigg[\sum_{n=1}^{\infty} \frac{{}_r(0,i|v_{12}|n,i+n)_r\, {}_r(0,j|v_{12}|n,j+n)_r}{2^n n} R^{m+n-i'}_{i+n,j'} R^{m+n-j'}_{j+n,i'} \cdots$$
$$\cdots - \frac{{}_r\langle 2,i|v_{12}|1,i-1\rangle_r\, {}_r\langle 1,j-1|v_{12}|2,j\rangle_r}{4} R^{m-1-i'}_{i-1,j'} R^{m-1-j'}_{j-1,i'}\Bigg], \quad \text{(A21)}$$

where we have introduced the notation,

$${}_r(0,i|v_{12}|n,i+n)_r \equiv {}_r\langle 0,i|v_{12}|n,i+n\rangle_r R^{n+2}_{n,1} \cdots$$
$$\cdots - {}_r\langle 2,i|v_{12}|n+2,i+n\rangle_r R^{n+2}_{n+2,1}. \quad \text{(A22)}$$

From Eq. (A21) one obtains the pseudopotentials for $S = 3/2$ states in the $n = 1$ LL listed in Table V.

To obtain the $S = 1/2$ states in the $n = 1$ LL we apply $a_1^\dagger a_2^\dagger a_3^\dagger$ to Eq. (A17). From the interaction appearing in Eq. (27) we obtain the analogue of Eq. (A15), now with h_0 and h_1 replaced by

$$\tilde{h}_0 = -v_{13}\mathcal{P}_\perp \frac{1}{\hat{n}-3}\mathcal{P}_\perp v_{12}, \quad \tilde{h}_1 = -v_{13}\mathcal{P}_\perp \frac{1}{\hat{n}-3}\mathcal{P}_\perp v_{23}. \quad \text{(A23)}$$

The matrix elements for $\tilde{h}_0$ are,

$$\bar{\Psi}^{1/2}_{\sigma'k'l'}\tilde{h}_0\Psi^{1/2}_{\sigma kl} = -\sum_{i=0}^{m}\sum_{i'=0}^{m-i} C^{13}_{\sigma'k'l',ii'} \sum_{j=0}^{m}\sum_{j'=0}^{m-j} C^{12}_{\sigma kl,jj'} \Bigg[\sum_{n=1}^{\infty} \frac{{}_r(0,i|v_{12}|n,i+n)_r\, {}_r(0,j|v_{12}|n,j+n)_r}{2^n n} R^{m+n-i'}_{i+n,j'} R^{m+n-j'}_{j+n,i'} \cdots$$
$$\cdots - \frac{{}_r\langle 2,i|v_{12}|1,i-1\rangle_r\, {}_r\langle 1,j-1|v_{12}|2,j\rangle_r}{4} R^{m-1-i'}_{i-1,j'} R^{m-1-j'}_{j-1,i'}\Bigg]. \quad \text{(A24)}$$

The expression for $\bar{\Psi}^{1/2}_{\sigma'k'l'}\tilde{h}_1\Psi^{1/2}_{\sigma kl}$ is the same after replacing $C^{12}_{\sigma kl,jj'} \to C^{32}_{\sigma kl,jj'}$, and $C^{13}_{\sigma'k'l',ii'} \to (-1)^i C^{13}_{\sigma'k'l',ii'}$, in analogy with Eq. (A20) for the $n = 0$ LL. The $S = 1/2$ Haldane pseudopotentials in the $n = 1$ LL, listed in Table V, are given $V_{1/2}(\sigma'k'l',\sigma kl) = 2\bar{\Psi}^{1/2}_{\sigma'k'l'}(\tilde{h}_0 + \tilde{h}_0^\dagger + \tilde{h}_1)\Psi^{1/2}_{\sigma kl}$, which is the analogue of Eq. (A15) for the $n = 0$ LL.

In spite of how cumbersome Eqs. (A21) and (A24) might look, its evaluation is very efficient, and each pseudopotential takes only a few seconds to evaluate using *Mathematica* in a conventional laptop computer.

We listed in Tables III and V pseudopotentials matrices up to 2×2 in size. For this case the two-dimensional subspaces for given m contain one state for $\sigma = 1$ and one for $\sigma = 2$. We have orthogonalized these states by rotating the $\sigma = 2$ state only. Let us call the matrix in the non-orthogonal basis $\mathbf{V}$, which is obtained from Eqs. (A15) and (A20). The matrix listed in Tables III and V corresponds to $(\mathbf{B}^T)^{-1}\mathbf{V}\mathbf{B}^{-1}$, with $\mathbf{B}$ the change of basis matrix,

$$\mathbf{B} = \begin{pmatrix} 1 & \frac{\langle\Psi_1|\Psi_2\rangle}{\sqrt{1-\langle\Psi_1|\Psi_2\rangle^2}} \\ 0 & \end{pmatrix}. \quad \text{(A25)}$$

$\langle\Psi_1|\Psi_2\rangle$ is a shorthand for the overlap between $\sigma = \{1,2\}$ states, appearing in Eq. (20).

Appendix B: Effective interactions in the $n = 1$ LL

In this appendix we describe in more detail how the interactions in the $n = 1$ LL were obtained. Consider the effective Hamiltonian to order κ^2 in the $n = 1$ LL,

$$\mathcal{H}_2 = \mathcal{H}_1 - \frac{\kappa^2}{4}\sum_{\substack{1\cdots 8\\ \mu\nu\lambda\sigma}} v_{12,34}v_{56,78} \times \cdots$$
$$P_0 c^\dagger_{5\nu}c^\dagger_{6\mu}c_{7\mu}c_{8\nu}P_\perp \frac{1}{\hat{n}-N_1}P_\perp c^\dagger_{1\lambda}c^\dagger_{2\sigma}c_{3\sigma}c_{4\lambda}P_0, \quad \text{(B1)}$$

where the integers abbreviate single-particle kinetic and guiding center quantum numbers (*e.g.* $\{1 \Leftrightarrow n_1 m_1\}$), N_1 is the number of particles in the partially filled $n = 1$ LL, P_0 is the projector into the many-body eigenspace with $\hat{n} = N_1$, and $P_\perp$ the projector into its orthogonal complement.

The second order term in Eq. (B1) can be viewed as a sequence of scattering process in which a pair of particles is taken from states “3” and “4”, contained either in the completely filled $n = 0$ LL or the partially filled $n = 1$ LL, and placed into states “1” and “2” with higher total kinetic energy. Subsequently, the particles are removed from these virtually excited states by operators “7” and “8”, to be finally placed back into states “5” and “6”, which are contained either in the $n=0$ LL or the $n=1$ LL.

Threrefore any term with non-vanishing matrix elements in the $\hat{n}=N_1$ eigenspace must have the outermost destruction and creation operators (*i.e.* $c^\dagger_{5\nu}$, $c^\dagger_{6\mu}$, $c_{3\sigma}$ and $c_{4\lambda}$) with Landau level indices either 0 or 1. Therefore, the kinetic energy of the "incoming" particles, *i.e.* n_3+n_4, or the kinetic energy of the "outgoing" particles, *i.e.* n_5+n_6, is allowed to take only the values $\{0,1,2\}$. It can be shown that there is a kinetic energy balance condition between "incoming" and "outgoing" labels, namely the terms with $n_3+n_4 \neq n_5+n_6$ vanish, and they correspond to scattering type diagrams that have tad-pole and self-energy insertions in their legs. In summary, we can label the allowed interactions by the total incoming/outgoing pair kinetic energy $n_3+n_4=n_5+n_6=\{0,1,2\}$.

As an example we discuss in detail how to obtain the interaction corresponding to incoming pairs with total kinetic energy $n_3+n_4=2$, labeled $\mathcal{V}^{2b}(2)$, and will leave the verification of the remaining terms to the interested reader. This term corresponds to all the two-body terms arising from Eq. (B1) with $n_3=n_4=n_5=n_6=1$, after normal ordering of the operators is performed. In order to have a non-vanising contribution the creation operators $c^\dagger_{1\lambda}$ and $c^\dagger_{2\sigma}$ must raise the kinetic energy of the incoming pair, thus $n_1+n_2>2$. Similarly we have that $n_7+n_8>2$. Since the $n=0$ LL is assumed to be completely full, these labels are additionally constrained to satisfy $n_i \geq 1$, for $i=\{1,2,7,8\}$.

With these constraints we can see that there are two possibilities for these middle operators. Either neither of them corresponds to the partially filled $n=1$ LL, *i.e.* $n_i>1$, or else only one creation and only one destruction do correspond to $n=1$ LL, for example $n_2=n_7=1$ and $n_1=n_8>1$. For the first possibility we see that the indices $\{1,2,7,8\}$ must be fully contracted in order for the virtually excited pair to go back into the $n=1$ LL. Let us name the terms arising from this first possibility as $\mathcal{V}^{2b}_a(2)$. We can write

$$\mathcal{V}^{2b}_a(2) = -\frac{\kappa^2}{4}\sum_{\substack{1\cdots 8\\ \mu\nu\lambda\sigma}} v_{12,34}v_{56,78}\langle c_{7\mu}c_{8\nu}c^\dagger_{1\lambda}c^\dagger_{2\sigma}\rangle \times \cdots$$
$$\frac{\vartheta(n_1)\vartheta(n_2)}{n_1+n_2-2}c^\dagger_{5\nu}c^\dagger_{6\mu}c_{3\sigma}c_{4\lambda}, \quad \text{(B2)}$$

where now we understand the Landau level index of the uncontracted operators to be in the $n=1$ LL, and $\vartheta(n)$ restricts $n\geq 2$.

For the second possibility an extra pair of uncontracted creation and destruction operators in Eq.(B1) will have indices in the $n=1$ LL. Let us name this term $\mathcal{V}_b(2)$, which reads as

$$\mathcal{V}_b(2) = -\frac{\kappa^2}{4}\sum_{\substack{1\cdots 8\\ \mu\nu\lambda\sigma}} v_{12,34}v_{56,78}c^\dagger_{5\nu}c^\dagger_{6\mu}\Big[\frac{\vartheta(n_1)}{n_1-1}\times\cdots$$
$$\Big(\langle c_{8\nu}c^\dagger_{1\lambda}\rangle c_{7\mu}c^\dagger_{2\sigma} - \langle c_{7\mu}c^\dagger_{1\lambda}\rangle c_{8\nu}c^\dagger_{2\sigma}\Big) + \{1\Leftrightarrow 2\}\Big]c_{3\sigma}c_{4\lambda}, \quad \text{(B3)}$$

where again we understand the Landau level index of the uncontracted operators to be in the $n=1$ LL.

It is clear from Eq. (B3), that $\mathcal{V}_b(2)$ has a contribution to the effective three body interactions, but it also contributes to the two body interactions after due normal ordering of the operators is performed. The three body piece in Eq. (B3) contributes to the terms in Eq. (26), with $n_7\geq 2$. On the other hand, the two body term, which we label $\mathcal{V}^{2b}_b(2)$, makes up the remainder of the interaction $\mathcal{V}^{2b}(2)$ appearing in Eq. (24), namely $\mathcal{V}^{2b}(2)=\mathcal{V}^{2b}_a(2)+\mathcal{V}^{2b}_b(2)$.

A very similar analysis leads to the forms for the remaining two- and three-body interactions given in Eqs. (24), (25), (26), and (31).

Appendix C: Two-body states and Haldane pseudopotentials in the $n=1$ LL

In this section we discuss several useful properties of the two-body states in the $n=1$ LL and their associated Haldane pseudopotentials.

We begin by discussing two-particle states. Consider a state for particles i and j in which each of them has well defined kinetic and guiding center quantum numbers:

$$|n_1m_1\rangle_i|n_2m_2\rangle_j = \frac{a_i^{\dagger n_1}a_j^{\dagger n_2}b_i^{\dagger m_1}b_j^{\dagger m_2}}{\sqrt{n_1!n_2!m_1!m_2!}}|0\rangle, \quad \text{(C1)}$$

where $a^\dagger$ and $b^\dagger$ are the kinetic energy and guiding center raising operators.[49,50] Although we often loosely refer to the guiding center quantum number m_i as the angular momentum of particle i, notice that the physical angular momentum is actually m_i-n_i.

An alternative representation for two-particle states is obtained by constructing states with well defined center of mass and relative coordinate quantum numbers, for both the kinetic energy and guiding center labels, as follows

$$|NM\rangle_c|nm\rangle_r = \frac{A^{\dagger N}B^{\dagger M}a^{\dagger n}b^{\dagger m}}{\sqrt{N!M!n!m!}}|0\rangle, \quad \text{(C2)}$$

where the four operators $A=(a_i+a_j)/\sqrt{2}$, $B=(b_i+b_j)/\sqrt{2}$, $a=(a_i-a_j)/\sqrt{2}$ and $b=(b_i-b_j)/\sqrt{2}$, commute with each other. These states are related to those in Eq. (C1) by the unitary transformation

$$|n_1 m_1\rangle_i |n_2 m_2\rangle_j = \sum_{\nu=0}^{n_1+n_2} \sum_{\mu=0}^{m_1+m_2} R^{n_1+n_2}_{n_2,\nu} R^{m_1+m_2}_{m_2,\mu} \times \cdots$$
$$|n_1+n_2-\nu, m_1+m_2-\mu\rangle_c |\nu\mu\rangle_r, \quad \text{(C3)}$$

where $R^L_{m,m'}$ is a real, orthogonal, and symmetric $L \times L$ matrix. The explicit form of $R^L_{m,m'}$ is

$$R^L_{m,m'} = \sqrt{\frac{\binom{L}{m}}{2^L \binom{L}{m'}}} \times \cdots$$
$$\sum_{\mu=Max(0,m+m'-L)}^{min(m',m)} \binom{L-m}{m'-\mu} \binom{m}{\mu} (-1)^\mu. \quad \text{(C4)}$$

The properties of $R^L_{m,m'}$ imply that the inverse transformation is still given by Eq. (C1) with the exchange of roles $\{i \Leftrightarrow c, j \Leftrightarrow r\}$.

For two isolated particles in vacuum the center of mass and relative basis of Eq. (C2) is the most natural basis to find eigenstates of the two-body problem in the presence of strong magnetic fields. The reason is that the interaction is almost diagonal in this basis:

$${}_c\langle N'M'|{}_r\langle n'm'|v_{ij}|NM\rangle_c|nm\rangle_r$$
$$= {}_r\langle n'm'|v_{ij}|nm\rangle_r \delta_{m-n,m'-n'} \delta_{N,N'} \delta_{M,M'}. \quad \text{(C5)}$$

For the Coulomb potential, $v_{ij} = 1/|r_i - r_j|$, with lengths measured in l_0 units, the non-vanishing matrix elements are:

$${}_r\langle n'm'|v_{ij}|nm\rangle_r = \frac{\Gamma(|j|+1/2)\Gamma(l'+1/2)}{2|j|!} \times \cdots$$
$$\times \sqrt{\frac{(l+|j|)!}{\pi(l'+|j|)!l'!l!}}\; {}_3F_2\left[\begin{matrix} -l, |j|+\frac{1}{2}, \frac{1}{2} \\ |j|+1, 1/2-l' \end{matrix}; 1\right], \quad \text{(C6)}$$

where $m \geq 0$, $m' \geq 0$, $n \geq 0$, $n' \geq 0$, $m' - n' = m - n$, $j = m - n$, $l = n + (j - |j|)/2$, and $l' = n' + (j - |j|)/2$.

Equation (C6) can be obtained by combining the explicit expressions for the relative coordinate wavefunctions[50] with the useful integrals computed in the Appendix of Ref. 60.

For the $N = fN_{LL} + 2$-body problem, in which the lowest Landau level is completely filled and only two particles are in the $n = 1$ LL, the basis of pure relative and center of mass quantum numbers is no longer the most convenient choice to describe the relative motion of these two particles. The reason is that generally states with well defined relative kinetic energy, $n > 1$, will have a probability amplitude for the individual particles to occupy the $n = 0$ LL, which is forbidden by the Pauli exclusion principle. The most natural states would look entangled in both the basis of Eq. (C1) and Eq. (C2), and have the form

$$|M, m\rangle_1 \equiv a_1^\dagger a_2^\dagger |M,0\rangle_c |m,0\rangle_r$$
$$= \frac{1}{\sqrt{2}} (|M,2\rangle_c |m,0\rangle_r - |M,0\rangle_c |m,2\rangle_r). \quad \text{(C7)}$$

These are the states that determine the Haldane pseudopotentials in the $n = 1$ LL[46,47] appearing in Eq. (29). To simplify notation, we have denoted them simply in Eq. (29) by $|m\rangle_1$, since the pseudopotentials are diagonal on and independent of the center of mass label M.

The computation of Haldane pseudopotentials in the $n = 1$ LL is simplified by transforming the two body interactions of Eqs. (24), (25), (26), and (31), into their first quantization version, which we describe in the remainder of this Appendix. For $\mathcal{V}^{2b}(2)$ appearing in Eq. (24), we have the equivalent first quantization form

$$\mathcal{V}^{2b}_{ij}(2) = -\kappa^2 v_{ij} \left[\sum_{1,2} |1_i 2_j\rangle\langle 1_i 2_j| \frac{\theta(n_1,n_2)}{n_1+n_2-2}\right] v_{ij}, \quad \text{(C8)}$$

where $|1_i 2_j\rangle$ is a shorthand for the state $|n_1 m_1\rangle_i |n_2 m_2\rangle_j$, and $\theta(n_1, n_2)$ restricts $n_1 + n_2 \geq 3$ and $n_1 \geq 1, n_2 \geq 1$. Combining Eqs. (C3), (C4), and (C7) with Eq. (C8) leads to the following semi-analytic expression for the Haldane pseudopotentials associated with $\mathcal{V}^{2b}(2)$:

$$V^{2b}(2)(m) \equiv \frac{1}{\kappa^2}\,{}_1\langle m|\mathcal{V}^{2b}_{ij}(2)|m\rangle_1 = -\sum_{n=1}^{\infty} \frac{1}{2n} \Big[{}_r\langle n, m+n|v_{12}|0,m\rangle_r^2 + {}_r\langle n+2, m+n|v_{12}|2,m\rangle_r^2$$
$$- \frac{1}{2^{n+1}} \Big({}_r\langle n, m+n|v_{12}|0,m\rangle_r \sqrt{(n+2)(n+1)/2} - {}_r\langle n+2, m+n|v_{12}|2,m\rangle_r \Big)^2 \Big]. \quad \text{(C9)}$$

For $\mathcal{V}^{2b}(0)$, appearing in Eq. (24), we have the equivalent first quantization form

$$\mathcal{V}^{2b}_{ij}(0) = -\frac{\kappa^2}{2} v_{ij} \left[\sum_{1,2} |1_i 2_j\rangle\langle 1_i 2_j| \delta_{n_1,0} \delta_{n_2,0}\right] v_{ij}. \quad \text{(C10)}$$

This form leads to the following analytic expression for the Haldane pseudopotentials:

$$V^{2b}(0)(m) \equiv \frac{1}{\kappa^2}{}_1\langle m|\mathcal{V}^{2b}_{ij}(0)|m\rangle_1 = \\ -\frac{1}{4}{}_r\langle 2,m|v_{12}|0,m-2\rangle^2_r. \quad \text{(C11)}$$

We convert the remaining two-body interactions into first quantization using a slightly different technique, along similar lines to the original computation of the conventional Haldane pseudopotentials in the $n=1$ LL.[46] We write the interaction as a Fourier sum

$$v_{ij} = \sum_q v_q e^{iq\cdot(r_i-r_j)}, \quad \text{(C12)}$$

where $\sum_q$ abbreviates $1/A\sum_q$.

For $\mathcal{V}^{2b}_a(1)$ the interaction matrix appearing in Eq. (25) can be seen to correspond to the following first quantized interaction:

$$\mathcal{V}^{2b}_{a,ij}(1) = -2\kappa^2 f \sum_{q_1,q_2} v_{q_1}v_{q_2}e^{iq_1\cdot r_i - iq_2\cdot r_j} \times \cdots \\ \left[\sum_{5,6}\frac{\tau(n_5,n_6)}{n_6}\langle 6|e^{iq_2\cdot r}|5\rangle\langle 5|e^{-iq_1\cdot r}|6\rangle\right], \quad \text{(C13)}$$

where $|5\rangle$ abbreviates single-particle state $|n_5 m_5\rangle$ (similarly for $|6\rangle$), $\tau(n_5,n_6)$ restricts $n_6\geq 1$ and $n_5=0$, and f is the spin multiplicity. Then using the properties of the matrix elements of the single-particle density operator,[49,50] one finds

$$\mathcal{V}^{2b}_{a,ij}(1) = -\frac{\kappa^2 f}{\pi}\int_q v_q^2\sum_{n=1}^{\infty}\frac{|\mathcal{F}_{0n}(q)|^2}{n}\, e^{iq\cdot(r_i-r_j)}, \quad \text{(C14)}$$

where $|\mathcal{F}_{0n}(q)|^2 = |q|^{2n}e^{-|q|^2/2}/(2^n n!)$ is the modulus squared of the density form factors between the 0 and $n>0$ Landau levels[49,50] (explicitly presented in Eq. (C21) below), and $\int_q = \int d^2q/(2\pi)^2$. The Haldane pseudopotentials are then found to be

$$V^{2b}_a(1)(m) = -\frac{f}{2\pi^2}\int_0^{\infty} dq\; q v_q^2[L_1(q^2/2)]^2 L_m(q^2)\times\cdots \\ e^{-3q^2/2}\left[-\gamma + Ei(q^2/2) - Ln(q^2/2)\right], \quad \text{(C15)}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, $Ei(x) = -\int_{-x}^{\infty}dte^{-t}/t$ is the exponential integral, L_n are the Laguerre polynomials, and Ln is the natural logarithm. We substituted the explicit form of the Coulomb potential, $v_q = 2\pi/q$, in this expression to obtain the numbers listed in Table IV.

It is interesting to note that this expression is the κ^2 term of the well-known RPA screened potential for a completely filled $n=0$ LL, as we demonstrate below. The static screened RPA potential is,

$$v_q^{RPA} = \kappa v_q + \kappa^2 v_q^2\chi_q^0 + \mathcal{O}(\kappa^3), \quad \text{(C16)}$$

with χ_q^0 the static density-density response function. For a completely filled $n=0$ LL, the static density-density response function is[50]

$$\chi_q^0 = \frac{f}{2\pi}\sum_{n\neq n'}\frac{f_n - f'_n}{n-n'}|\mathcal{F}_{nn'}(q)|^2 = -\frac{f}{2\pi}\sum_{n=1}^{\infty}\frac{|\mathcal{F}_{0n}(q)|^2}{n}, \quad \text{(C17)}$$

where the Fermi occupation factor, f_n, is 1 for $n=0$ and 0 otherwise. The density form factors $\mathcal{F}_{nn'}(q)$ are given in Eq. (C21) below. It is clear from Eq. (C17), that the κ^2 term of v_q^{RPA} is equivalent to $\mathcal{V}^{2b}_{a,ij}(1)$ appearing in Eq. (C14).

$\mathcal{V}^{2b}_b(1)$, appearing in Eq. (25), can be seen to correspond to the following first quantized interaction,

$$\mathcal{V}^{2b}_{b,ij}(1) = \sum_{q_1,q_2} v_{q_1}v_{q_2}e^{iq_2\cdot r_i}\left[\sum_{6,n_6\geq 1}\frac{|6_i\rangle\langle 6_i|}{n_6}\right]e^{iq_1\cdot r_i}\times\cdots \\ e^{-iq_1\cdot r_j}\left[\sum_{5,n_5=0}|5_j\rangle\langle 5_j|\right]e^{-iq_2\cdot r_j} + \{i\Leftrightarrow j\}, \quad \text{(C18)}$$

where $|6_i\rangle\langle 6_i|$ operates only on particle i, and $|5_j\rangle\langle 5_j|$ only on j. It is not transparent in this representation that $\mathcal{V}^{2b}_{b,ij}(1)$ has translational and rotational invariance. These properties can be made manifest if we write the position operators, r, in terms of the mechanical momentum, π, and guiding center, c, coordinates, from the following equations:

$$\begin{aligned}\pi &= p + \frac{e}{c}A(r), \\ c &= r - \hat{z}\times\pi.\end{aligned} \quad \text{(C19)}$$

The mechanical momentum and guiding center coordinates are related to the lowering operators as $a = (\pi_x + i\pi_y)/\sqrt{2}$, $b = (c_x - ic_y)/\sqrt{2}$.[49,50]

The interaction in Eq. (C18), projected into the $n=1$ LL, can be written as

$$\mathcal{V}^{2b}_{b,ij}(1) = \int_{q_1}\int_{q_2} v_{q_1}v_{q_2}e^{i(q_1-q_2)\cdot(c_i-c_j)}\times\cdots \\ \mathcal{F}_{10}(-q_1)\mathcal{F}_{01}(q_2)\left[\sum_{n=1}^{\infty}\frac{\mathcal{F}_{1n}(-q_2)\mathcal{F}_{n1}(q_1)}{n}\right] + \{i\Leftrightarrow j\}, \\ \text{(C20)}$$

where $\mathcal{F}_{n'n}(q) \equiv \langle n'|\exp(iq \cdot \hat{z} \times \pi)|n\rangle$ are the density form factors, given by[49,50]

$$\mathcal{F}_{n'n}(q) = \sqrt{\frac{n!}{n'!}}\Big(\frac{q_x + iq_y}{\sqrt{2}}\Big)^{n'-n} e^{-\frac{|q|^2}{4}} L_n^{n'-n}\Big(\frac{|q|^2}{2}\Big), \tag{C21}$$

for $n' \geq n$, and $\mathcal{F}_{n'n}(-q) = \mathcal{F}^*_{nn'}(q)$. Combinig this with the analogue relation for the guiding center coordinates,

$$_r\langle 0, m|e^{iq\cdot(c_i - c_j)}|0, m\rangle_r = e^{-\frac{|q|^2}{2}} L_m(|q|^2), \tag{C22}$$

one finds the Haldane pseudopotentials for $\mathcal{V}_b^{2b}$ listed in Table IV.

For $\mathcal{V}_c^{2b}(1)$, appearing in Eq. (25), following an analogous procedure as the one just outlined for $\mathcal{V}_b^{2b}(1)$, one finds the following first quantized representation for the interaction projected into the $n = 1$ LL:

$$\mathcal{V}^{2b}_{c,ij}(1) = 4 \int\limits_{q_1}\int\limits_{q_2} v_{q_1} v_{q_2} e^{iq_1\cdot(c_i - c_j)} \times \cdots$$
$$\Re\Big\{e^{i\hat{z}\cdot q_2\times q_1} \mathcal{F}_{01}(-q_2)\mathcal{F}_{11}(-q_1)\Big[\sum_{n=1}^{\infty}\frac{\mathcal{F}_{1n}(q_2)\mathcal{F}_{n0}(q_1)}{n}\Big]\Big\}. \tag{C23}$$

Combining Eq. (C23) with Eq. (C22) one obtains the Haldane pseudopotentials listed in Table IV for $\mathcal{V}_c^{2b}(1)$.

Finally, the pseudopotentials corresponding to the effective interaction in the spin polarized $n = 0$ LL, appearing in Eq. (31), can be obtained similarly to those for $\mathcal{V}_a^{2b}(1)$, and read as,

$$V_o^{2b}(m) = -\frac{f_o}{2\pi^2}\int\limits_0^{\infty} dq\; q v_q^2 L_m(q^2) \times \cdots$$
$$e^{-3q^2/2}\big[-\gamma + Ei(q^2/2) - Ln(q^2/2)\big], \tag{C24}$$

where f_o is the number of fully occupied flavors ($f_o < f$). The numbers associated with this interaction are listed in Table VI.

As an independent consistency check, we have computed the Haldane pseudopotentials for the $n = 1$ LL appearing in Table IV, for the leading spin triplet states (odd m), in a more direct numerical approach using the two-body interaction matrices that appear in Eqs. (24) and (25), and verified they coincide with those computed through the semi-analytical formulae discussed in this Appendix.

We have presented our formulae in a manner that can be easily modified to incorporate cases other than pure Coulomb interactions, with the only requirement of rotational invariance. Only Eq. (C6) makes explicit use of the form of the Coulomb interaction.

[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2] H. Fukuyama, P. M. Platzman, and P. W. Anderson, Phys. Rev. B **19**, 5211 (1979).
[3] D. Yoshioka and H. Fukuyama, Journal of the Physical Society of Japan **47**, 394 (1979).
[4] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[5] J. Eisenstein and A. MacDonald, Nature **432**, 691 (2004).
[6] A. Kumar, N. Samkharadze, G. A. Csáthy, M. J. Manfra, L. N. Pfeiffer, and K. W. West, Phys. Rev. B **83**, 201305 (2011).
[7] M. Padmanabhan, T. Gokmen, and M. Shayegan, Phys. Rev. B **81**, 113301 (2010).
[8] D. Maryenko, J. Falson, Y. Kozuka, A. Tsukazaki, M. Onoda, H. Aoki, and M. Kawasaki, Phys. Rev. Lett. **108**, 186803 (2012).
[9] M. R. Peterson and C. Nayak, arXiv:1303.1541.
[10] T. M. Kott, B. Hu, S. H. Brown, and B. E. Kane, arXiv:1210.2386.
[11] R. L. Willett, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. B **37**, 8476 (1988).
[12] R. Morf and N. d'Ambrumenil, Phys. Rev. B **68**, 113309 (2003).
[13] X. Wan, D. N. Sheng, E. H. Rezayi, K. Yang, R. N. Bhatt, and F. D. M. Haldane, Phys. Rev. B **72**, 075325 (2005).
[14] V. Melik-Alaverdian and N. E. Bonesteel, Phys. Rev. B **52**, R17032 (1995).
[15] D. Yoshioka, Journal of the Physical Society of Japan **53**, 3740 (1984).
[16] D. Yoshioka, Surf. Sci. **170**, 125 (1986).
[17] D. Yoshioka, Journal of the Physical Society of Japan **55**, 885 (1986).
[18] G. Murthy and R. Shankar, Phys. Rev. B **65**, 245309 (2002).
[19] G. Moore and N. Read, Nuclear Physics B **360**, 362 (1991).
[20] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[21] W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **83**, 3530 (1999).
[22] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. **99**, 236807 (2007).
[23] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. **99**, 236806 (2007).
[24] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[25] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).
[26] G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).
[27] M. R. Peterson, T. Jolicoeur, and S. Das Sarma, Phys. Rev. B **78**, 155308 (2008).
[28] M. R. Peterson, T. Jolicoeur, and S. Das Sarma, Phys. Rev. Lett. **101**, 016807 (2008).
[29] M. R. Peterson, K. Park, and S. Das Sarma, Phys. Rev. Lett. **101**, 156803 (2008).
[30] A. Wójs, Phys. Rev. B **80**, 041104 (2009).
[31] H. Wang, D. N. Sheng, and F. D. M. Haldane, Phys. Rev. B **80**, 241311 (2009).
[32] M. Storni, R. H. Morf, and S. Das Sarma, Phys. Rev. Lett. **104**, 076803 (2010).
[33] H. Lu, S. Das Sarma, and K. Park, Phys. Rev. B **82**, 201303 (2010).
[34] A. Wójs, C. Tőke, and J. K. Jain, Phys. Rev. Lett. **105**, 196801 (2010).
[35] A. Wójs, C. Tőke, and J. K. Jain, Phys. Rev. Lett. **105**, 096802 (2010).
[36] E. H. Rezayi and S. H. Simon, Phys. Rev. Lett. **106**, 116801 (2011).
[37] Z. Papic, F. D. M. Haldane, and E. H. Rezayi, Phys. Rev. Lett. **109**, 266806 (2012).
[38] Y. Liu, D. Kamburov, M. Shayegan, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, Phys. Rev. Lett. **107**, 176805 (2011).
[39] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Solid State Communications **149**, 1867 (2009).
[40] M. Padmanabhan, T. Gokmen, and M. Shayegan, Phys. Rev. Lett. **104**, 016805 (2010).
[41] A. H. MacDonald, Phys. Rev. B **30**, 4392 (1984).
[42] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. B **42**, 4532 (1990).
[43] W. Bishara and C. Nayak, Phys. Rev. B **80**, 121302 (2009).
[44] S. H. Simon and E. H. Rezayi, arXiv:1303.2809.
[45] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[46] F. D. M. Haldane, *The quantum Hall effect*, edited by R. Prange and S. Girvin, Vol. 1 (Springer-Verlag New York, 1987) Chap. 8.
[47] S. H. Simon, E. H. Rezayi, and N. R. Cooper, Phys. Rev. B **75**, 195306 (2007).
[48] S. C. Davenport and S. H. Simon, Phys. Rev. B **85**, 075430 (2012).
[49] A. H. MacDonald, arXiv:cond-mat/9410047.
[50] G. Giuliani and G. Vignale, *Quantum theory of the electron liquid* (Cambridge University Press, 2005).
[51] A. H. MacDonald and D. S. Ritchie, Phys. Rev. B **33**, 8336 (1986).
[52] R. B. Laughlin, Phys. Rev. B **27**, 3383 (1983).
[53] M. Hamermesh, *Group theory and its application to physical problems* (Dover publications, 1989).
[54] As in the $n = 0$ LL case, the derivation is independent of the presence of the Zeeman energy term, because virtual transitions into states with different S_z are forbidden. This conclusion still holds for f flavors and SU(f)-invariant interactions in the presence of single particle Zeeman-like terms.
[55] This decrease is not that fast for the $m = 0$ state, but it is very fast for $m \geq 1$.
[56] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).
[57] M. Greiter, X. Wen, and F. Wilczek, Nuclear Physics B **374**, 567 (1992).
[58] X.-G. Wen, Phys. Rev. Lett. **70**, 355 (1993).
[59] In the notation of Ref. 48, $\beta_1 = \sqrt{2} z_b$ and $\beta_2 = 2 z_a z_b / \sqrt{3}$.
[60] M. L. Glasser and N. J. M. Horing, Phys. Rev. B **31**, 4603 (1985).