Projective construction of two-dimensional symmetry-protected topological phases with U(1), SO(3), or SU(2) symmetries

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Projective construction of two-dimensional symmetry-protected topological phases with U(1) / SO(3) / SU(2) symmetries

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We propose a general approach to construct symmetry protected topological (SPT) states (*ie* the short-range entangled states with symmetry) in 2D spin/boson systems on lattice. In our approach, we fractionalize spins/bosons into different fermions, which occupy nontrivial Chern bands. After the Gutzwiller projection of the free fermion state obtained by filling the Chern bands, we can obtain SPT states on lattice. In particular, we constructed a U(1) SPT state of a spin-1 model, a SO(3) SPT state of a boson system with spin-1 bosons and spinless bosons, and a SU(2) SPT state of a spin-1/2 boson system. By applying the "spin gauge field" which directly couples to the spin density and spin current of S^z components, we also calculate the quantum spin Hall conductance in each SPT state. The projective ground states can be further studied numerically in the future by variational Monte Carlo etc.

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I. INTRODUCTION

Recently, quantum entanglement becomes an important concept and a powerful tool in the fields of quantum information and quantum many-body systems.[1,2] A long-range entangled state (LRE) can not be deformed to a direct product state by any local unitary (LU) transformation.[3–5] The fractional quantum Hall effect (FQHE)[6] is a concrete example of LRE and has been considered as an example of intrinsic "topological order".[7–9] If a gapped quantum state is short-range entangled (SRE), one can always deform it to a spatially direct product state by a LU transformation[3]. So a SRE state is trivial. But if a SRE state is invariant under some symmetry group G, then it can be non-trivial (even if it does not break any symmetry). This because for some symmetric SRE states, there is no way to locally deform the states into direct product states unless one explicitly breaks the symmetry in LU procedure. We call such SRE states "Symmetry-Protected Topological Phases" (SPT).[3–5] Topological insulators[10] (TI) belongs to fermionic SPT protected by $\mathbb{Z}_2$ time reversal symmetry and U(1) particle number conservation. On the other hand, if a gapped symmetric quantum state is long-range entanglements, it will be called "Symmetry-Enriched Topological state" (SET).

Group theory has been successfully utilized in analyzing symmetry-breaking phases. Recently, it was shown that group cohomology with U(1) coefficient can be utilized in describing SPT phases systematically.[3] A concrete SPT phase is the 1D Haldane phase with SO(3) symmetry where integer spin is a faithful representation of SO(3) symmetry. It is known that the spin-1 antiferromagnetic chain is semiclassically (in large spin-S expansion) described by a nonlinear sigma model (NLSM) with 2π topological theta term. The two boundaries are occupied by two dangling free 1/2 spins which are protected by SO(3) symmetry and contributes to fourfold degeneracy.

It is interesting that, by using group cohomology theory, one can construct exactly solvable lattice models which realize specific SPT phases respecting given symmetry groups[3,11,12]. Along this line, some interesting lattice models of bosonic SPT phases with discrete symmetry groups have been proposed in literatures. For continuous symmetry group, however, the solution of cocycle construction in group cohomology is more difficult mathematically. 2D SPT phases with U(1) symmetry[13,14] and with SO(3) and with SU(2) symmetries[15] are studied using Chern-Simons field theory and Wess-Zumino-Witten (WZW) field theory.[16–18] But, a concrete lattice model for realizing those SPT phases is much desired, which is more physical than continuum field theory description.

Instead of directly constructing exactly solvable models, in this paper, we present an effective approach to realize SPT phases on 2D lattice. Our approach is based on the projective construction of strongly correlated bosonic or spin systems,[19–30]. In this construction, bosons/spins are fractionalized into several fermions which form some meanfield states before projection. The projection can be done by considering the confined phase of internal gauge fields which glue the fermions back into bosons/ spins. Recently, bosonic topological insulators (a bosonic SPT with $U(1) \rtimes Z_2^T$ symmetry group where U(1) is particle number conservation and Z_2^T is time-reversal symmetry) on 3D lattice are constructed in this way[31]. In the present paper, we shall consider 2D SPT systems with continuous symmetry groups. We assume that fermions occupy several independent nontrivial Chern bands respectively. After the Gutzwiller projection of the free fermion state obtained by filling the Chern bands, we can obtain SPT states on lattice. We have constructed a U(1) SPT state, a SO(3) SPT state, and a SU(2) SPT state using the projective construction.

II. $U(1)$ SPT STATES IN SPIN-1 LATTICE MODEL

A. Meanfield ansatz

In this section, we are going to construct a SPT phase with $U(1)$ symmetry on lattice.[3,13,14,32] Such a state has been obtained by several other constructions.[33–36] Our lattice model is a spin-1 model on a honeycomb lattice, with 3 states, $|m\rangle$, $m = 0, \pm 1$, on each site. We can view our spin-1 model as spin-1/2 hardcore-boson model with $|m = 0\rangle$ state as the no-boson state (or two bosons with different flavors and different spins) and $|m = \pm 1\rangle$ state as the one-boson state with spin-up or spin-down.

In order to ensure the particle number constraint as to be explained below, we must introduce two flavors of hard-core bosons. In a fermionic projective construction, we write the bosonic operators as

$$b_{\alpha,i}^{(s)} = f_{\alpha,i} c_i^{(s)}, \quad \alpha = \uparrow, \downarrow \tag{1}$$

where, $f_\uparrow, f_\downarrow$ are two fermions with spin-$\uparrow$ and spin-$\downarrow$. $s = 1, 2$ is the flavor index of bosons and also the orbital index of c-fermions. $c_i^{(1)}$ and $c_i^{(2)}$ are annihilation operators of c-fermions in two orbitals of c-fermions at site-i, respectively. The nature of hard-core is implied by $(b_\alpha^{(s)})^2 = 0$.

The identity (1) is not an operator identity. Rather, the matrix elements of the two sides in the physical Hilbert space are identical to each other. The physical Hilbert space we consider is three-dimensional spin-1 space ($|m_i = 0, \pm 1\rangle$) at any site-i. In terms of fermions, each spin state can be constructed by two different fermionic states:

$$m = 0 \text{ state } : |0\rangle \text{ and } f_{i\uparrow}^\dagger c_i^{(1)\dagger} f_{i\downarrow}^\dagger c_i^{(2)\dagger} |0\rangle, \tag{2}$$

$$m = 1 \text{ state } : f_{i\uparrow}^\dagger c_i^{(1)\dagger} |0\rangle \text{ and } f_{i\uparrow}^\dagger c_i^{(2)\dagger} |0\rangle, \tag{3}$$

$$m = -1 \text{ state } : f_{i\downarrow}^\dagger c_i^{(1)\dagger} |0\rangle \text{ and } f_{i\downarrow}^\dagger c_i^{(2)\dagger} |0\rangle, \tag{4}$$

where, $|0\rangle$ denotes vacuum with no particles. For example, the state at site-i is $|m_i = 1\rangle$ if one $f_\uparrow$ fermion and one $c^{(1)}$-fermion occupy the site, or, if one $f_\uparrow$-fermion and one $c^{(2)}$-fermion occupy the site. Therefore, the three-dimensional spin space is actually a six-dimensional Hilbert space in terms of fermions and each of spin states is constructed by equal number of f-fermions and c-fermions, which is the particle number constraint implemented by the present fermionic projective construction. This can be also seen clearly by noting that the projective construction in Eq. (1) introduces an internal gauge field $\mathcal{A}_\mu$ that is responsible for gluing f- and c-fermions together to form a hard-core boson. By choice of convention, both of $f_\uparrow$ and $f_\downarrow$ ($c^{(1)}$ and $c^{(2)}$) carry $+1$ (-1) gauge charge of $\mathcal{A}_\mu$. As a result, by integrating out the temporal component $\mathcal{A}_0$, the above particle number constraint on particle numbers is obtained.

Therefore, we can start with a many-fermion state of $f_{i,\alpha}$ and c_i, $|\Psi\rangle$, and obtain a physical spin state $|\Phi\rangle$ by projecting into the sub-space with equal f-fermions and c-fermions on each site:

$$|\Phi\rangle = P|\Psi\rangle. \tag{5}$$

Using such a projective construction, we would like to construct a U(1) SPT state on a honeycomb lattice. The projection operator P is given by:

$$\begin{aligned} P = \prod_i \Big(& |m_i = +1\rangle (\langle i, \uparrow, (1)| + \langle i, \uparrow, (2)|) \\ &+ |m_i = -1\rangle (\langle i, \downarrow, (1)| + \langle i, \downarrow, (2)|) \\ &+ |m_i = 0\rangle (\langle i, 0| + \langle i, \uparrow, (1), \downarrow, (2)|) \Big) \end{aligned} \tag{6}$$

where, $|i, \alpha, (s)\rangle$ denotes $f_{i\alpha}^\dagger c_i^{(s)\dagger} |0\rangle$.

Let us consider the following free fermion Hamiltonian on a honeycomb lattice

$$H_{MF} = \sum_{i,j} \left[\mathbf{f}_j^\dagger U_{ji} \mathbf{f}_i + \mathbf{c}_j^\dagger V_{ji} \mathbf{c}_i \right], \tag{7}$$

where, $\mathbf{f} \equiv (f_\uparrow, f_\downarrow)^T$, $\mathbf{c} \equiv (c^{(1)}, c^{(2)})^T$, and, U_{ji} are 2×2 matrices satisfying

$$U_{ji}^\dagger = U_{ij}, \tag{8}$$

and V_{ji}:

$$V_{ji}^\dagger = V_{ij}. \tag{9}$$

Let $|\Psi\rangle$ be the lowest energy state of the above free fermion Hamiltonian. Then $|\Phi\rangle = P|\Psi\rangle$ will be a spin/boson state induced by the above free fermion Hamiltonian. We say that such a spin/boson state is described by the ansatz U_{ji} and V_{ji}.

To construct a SPT state using the above projective construction, we choose the ansatz U_{ij} to be[37]

$$U_{ij} = t_0 \sigma^0, \quad ij = \text{ nearest neighbour (NN) links} \tag{10}$$
$$U_{ij} = i\nu_{ij} t \sigma^3, \quad ij = \text{next nearest neighbour (NNN) links},$$

where, the complex number v_{ij} is

$$v_{ij} = t_0, \quad ij = \text{NN links} \tag{11}$$
$$v_{ij} = i\nu_{ij} t, \quad ij = \text{NNN links},$$

Here, t_0, t are hopping energies of NN and NNN respectively. σ^0 is a two-dimensional identical matrix, $\sigma^{1,2,3}$ are three usual Pauli matrices. $\nu_{ij} = +$ if the fermion f makes a right turn going from j to i on the honeycomb lattice, and $\nu_{ij} = -$ if the fermion f makes a left turn. In this ansatz, each of $f_\uparrow, f_\downarrow$ contains two bands (due to the two independent sites per unit cell of the honeycomb lattice) with a gap at zero energy. Each lower band has a

Chern number $+1$ for the $f_\uparrow$-fermions and a Chern number -1 for the $f_\downarrow$-fermions. Let's fill the lower band of $f_\uparrow$ and the lower band of $f_\uparrow$ completely, such that each unit cell is filled with one $f_\uparrow$ and one $f_\downarrow$ fermion respectively. For c-fermions, each unit cell contains four bands due to two independent sites of honeycomb lattice and two orbitals at each site. In order that the particle number constraint between f and c fermions is satisfied at least in meanfield level, an appropriate design of the two by two matrix V_{ij} leads that the two lower bands are filled and the total Chern number of these two bands (i.e. summation of the two Chern numbers of the two bands) is equal to $+1$. Now, each unit cell contains one $f_\uparrow$, one $f_\downarrow$, and two c fermions.

B. Effective field theory

In the following, we will show that $|\Phi\rangle = P|\Psi\rangle$ is a bosonic SPT state protected by the $U(1)$ symmetry (generated by S^z). First, from the free fermion Hamiltonian, we see that the ground state state $|\Psi\rangle$ respects the S_z spin rotation symmetry generated by S^z. The f_α-fermions form a "spin Hall" state described by $U(1) \times U(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi}(a_{\uparrow\mu}\partial_\nu a_{\uparrow\lambda}\epsilon^{\mu\nu\lambda} - a_{\downarrow\mu}\partial_\nu a_{\downarrow\lambda}\epsilon^{\mu\nu\lambda})$$
$$+ \frac{1}{2\pi}A^{\text{spin}}_\mu\partial_\nu(a_{\uparrow\lambda} - a_{\downarrow\lambda})\epsilon^{\mu\nu\lambda}$$
$$= \frac{1}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}q_I A^{\text{spin}}_\mu\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda} \quad (12)$$

with $I, J = 1, 2$ and

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (13)$$

The coupling gauge charges of $f_\uparrow$ and $f_\downarrow$ in q-vector are $+1$ and -1 since $\uparrow$ and $\downarrow$ are related to the two spinful states of the physical spin-1 operator at each site. Here the "spin gauge field" A^{spin}_μ is the gauge potential that *directly* couples to the S_z spin density and current, which is different from the definition of conventional spin Hall effect where the applied external gauge field is the electromagnetic field. Therefore, this "spin Hall" meanfield ansatz for f fermions has no spin Hall conductance ($\frac{1}{2\pi}q^T K^{-1}q = 0$) in the present definition, although it is nonzero ($\frac{1}{2\pi}(1,1)K^{-1}q = 2 \times \frac{1}{2\pi}$) if external electric field (composed by electromagnetic gauge potential) is the probe field instead. Since the U(1) global symmetry which defines the U(1) SPT discussed here is the spin rotational symmetry, it is physically natural to gauge this U(1) global symmetry (the corresponding gauge potential is nothing but A^{spin}_μ) and discuss its response theory to diagnose SPT physics. In the whole paper, we shall apply the spin gauge field A^{spin}_μ to probe the *intrinsic* quantum spin Hall response phenomenon.

The c-fermions form an "integer quantum Hall" state described by $U(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi}b_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda} \quad (14)$$

Thus the total effective theory is given by

$$\mathcal{L} = \frac{1}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}q_I A^{\text{spin}}_\mu\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}$$
$$+ \frac{1}{4\pi}b_\mu\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda} \quad (15)$$

The projection P is done by setting the total f_α-fermion density-current, $J^f_\mu = \frac{1}{2\pi}\partial_\nu(a_{\uparrow\lambda} + a_{\downarrow\lambda})\epsilon^{\mu\nu\lambda}$, equal to the c-fermion density-current, $J^c_\mu = \frac{1}{2\pi}\partial_\nu b_\lambda\epsilon^{\mu\nu\lambda}$.[38,39] After setting $b_\lambda = (a_{\uparrow\lambda} + a_{\downarrow\lambda})$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi}\bar{K}_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}q_I A^{\text{spin}}_\mu\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda} \quad (16)$$

with

$$\bar{K} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{K}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \quad (17)$$

Since $|\det(\bar{K})| = |-1| = 1$, Eqn. (16) is the low energy effective theory for the spin/boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. Such a low energy effective theory describes a $U(1)$ SPT state with an integer Hall conductance (U(1) is spin rotation symmetry of spin-1 along z-axis)

$$\sigma_{xy} = q^T\bar{K}^{-1}q\frac{1}{2\pi} = -4\frac{1}{2\pi}. \quad (18)$$

We note that the spin Hall conductance is an even integer in the unit of $\frac{1}{2\pi}$.

III. SO(3) SPT STATES IN A LATTICE MODEL WITH SPIN-1 BOSONS AND SPINLESS BOSONS

A. Meanfield ansatz

SO(3) SPT phases were firstly studied in Ref.[15] where field theoretic approach is applied. The bulk is described by SO(3) principal chiral nonlinear sigma model (the field variable g is 3×3 SO(3) matrix) with $2\pi K$ topological theta term where K ($K/4 \in \mathbb{Z}$) completely labels distinct SPT phases. The boundary is described by a $SO(3)_K$ WZW term where SO(3) transformation is defined as left-multiplication, *ie* $g \to hg$ with $h \in SO(3)_L$, such that only left mover carries SO(3) charge and right mover is SO(3) charge-neutral. Due to this chiral SO(3) transformation, the boundary can not be simply replaced by a $SO(3)_K$ WZW critical spin chain that is nonchiral.

In this section, we are going to construct a SPT phase protected by $SO(3)$ symmetry on lattice. Our lattice model contain spin-1 bosons and three spin-0 bosons on

a honeycomb lattice. Therefore, this model is a model of many-boson, such as a cold-atom system in 2D.

In a fermionic projective construction, we write the spin-1 bosonic operators (labeled by a spin index $m = 0, \pm 1$ and a flavor index $s = 1, \cdots 4$) as

$$b_{m,i}^{(s)} = f_{m,i} c_i^{(s)}, \quad m = 0, \pm 1 \tag{19}$$

and the three ($m = 0, \pm 1$) spin-0 bosons each of which admits four flavors ($s = 1, \cdots, 4$) as

$$\tilde{b}_{m,i}^{(s)} = \tilde{f}_{m,i} c_i^{(s)}, \quad m = 0, \pm 1 \tag{20}$$

Here, $s = 1, \cdots, 4$ is also the orbital index for c-fermions at each site.

Again, we can start with a many-fermion state of $f_{m,i}$, $\tilde{f}_{m,i}$ and c_i, $|\Psi\rangle$, and obtain a physical spin state $\Phi\rangle$ by projecting into the sub-space where the sum of the f-fermion number and the $\tilde{f}$-fermion number is equal to the c-fermion number on each site:

$$|\Phi\rangle = P|\Psi\rangle. \tag{21}$$

Using such a projective construction, we can construct a SO(3) SPT state on a honeycomb lattice.

Let us consider the following free fermion Hamiltonian on a honeycomb lattice

$$H = \sum_{i,j} \left[\mathbf{f}_j^\dagger u_{ji} \mathbf{f}_i + \tilde{\mathbf{f}}_j^\dagger u_{ji}^* \tilde{\mathbf{f}}_i, + \mathbf{c}_j^\dagger V_{ji} \mathbf{c}_i, \right] \tag{22}$$

where, $\mathbf{f} = (f_{+1}, f_0, f_{-1})^T, \tilde{\mathbf{f}} = (\tilde{f}_{+1}, \tilde{f}_0, \tilde{f}_{-1})^T, \mathbf{c} = (c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)})^T$, and, u_{ji} are complex numbers satisfying

$$u_{ji}^* = u_{ij}, \tag{23}$$

To construct a SO(3) SPT state using the above projective construction, we choose the ansatz to be

$$u_{ij} = 1, \quad ij = \text{NN links} \tag{24}$$
$$u_{ij} = i\nu_{ij} t, \quad ij = \text{NNN links},$$

where $\nu_{ij} = +$ if the fermion f makes a right turn going from j to i on the honeycomb lattice, and $\nu_{ij} = -$ if the fermion f makes a left turn. For such an ansatz, each of the six fermions (f_m and $\tilde{f}_m$) forms two energy bands with a gap at zero energy and the lower one is filled. Each lower band has a Chern number $+1$ for the f-fermions and a Chern number -1 for the $\tilde{f}$-fermions.

The four by four matrix V_{ij} leads to an eight by eight single particle Hamiltonian in momentum space for c-fermions in the honeycomb unit cell. Let's fill the six bands from the bottom and assume that the band gap exists between the sixth and seventh bands. The total Chern numbers contributed by the six bands is equal to $+1$. As a result, in each unit cell, there are three f fermions (one for each of the three components), three $\tilde{f}$ fermions (one for each of the three components), and six c fermions, which satisfy the particle number constraint.

B. Effective field theory

In the following, we like to show that $|\Phi\rangle = P|\Psi\rangle$ will be a bosonic SPT state protected by the SO(3) symmetry (generated by $f^\dagger T^l f$, where T^l, $l = 1, 2, 3$, are the 3×3 matrices that generate the SO(3) Lie algebra). From the free fermion Hamiltonian, we see that the ground state state $|\Psi\rangle$ respects the SO(3) spin rotation symmetry generated by $f^\dagger T^l f$. The f-fermions form an "integer quantum Hall" state described by $U^3(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu^{\text{spin}} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \tag{25}$$

with $I, J = 1, 2, 3$ and

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{26}$$

Here A_μ^{spin} is the gauge potential that couple to the S_z spin density and current. The $\tilde{f}$-fermions also form an "integer quantum Hall" state with an opposite Hall conductance, which is described by $U^3(1)$ Chern-Simons theory

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda} \tag{27}$$

The c-fermions form an "integer quantum Hall" state described by $U(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} b_\mu \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda} \tag{28}$$

Thus the total effective theory is given by

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} - \frac{1}{4\pi} K_{IJ} \tilde{a}_{I\mu} \partial_\nu \tilde{a}_{J\lambda} \epsilon^{\mu\nu\lambda}$$
$$+ \frac{1}{2\pi} q_I A_\mu^{\text{spin}} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{4\pi} b_\mu \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda} \tag{29}$$

The projection P is done by setting the total f-fermion and $\tilde{f}$-fermion density-current, $J_\mu^f = \sum_I \frac{1}{2\pi} \partial_\nu (a_{I\lambda} + \tilde{a}_{I\lambda}) \epsilon^{\mu\nu\lambda}$, equal to the c-fermion density-current, $J_\mu^c = \frac{1}{2\pi} \partial_\nu b_\lambda \epsilon^{\mu\nu\lambda}$. After setting $b_\lambda = \sum_I (a_{I\lambda} + \tilde{a}_{I\lambda})$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi} \bar{K}_{IJ} \bar{a}_{I\mu} \partial_\nu \bar{a}_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} \bar{q}_I A_\mu^{\text{spin}} \partial_\nu \bar{a}_{I\lambda} \epsilon^{\mu\nu\lambda} \tag{30}$$

with $I, J = 1, 2, 3, 4, 5, 6$ and

$$\bar{K} = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{31}$$

Also $\bar{a}_{I,\mu} = a_{I\mu}$ and $\bar{a}_{I+3,\mu} = \tilde{a}_{I\mu}$, $I = 1,2,3$. Since $|\det(K)| = |-1| = 1$, Eqn. (30) is the low energy effective theory for the spin-1 boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. Such a low energy effective theory describes a SO(3) SPT state with an integer Hall conductance for the S_z "charge"

$$\sigma_{xy} = \bar{q}^T \bar{K}^{-1} \bar{q} \frac{1}{2\pi} = 2\frac{1}{2\pi}. \tag{32}$$

Again, we note that the spin Hall conductance is an even integer in the unit of $\frac{1}{2\pi}$. This nontrivial quantum spin Hall effect indicates that gapless edge Hall current is protected by U(1) subgroup of SO(3) spin rotational symmetry. Beyond this K matrix formulation, Ref. (15) proved that edge can not be further gapped out by keeping nonabelian SO(3) symmetry, because the mass term which gaps out the excitations will mix the left mover and right mover and hence breaks the SO(3) / SU(2) symmetry. To realize this result in our projective construction, we need to further construct the same WZW theory description for edge states. We shall discuss SU(2) case in Sec. IV C and SO(3) is similar.

IV. SU(2) SPT STATE IN A SPIN-1/2 BOSON MODEL ON LATTICE

A. Meanfield ansatz

SU(2) SPT phases were also firstly studied in Ref.15 where field theoretic approach is applied. The bulk is described by SU(2) principal chiral nonlinear sigma model (the field variable g is 2×2 SU(2) matrix) with $2\pi K$ topological theta term where integer K labels different SPT phases. The boundary is described by a $SU(2)_K$ WZW term where SU(2) transformation is defined as left-multiplication, ie $g \to hg$ with $h \in SU(2)_L$, such that only left mover carries SU(2) charge and right mover is SU(2) charge-neutral. Due to this chiral SU(2) transformation, the boundary can not be simply replaced by a $SU(2)_K$ WZW critical spin chain that is nonchiral.

In the following, we are going to construct a SPT phase with a SU(2) symmetry on lattice. Our lattice model contains spin-1/2 bosons. Using the fermionic projective construction, we write the spin-1/2 bosonic operators (labeled by a spin index α and a flavor index s) as

$$b_{\alpha,i}^{(s)} = f_{\alpha,i} c_i^{(s)}, \quad \alpha = \uparrow, \downarrow \tag{33}$$

where, s is the flavor index as well as the orbital index of c-fermion at each site. Let's consider a general construction. N_O and N_L denote the orbital number per site ($s = 1, 2, \cdots, N_O$) and independent site number per unit cell, respectively. For honeycomb lattice, $N_L = 2$. Again, we can start with a many-fermion state of $f_{\alpha,i}$ and c_i, $|\Psi\rangle$, and obtain a physical spin state $|\Phi\rangle$ by projecting into the sub-space where the the f-fermion number

is equal to the c-fermion number on each site:

$$|\Phi\rangle = P|\Psi\rangle. \tag{34}$$

Using such a projective construction, we can construct a SU(2) SPT state on lattice.

Let us consider the following free fermion Hamiltonian

$$H = \sum_{i,j} \left[\mathbf{f}_{\alpha j}^\dagger u_{ji} \mathbf{f}_{\alpha i} + \mathbf{c}_j^\dagger V_{ji} \mathbf{c}_i, \right] \tag{35}$$

where u_{ji} and v_{ji} are complex numbers satisfying

$$u_{ji}^* = u_{ij}, \tag{36}$$

To construct a SU(2) SPT state using the above projective construction, we choose the ansatz u_{ji} such that each of $f_\uparrow$ and $f_\downarrow$-fermions forms a band insulator (with N_L bands) where filled bands (say, N_F, $N_F < N_L$) admit a total Chern number $+1$. The N_O by N_O (N_O is orbital number per site) matrix V is appropriately designed such that c-fermons form a band insulator (with $N_L \times N_O$ bands) where $2N_F$ bands from bottom are filled ($2N_F < N_L \times N_O$). These $2N_F$ bands admit a total Chern number -2 and band gap is open at chemical potential of c-fermions. Recently, some efforts have been made in lattice models with nontrivial Chern number larger than one, such as Refs. 40–43. As a result, there are in total N_F $f_\uparrow$ fermions, N_F $f_\downarrow$ fermions and $2N_F$ c-fermions in each unit cell, required by the particle number constraint.

B. Spinful SU(2) SPT state with charged superfluid

In the following, we like to show that $|\Phi\rangle = P|\Psi\rangle$ will be a bosonic SPT state with the SU(2) symmetry. First, from the free fermion Hamitonian, we see that the ground state state $|\Psi\rangle$ respects the SU(2) spin rotation symmetry generated by $f^\dagger \sigma^l f$. The f-fermions form an "integer quantum Hall" state described by $U^2(1)$ Chern-Simons theory

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu^{\text{spin}} \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \tag{37}$$

with $I, J = 1, 2$ and

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}. \tag{38}$$

Here $A_\mu^{S_z}$ is the gauge potential that couples to the S_z spin density and current. The c-fermions form an "integer quantum Hall" state described by $U^2(1)$ Chern-Simons theory

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} b_{I\mu} \partial_\nu b_{J\lambda} \epsilon^{\mu\nu\lambda} \tag{39}$$

Thus the total effective theory is given by

$$\mathcal{L} = \frac{1}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} - \frac{1}{4\pi}K_{IJ}b_{I\mu}\partial_\nu b_{J\lambda}\epsilon^{\mu\nu\lambda}$$
$$+ \frac{1}{2\pi}q_I A_\mu^{\rm spin}\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda} \qquad (40)$$

The projection P is done by setting the total f-fermion density-current, $J_\mu^f = \sum_I \frac{1}{2\pi}\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}$, equal to the c-fermion density-current, $J_\mu^c = \sum_I \frac{1}{2\pi}\partial_\nu b_{I\lambda}\epsilon^{\mu\nu\lambda}$. After setting $b_{2\lambda} = -b_{1\lambda} + \sum_I a_{I\lambda}$, we reduce the effective theory to

$$\mathcal{L} = \frac{1}{4\pi}\bar{K}_{IJ}\bar{a}_{I\mu}\partial_\nu \bar{a}_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}\bar{q}_I A_\mu^{\rm spin}\partial_\nu \bar{a}_{I\lambda}\epsilon^{\mu\nu\lambda} \quad (41)$$

with $I, J = 1, 2, 3$ and

$$\bar{K} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}. \qquad (42)$$

Also $\bar{a}_{I,\mu} = a_{I\mu}$ $I = 1, 2$, and $\bar{a}_{3,\mu} = b_{1\mu}$. Using an invertible integer matrix

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad (43)$$

we can rewrite $\bar{a}_{I\mu}$ as $\bar{a}_{I\mu} = U^T \tilde{a}_{I\mu}$, and rewrite Eqn. (41) as

$$\mathcal{L} = \frac{1}{4\pi}\tilde{K}_{IJ}\tilde{a}_{I\mu}\partial_\nu \tilde{a}_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}\tilde{q}_I A_\mu^{\rm spin}\partial_\nu \tilde{a}_{I\lambda}\epsilon^{\mu\nu\lambda} \quad (44)$$

where

$$\tilde{K} = U\bar{K}U^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{q} = U\bar{q} = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}. \quad (45)$$

Eqn. (44) is the low energy effective theory for the spin-1/2 boson state $|\Phi\rangle = P|\Psi\rangle$ after the projection. We see that the mode $\tilde{a}_{3\mu}$ is gapless. Such a gapless mode corresponds to the total density fluctuations of the spin-1/2 bosons, indicating that the bosons are in a superfluid phase. But this is an unusual superfluid phase where the spin degrees of freedom form an SU(2) SPT phase.

To see this point, let us assume that the unit cell is large enough so that there are, on average, two spin-1/2 bosons per unit cell. In this case, when the repulsion between the bosons is large enough, the bosons may form a Mott insulator state. Such a Mott insulator state is described by the confinement of $\tilde{a}_{3\mu}$ U(1) gauge field. So we can drop the $\tilde{a}_{3\mu}$ U(1) gauge field and obtain the following low energy effective theory in the Mott insulator phase:

$$\mathcal{L} = \frac{1}{4\pi}\tilde{K}_{2,IJ}\tilde{a}_{I\mu}\partial_\nu \tilde{a}_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}\tilde{q}_{2,I} A_\mu^{\rm spin}\partial_\nu \tilde{a}_{I\lambda}\epsilon^{\mu\nu\lambda} \qquad (46)$$

where

$$\tilde{K}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{q}_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}. \qquad (47)$$

Since $\det(\tilde{K}_2) = -1$, the state $|\Phi\rangle$ (in the Mott insulator phase) is a SPT state. So, such a low energy effective theory describes a SU(2) SPT state, which has a spin Hall conductance for the S_z "charge"

$$\sigma_{xy} = \tilde{q}_2^T \tilde{K}_2^{-1} \tilde{q}_2 \frac{1}{2\pi} = 2\frac{1}{4}\frac{1}{2\pi}. \qquad (48)$$

We note that the spin Hall conductance is an even integer in the unit of $\frac{1}{4}\frac{1}{2\pi}$. Although the above discussion is for spin-1/2 bosons, a similar construction can be done for more physical spin-1 bosons, which give us a SO(3) SPT state.

C.   Projective Kac-Moody algebra of the edge profile

Above quantum spin Hall effect gives the edge gapless Hall current protected by U(1) subgroup of SU(2). In the following, we shall construct the Kac-Moody algebra of edge profile which is actually described by SU(2)$_1$ WZW theory as same as Ref. (15) where bulk non-linear sigma model is applied.

Let's define the U(1)$_{L,R}$ scalar density operators $J_{L,R}$ and SU(2)$_{L,R}$ vector density operators $\mathbf{J}_{L,R}$ as

$$J_{L,R} =: \Psi_{L,R;a}^\dagger \Psi_{L,R;a} :, \qquad (49)$$

$$\mathbf{J}_{L,R} =: \Psi_{L,R;a}^\dagger \frac{\vec{\sigma}_{ab}}{2} \Psi_{L,R;b} :, \qquad (50)$$

respectively, where, $\vec{\sigma}$ denotes a matrix vector formed by three Pauli matrices and $a, b, \ldots = 1, 2$, and summation of common indices is implicit. Ψ_L is a two-component gapless chiral fermion from the edge of f-fermions; Ψ_R is also a two-component gapless chiral fermion but from the edge of c-fermions. "::" is usual normal ordering operator. The bosonized[18] edges admit U(1)$_{L,R}$×SU(2)$_{L,R}$ Kac-Moody algebras as following commutation relations ($x, x'\ldots$ are spatial coordinates of edge):

$$[J_{L,R}(x), J_{L,R}(x')] = \frac{1}{i\pi}\delta'(x - x'), \qquad (51)$$

$$[J_{L,R}^\alpha(x), J_{L,R}^\beta(x')] = i\epsilon^{\alpha\beta\gamma}J_{L,R}^\gamma(x)\delta(x - x')$$
$$- \frac{i}{4\pi}\delta^{\alpha\beta}\delta'(x - x') \qquad (52)$$

The bosonized edge Hamiltonian consists of the energy of density-density interactions:

$$\mathcal{H}_{\rm edge} = \sum_s \int dx v_s \left[\frac{\pi}{2} : J_s J_s : + \frac{2\pi}{3} : \mathbf{J}_s \cdot \mathbf{J}_s : \right],$$

where, $s = L, R$. v_L and v_R are the Fermi velocities of χFL (chiral Fermi liquid) of f and c systems, respectively, which are nonuniversal constants depending on

microscopic details of projective Chern band parameters. The U(1) scalar density fluctuations must be gapped out by considering the Gutzwiller projection.[44] The resultant edge theory at low-energies only consists the SU(2) vector density-density interaction terms with both of left and right movers:

$$\mathcal{H}_{\text{edge}}^{\text{Projected}} = \sum_s \int dx \frac{2\pi v_s}{3} : \mathbf{J}_s \cdot \mathbf{J}_s : ,$$

Therefore, the Kac-Moody algebra is reconstructed after the projection as same as the result from non-linear sigma model in Ref. (15).

V. CONCLUSION

In conclusion, a general approach is proposed to construct bosonic SPT phases on 2D lattice with various symmetries. Our approach is based on the projective construction where the bosons/spins are fractionalized into several fermions which occupy nontrivial Chern bands. The bosonic SPT phases then can be constructed from the fermion state by Gutzwiller projection. We can calculate the low energy effective Chern-Simons theory of the projected states, which allows us to determine what kinds of SPT states are obtained after the Gutzwiller projection. We have constructed a U(1) SPT state, a SO(3) SPT state, and a SU(2) SPT state for spin-1 and spin-1/2 bosons. In particular, in the SU(2) SPT phases, the U(1) charge is in gapless superfluid phase while the spin degree of freedom is in SU(2) SPT phase.

The present approach can be generalized to more complex continuous nonabelian symmetry group, such as SU(N) SPT magnets. Moreover, generally the same approach can be applied to construct SET phases which have intrinsic topological order (with $|\det K| \neq 1$) protected by symmetry. Finally, the present general approach provides a systematical way to construct the trial projective wavefunctions of 2D SPT states. In the future work, it is quite interesting to search concrete microscopic spin/boson Hamiltonians that realize the mean-field ansatzes and trial wavefunctions. And, many projective wavefunctions can be constructed in the approach such that it will be attractive to numerically study the ground state properties via numerical methods such as variational Monte Carlo.

1. M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006), cond-mat/0510613.
2. A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
3. Xie Chen, Z. C. Gu, Z. X. Liu, and X. G. Wen, arxiv:1106.4772.
4. Xie Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B **82**, 155138 (2010).
5. X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen, Science 338, 1604 (2012); X. L. Qi, Science 338, 1550 (2012).
6. D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
7. X.-G. Wen, Phys. Rev. B **40**, 7387 (1989).
8. X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
9. X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
10. M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010); J. E. Moore, Nature 464, 194 (2010); X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
11. Xie Chen, Z. X. Liu, and X. G. Wen, Phys. Rev. B **84**, 235141 (2011).
12. Z. X. Liu, X. Chen, and X. G. Wen, Phys. Rev. B **84**, 195145 (2011).
13. Y.-M. Lu and A. Vishwanath, Phys. Rev. **86**, 125119 (2012), arXiv:1205.3156.
14. M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009), arXiv:0906.2769.
15. Zheng-Xin Liu and Xiao-Gang Wen, arxiv:1205.7024.
16. J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).
17. E. Witten, Nucl. Phys. B **223**, 422 (1983).
18. E. Witten, Commun. Math. Phys. **92**, 455 (1984).
19. G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Comm. **63**, 973 (1987).
20. G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).
21. I. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).
22. G. Kotliar and J. Liu, Phys. Rev. B **38**, 5142 (1988).
23. Y. Suzumura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. **57**, 2768 (1988).
24. I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988).
25. E. Dagotto, E. Fradkin, and A. Moreo, Phys. Rev. B **38**, 2926 (1988).
26. X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
27. X.-G. Wen, Phys. Rev. B **44**, 2664 (1991).
28. P. A. Lee and N. Nagaosa, Phys. Rev. B **45**, 5621 (1992).
29. C. Mudry and E. Fradkin, Phys. Rev. B **49**, 5200 (1994).
30. X.-G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996), cond-mat/9506065.
31. Peng Ye and X.-G. Wen, arxiv:1303.3572.
32. X. Chen and X.-G. Wen (2012), arXiv:1206.3117.
33. T. Senthil and M. Levin (2012), arXiv:1206.1604.
34. T. Grover and A. Vishwanath (2012), arXiv:1210.0907.
35. Y.-M. Lu and D.-H. Lee (2012), arXiv:1210.0909.

[36] Y.-M. Lu and D.-H. Lee (2012), arXiv:1212.0863.

[37] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005), cond-mat/0411737.

[38] X.-G. Wen, Advances in Physics **44**, 405 (1995).

[39] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999), cond-mat/9811111.

[40] Shuo Yang, Zheng-Cheng Gu, Kai Sun, and S. Das Sarma, arxiv:1205.5792.

[41] F. Wang and Y. Ran, Phys. Rev. B **84**, 241103 (2011).

[42] Yi-Fei Wang, Hong Yao, Chang-De Gong, and D. N. Sheng, Phys. Rev. B 86, 201101 (2012).

[43] Maximilian Trescher and Emil J. Bergholtz, Phys. Rev. B 86, 241111 (2012).

[44] X.-G. Wen, Phys. Rev. B **43** 11025, (1991).