Frequency-dependent admittance of a short superconducting weak link

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Frequency-dependent admittance of a short superconducting weak link

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We consider the linear and non-linear electromagnetic responses of a nanowire connecting two bulk superconductors. Andreev states appearing at a finite phase bias substantially affect the finite-frequency admittance of such a wire junction. Electron transitions involving Andreev levels are easily saturated, leading to the nonlinear effects in photon absorption for the sub-gap photon energies. We evaluate the complex admittance analytically at arbitrary frequency and arbitrary, possibly non-equilibrium, occupation of Andreev levels. Special care is given to the limits of a single-channel contact and a disordered metallic weak link. We also evaluate the quasi-static fluctuations of admittance induced by fluctuations of the occupation factors of Andreev levels. In view of possible qubit applications, we compare properties of a weak link with those of a tunnel Josephson junction. Compared to the latter, a weak link has smaller low-frequency dissipation. However, because of the deeper Andreev levels, the low-temperature quasi-static fluctuations of the inductance of a weak link are exponentially larger than of a tunnel junction. These fluctuations limit the applicability of nanowire junctions in superconducting qubits.

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The search for longer coherence times of superconducting qubits brought the study of finite-frequency electromagnetic properties of mesoscopic superconductors to the forefront of experimental research [1–6]. The majority of experiments until recently was performed on structures using Josephson junctions as "weak" superconductors, and substantial progress in recognizing the coherence-limiting mechanisms was achieved. One may view a number of mechanisms causing energy or phase relaxation as extrinsic ones. These involve, e.g., imperfections in the tunnel barriers comprising junctions[7], charge trapping[8], and interaction with stray photons[9,10]. Along with them, there are intrinsic mechanisms associated with the kinetics of quasiparticles in the superconductors[11–13]. These mechanisms provide fundamental limitations to the coherence. The majority of effects of quasiparticles on the finite-frequency properties of Josephson junctions can be derived[12,14] from the electromagnetic admittance of the junction $Y(\omega)$. This property was extensively studied theoretically, starting from the seminal phenomenological paper of Josephson [15] and microscopic evaluation[16,17] based on the BCS theory.

The use of weak superconducting links instead of Josephson junctions in qubits was proposed recently as a way to avoid extrinsic decoherence mechanisms (such as imperfections of the tunnel barriers)[18]. An apparent observation of a coherent phase slip in a conducting weak link[19] may be viewed as an incipient experimental step in that direction. That makes the question about the intrinsic mechanisms of decoherence in weak links important. Like with Josephson junction devices[12], this question is directly related to the finite-frequency admittance of a weak link. Surprisingly, this property was given relatively little attention to. The admittance of a short SNS contact was investigated, mostly numerically, in the recent papers[20,21]. Some qualitative aspects of the AC response of a single-channel point contact can be extracted from two other papers devoted to the theory of enhancement of supercurrent by microwave radiation[22,23].

Here we perform a fully-analytical evaluation of the admittance of a weak link connecting two bulk superconductors, valid at arbitrary frequency ω, quasiparticle distribution function, and normal-state conductance of the link. Compared to the Josephson junction case, the dissipative part of the weak link admittance exhibits a number of new thresholds in its frequency dependence, associated with the presence of Andreev levels. The complex admittance close to these new threshold frequencies is sensitive to the occupation of the discrete Andreev states. Fluctuations of the equilibrium or non-equilibrium occupation factors result in fluctuations of the admittance. We analyze the average values and fluctuations of the linear electromagnetic response, giving special attention to the practically important limits of a single-channel contact[24] and a disordered metallic wire[3,25].

The discrete nature of Andreev states is responsible for a low threshold for the nonlinear absorption. In the nonlinear regime, we find a suppression of the absorption coefficient in a disordered metallic link at radiation frequency $\omega \leq 2\Delta/3$, while at higher ω dissipation power depends non-linearly on the radiation intensity (here Δ is the BCS gap in the leads).

The paper is organized as follows: the model used in the derivation of the admittance of a point contact with an arbitrary transmission coefficient is formulated in Section I. The linear response theory for the AC perturbation of the point contact is developed in Section II. In Section III we discuss the results for the admittance of the point contact at zero temperature and no quasiparticles present. In Section IV we study the changes in the admittance caused by the arbitrary distribution of quasiparticles in the junction. These results are used in Section V to find the admittance of a disordered weak link. The fluctuations of the admittance are analyzed in Section VI, both for the case of point contact and of a weak link. In Section VII we consider the absorption rate in a non-linear regime for the radiation frequencies close to the Andreev level resonance. We conclude with the final remarks in Section VIII.

I. POINT CONTACT HAMILTONIAN

We start by considering a point contact between two leads. It can be described by the tunnel Hamiltonian

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_T\,, \tag{1.1}$$

where $\hat{H}_{L(R)}$ are the BCS Hamiltonians of the left (right) leads:

$$\hat{H}_L = \sum_k \xi_k c_k^\dagger c_k + \Delta_L \sum_k c_k^\dagger c_{\bar{k}}^\dagger + \Delta_L^* \sum_k c_{\bar{k}} c_k\,, \tag{1.2}$$

$$\hat{H}_R = \sum_p \xi_p c_p^\dagger c_p + \Delta_R \sum_p c_p^\dagger c_{\bar{p}}^\dagger + \Delta_R^* \sum_p c_{\bar{p}} c_p\,, \tag{1.3}$$

and

$$\hat{H}_T = w\sum_{kp}\left(c_k^\dagger c_p + c_p^\dagger c_k\right) + w\sum_{k_1 k_2} c_{k_1}^\dagger c_{k_2} + w\sum_{p_1 p_2} c_{p_1}^\dagger c_{p_2} \tag{1.4}$$

is the tunneling term. Here $c_{k(p)}$ and $c_{\bar{k},(\bar{p})}$ are electron operators in the left (right) lead corresponding to states $k(p)$ and its time-reversed pairs $\bar{k}(\bar{p})$, and $\Delta_{L(R)} = \Delta e^{i\phi_{L(R)}}$ are the BCS gap functions. The last two terms in Eq. (1.4) describe the back-scattering processes in left and right lead, respectively. The validity of the Hamiltonian description of the point contact systems was discussed in in Refs. 26 and 27, where the real-space counterpart of the Hamiltonian (1.1) was studied.

The tunneling amplitude w is assumed to be momentum-independent near the Fermi level. It is related to the transmission coefficient, $\tau = (2\pi\nu_0 w)^2/[1 + (2\pi\nu_0 w)^2]$, where ν_0 is normal-state density of states. Keeping the back-scattering terms in Eq. (1.4) allows for a consistent description of the unitary limit, $\tau \to 1$, which corresponds to the strong tunneling, $w \to \infty$. The conductance of the junction G in the normal state is proportional to τ (hereinafter we set $\hbar = 1$). A point contact between superconducting leads hosts a single Andreev level[28,29] with energy $E_A(\tau, \phi)$ depending on G:

$$G = e^2\tau/\pi\,, \quad E_A(\tau,\phi) = \Delta(1 - \tau\sin^2\phi/2)^{1/2}\,; \tag{1.5}$$

here the phase difference between the leads order parameters, $\phi = \phi_R - \phi_L$, is assumed to be time-independent.

II. LINEAR RESPONSE TO AC PERTURBATION

We may account for an applied small, time-dependent voltage $V(t)$ by modifying $\phi_L \to \phi_L + 2\phi_1(t)$ in Eq. (1.2), with $\dot{\phi}_1 = eV(t)$, and adding the term $-eV(t)\hat{N}_L$ to Eq. (1.1):

$$\mathcal{H} = \hat{H}(t) - eV\hat{N}_L\,, \quad \hat{N}_L = \sum_k c_k^\dagger c_k\,. \tag{2.1}$$

We want to find the current $\langle \hat{I} \rangle$,

$$\hat{I} = e\dot{\hat{N}}_L = -iew\sum_{kp}\left[c_k^\dagger c_p - c_p^\dagger c_k\right]\,, \tag{2.2}$$

induced by an applied voltage to linear order in V and at arbitrary transmission τ. The validity of linear response in V requires at least the smallness of the perturbation to the dynamics of the system, $|\phi_1| = |eV/\omega| \ll 1$, where ω is the frequency of perturbation. Further limitations on the parameters, which may come from the effect of V on occupation factors, will be discussed later.

It is convenient to do the gauge transformation $c_k \to c_k e^{i\phi_1}$ before performing the perturbation theory. This moves the ϕ_1–dependence to the tunneling terms. Using the Kubo formula for linear response, we get

$$\langle \hat{I}(t) \rangle = I_J + \int_{-\infty}^{\infty} dt' \chi(t - t')\phi_1(t')\,. \tag{2.3}$$

Here I_J is the Josephson current which is present even without applied voltage:

$$I_J = ew\,\mathrm{Im}\left\langle \sum_{kp} c_k^\dagger c_p \right\rangle. \tag{2.4}$$

The response function $\chi(t)$ is given by

$$\chi(t) = iew^2\theta(t)\sum_{k_1 p_1}\sum_{k_2 p_2}\Big\langle\Big[c_{k_1}^\dagger(t)c_{p_1}(t) - c_{p_1}^\dagger(t)c_{k_1}(t), c_{k_2}^\dagger(0)c_{p_2}(0) - c_{p_2}^\dagger(0)c_{k_2}(0)\Big]\Big\rangle - ew\delta(t)\,\mathrm{Re}\left\langle\sum_{kp} c_k^\dagger c_p\right\rangle. \tag{2.5}$$

Averages $\langle \ldots \rangle$ are taken over the Gibbs ensemble of the original Hamiltonian H. We can use Wick's theorem to evaluate averages. They can be expressed in terms of Green's functions of the unperturbed system. Green's functions satisfy a system of linear integral equations, but the corresponding kernels are separable due to the form of the tunneling term Eq. (1.4). Therefore, that system reduces to a system of algebraic equations which can be solved exactly. This response function is related to the admittance in frequency domain:

$$Y(\omega) = i\frac{e}{\omega}\chi(\omega)\,. \tag{2.6}$$

It is convenient to split the admittance into a sum

$$Y = \sum_{i=1}^{5} Y_i + \frac{i}{\omega L_J}\,, \tag{2.7}$$

each term of which has a clear physical origin. The purely inductive term L_J comes from the $\omega \to 0$ response of the condensate,

$$\frac{1}{L_J} = 2e\frac{\partial I_J}{\partial \phi}\,. \tag{2.8}$$

The other five parts of Eq. (2.7) originate from the quasiparticles transitions. To better understand the structure of these parts, recall that $\mathrm{Re}\, Y$ (dissipative part of admittance) is related to the linear absorption rate W of the radiation by:

$$W = \frac{\phi_1^2}{2e^2}|\omega|\,\mathrm{Re}\, Y(\omega)\,. \tag{2.9}$$

The elementary processes leading to the absorption are depicted in Fig. 1. The $Y_1(\omega)$ term corresponds to a process in which two quasiparticles are created in the band, leading to the energy threshold 2Δ. The contribution $Y_2(\omega)$ in (2.7) comes from creating one quasiparticle in the bound state and one in the band, the corresponding threshold energy is $\Delta + E_A$. Creation of a pair of quasiparticles in the bound state[30], which costs energy $2E_A$, leads to the term $Y_3(\omega)$. In addition to these three contributions which exist even in the absence of quasiparticles, there are two more associated with the promotion of an existing quasiparticle to a higher energy in the absorption process: Y_4 is the intra-band contribution, and Y_5 corresponds to an ionization of an occupied Andreev level.

III. ADMITTANCE OF A SINGLE-CHANNEL JUNCTION AT $T = 0$ (EQUILIBRIUM STATE WITH NO QUASIPARTICLES)

Evaluating averages in Eq. (2.4) for system at zero temperature one gets[31] for the Josephson current:

$$I_J^{(0)} = \pi G \frac{\Delta^2 \sin\phi}{2eE_A(\phi)}\,, \tag{3.1}$$

in agreement with the result one obtains[29] from Eq. (1.5) by differentiating energy over ϕ, i.e. $I_J^{(0)} = -2e\partial E_A/\partial\phi$. Using the above expression for $I_J^{(0)}$ and Eq. (2.8) we find the inductive term at zero temperature:

$$\frac{1}{L_J^{(0)}} = \pi G\Delta \frac{\cos\phi + \frac{\pi G}{e^2}\sin^4\frac{\phi}{2}}{\left(1 - \frac{\pi G}{e^2}\sin^2\frac{\phi}{2}\right)^{3/2}}\,. \tag{3.2}$$

Evaluating averages in Eq. (2.5) we find the contributions $Y_i^{(0)}$ to the admittance. With no quasiparticles present, there can be no processes of type 3 or 4 in Figure 1. Therefore, $Y_4^{(0)}(\omega) = Y_5^{(0)}(\omega) = 0$. The contribution $Y_1^{(0)}(\omega)$ comes from the creation of pairs of quasiparticles in the band (two excitations of type 1 in Figure 1), and its real part is given by:

$$\frac{\mathrm{Re}\, Y_1^{(0)}(\omega)}{G} = \frac{\theta(\omega - 2\Delta)}{\omega}\int_\Delta^{\omega-\Delta} d\epsilon\,\rho(\epsilon)\rho(\omega-\epsilon)|z_{\epsilon,\omega}|^2\,, \tag{3.3}$$

where $\rho(\omega)$ is the density of states in the continuum normalized to normal-state density of states ν_0:

$$\rho(\epsilon) = \frac{\epsilon\sqrt{\epsilon^2 - \Delta^2}}{\epsilon^2 - E_A^2}\,, \tag{3.4}$$

and the matrix element z is given by:

$$|z_{\epsilon,\omega}|^2 = 1 - \frac{\Delta^2\cos\phi + \Delta^2 - E_A^2}{\epsilon(\omega-\epsilon)}\,. \tag{3.5}$$

We assume $\omega > 0$ throughout this section. The result for negative frequencies can be found using the fact that $\mathrm{Re}\, Y^{(0)}(\omega)$ is an even function. The theta function in Eq. (3.3) shows that there can be no creation of pairs in the continuum for frequencies less than 2Δ.

The $Y_2^{(0)}(\omega)$ term comes from processes in which one quasiparticle is created in the band and another one in the Andreev level. These processes are represented by one arrow of type 1 and one of type 2 in the Figure 1. The real part of $Y_2^{(0)}(\omega)$ is given by:

$$\frac{\mathrm{Re}\, Y_2^{(0)}(\omega)}{G} = \pi\theta(\omega - E_A - \Delta)\frac{\sqrt{\Delta^2 - E_A^2}}{\omega}\rho(\omega - E_A)|z_{E_A,\omega}|^2\,, \tag{3.6}$$

and it vanishes for $\omega < \Delta + E_A$, as for these frequencies the processes "1+2" are energetically not allowed.

Finally, there are processes in which two quasiparticles on Andreev level are created. Those are represented by two excitations of type 2 in Figure 1. In this case, the frequency must be equal to $2E_A$. The $Y_3^{(0)}(\omega)$ term comes from such processes and its real part is given by:

$$\frac{\mathrm{Re}\, Y_3^{(0)}(\omega)}{G} = \pi^2\frac{(\Delta^2 - E_A^2)(E_A^2 - \Delta^2\cos^2\frac{\phi}{2})}{2E_A^3}\delta(\omega - 2E_A)\,. \tag{3.7}$$

Note that the RHS of Eqs. (3.3)-(3.7) depend on G and ϕ through E_A, see Eq. (1.5).

The admittance exhibits non-analytical behavior at threshold frequencies $\omega = 2E_A$, $\Delta + E_A$ and 2Δ. For $\omega \approx 2\Delta$ we have $\mathrm{Re}\, Y_1^{(0)} \propto (\omega - 2\Delta)^2\theta(\omega - 2\Delta)$ according to (3.3). Similarly, for frequencies $\omega \approx \Delta + E_A$ we get $\mathrm{Re}\, Y_2^{(0)} \propto \sqrt{\omega - (\Delta + E_A)}\theta(\omega - (\Delta + E_A))$ from Eq. (3.6).

The imaginary parts of $Y_i(\omega)$'s can be obtained from their real parts using Kramers-Kronig relations since $Y(\omega)$ is analytic in the upper half of the complex ω-plane. The complete expression for $\mathrm{Im}\, Y(\omega)$ is given in Appendix A. At threshold frequencies $\mathrm{Im}\, Y(\omega)$ exhibits non-analytical behavior which parallels threshold behavior of $\mathrm{Re}\, Y(\omega)$. At $\omega \approx 2\Delta$ the non-analytical contribution behaves as $\mathrm{Im}\, Y_1^{(0)} \propto (\omega - 2\Delta)^2\ln|\omega - 2\Delta|$ and at $\omega \approx \Delta + E_A$ it behaves as $\mathrm{Im}\, Y_2^{(0)} \propto \sqrt{(\Delta + E_A) - \omega}\theta((\Delta + E_A) - \omega))$. The coefficients omitted from the asymptotes of $\mathrm{Re}\, Y^{(0)}(\omega)$ and $\mathrm{Im}\, Y^{(0)}(\omega)$ equal each other, confirming that the complex function $Y^{(0)}(\omega)$ is analytical.

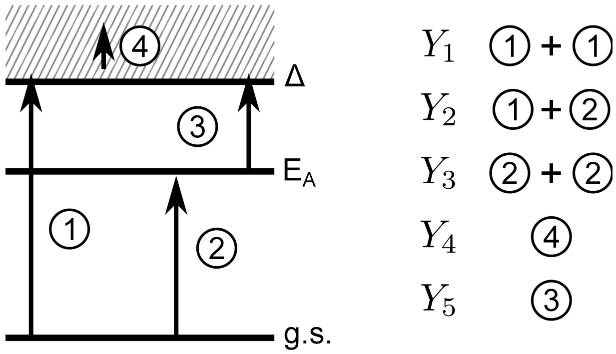

FIG. 1: Schematic representation of contributions to the admittance, Eqs. (3.3)-(3.7),(4.5) and (4.7). Horizontal bars are energy levels and arrows depict possible excitations. The picture shows that $Y_4 = Y_5 = 0$ in the absence of quasiparticles. Since $\operatorname{Re} Y(\omega)$ is proportional to the absorption rate, we see that $\operatorname{Re} Y_{1,2,3,5}$ will have threshold frequencies of 2Δ,$\Delta + E_A$, $2E_A$, $\Delta - E_A$, respectively.

IV. ADMITTANCE OF A SINGLE-CHANNEL JUNCTION IN THE PRESENCE OF QUASIPARTICLES

The admittance changes once there are quasiparticles present. Each term in Eq. (2.7) acquires an additional factor depending on the quasiparticle occupation numbers. We introduce occupation factors p_0, $p_\uparrow$, $p_\downarrow$ and p_2 denoting probabilities of having zero, one or two quasiparticles in the bound state; $p_0 + p_\uparrow + p_\downarrow + p_2 = 1$. The inductance in Eq. (2.7) then becomes:

$$\frac{1}{L_J} = \frac{1}{L_J^{(0)}}[p_0 - p_2]\,. \tag{4.1}$$

The Y_1 term acquires a factor depending on the occupation factors $f(\epsilon)$ of the continuum states,

$$\begin{aligned}\frac{\operatorname{Re} Y_1^{(0)}(\omega)}{G} =& \frac{\theta(\omega - 2\Delta)}{\omega}\int_\Delta^{\omega-\Delta} d\epsilon \rho(\epsilon)\rho(\omega-\epsilon)|z_{\epsilon,\omega}|^2 \\ &\times [1 - f(\epsilon) - f(\omega-\epsilon)]\,.\end{aligned} \tag{4.2}$$

This expression is different from Eq. (3.3) by a factor equal to the difference of probabilities for having the initial and the final state occupied in the transition from the ground state to the band, see Fig. 1. Similarly, the $\operatorname{Re} Y_2$ term is given by

$$\operatorname{Re} Y_2(\omega) = \operatorname{Re} Y_2^0(\omega)(p_0 + \frac{p_\uparrow + p_\downarrow}{2} - f(\omega - E_A))\,, \tag{4.3}$$

and Y_3 term by

$$Y_3(\omega) = Y_3^0(\omega)(p_0 - p_2)\,. \tag{4.4}$$

The part of the admittance coming from transitions to Andreev level, equations (3.7) and (4.4), can be inferred from the results of Ref. 30 for the frequency noise spectrum, since noise spectrum and the real part of the admittance are related by the fluctuation-dissipation theorem.

At non-zero occupancies, there are two additional contributions to absorption, $Y_4(\omega)$ and $Y_5(\omega)$. The former one comes from the band-to-band transitions, represented by the arrow 4 in Figure 1. Its real part is given by:

$$\frac{\operatorname{Re} Y_4(\omega)}{G} = \frac{2}{\omega}\int_\Delta^\infty d\epsilon\rho(\epsilon)\rho(\omega+\epsilon)|z_{-\epsilon,\omega}|^2[f(\epsilon) - f(\omega+\epsilon)]\,. \tag{4.5}$$

The other term, $Y_5(\omega)$, is generated by the Andreev level-to-band transitions[22]. These transitions are represented by the arrow 3 in Figure 1. We can write it in the form resembling that of $Y_2(\omega)$:

$$\operatorname{Re} Y_5(\omega) = \operatorname{Re} Y_5^{(0)}(\omega)[p_\uparrow + p_\downarrow + 2p_2 - 2f(\omega + E_A)]\,, \tag{4.6}$$

where the real part of $Y_5^{(0)}(\omega)$ is given by:

$$\frac{\operatorname{Re} Y_5^{(0)}(\omega)}{G} = \pi\theta(\omega + E_A - \Delta)\frac{\sqrt{\Delta^2 - E_A^2}}{2\omega}\rho(\omega + E_A)|z_{-E_A,\omega}|^2\,, \tag{4.7}$$

The occupation factors p and f in all of the above expressions may, but need not to be the equilibrium ones. The density of states $\rho(\epsilon)$ and matrix element $z_{\epsilon,\omega}$ in Eqs. (4.2) and (4.5)-(4.7) are defined in Eqs. (3.4) and (3.5). The final expression for $\operatorname{Re} Y(\omega)$ at arbitrary frequency can be related to the results of Refs. 30,32 for the noise spectrum of a weak link using the fluctuation-dissipation theorem.

Note that the transitions involving the Andreev level vanish at $E_A \to \Delta$. This is achieved if either $G = 0$ or $\phi = 0$. Expanding to the lowest order in G for $G \ll e^2/\pi$, the contributions $Y_{1,4}$ are linear in G and they reduce to the familiar perturbative result for the admittance of a Josephson junction[33]. The other contributions are higher order in G, with $Y_{2,5} \propto G^{3/2}$, and $\operatorname{Re} Y_3 \propto G^2$.

In the limit $\phi \to 0$, the terms involving transitions to Andreev level vanish as $\operatorname{Re} Y_{2,5} \propto |\phi|$, and $\operatorname{Re} Y_3 \propto \phi^2$. Therefore, at $\phi = 0$ only terms contributing to Y are again Y_1 and Y_4. In that case, the expression for Y coincides with the one found perturbatively[33] in the limit of weak tunneling $G \ll e^2/\pi$. Thus, at small phase bias ϕ we don't expect much difference from the simple Josephson junction.

Now we analyze the behavior of admittance in the limits of low frequencies and low temperatures, as these are the conditions often encountered in the application of the superconducting junctions. At frequencies below the threshold for Andreev level ionization, $\omega < \Delta - E_A$, and away from the bound pair creation resonance, $\omega \neq 2E_A$, the only contribution to the dissipative part of admittance comes from Eq. (4.5). We assume the quasiparticle occupation factors are distributed according to Boltzmann distribution $f(\epsilon) = Ae^{-\epsilon/T}$. At low temperatures

$T \ll \Delta - E_A$, where $\Delta - E_A$ is characteristic scale for the energy dependence of the density of states above the gap, the dominant contribution to $\mathrm{Re}\, Y_4(\omega)$ comes from the transitions between the states near the bottom of the band. In that limit, we get for the asymptotic form of $\mathrm{Re}\, Y_4(\omega)$:

$$\frac{\mathrm{Re}\, Y_4(\omega)}{G} \approx \sqrt{\frac{2}{\pi}} x_{qp} \frac{1+\cos\phi + \frac{\pi G}{e^2}\sin^2\frac{\phi}{2}}{(\frac{\pi G}{e^2})^2 \sin^4\frac{\phi}{2}} (1-e^{-\omega/T})\sqrt{\frac{T}{\Delta}} e^{\omega/4T} K_1(\frac{\omega}{4T}), \qquad (4.8)$$

where x_{qp} is the density of quasiparticles, n_{qp}, in the bulk normalized to the "Cooper pair density", $x_{qp} = n_{qp}/\nu_0\Delta$, and $K_1(x)$ is the modified Bessel function of the second kind. Note that at small frequencies, $\omega \ll T$, it follows from Eq. (4.8) that $\mathrm{Re}\, Y_4(\omega)$ is frequency-independent and proportional to $\sqrt{T/\Delta}$.

In the limit $G \ll e^2/\pi$, the Andreev level is shallow, $\Delta - E_A \ll \Delta$. If now $T \gg \Delta - E_A$, the main contribution to Y_4 comes from transitions involving states far above the gap where the density of states is described by the usual BCS result. In this limit, Eq. (4.5) is reduced to the known result[12] for a Josephson junction,

$$\frac{\mathrm{Re}\, Y(\omega)}{G} \approx \frac{1}{2\sqrt{2}} \frac{n_{qp}}{\nu_0 \Delta}(1+\cos\phi)\left(\frac{\Delta}{T}\right)^{3/2} \ln\frac{4T}{\omega}. \qquad (4.9)$$

This limit is the opposite to the one of Eq. (4.8). The two asymptotes match each other at $T \sim \Delta - E_A$ up to the logarithmic factor.

It is interesting to compare the dissipation in a large-area Josephson junction of $G \lesssim e^2/\pi$ with the dissipation in a single-channel weak link of the same G. The weak-link quasiparticle density of states in the continuum, see Eq. (3.4), is suppressed compared to the singular tunneling density of states in a Josephson junction. As a result, at frequencies and temperatures $\omega, T \lesssim \Delta - E_A$ a weak link is less dissipative than a Josephson junction with small-transparency but large-area tunnel barrier of the same G. Using Eqs. (4.8) and (4.9) we find, e.g., that the dissipation is smaller by a factor T^2/Δ^2 in the case of a weak link.

V. DISORDERED WEAK LINK

For a multi-channel junction one needs to sum the contributions to the admittance from each channel. We consider the case of a disordered weak link for which we can assume the transmission coefficients are continuously distributed according to Dorokhov distribution[34] $\rho(\tau) = \pi G/2e^2\tau\sqrt{1-\tau}$. The admittance is then given by:

$$\bar{Y}(\omega) = \int_0^1 d\tau \rho(\tau) Y(\omega,\tau). \qquad (5.1)$$

We can write $\bar{Y}(\omega)$ as a sum of five terms, in the same way we did it for the single-channel junction in Eq (2.7). Transitions between Andreev levels are ignored, which is justified in the limit of short junction with $\Delta/E_T \to 0$, where E_T is Thouless energy[35].

The Josephson current of the disordered weak link can be found using the same averaging procedure as in Eq. (5.1). In the absence of quasiparticles it is given by:[36]

$$\bar{I}_J^{(0)} = \frac{\pi G \Delta}{e} \cos\frac{\phi}{2} \,\mathrm{artanh} \sin\frac{\phi}{2}. \qquad (5.2)$$

Similarly, averaging $1/L_J^{(0)}$ from Eq. (3.2) we get:

$$\frac{1}{\bar{L}_J^{(0)}} = \pi G \Delta \left[1 - \sin\frac{\phi}{2}\,\mathrm{artanh}\sin\frac{\phi}{2}\right]. \qquad (5.3)$$

Evaluating the integral in Eq. (5.1) results in expressions for $\bar{Y}_i$. If there are no quasiparticles present, the only non-vanishing terms are $\bar{Y}_{1,2,3}$. Their real parts exhibit threshold behavior at frequencies 2Δ, $\Delta + \Delta|\cos\frac{\phi}{2}|$ and $2\Delta|\cos\frac{\phi}{2}|$. The latter two thresholds correspond to the fully transmitting channel which has the lowest possible Andreev level energy at given phase ϕ. The term corresponding to the creation of a pair of quasiparticles in the continuum is given by:

$$\frac{\mathrm{Re}\,\bar{Y}_1^{(0)}(\omega)}{G} = \theta(\omega-2\Delta)\mathrm{P.V.}\int_\Delta^{\omega-\Delta} d\omega_1 \frac{\omega_1\omega - \Delta^2(1+\cos\phi)}{\Delta^2\omega^2|\sin\frac{\phi}{2}|(\omega-2\omega_1)} \frac{\sqrt{\omega_1^2-\Delta^2}\sqrt{(\omega-\omega_1)^2-\Delta^2}}{\sqrt{\omega_1^2/\Delta^2-\cos^2\frac{\phi}{2}}} \ln\frac{\sqrt{\omega_1^2/\Delta^2-\cos^2\frac{\phi}{2}}+|\sin\frac{\phi}{2}|}{\sqrt{\omega_1^2/\Delta^2-\cos^2\frac{\phi}{2}}-|\sin\frac{\phi}{2}|}. \qquad (5.4)$$

It has the threshold frequency 2Δ, just like the single channel admittance, Eq. (3.3). At frequencies higher than this threshold, $\mathrm{Re}\,\bar{Y}_1(\omega)$ starts to grow linearly, $\mathrm{Re}\,\bar{Y}_1(\omega) \propto \omega - 2\Delta$. The $\mathrm{Re}\,\bar{Y}_2(\omega)$ term, corresponding to the creation of one quasiparticle in the Andreev level and one in the continuum, is given by:

$$\frac{\mathrm{Re}\,\bar{Y}_2^{(0)}(\omega)}{G} = \pi \int_{\Delta|\cos\frac{\phi}{2}|}^{\Delta} d\omega_1 \theta(\omega-\omega_1-\Delta) \frac{\omega\omega_1 - \Delta^2(1+\cos\phi)}{\omega^2\Delta|\sin\frac{\phi}{2}|(\omega-2\omega_1)} \frac{\sqrt{(\omega-\omega_1)^2-\Delta^2}\sqrt{\Delta^2-\omega_1^2}}{\sqrt{\omega_1^2-\Delta^2\cos^2\frac{\phi}{2}}}\,. \tag{5.5}$$

The threshold frequency of this term is $\Delta + \Delta|\cos\frac{\phi}{2}|$, same as the threshold frequency of the fully transmitting channel for this process. Behavior near threshold is given by $\mathrm{Re}\,\bar{Y}_2(\omega) \propto \omega - \Delta - \Delta|\cos\frac{\phi}{2}|$. Finally, there is a term coming from the processes in which a pair of quasiparticles is created in the Andreev level. It has the threshold frequency of $\omega_{th}(\phi)$ and is given by:

$$\frac{\mathrm{Re}\,\bar{Y}_3^{(0)}(\omega)}{G} = \theta(\omega-\omega_{th}(\phi))\theta(2\Delta-\omega)\frac{\pi^2}{|\sin\frac{\phi}{2}|}\frac{\Delta}{\omega}\left(1-\frac{\omega^2}{4\Delta^2}\right)\sqrt{1-\frac{\omega_{th}^2(\phi)}{\omega^2}}, \quad \omega_{th}(\phi) = 2\Delta|\cos\frac{\phi}{2}|\,. \tag{5.6}$$

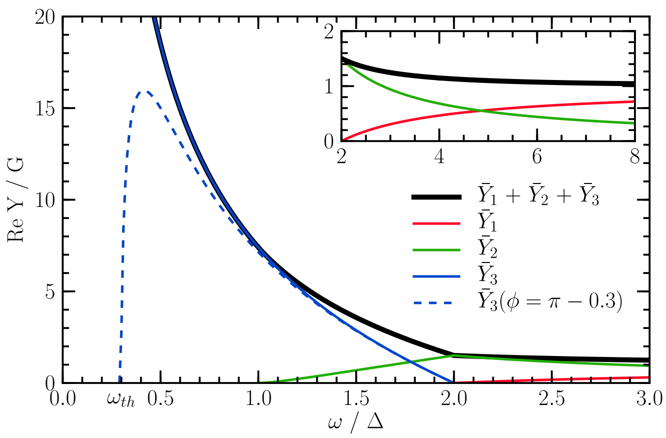

FIG. 2: Real part of admittance of a weak link. Solid lines correspond to $\phi=\pi$ case. The explicit formulas are given by Eqs. (5.4), (5.5) and (5.6). $\mathrm{Re}\,\bar{Y}_2$ and $\mathrm{Re}\,\bar{Y}_1$ exhibit threshold behavior at $\omega=\Delta$ and $\omega=2\Delta$ where they start to grow as $\propto(\omega-\Delta)^{3/2}$ and $\propto \omega-2\Delta$, respectively. $\mathrm{Re}\,\bar{Y}_3(\omega)$ is diverging for $\omega\to 0$. Dashed line shows $\mathrm{Re}\,\bar{Y}_3(\omega)$ at phase different than π, with threshold frequency ω_{th}, see Eq. (5.6). As $\phi\to\pi$ its maximum grows and shifts towards the $\omega=0$.

In the presence of quasiparticles, the above expressions for the admittance acquire additional factors reflecting the quasiparticles distribution function, similar to the single-channel junction. In addition, there are two other terms, $\bar{Y}_4$ and $\bar{Y}_5$, coming from band-to-band transitions and ionization of Andreev level, respectively. These are obtained by averaging Eqs. (4.5) and (4.6) over transmission coefficients. The complete expression for the dissipative part of the admittance in the presence of quasiparticles is given in Appendix B.

We expect the greatest change in admittance from the simple Josephson junction at $\phi\approx\pi$, when Andreev level energies of the channels contributing to the admittance fill the whole range of energies between 0 and Δ. In that case, for $\omega_{th}<\omega<\Delta$ and no quasiparticles present, the only contribution to dissipative part of the admittance comes from $\bar{Y}_3(\omega)$ and is given by Eq. (5.6) with $\omega_{th} = \Delta|\phi-\pi|$. At this threshold $\mathrm{Re}\,\bar{Y}_3(\omega)$ grows as $(\omega-\omega_{th})^{1/2}$ and reaches maximum for $\omega=\sqrt{2}\omega_{th}$. The height of the maximum scales as Δ/ω_{th}. Frequency dependence of $\mathrm{Re}\,\bar{Y}_3$ for ϕ close to π is shown on Fig. 2.

When $\phi=\pi$ exactly, there is no low-frequency cutoff. For low frequencies $\mathrm{Re}\,\bar{Y}_3(\omega)$ diverges as $1/\omega$. At frequencies $\Delta<\omega<2\Delta$, in addition to $\mathrm{Re}\,\bar{Y}_3$, there is the contribution $\mathrm{Re}\,\bar{Y}_2$ given by Eq. (5.5). Its behavior near threshold for $\phi=\pi$ is different than for any other ϕ and is given as $\mathrm{Re}\,\bar{Y}_2\propto(\omega-\Delta)^{3/2}$. At frequencies higher than 2Δ, the $\mathrm{Re}\,\bar{Y}_3$ contribution vanishes and $\mathrm{Re}\,\bar{Y} = \mathrm{Re}\,\bar{Y}_1+\mathrm{Re}\,\bar{Y}_2$. The frequency dependence of $\mathrm{Re}\,\bar{Y}(\omega)$ for $\phi=\pi$ is also shown on Fig. 2.

If $\phi\neq\pi$, the dissipative part of the admittance is zero for $\omega<\omega_{th}$ and vanishing occupation factors. Assuming Boltzmann distribution $f(E)=Ae^{-E/T}$ for quasiparticles and considering frequencies $\omega<\omega_{th}$, the only contribution to $\mathrm{Re}\,\bar{Y}(\omega)$ comes from $\bar{Y}_4$ and $\bar{Y}_5$ terms – Eqs. (B6) and (B7). The former comes from transitions within the continuum band. In the limit $\omega, T \ll \Delta(1-|\cos\frac{\phi}{2}|)$ the most important are transitions from the bottom of the band and we get:

$$\frac{\mathrm{Re}\,\bar{Y}_4(\omega)}{G} \approx \frac{x_{qp}}{\sqrt{2\pi}}\cot^2\frac{\phi}{2}(1-e^{-\omega/T})\sqrt{\frac{\Delta}{T}}U(\frac{\omega}{T})\,, \tag{5.7}$$

where $U(x) = \int_0^\infty dt e^{-xt}\sqrt{t(1+t)}\ln(1+1/t)$. The $\mathrm{Re}\,\bar{Y}_5(\omega)$ term is due to transitions from Andreev levels to the continuum. In the same limit of small frequency and temperature, it is given by:

$$\frac{\mathrm{Re}\,\bar{Y}_5(\omega)}{G} \approx \frac{\pi^2 x_{qp}}{\sqrt{2\pi}} \cot^2\frac{\phi}{2}\frac{\sqrt{\Delta T}}{\omega}\sinh\frac{\omega}{2T} I_1(\frac{\omega}{2T}), \quad (5.8)$$

where $I_1(x)$ is the modified Bessel function of the first kind. The dissipative part of admittance is given by the sum of the two terms: $\mathrm{Re}\,\bar{Y}(\omega) = \mathrm{Re}\,\bar{Y}_4(\omega) + \mathrm{Re}\,\bar{Y}_5(\omega)$. For $\omega \ll T$ the leading term comes from Eq. (5.7) and is frequency-independent, $\mathrm{Re}\,\bar{Y}_4 \propto \sqrt{\Delta/T}$. Comparing the considered case of a weak link to a tunnel junction of the same conductance G, $\mathrm{Re}\,\bar{Y}_4(\omega)$ has an additional, small factor of $T/(\Delta\sin^2\frac{\phi}{2}\ln(T/\omega))$, suggesting that $\mathrm{Re}\,\bar{Y}_4(\omega)$ is reduced in the case of a weak link at low frequencies. In the opposite limit, $\omega \gg T$, the leading term comes from Eq. (5.8) due to higher population of low energy Andreev levels. In that case, $\mathrm{Re}\,\bar{Y}_5(\omega) \propto x_{qp} e^{\omega/T}\sqrt{\Delta T}\omega^{-3/2}$. Because of the large exponential factor, the dissipation in the weak link is greater than in the tunnel junction of the same conductance at high frequencies.

VI. FLUCTUATIONS OF ADMITTANCE

Superconducting junctions are crucial elements of superconducting qubits. The admittance of a junction affects the properties of such qubits (i.e. their frequency and relaxation rates)[12]. As shown above, the admittance depends on the number of quasiparticles in the junction. Fluctuations of the occupation numbers cause fluctuation of the admittance. Consequently, the resonant frequency of a qubit containing the junction will fluctuate.

The inductive $1/L_J$ term (which determines the frequency of the qubit) depends only on the occupation numbers of Andreev level. Therefore, its variance depends only on the variance of the occupation numbers of Andreev level

$$\left(L_J^{(0)}\right)^2 \mathrm{Var}\,\frac{1}{L_J} = (p_0 + p_2 - (p_0 - p_2)^2), \quad (6.1)$$

The relative fluctuations of I_J and $1/L_J$ are significant unless p_0, p_2 or $p_\uparrow + p_\downarrow$ are close to 1. Assuming equilibrium between quasiparticles in the band and in the Andreev level, as well as low quasiparticle occupation numbers of the Andreev level, Eq. (6.1) reduces to:

$$\left(L_J^{(0)}\right)^2 \mathrm{Var}\frac{1}{L_J} = \frac{x_{qp}}{\sqrt{2\pi}}(\Delta/T)^{1/2} e^{(\Delta - E_A)/T}, \quad (6.2)$$

The Y_i terms in the expression (2.7) depend on the occupation numbers of the continuum states as well. However, the fluctuations of admittance caused by the fluctuations of these occupation numbers are inversely proportional to the volume of the system and therefore are negligible in the macroscopic limit. The expression for the variance of $Y(\omega)$ is then similar to the one for $\mathrm{Var}\,1/L_J$. At frequencies $\omega < \Delta + E_A$ and $\omega \neq 2E_A$ to avoid the resonance, we get:

$$\mathrm{Var}\,\mathrm{Re}\,Y(\omega) \approx (p_0 + p_2 - (p_0 - p_2)^2)(\mathrm{Re}\,Y_5^{(0)}(\omega))^2, \quad (6.3)$$

where $\mathrm{Re}\,Y_5^{(0)}(\omega)$ is given by (4.7). Note that at low frequencies $\mathrm{Re}\,Y_5(\omega) = 0$ as it has a phase-dependent threshold.

To calculate the fluctuations in disordered weak links one must integrate the above expressions for variances over the distribution of the transmission coefficients. Assuming again equilibrium between band states and Andreev levels, the fluctuations of the Josephson current are given by:

$$\mathrm{Var}\,I_J = \frac{\pi}{2} G\Delta^2 x_{qp} \frac{|\cos\frac{\phi}{2}|^{1/2}(1 - |\cos\frac{\phi}{2}|)}{|\sin\frac{\phi}{2}|} e^{(1-|\cos\frac{\phi}{2}|)\Delta/T}. \quad (6.4)$$

Here we also assumed $T \ll \Delta(1 - |\cos\frac{\phi}{2}|)$, so that the main contribution comes from the channels with lowest E_A (fully transmittive channels), and $T \ll \Delta|\cos\frac{\phi}{2}|$, see also Eq. (5.3). Under the same assumptions, the variance of the mean inductance $1/\bar{L}_J$ is:

$$\left(\bar{L}_J^{(0)}\right)^2 \mathrm{Var}\,\frac{1}{\bar{L}_J} = \frac{e^2}{2\pi G} x_{qp} e^{(1-|\cos\frac{\phi}{2}|)\Delta/T} \cdot g(\phi), \quad (6.5)$$

$$g(\phi) = \frac{|\cos\frac{\phi}{2}|^{5/2}}{|\sin\frac{\phi}{2}|(1 - \sin\frac{\phi}{2}\,\mathrm{artanh}\sin\frac{\phi}{2})^2}. \quad (6.6)$$

The factor $e^2/2\pi G$ can be interpreted as $1/N_e$, where N_e is the effective number of channels in the weak link. Comparing to the case of a weak tunneling junction with the same number of channels, the relative fluctuations of $1/\bar{L}_J$ have a factor $\exp((1 - |\cos\frac{\phi}{2}|)\Delta/T)$. This large exponential factor suggests that the fluctuations are greater in the weak link. Such shot-to-shot fluctuations contribute to inhomogeneous broadening and limit the usefulness of weak links in superconducting qubits.

VII. NON-LINEAR ABSORPTION RATE AT RESONANT FREQUENCY

For frequencies $\omega \approx 2E_A$ we found that the admittance of a single-channel junction has a resonant delta-function peak corresponding to creation of quasiparticles at the Andreev level. Using Eq. (3.7), we may re-cast the absorption rate Eq. (2.9) in the form

$$W = \frac{\pi}{2}\Omega_R^2\delta(\omega - 2E_A). \quad (7.1)$$

Here

$$\Omega_R = |\phi_1| \frac{(\Delta^2 - E_A^2)\sqrt{E_A^2 - \Delta^2\cos^2\frac{\phi}{2}}}{\Delta E_A|\sin\frac{\phi}{2}|} \quad (7.2)$$

has the meaning[23] of Rabi frequency for the transitions in an effective two-level system driven by AC perturbation ϕ_1. The two levels correspond, respectively, to the empty and doubly-occupied Andreev state. In the linear response, we neglect the effect of Rabi oscillations on the dynamics of the two-level system. This is possible as long as Ω_R is smaller than some "natural", independent of ϕ_1 width η_0 of the levels. Such natural width coming, *e.g.*, from inelastic scattering of quasiparticles[30] leads to a replacement $\delta(\omega - 2E_A) \to \eta_0/\pi[(\omega - 2E_A)^2 + \eta_0^2]$.

The effect of a stronger AC perturbation is two-fold. First, it may result in $\Omega_R > \eta_0$ affecting the dynamics of the two-level system. Second, it may make the levels lifetimes dependent on ϕ_1 by introducing new processes in the kinetics of quasiparticles. Indeed, the AC field may "ionize" the Andreev state, transferring a quasiparticle from that state into the continuum. One needs $\omega > \Delta - E_A$ for that. Using the resonance condition, $\omega \approx 2E_A$, we find that the kinetics of the Andreev states is sensitive to the AC perturbation at $E_A > \Delta/3$.

To address these two effects, we truncate the time-dependent part of the original Hamiltonian (2.1) retaining only terms responsible for the Rabi oscillations between the empty and doubly-occupied Andreev state and terms causing the ionization of that state,

$$\begin{aligned} H =& \sum_\sigma E_A \alpha_\sigma^\dagger \alpha_\sigma + \sum_{k\sigma} E_k \alpha_{k\sigma}^\dagger \alpha_{k\sigma} + \Omega_R \cos\omega t [\alpha_\uparrow \alpha_\downarrow \\ &+ \alpha_\downarrow^\dagger \alpha_\uparrow^\dagger] + \cos\omega t \sum_{k\sigma} [\lambda_k \alpha_{k\sigma}^\dagger \alpha_\sigma + \lambda_k^* \alpha_\sigma^\dagger \alpha_{k\sigma}] \,. \end{aligned} \tag{7.3}$$

Here α_σ and $\alpha_{k\sigma}$ are annihilation operators of quasiparticles in the Andreev level and the band, respectively. The last sum in Hamiltonian (7.3) is responsible for the transitions between the Andreev state and continuum. The corresponding ionization rate is

$$\eta = \frac{\pi}{2} \sum_k |\lambda_k|^2 \delta(\omega - E_k + E_A) \,. \tag{7.4}$$

The very same term leads to the $\mathrm{Re}\, Y_5$ part of admittance in the linear response theory, allowing us to relate η to $\mathrm{Re}\, Y_5$,

$$\eta = \frac{\phi_1^2}{e^2} E_A \,\mathrm{Re}\, Y_5^{(0)}(2E_A) \,. \tag{7.5}$$

The Hamiltonian (7.3) is quadratic, so the equations of motion for operators α reduce to a linear system of differential equations. Assuming frequencies close to the resonance, $\omega - 2E_A \ll \Delta - E_A$, we can find the behavior of the solutions to the equations of motion after a long period of time, $t \gg 1/\eta$. The system then reaches the stationary state in which the energy absorption rate P is given by:

$$P = \langle \dot{H} \rangle = \frac{3}{2} n_A \omega \eta \,, \tag{7.6}$$

with n_A being the average number of quasiparticles in the Andreev level in the process of Rabi oscillations,

$$n_A = \frac{\Omega_R^2}{(\omega - 2E_A)^2 + \Omega_R^2 + \eta^2} \,. \tag{7.7}$$

Hereinafter we neglected a shift of the resonant frequency, $|2\tilde{E}_A - 2E_A| \propto \phi_1^2$, which is parametrically smaller than the broadening due to the $\Omega_R^2 + \eta^2$ term in the denominator of Eq. (7.7). The expression for the absorption power Eq. (7.6) has a simple interpretation: η is the transition rate from the level to the band, so $n_A\eta$ is the rate at which the Andreev level loses quasiparticles. To keep the number of quasiparticles in the level stationary, for each particle that left, a new one must be created in the level. This amounts to energy of $\omega + E_A = 3\omega/2$ for each transition, explaining the factor of 3/2 in Eq. (7.6). The condition $E_A > \Delta/3$ needed for $\eta \neq 0$ and the resonant condition $\omega \approx 2E_A$ imply $\omega > 2\Delta/3$ for the absorption power Eq. (7.6) to be finite.

Using Eqs. (7.7), (7.5), (4.7), (3.4), (3.5), and (7.2) for n_A, η, and Ω_R we can write P in terms of the Rabi frequency,

$$\begin{aligned} P =& \frac{3\eta E_A \Omega_R^2}{(\omega - 2E_A)^2 + \Omega_R^2 + \eta^2} \,, \\ \eta =& \frac{\Omega_R^2 \sqrt{9E_A^2 - \Delta^2}(E_A^2 + \Delta^2 \cos^2 \frac{\phi}{2})}{16 E_A \sqrt{\Delta^2 - E_A^2}(E_A^2 - \Delta^2 \cos^2 \frac{\phi}{2})} \,. \end{aligned} \tag{7.8}$$

At generic values of static phase bias ϕ and transmission coefficient τ, one has $\Omega_R \gg \eta$ as long as the perturbation is reasonably weak, $\phi_1 \ll 2\pi$. In that case n_A exhibits saturation at resonance, while P grows linearly with the perturbation intensity $\propto \phi_1^2$. At a small static phase bias η takes form

$$\eta = \frac{\Omega_R^2}{\Delta} \frac{\sqrt{8}}{\sqrt{\tau}(1-\tau)|\phi|^3} \,. \tag{7.9}$$

It indicates that an increase of the excitation amplitude may result in a non-monotonic n_A vs. Ω_R dependence and in saturation of P at fairly low excitation strength, $\Omega_R \sim |\phi|^3 \Delta \sqrt{\tau}(1-\tau)$.

The population of the Andreev states by quasiparticles drastically alters the critical current of the junction and its low-frequency properties due to the changes in the inductance. Using Eqs. (4.1) and (7.7) we find

$$\frac{1}{L_J} = \frac{1}{L_J^{(0)}} \cdot \frac{(\omega - 2E_A)^2 + \eta^2}{(\omega - 2E_A)^2 + \Omega_R^2 + \eta^2} \,. \tag{7.10}$$

Therefore, Ω_R may be inferred experimentally from a measurement of the critical current[24] or from a two-tone experiment of the type[37].

In the case of a disordered weak link, an AC voltage at frequencies $2\Delta|\cos\frac{\phi}{2}| < \omega < 2\Delta$ also populates Andreev levels with quasiparticles. If the applied voltage is low and the ionization processes are negligible, levels with energies within an interval $|E_A - \omega/2| \lesssim \Omega_R$ are substantially populated, cf. Eq. (7.7). The resulting absorption power,

$$P = \theta(\omega - \frac{2}{3}\Delta)\frac{3\pi^2 G}{40e^2}|\phi_1|^3 \frac{(4\Delta^2-\omega^2)^{3/2}\sqrt{9\omega^2-4\Delta^2}}{\omega^2\Delta^2\sin^2\frac{\phi}{2}} \times [\frac{1}{2}\omega^2 + \Delta^2\cos\phi + \Delta^2], \quad (7.11)$$

scales as $|\phi_1|^3$ reflecting the growing with $|\phi_1|$ number of states involved in the absorption. As before, it is required that $\omega > 2\Delta/3$ to allow the AC-field-induced ionization of the excited Andreev levels.

VIII. CONCLUSION

The motivation for this study was two-fold. First, it came from the prospects[18] of using nanowires instead of tunnel junctions in qubits and other microwave devices[3]. Additional impetus for the study came from experiments[24] with nano-scale junctions, pointing to their extreme sensitivity to the presence of quasiparticles in Andreev level. The effect of Andreev states on electromagnetic properties has also been observed in recent experiment[38] with NS ring. However, these junctions do not fall into the short weak link limit considered in this work.

We obtained an analytical expression for a frequency-dependent admittance of a point contact of arbitrary transmission coefficient and arbitrary quasiparticle occupation factors. The results are valid even for non-equilibrium distribution of quasiparticles (see Section IV). The generalization to a short weak link (shorter than coherence length), is presented in Section V. We found that at low frequencies and temperatures, which are of interest in qubit devices, the dissipation of a point contact and a disordered weak link may indeed be lower than in a tunnel junction of a similar conductance. The lower dissipation is the result of the suppressed density of states, see Eqs. (4.8), (5.7) and (5.8) and the discussion following these equations.

On the other hand, we have shown that at low temperatures the fluctuations of the admittance caused by the fluctuations of the Andreev level occupation can become large (see Section VI). At fixed number of conducting channels N_e, they are larger than the admittance fluctuations of a tunnel junction by a factor $\exp[(\Delta - E_A)/T]$, where E_A is the energy of the lowest Andreev level, see Eqs. (6.2) and (6.5). In addition to that factor, already enhancing fluctuations, their amplitude scales as $N_e^{-1/2}$. Josephson junctions in the existing qubit devices have conductance $G \sim e^2/\pi$. An all-metallic link replacing such junction would have $N_e \sim 1$ leading to gigantic fluctuations. Situation is better for resonant devices designed for different applications[3] where $N_e \sim 100$, and at the same time the demand on the resonance frequency stability may be milder.

The admittance of a single-channel junction exhibits a resonant behavior at frequencies $\omega \approx 2E_A$ corresponding to the creation of pair of quasiparticles in the Andreev level. We studied in more detail the effect of the AC perturbation of such frequencies on a quasiparticle dynamics, see Section VII. If $E_A < \Delta/3$, the system goes through Rabi oscillations between the empty and doubly-occupied Andreev level without dissipation. For $E_A > \Delta/3$ the AC perturbation also causes excitations of quasiparticles from the level to the band. The dissipation power is then non-zero and has resonant behavior, with the resonance width depending on the amplitude of the AC perturbation, Eq. (7.8). We found that the junction inductance follows the same behavior, therefore the Rabi frequency can be measured in a two-tone experiment. In the case of a disordered weak link, there is no dissipation at the AC perturbation frequencies lower than $2\Delta/3$. At higher frequencies, the dissipation power depends non-linearly on the AC perturbation intensity, see Eq. (7.11). Finally, it is worth noting that the population of Andreev level may depend non-monotonically on the intensity of perturbation, see Eqs. (7.7) and (7.9). The population of separate Andreev levels may be studied in experiments[24] with break junctions.

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Appendix A: The complete expression for $\operatorname{Im} Y(\omega)$

From Eq. (2.5) we can get the complete expression for the admittance, including the imaginary part. Since $Y(\omega)$ is analytical in the upper half-plane, $\operatorname{Im} Y(\omega)$ can also be obtained from the expressions for $\operatorname{Re} Y(\omega)$ by Kramers-Kronig relations. At zero temperature, the contributions to the $\operatorname{Im} Y(\omega)$ corresponding to the real parts from Eqs. (3.3)-(3.7) are given by

$$\frac{\mathrm{Im}\, Y_1(\omega)}{G} = \frac{1}{\pi\omega}\mathrm{P.V.}\int_\Delta^\infty d\epsilon_1 \int_\Delta^\infty d\epsilon_2 \rho(\epsilon_1)\rho(\epsilon_2)|z_{\epsilon_1,\epsilon_1+\epsilon_2}|^2 \times [1-f(\epsilon_1)-f(\epsilon_2)] \left[\frac{1}{\omega-\epsilon_1-\epsilon_2} - \frac{1}{\omega+\epsilon_1+\epsilon_2} + \frac{2}{\epsilon_1+\epsilon_2}\right], \tag{A1}$$

$$\frac{\mathrm{Im}\, Y_2(\omega)}{G} = \frac{\sqrt{\Delta^2 - E_A^2}}{\omega}\mathrm{P.V.}\int_\Delta^\infty d\epsilon \rho(\epsilon)|z_{\epsilon,\epsilon+E_A}|^2 \times [p_0 + \frac{p_\uparrow + p_\downarrow}{2} - f(\epsilon)] \left[\frac{1}{\omega-\epsilon-E_A} - \frac{1}{\omega+\epsilon+E_A} + \frac{2}{\epsilon+E_A}\right], \tag{A2}$$

$$\frac{\mathrm{Im}\, Y_3(\omega)}{G} = \pi \frac{(\Delta^2 - E_A^2)(E_A^2 - \Delta^2\cos^2\frac{\phi}{2})}{\omega E_A^2} \times [p_0 - p_2] \left[\frac{1}{\omega - 2E_A} - \frac{1}{\omega + 2E_A} + \frac{1}{E_A}\right]. \tag{A3}$$

These expressions are valid for any distribution of quasiparticles. The case of no quasiparticles present corresponds to $p_0 = 1$, $p_{\uparrow,\downarrow,2} = 0$ and $f(\epsilon) = 0$. The imaginary parts of the last two contributions, $Y_4(\omega)$ and $Y_5(\omega)$ are given by:

$$\frac{\mathrm{Im}\, Y_4(\omega)}{G} = \frac{1}{\pi\omega}\mathrm{P.V.}\int_\Delta^\infty d\epsilon_1 \int_\Delta^\infty d\epsilon_2 \rho(\epsilon_1)\rho(\epsilon_2)|z_{\epsilon_1,\epsilon_1-\epsilon_2}|^2 \times [f(\epsilon_2)-f(\epsilon_1)] \left[\frac{1}{\omega-\epsilon_1+\epsilon_2} - \frac{1}{\omega+\epsilon_1-\epsilon_2} + \frac{2}{\epsilon_1-\epsilon_2}\right], \tag{A4}$$

$$\mathrm{Im}\, Y_5(\omega) = \frac{\sqrt{\Delta^2 - E_A^2}}{\omega}\mathrm{P.V.}\int_\Delta^\infty d\epsilon \rho(\epsilon)|z_{\epsilon,\epsilon-E_A}|^2 \times [\frac{p_\uparrow + p_\downarrow}{2} + p_2 - f(\epsilon)] \left[\frac{1}{\omega-\epsilon+E_A} - \frac{1}{\omega+\epsilon-E_A} + \frac{2}{\epsilon-E_A}\right]. \tag{A5}$$

Appendix B: Admittance of a weak link

Let $p_0(\epsilon)$, $p_{\uparrow,\downarrow}(\epsilon)$ and $p_2(\epsilon)$ be the probabilities to have zero, one or two quasiparticles in the Andreev level with energy ϵ. The occupation factor of the continuum state with energy ϵ is denoted by $f(\epsilon)$. The admittance of a disordered weak link for general occupation numbers is given by:

$$\bar{Y}(\omega) = \sum_{i=1}^{5} \bar{Y}_i(\omega) + \frac{i}{\omega \bar{L}_J}, \tag{B1}$$

where the inductance term is:

$$\frac{1}{\bar{L}_J} = \frac{\pi G \Delta}{|\sin\frac{\phi}{2}|} \int_{\Delta|\cos\frac{\phi}{2}|}^{\Delta} d\epsilon \frac{\Delta^2\cos^2\frac{\phi}{2} - \epsilon^2\sin^2\frac{\phi}{2}}{\epsilon^2\sqrt{\epsilon^2 - \Delta^2\cos^2\frac{\phi}{2}}} [p_0(\epsilon) - p_2(\epsilon)]. \tag{B2}$$

The real parts of the $\bar{Y}_i$ terms are given by:

$$\frac{\mathrm{Re}\, \bar{Y}_1(\omega)}{G} = \theta(\omega - 2\Delta)\mathrm{P.V.}\int_\Delta^{\omega-\Delta} d\epsilon \frac{\epsilon\omega - \Delta^2(1+\cos\phi)}{\Delta^2\omega^2|\sin\frac{\phi}{2}|(\omega-2\epsilon)} \frac{\sqrt{\epsilon^2-\Delta^2}\sqrt{(\omega-\epsilon)^2-\Delta^2}}{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}}} \ln\frac{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}} + |\sin\frac{\phi}{2}|}{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}} - |\sin\frac{\phi}{2}|} [1 - f(\epsilon) - f(\omega-\epsilon)], \tag{B3}$$

$$\frac{\mathrm{Re}\, \bar{Y}_2(\omega)}{G} = \pi \int_{\Delta|\cos\frac{\phi}{2}|}^{\Delta} d\epsilon \theta(\omega - \epsilon - \Delta) \frac{\omega\epsilon - \Delta^2(1+\cos\phi)}{\omega^2\Delta|\sin\frac{\phi}{2}|(\omega-2\epsilon)} \frac{\sqrt{(\omega-\epsilon)^2-\Delta^2}\sqrt{\Delta^2-\epsilon^2}}{\sqrt{\epsilon^2 - \Delta^2\cos^2\frac{\phi}{2}}} [p_0(\epsilon) + \frac{p_\uparrow(\epsilon)+p_\downarrow(\epsilon)}{2} - f(\omega-\epsilon)], \tag{B4}$$

$$\frac{\mathrm{Re}\, \bar{Y}_3(\omega)}{G} = \theta(\omega - 2\Delta|\cos\frac{\phi}{2}|)\theta(2\Delta - \omega) \frac{\pi^2}{|\sin\frac{\phi}{2}|} \left(\frac{\Delta^2}{\omega^2} - \frac{1}{4}\right) \sqrt{\frac{\omega^2}{\Delta^2} - 4\cos^2\frac{\phi}{2}} [p_0(\omega/2) - p_2(\omega/2)], \tag{B5}$$

$$\frac{\mathrm{Re}\, \bar{Y}_4(\omega)}{G} = \int_\Delta^\infty d\epsilon \theta(\epsilon + \omega - \Delta)[f(\epsilon) - f(\epsilon+\omega)] \frac{\sqrt{\epsilon^2-\Delta^2}\sqrt{(\epsilon+\omega)^2-\Delta^2}}{\omega^2\Delta^2|\sin\frac{\phi}{2}|(2\epsilon+\omega)} \times \left\{ \frac{\omega(\epsilon+\omega) - \Delta^2(1+\cos\phi)}{\sqrt{(\epsilon+\omega)^2/\Delta^2 - \cos^2\frac{\phi}{2}}} \ln\frac{\sqrt{(\epsilon+\omega)^2/\Delta^2 - \cos^2\frac{\phi}{2}} + |\sin\frac{\phi}{2}|}{\sqrt{(\epsilon+\omega)^2/\Delta^2 - \cos^2\frac{\phi}{2}} - |\sin\frac{\phi}{2}|} + \frac{\omega\epsilon + \Delta^2(1+\cos\phi)}{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}}} \ln\frac{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}} + |\sin\frac{\phi}{2}|}{\sqrt{\epsilon^2/\Delta^2 - \cos^2\frac{\phi}{2}} - |\sin\frac{\phi}{2}|} \right\}, \tag{B6}$$

$$\frac{\mathrm{Re}\,\bar{Y}_5(\omega)}{G} = \pi \int_{\Delta|\cos\frac{\phi}{2}|}^{\Delta} d\epsilon\theta(\epsilon+\omega-\Delta)\frac{\omega\epsilon+\Delta^2(1+\cos\phi)}{\omega^2\Delta|\sin\frac{\phi}{2}|(2\epsilon+\omega)}\frac{\sqrt{(\omega+\epsilon)^2-\Delta^2}\sqrt{\Delta^2-\epsilon^2}}{\sqrt{\epsilon^2-\Delta^2\cos^2\frac{\phi}{2}}}\left[\frac{p_\uparrow(\epsilon)+p_\downarrow(\epsilon)}{2}+p_2(\epsilon)-f(\omega+\epsilon)\right]. \quad \text{(B7)}$$

From these we can also find imaginary parts using Kramers-Kronig relations.

Assuming Boltzmann distribution $f(\epsilon) = Ae^{-\epsilon/T}$ both below and above the gap, for frequencies less than $2\Delta|\cos(\phi/2)|$ the only contribution to $\mathrm{Re}\,Y(\omega)$ comes from (B6) and (B7). In the limit $\omega, T \ll \Delta$ we get:

$$\mathrm{Re}\,\bar{Y}_4(\omega) \approx \frac{G}{\sqrt{2\pi}}x_{qp}\cot^2\frac{\phi}{2}\sqrt{\frac{\Delta}{T}}(1-e^{-\omega/T})U(\frac{\omega}{T}),$$

$$\mathrm{Re}\,\bar{Y}_5(\omega) \approx \frac{\pi^2 G}{\sqrt{2\pi}}x_{qp}\cot^2\frac{\phi}{2}\frac{\sqrt{T\Delta}}{\omega}\sinh\frac{\omega}{2T}I_1(\frac{\omega}{2T}),$$

where $U(x) = \int_0^\infty dte^{-xt}\sqrt{x(1+x)}\ln(1+1/x)$ and $I_1(x)$ is the modified Bessel function of the first kind. Now consider two opposite limits, $\omega \ll T$:

$$\mathrm{Re}\,\bar{Y}_4(\omega) \approx \frac{G}{\sqrt{2\pi}}x_{qp}\cot^2\frac{\phi}{2}\sqrt{\Delta/T}, \quad \text{(B8)}$$

$$\mathrm{Re}\,\bar{Y}_5(\omega) \approx \frac{\pi^2 G}{8\sqrt{2\pi}}x_{qp}\cot^2\frac{\phi}{2}\frac{\omega}{\sqrt{T\Delta}}, \quad \text{(B9)}$$

and $\omega \gg T$:

$$\mathrm{Re}\,\bar{Y}_4(\omega) \approx \frac{G}{2\sqrt{2}}x_{qp}\cot^2\frac{\phi}{2}\frac{T\sqrt{\Delta}}{\omega^{3/2}}\ln\frac{\omega}{T}, \quad \text{(B10)}$$

$$\mathrm{Re}\,\bar{Y}_5(\omega) \approx \frac{\pi G}{2\sqrt{2}}x_{qp}\cot^2\frac{\phi}{2}\frac{\sqrt{T}\Delta}{\omega^{3/2}}e^{\omega/T}. \quad \text{(B11)}$$

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