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Relaxation of weakly interacting electrons in one dimension

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We consider the problem of relaxation in a one-dimensional system of interacting electrons. In the limit of weak interactions, we calculate the decay rate of a single-electron excitation, accounting for the nonlinear dispersion. The leading processes which determine the relaxation involve scattering of three particles. We elucidate how particular forms of Coulomb interaction, unscreened and screened, lead to different results for the decay rates and identify the dominant scattering processes responsible for relaxation of excitations of different energies. Interestingly, temperatures much smaller than the excitation energy strongly affect the rate. At higher temperatures the quasiparticle relaxes by exciting co-propagating electron-hole pairs, whereas at lowest temperatures the relaxation proceeds via excitations of both co-propagating and counter-propagating pairs.

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Low energy excitations of a three-dimensional interacting electron system are fermionic quasiparticles that in many respects resemble bare electrons [1]. A quasiparticle excitation of energy ε has finite decay rate $\tau^{-1} \propto \varepsilon^2$, where ε is measured from the Fermi level. This fact is the foundation of the Fermi liquid theory and was confirmed experimentally by measuring the broadening of the Lorentzian-shaped spectral function [2].

One-dimensional interacting fermions are conventionally described within the exactly solvable Tomonaga-Luttinger model where particles are assumed to have a linear dispersion. This model can be diagonalized in terms of noninteracting bosonic excitations [3, 4] which have infinite lifetimes. This feature reveals an important limitation of the Tomonaga-Luttinger model, because in general an excited system is expected to relax to equilibrium. To study relaxation, one should therefore consider models that take into account the curvature of the spectrum. Recent experimental observation of different equilibration rates of hot electrons and holes in quantum wires [5] has confirmed the importance nonlinear dispersion of electrons. Study of interacting electrons with nonlinear spectrum is a subject of intense theoretical interest [6–13]. The area of new physics beyond the Luttinger liquid formalism has been recently reviewed in Ref. [14].

In this paper we consider a system of spinless fermions and study the decay rate of a quasiparticle excitation placed above the Fermi level, see Fig. 1. Since the complete study of the effects of nonlinear dispersion is very difficult, here we analyze the limit of weak interactions. In this case, the scattering processes can be classified by the number of colliding particles. Unlike in higher dimensions, two-particle processes do not lead to relaxation due to the conservation laws of energy and momentum. Therefore, the leading mechanism which provides finite relaxation rate involves scattering of three particles [6].

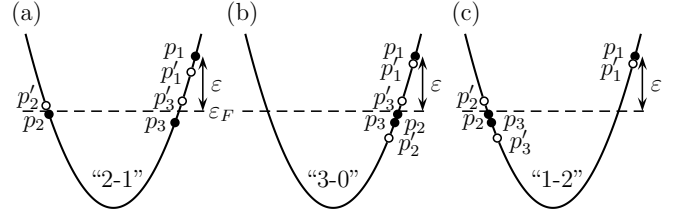


FIG. 1. A single electron excitation of energy ε and momentum p_1 relaxes via scattering off two other electrons. Filled and empty circles represent, respectively, incoming and outgoing momenta. The one-dimensional topology of the Fermi surface determines the three cases (a), (b), and (c), which we refer to as the “2-1”-, “3-0”-, and “1-2”-type processes. While at zero temperature only the “2-1” processes are allowed, all three types contribute to the quasiparticle decay at nonzero temperatures.

At zero temperature, the problem of relaxation was studied in Ref. [7], where the quasiparticle decay rate was found to behave as $\tau^{-1} \propto \varepsilon^8$. Here we study the effect of temperature T , and find dramatic departures from the $T = 0$ case. We consider the situation when the temperature is much smaller than the energy of the excitation. Denoting the momentum of the excitation by p_1 , we find the expression for its decay rate

$$\frac{1}{\tau} = \sum'_{\substack{p_2, p_3 \\ p_1', p_2', p_3'}} W_{123}^{1'2'3'} n_2 n_3 (1 - n_{1'}) (1 - n_{2'}) (1 - n_{3'}). \quad (1)$$

This equation accounts for scattering of the quasiparticle of momentum p_1 and two others with momenta $\{p_2, p_3\}$ into three outgoing states $\{p_1', p_2', p_3'\}$, see Fig. 1. In Eq. (1) by $n_i = n_{p_i}$ we denote the Fermi occupation numbers, while Σ' indicates summation over distinct states. The scattering rate $W_{123}^{1'2'3'}$ is determined by the Fermi

golden rule expression, and reads

$$W_{123}^{1'2'3'} = \frac{2\pi}{\hbar} |\mathcal{A}_{123}^{1'2'3'}|^2 \delta(E - E'). \quad (2)$$

Here the delta function imposes conservation of the total energy, defined as $E = \varepsilon_{p_1} + \varepsilon_{p_2} + \varepsilon_{p_3}$ and similarly for the outgoing momenta. The three particle scattering amplitude is defined as a vacuum expectation value

$$\mathcal{A}_{123}^{1'2'3'} = \langle a_{p_1'}^\dagger a_{p_2'}^\dagger a_{p_3'}^\dagger | V \frac{1}{E - H_0 + i0^+} V | a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger \rangle. \quad (3)$$

It is the central object which determines the relaxation rate (1). The unperturbed Hamiltonian H_0 and the perturbation V are taken in the form

$$H_0 = \sum_p \varepsilon_p a_p^\dagger a_p, \quad V = \frac{1}{2L} \sum_{p_1, p_2, q} V_q a_{p_1+q}^\dagger a_{p_2-q}^\dagger a_{p_2} a_{p_1},$$

where a and a^\dagger are the fermionic operators, $\varepsilon_p = p^2/2m$, and L is the system size. The two-body interaction enters the Hamiltonian via its Fourier transform V_q .

The scattering amplitude (3) is very sensitive to the form of the two-body interaction $V(x)$. It vanishes for $V(x) \propto \delta(x)$, i.e., $V_q = \text{const}$, corresponding to the contact interaction between spinless fermions. Nullification of the amplitude in this case arises because Pauli principle prevents two electrons from sharing the same position. The amplitude (3) also vanishes for the Cheon-Shigehara model [14, 15], defined by $V(x) \propto \delta''(x)$, i.e., $V_q \propto q^2$. The latter belongs to the class of the so-called integrable models [16] for which there is no relaxation and therefore $\tau^{-1} = 0$.

In quantum wires, the interaction between electrons is of a longer range. As an example of the most practical use, we consider the Coulomb interaction defined by $V^c(x) = e^2/|x|$, which has the Fourier transform $2e^2 \ln(1/|p|)$. Interestingly, this logarithmic form gives a vanishing three-particle amplitude (3), although the model describing fermions interacting via Coulomb interaction is not expected to be integrable. It is therefore important to cut off the short distance singularity. This is done by accounting for the finite width of the wire w . One then obtains

$$V_p^c = 2e^2 \ln \left(\frac{\hbar}{|p|w} \right) \left(1 + \frac{p^2 w^2}{\hbar^2} \right), \quad (4)$$

keeping the first two leading-order terms. Here and in the following we neglect numerical factors under the logarithm because we consider small momenta $|p| \ll \hbar/w$.

Calculation of the three particle amplitudes is rather tedious [17]. For unscreened Coulomb interaction (4), one obtains the leading order result on the mass shell

$$(\mathcal{A}_{123}^{1'2'3'})_c = -\frac{4e^4}{L^2} \frac{mw^2}{\hbar^2} \ln \left(\frac{\hbar}{p_r w} \right) [f_c((\varphi' + \varphi)/2) - f_c((\varphi' - \varphi)/2)] \delta_{P, P'}, \quad (5)$$

with the even periodic function $f_c(\theta) = f_c(\theta + \pi/3)$

$$f_c(\theta) = -\sum_{j=1}^3 \frac{9 \sin(\theta + 2\pi j/3) \ln |\sin(\theta + 2\pi j/3)|}{\sin(3\theta)}. \quad (6)$$

Instead of using the incoming momenta $\{p_j\}$ and the outgoing ones $\{p'_j\}$, for convenience here we have introduced Jacobi coordinates P, p_r, φ , defined as

$$p_j = P/3 + \sqrt{2/3} p_r \cos(\varphi - 2\pi j/3), \quad j = 1, 2, 3, \quad (7)$$

and similarly for the outgoing momenta. Here $P = p_1 + p_2 + p_3$ has a simple meaning of the total momentum of the three colliding particles. The momentum p_r is given by

$$p_r = \sqrt{(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2} / \sqrt{3}, \quad (8)$$

and measures the typical separation between the momenta. We note that the conservation laws impose $p'_r = p_r$.

We consider scattering at low temperatures, when all scattering particles should be in the vicinity of the two Fermi points. This enables us to classify particles as being right- or left-moving. Throughout this paper we study the decay of a right-moving excitation. The two other incoming particles can be classified in one of three ways: (i) one particle has positive momentum and the other negative, (ii) both have positive momenta, and (iii) both have negative momenta. These three configurations, respectively, have the values of the total momentum near p_F , $3p_F$, and $-p_F$, where p_F is the Fermi momentum. Therefore, the incoming and the outgoing states must be in the same momentum configuration. Thus, we can distinguish between three different cases, which we call “2-1”, “3-0”, and “1-2” processes, see Fig. 1. We choose notations where the momenta p_j and p'_j are always on the same branch of the Fermi surface, while the scattering amplitude takes into account all possible exchange processes.

At zero temperature, only processes of “2-1” type lead to relaxation [7]. On the contrary, at nonzero temperatures all three processes have nonzero rates. In the following we will calculate the rates and identify the dominant processes as the temperature increases.

We start our analysis by considering the “2-1” scattering process, Fig. 1(a). In order to understand the energy dependence of the decay rate, let us consider the momentum change of the left-moving particle-hole pair. It is easily obtained from the conservation laws and reads $p'_2 - p_2 = (p'_1 - p_1)(p'_1 - p_3)/(p'_3 - p_2)$. Using the estimate $p'_3 - p_2 \approx 2p_F$, we can express the conservation laws contained in the scattering rate (2) as

$$\delta(E - E') \delta_{P, P'} \approx \delta \left(p'_2 - p_2 - \frac{(p_1 - p'_1)(p'_1 - p_3)}{2p_F} \right) \times \frac{m}{2p_F} \delta_{p_1 + p_2 + p_3, p'_1 + p'_2 + p'_3}, \quad (9)$$

where the momentum conservation comes from the amplitude (5). We can now employ the last equation to perform the summations over p'_2 and p'_3 in the expression for the rate (1). The summation over the remaining three momenta p_2, p_3 , and p'_1 then determines the rate. From Eq. (9) we conclude that the typical energy of the left moving particle-hole pair is of the order of $\varepsilon^2/\varepsilon_F$, where $\varepsilon \simeq v_F(p_1 - p_F)$, and v_F is the Fermi velocity. Therefore, the integration over p_2 is restricted to an energy range of that width, while for both p_3 and p'_1 that range is of order ε . As a result, at $T = 0$ we find the phase space volume available for scattering to be proportional to $\varepsilon^2(\varepsilon^2/\varepsilon_F)$, where ε_F denotes the Fermi energy [7]. Since the amplitude (5) depends on momenta only logarithmically, we infer the scattering rate $\tau^{-1} \propto \varepsilon^4$. This result still applies at very low temperatures $T \ll \varepsilon^2/\varepsilon_F$, since then the thermal smearing of the occupation numbers is not significant. However, a new behavior of the decay rate emerges in the range of temperatures $\varepsilon^2/\varepsilon_F \ll T \ll \varepsilon$, because the occupation numbers $n_2, n_{2'}$ in Eq. (1) of the left-moving pair become thermally smeared. As a result, the integration over momentum p_2 covers the energy range of order T , and the phase space volume is proportional to $\varepsilon^2 T$ [14]. After careful calculation, using the amplitude (5) and (6) for the unscreened Coulomb interaction, one finds

$$\frac{1}{(\tau_{2-1})_c} = \left(\frac{e^2}{\hbar v_F}\right)^4 \left(\frac{p_F w}{\hbar}\right)^4 \left(\ln \frac{\hbar}{p_F w}\right)^2 \times \begin{cases} c_1 \varepsilon^4 / \hbar \varepsilon_F^3, & T \ll \varepsilon^2 / \varepsilon_F \\ c_2 \varepsilon^2 T / \hbar \varepsilon_F^2, & \varepsilon^2 / \varepsilon_F \ll T \ll \varepsilon \end{cases} \quad (10)$$

where the numerical prefactors are $c_1 = (15 - \pi^2)/1024\pi^3$ and $c_2 = 3(12 - \pi^2)/64\pi^3$. The logarithm of the expression (10) originates from the first term in the amplitude (5), using $p_r \sim p_F$. Compared to the decay rate $\tau^{-1} \propto \varepsilon^8$ of Ref. [7], our result (10) is significantly larger due to long range nature of unscreened Coulomb interaction. On the other hand, the rate (10) is smaller than the one for electrons with spin [9], because interaction between spinless fermions is weaker as a consequence of the Pauli principle.

At zero temperature the “3-0” processes, Fig. 1(b), are not allowed by the conservation laws [7]. However, they do contribute to the decay rate at finite temperatures. To evaluate the rate one can employ a similar strategy to the one used above for the “2-1” processes. The momentum change of the initial excitation is $p_1 - p'_1 = (p'_3 - p_3)(p'_3 - p_2)/(p_1 - p'_2)$, which enables us to express the conservation laws as

$$\delta(E - E')\delta_{P,P'} \approx \delta\left(p'_1 - p_1 - \frac{(p'_3 - p_3)(p'_3 - p_2)}{p_1 - p_F}\right) \times \frac{m}{p_1 - p_F} \delta_{p_1 + p_2 + p_3, p'_1 + p'_2 + p'_3}. \quad (11)$$

We can now use Eq. (11) to perform the summation over the momenta p'_1 and p_2 in Eq. (1), which gives rise to a factor of $1/\varepsilon$ in the rate. The remaining summation over p'_2, p_3, p'_3 is over the typical range of momenta of the order T/v_F and delivers the factor T^3 . The detailed calculation reveals the final result for the unscreened case

$$\frac{1}{(\tau_{3-0})_c} = c_3 \left(\frac{e^2}{\hbar v_F}\right)^4 \left(\frac{p_F w}{\hbar}\right)^4 \left[\ln \frac{\hbar}{(p_1 - p_F)w}\right]^2 \frac{T^3}{\hbar \varepsilon_F \varepsilon}, \quad (12)$$

where $c_3 \approx 1.13$ [17]. The logarithmic prefactor arises from the amplitude (5), using $p_r \sim p_1 - p_F$.

The processes of “1-2” type are similar to the “3-0” ones. In order to evaluate their decay rate, one can use the expression for the momentum change $p_1 - p'_1$ of the initial excitation of the same form as for the “3-0” case. However, in the present situation the denominator $p_1 - p'_2$ should be replaced by $2p_F$, rather than by $p_1 - p_F$. Therefore, the final result for the rate should be the same as in Eq. (12), provided one replaces ε by $4\varepsilon_F$, which we confirmed by a careful calculation. Since $\varepsilon \ll \varepsilon_F$, the contribution of the “1-2” processes is always subdominant.

We now turn to the case of screened Coulomb interaction. We model the two-body potential as $V^{sc}(x) = e^2/|x| - e^2/\sqrt{x^2 + 4d^2}$, where d is the distance between the wire and a conducting plane representing nearby gates. At $|p| \ll \hbar/d \ll \hbar/w$, its Fourier transform is

$$V_p^{sc} = 2e^2 \ln\left(\frac{d}{w}\right) - 2e^2 \frac{p^2 d^2}{\hbar^2} \ln\left(\frac{\hbar}{|p|d}\right). \quad (13)$$

The first term in the right hand side of Eq. (13) corresponds to the contact interaction $V \propto \delta(x)$, which does not affect spinless fermions. Therefore, the three-particle amplitude is determined by the second term of Eq. (13) and reads [17]

$$(\mathcal{A}_{123}^{1'2'3'})_{sc} = -\frac{8e^4}{3L^2} \frac{md^4 p_r^2}{\hbar^4} \ln\left(\frac{\hbar}{p_r d}\right) [f_{sc}((\varphi' + \varphi)/2) - f_{sc}((\varphi' - \varphi)/2)] \delta_{P,P'}, \quad (14)$$

with the even periodic function $f_{sc}(\theta) = f_{sc}(\theta + \pi/3)$

$$f_{sc}(\theta) = -\sum_{j=1}^3 \frac{9 [5 \sin(\theta + 2\pi j/3) - \sin(5\theta - 2\pi j/3)]}{4 \sin(3\theta)} \times \ln |\sin(\theta + 2\pi j/3)|. \quad (15)$$

Compared to the unscreened interaction (4), the screened interaction (13) has two additional powers of momentum. This is reflected in the corresponding amplitude (14), up to the logarithmic terms. As a result, the decay rates will have four additional powers of energy with respect to the unscreened case. For the “2-1” processes, the characteristic energy change is ε , which determines the decay

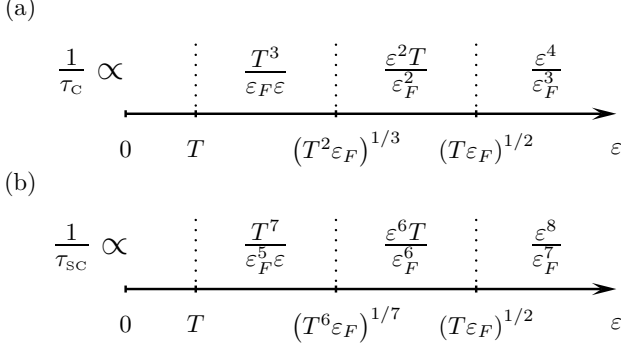


FIG. 2. Leading behavior of the decay rate of a quasiparticle of energy ε for (a) unscreened Coulomb interaction (4), and for (b) screened Coulomb interaction (13).

rate at zero temperature $\tau^{-1} \propto \varepsilon^8$ [7]. After a careful calculation one obtains

$$\frac{1}{(\tau_{2-1})_{sc}} = \left(\frac{e^2}{\hbar v_F} \right)^4 \left(\frac{p_F d}{\hbar} \right)^8 \left(\ln \frac{\hbar}{p_F d} \right)^2 \left(\ln \frac{\varepsilon}{\varepsilon_F} \right)^2 \times \begin{cases} c_4 \varepsilon^8 / \hbar \varepsilon_F^7, & T \ll \varepsilon^2 / \varepsilon_F \\ c_5 \varepsilon^6 T / \hbar \varepsilon_F^6, & \varepsilon^2 / \varepsilon_F \ll T \ll \varepsilon \end{cases} \quad (16)$$

where $c_4 = 15/32768\pi^3$ and $c_5 = 55/9216\pi^3$. For the “3-0” processes, the typical momentum change is T/v_F and therefore one obtains four additional powers of temperature compared to Eq. (12),

$$\frac{1}{(\tau_{3-0})_{sc}} = c_6 \left(\frac{e^2}{\hbar v_F} \right)^4 \left(\frac{p_F d}{\hbar} \right)^8 \left[\ln \frac{\hbar}{(p_1 - p_F)d} \right]^2 \times \left(\ln \frac{\varepsilon}{T} \right)^2 \frac{T^7}{\hbar \varepsilon_F^5 \varepsilon}, \quad (17)$$

where $c_6 \approx 22.1$. The decay rates given by Eqs. (10) and (12) for the Coulomb interaction as well as Eqs (16) and (17) for the screened Coulomb interaction are the main results of this paper. Now we comment on their physical meaning.

The spectral function of a system of interacting electrons described by the Tomonaga-Luttinger model displays a power-law edge singularity on the mass shell [3, 14]. This divergence is a signature of the infinite lifetime of excitations. However, once one accounts for the curvature of the spectrum, the divergence disappears and the spectral function becomes broadened [7]. Therefore, the quasiparticles on the mass shell are subject to decay. The above-calculated decay rate describes broadening of the spectral function in the vicinity of the particle mass shell. Our result (16) taken at zero temperature is consistent with Ref. [7]. It is worth noting, however, that unlike our paper, where for the screened interaction all the relevant momentum scales are assumed to be small compared with \hbar/d , Ref. [7] assumes $p_F \gg \hbar/d$, i.e., the Fourier components of interaction potential for momenta of the order of the Fermi momentum were neglected.

The leading behavior of the decay rate is summarized in Fig. 2. Excitations of energies much larger than $(T\varepsilon_F)^{1/2}$ decay with a temperature independent rate. For the unscreened interaction (4), we infer a new energy scale $\varepsilon^* \sim (T^2\varepsilon_F)^{1/3}$. Quasiparticles of energies lower than ε^* decay by exciting co-propagating particle-hole pairs, while quasiparticles of energies larger than ε^* decay by exciting both co-propagating and counter-propagating pairs. The same general picture applies in the case of the screened Coulomb interaction (13), but the crossover energy scale ε^* is of order $(T^6\varepsilon_F)^{1/7}$. Interestingly, because the decay rate decreases with energy for the “3-0” processes but increases for the “2-1” ones, the rate has a minimum near the crossover energy scale ε^* .

To summarize, motivated by recent experiment [5] we have calculated the decay rate of quasiparticles in weakly interacting one-dimensional electron systems. The dominant mechanism of quasiparticle decay involves three electrons and is illustrated in Figs. 1(a) and 1(b). The decay rate shows nontrivial temperature dependence even at $T \ll \varepsilon$, see Fig. 2.

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