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Approaching the RSOS critical points through entanglement: one model for many universalities

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We analytically compute the Renyi entropies for the RSOS models, representing a wide class of exactly solvable models with multicritical conformal points described by unitary minimal models and \mathbb{Z}_n parafermions. The exact expressions allow for an explicit comparison of the expansions around the critical points with the predictions coming from field theory. In this way it is possible to point out the nature of the so-called “unusual corrections”, clarifying the link with the operator content, the role of the symmetries and the boundary conditions. By choosing different boundary conditions, we can single out the ground states as well as certain combinations of high energy states. We find that the *entanglement spectrum* is given by operators different from those responsible for the off-critical perturbation, although they belong to the same representation of a Virasoro Algebra. In the parafermionic case we observe unexpected logarithmic corrections.

Keywords: entanglement entropy, RSOS, conformal field theory, correlation length, parafermion

I. INTRODUCTION

Entanglement is the unique feature distinguishing a quantum system from a classical one^{1,2}. While we still lack a fundamental, general definition of what entanglement is, we can characterize it well when we consider the mutual entanglement of two complementary components of a system in a definite state, the so-called bipartite-entanglement³. A popular way to quantify it is given by the *entanglement entropies*. In the recent years, it has become increasingly important to be able to compute them, either numerically or analytically⁴. Typically, one considers the ground state $|0\rangle$ of a quantum Hamiltonian. Once the system has been divided into two parts A, B , it is possible to introduce the reduced density matrix, tracing out one of the two subsystems

$$\rho_A \equiv \text{Tr}_B |0\rangle\langle 0| \quad (1)$$

and then the Renyi entropies are defined as

$$\mathcal{S}_\alpha \equiv \frac{1}{1-\alpha} \ln \text{Tr} \rho_A^\alpha, \quad (2)$$

We notice that, thanks to the free parameter α , the knowledge of the Renyi entropies is equivalent to the knowledge of the full spectrum of the reduced density matrix⁵, whose logarithm is known as the *entanglement spectrum*⁶. A particularly important point is the $\alpha \rightarrow 1$ limit, known as Von Neumann entropy

$$\mathcal{S} = \lim_{\alpha \rightarrow 1} \mathcal{S}_\alpha = -\text{Tr} \rho_A \ln \rho_A, \quad (3)$$

which provides a good quantification for the entanglement in terms of a single number.

Gapped $d+1$ -dimensional systems obey the so-called *area-law*⁷: at the leading order in the thermodynamic limit of large subsystem sizes, the entanglement entropy is proportional to the area of the boundary separating A and B . In $d=1$, such law predicts a saturation to a constant of the entanglement entropy when A is composed of large intervals. For $d>1$ the area law remains true for most gapless systems, with possible logarithmic corrections⁸. These logarithmic contributions are a signature of $d=1$ -dimensional physics. In fact, exploiting the conformal invariance of gapless $1+1$ models, it is known that the entropy grows logarithmically with the length of the A interval ℓ , with a proportionality given by the conformal anomaly⁹. In¹⁰, the sub-leading contributions were analyzed and the emergence of *unusual corrections* was linked to the effect of relevant (and irrelevant) operators of the critical theory:

$$\mathcal{S}_\alpha = \frac{c}{6} \left(\frac{1+\alpha}{\alpha} \right) \ln \ell + c'_\alpha + b_\alpha \ell^{-2x/\alpha} + \dots, \quad (4)$$

where c is the central charge of the CFT, c'_α and b_α are non-universal constants and $x = \Delta + \bar{\Delta}$ is the dimension of the operator “responsible” for the correction. Marginal operators act differently (renormalizing the central charge) and give rise to logarithmic corrections of the form $(\log \ell)^{-2n}$.

Similar results have been observed^{11–13} close to the critical points, where now the correlation length ξ , being large but finite, becomes the relevant length scale (that is $\ell \gg \xi$), and have led to the conjectured form

$$\mathcal{S}_\alpha = \frac{c}{12} \left(\frac{1+\alpha}{\alpha} \right) \ln \frac{\xi}{a_0} + A_\alpha + B_\alpha \xi^{-h/\alpha} + \dots, \quad (5)$$

where a_0 is a short distance cutoff, A_α and B_α are again non-universal constants and h can be interpreted as the dimension of a relevant operator⁵². In general, one should not expect the entropy to be the same scaling function in ℓ and in ξ : although the coefficients of the leading term have to coincide, the operators acting in the two cases can be different and x and h can be different. The crossover function between the two scaling regime might be accessible using techniques similar to^{14–16}.

Thus, while the entanglement entropy of a gapped $d = 1$ system could seem not very interesting, since it saturates to a constant in the thermodynamic limit, its study close to a critical point could shed light on the scaling theory governing the lattice models and its universal features. Moreover, a recently proposed protocol^{17,18} would allow the measurement of the Renyi entropies only for gapped systems, thus rendering the theoretical computation of the limiting value (5) amenable to cold-atom experimental confirmation.

In this paper we will focus on the analytic computation of the Renyi entropy in this thermodynamic limit for the quantum systems obtained from a class of integrable lattice models known as *Restricted Solid-on-solid* (RSOS)¹⁹. These models and their structures have inspired the discussion in the last section of¹¹, on a general relation between the entanglement entropy of quantum (integrable) models and Virasoro characters. Inspired by these considerations, we expand and detail the calculation sketched in¹¹, and extend it to the parafermionic case. The importance of the RSOS models is multifold: first of all, they provided the first lattice realization of the unitary conformal models²⁰ as pointed out in^{21,22}. In a different phase, they also realize parafermionic models and thus give access to consistent $c > 1$ CFT's²³. Moreover, thank to the rich underlying mathematical structure, they appeared as a fascinating link between integrable lattice models and number theory.

While entanglement is associated to a quantum state, here we will take advantage of the well-known link that allows to derive a quantum Hamiltonian from the row-to-row transfer matrix of an integrable classical model. In this way, the classical configuration with the lowest free energy corresponds to the ground state of the quantum model. Moreover, by a proper choice of the boundary conditions, one can select higher energy configurations, which correspond to the lowest quantum state within a given sector of the Hilbert space, and is thus a way to investigate the entanglement entropy of states other than the ground state. Without showing explicitly the quantum Hamiltonian associated to the RSOS transfer matrix, it is worth saying that it naturally arises in the context of loop models²⁴. More recently, an explicit realization of these Hamiltonians has been obtained from a very different perspective as a chain of interacting non-abelian anyons²⁵. Another possible approach, whose terminology we decide to adopt here, is to interpret the RSOS models as the lattice realizations of an integrable thermal perturbation of a class of rational CFTs. This allows to write the action as

$$\mathcal{A} = \mathcal{A}_{\text{CT}} + \lambda \int d^2x \, \epsilon(x) \quad (6)$$

where $\epsilon(x)$ is the operator representing the thermal perturbation and λ is the coupling constant measuring the distance from criticality.

The RSOS models being ubiquitous and integrable makes the computation of the Renyi entropy not only interesting, but also possible analytically by means of the *Corner Transfer Matrix* (CTM) approach. In fact, the reduced density matrix of a half-interval in the thermodynamic limit can be shown to be equal (except for the normalization factor) to the CTM operator^{26,27}

$$\hat{\rho}_A = \mathcal{Z}_1^{-1} \rho_{\text{CTM}}, \quad \mathcal{Z}_\alpha \equiv \text{Tr} \rho_{\text{CTM}}^\alpha, \quad (7)$$

and one can therefore compute the Renyi entropy as

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha - 1} \ln \mathcal{Z}_1 + \frac{1}{1 - \alpha} \ln \mathcal{Z}_\alpha. \quad (8)$$

Even though this procedure looks similar to the replica trick exploited in the conformal case, here α can be an arbitrary real (or even complex) parameter, thus avoiding all the subtleties of the analytic continuation from $\alpha = n \in \mathbb{N}$, necessary to compute, for instance, the Von Neumann entropy. Therefore, beyond checking that the conjectured form of Eq.(5) applies for the RSOS, our results provide the umpteenth check to the Cardy-Calabrese formula²⁸, both for minimal models, where the replica trick introduces operators not present in the original Kac table, and also in systems with central charges greater than unity. For minimal models, we identify the leading *unusual correction* of (5) as coming from the second most relevant operator in the model, that is $\Delta_{3,3}$, since the most relevant one, $\Delta_{2,2}$, is odd under the \mathbb{Z}_2 symmetry of the ground state. However, we find that, by varying the boundary conditions, different sectors can be traced out and the leading correction to the entropy can come from other operators as well. Let us stress that this result is different from the correction one could naïvely expect taking the anomalous dimension of the perturbation in (6). For parafermionic models, we find the leading correction to come from the first

thermal field, which is the most relevant only among the \mathbb{Z}_{r-2} neutral fields. In addition, some boundary conditions at infinity turn on logarithmic corrections, different from those in (5). It would be tempting to interpret these terms as due to a marginal operator in accordance with^{10,29}; however these corrections are present even when the theory does not seem to support a marginal field (which is normally related to the existence of a free boson and present only for certain given values of r). Thus, the origin of these terms still needs a full explanation and is probably rooted in a choice of boundary conditions which has no conformal counterpart in the continuum limit.

The paper is organized as follows: in section II we introduce the RSOS models, their phase diagrams and some details about the exact solution in Regime III and I on which we will focus. In section III, we will concentrate on the computation of the Renyi entropy for regime III, corresponding to unitary minimal models and in section IV on regime I corresponding to parafermionic CFT. To better elucidate the meaning of our formulae, we will conclude the analysis with the specific examples of the Ising and 3-state Potts model in section V. Finally in section VI we will discuss our results and their meaning. We collect some useful definitions and identities on elliptic functions in A.

II. THE MODEL

A. Definition

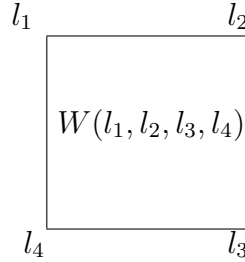
We consider the restricted solid-on-solid (RSOS) on a square lattice, first introduced in¹⁹. The variables on each node are called “heights” and are integer numbers restricted to the interval:

$$1 \leq l_i \leq (r-1) \quad (9)$$

A local constraints is imposed to every configuration

$$|l_i - l_j| = 1 \quad (10)$$

for each pair of nearest-neighbor i, j . The model belongs to the family of *interaction round-a-face* (IRF) models, introduced by Baxter³⁰. Each plaquette is given a Boltzmann weight according to the configuration of the 4 sites enclosing it: $W(l_1, l_2, l_3, l_4)$. Here the four sites around the plaquette are counted clockwise from the northwest l_1, l_2, l_3, l_4 . The model can be exactly solved



for a proper choice of the weights W where it appears as a consistent restriction of the solid-on-solid (SOS) model and hence with the same Yang-Baxter algebra of an eight-vertex model. Weights are thus parameterized in terms of elliptic functions and for the details we refer to the original work¹⁹.

At fixed maximum-height r , the phase-space of the model can be characterized by two parameters p, v . The requirement of real and positive Boltzmann weights gives the constraints

$$-1 < p < 1 \quad -\eta < v < 3\eta$$

naturally arranged in four, physically distinct, regimes

$$\begin{aligned} \text{I} \quad & -1 < p < 0 \quad \eta < v < 3\eta \\ \text{II} \quad & 0 < p < 1 \quad \eta < v < 3\eta \\ \text{III} \quad & 0 < p < 1 \quad -\eta < v < \eta \\ \text{IV} \quad & -1 < p < 0 \quad -\eta < v < \eta \end{aligned}$$

The parameter η is related to p by

$$\eta \equiv \frac{K(p)}{r}$$

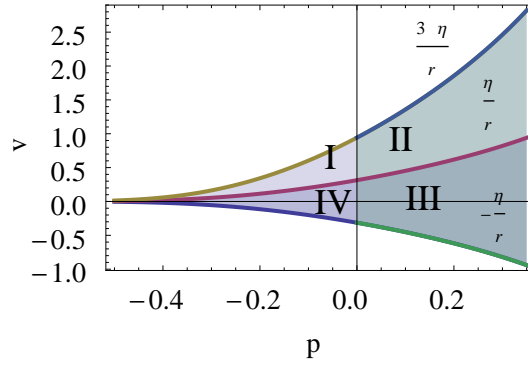


FIG. 1: The four regimes in the (p, v) plane.

where $K(p)$ is the complete elliptic integral with elliptic “nome” p . The regimes I, II and III, IV are separated by a line of critical points at $p = 0$. The parameter v can be considered, roughly speaking, as the spatial anisotropy of the interactions in the model and does not enter in the order parameters and the critical behavior. So, for fixed regime we will ignore it. The manifolds of exact solution will be simply lines parameterized by $p \in (-1, 1)$. By comparison with (6), we have that close to criticality, i.e. when $|p| \ll 1$, $p \simeq \lambda$.

B. Exact solution

The exact solution in¹⁹ consists of three parts.

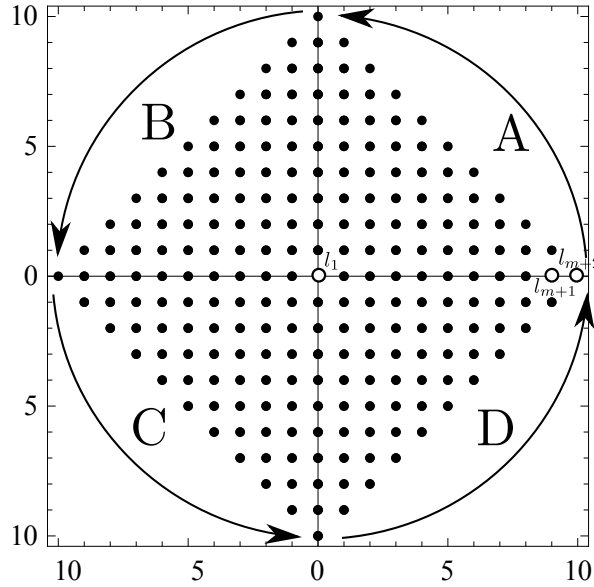


FIG. 2: A representation of the lattice for $N = m + 2 = 11$. The action of the four CTM generates the full partition function.

1. introduce the corner transfer matrix (CTM) that, once diagonalized, allows reducing the 2D configuration sum into a 1D sum already at finite size;
2. perform the thermodynamic limit by transforming the finite size expressions into series involving gaussian polynomials and then taking the limit as modular functions;
3. sum up the partial traces (for fixed value of the central height) of the CTM obtaining the full partition function.

Here we will briefly review the first two steps, that are functional to our derivation. For the last step, we will use a slightly different approach with respect to the traditional one, formulated in terms of the dual variables.

Corner transfer matrix

The method of the corner transfer matrix (CTM), introduced by Baxter³⁰, allows the exact solution of lattice integrable models, computing both the partition function and the one-point correlation function (e.g. the magnetization). As shown in fig. II B, four operators **A**, **B**, **C**, **D** are introduced. **A** is the partition function of the system restricted to the first quadrant and with fixed boundary conditions on the positive x and y axis. Similarly the other operators **B**, **C**, **D** are defined in the other quadrants and it follows that

$$(\rho_{\text{CTM}})_{l_1 \dots l_N}^{l'_1 \dots l'_N} = (\mathbf{ABCD})_{l_1 \dots l_N}^{l'_1 \dots l'_N} \Rightarrow \mathcal{Z} = \text{Tr} \rho_{\text{CTM}} \quad (11)$$

The local height probability (LHP) for the height l_1 at the origin can be written as

$$P_l \equiv \text{Prob}(l_1 = l) = \mathcal{Z}^{-1} \text{Tr}(\delta_{l_1, l} \mathbf{ABCD}) \quad (12)$$

The CTM formalism becomes particularly useful in integrable lattice models, where it becomes possible to fully diagonalize the operator ρ_{CTM} , hence computing the exact spectrum and, thus, its trace. In the RSOS case, the last two sites of every line $m+1, m+2$, determine the boundary conditions and we will take them as fixed and equal among every line. Once in the eigenbasis, the corresponding diagonal operator ρ_{diag} can be decomposed as

$$\rho_{\text{diag}} = \mathbf{RT} \quad (13)$$

where both **R**, **T** are diagonal, but **R** is a weight that depends only on the height at the origin l_1 , while **T** takes into account the configuration on the whole line $\underline{l} = \{l_1, \dots, l_m, l_{m+1}, l_{m+2}\}$ ($m+2 = N$). They can be summarized in the four regimes as follows

Regime	t	$\ln x$	$R_{\underline{l}, \underline{l}}$	$T_{\underline{l}, \underline{l}}$
II	$2-r$	$\frac{4\pi^2}{r \ln p }$	$x^{(2-t)(2l_1-r)^2/16r} E(x^{l_1}, x^r)$	$x^{t\phi[\underline{l}]}$
III	2			
I	$2-r$	$\frac{2\pi^2}{r \ln p }$	$x^{1/4+(2-t)(2l_1-r)^2/8r} E(x^{l_1}, x^{-r/2})$	$x^{t\psi[\underline{l}]}$
IV	2			

where $E(z, x)$ is the elliptic function defined in (A6) and we introduced the *CTM Hamiltonians*:

$$\phi[\underline{l}] = \sum_{j=1}^m j \frac{|l_{j+2} - l_j|}{4}, \quad (14)$$

$$\psi[\underline{l}] = \sum_{j=1}^m j \delta_{l_j, l_{j+2}} \left\{ \delta_{l_{j+1}, l_j+1} \theta\left(l_j - \frac{r}{2}\right) + \delta_{l_{j+1}, l_j-1} \left[1 - \theta\left(l_j - \frac{r}{2}\right)\right] \right\}, \quad (15)$$

where $\theta(x)$ is the step-function with $\theta(x \leq 0) = 0$ and $\theta(x > 0) = 1$.

C. Groundstate structure and critical points

The two functions (14,15), can be considered as Hamiltonians related to the CTM. In fact, they appear as energies for the 1d configurations in the trace sum of (11). We can therefore use such expressions to deduce the form of the groundstate in each regime, as the configurations having the maximum contribution in the trace. It can be shown explicitly that, in the thermodynamic limit, each $2d$ groundstate is invariant under a southwest to northeast translation, i.e. the height on the site (x, y) is the same as that on site $(x+1, y+1)$. It follows that once the configuration is known on the x -axis, it is known everywhere. The determination of the groundstate structure allows to macroscopically characterize the phase diagram: infact, in a disordered phase, the boundary conditions do not affect the behavior of the groundstate in the bulk, while the contrary happens in an ordered phase. Moreover, in the RSOS model, the constraint (10) naturally divides the system into two sub-lattices, one with even heights and one with odd ones. By specifying boundary conditions (at infinity and at the origin, in a consistent way), we

assign a given parity to each sub-lattice. But a translation of the whole system by a lattice site gives an equivalent configuration, with opposite parity. Thus, we can take the central height l_1 and use its parity to classify each ground state out of this trivial Z_2 degeneracy.

In the different regimes we have the following structures⁵³:

- I: There is only one groundstate per each parity of l_1 : $(l_i, l_{i+1}) = (n, n+1)$ and $(l_i, l_{i+1}) = (n+1, n)$ with $n \equiv \frac{r-1}{2}$.
- II: There are $2r-4$ groundstates ($r-2$ for each parity) of the form of ascending and descending sequence from 1 to $r-1$; more explicitly, for any value of $j_0 = 1, \dots, 2r-4$, we can build a groundstate configuration as

$$l_j = 1 + |k - r + 2|$$

where $k = (j - j_0) \bmod 2r - 4$ (thus $0 \leq k < 2r - 4$).

- III: Also in this case we have $2r-4$ groundstates, where all the odd/even sites have the same height: $l_{2i} = X, l_{2i+1} = Y$ with $|X - Y| = 1$.
- IV: As for regime III we have a groundstate for each couple of available nearest-neighbor values except for the regime I groundstate values: thus $2r-4-1-1 = 2r-6$ groundstates.

As we stressed, it is clear that if there is only one groundstate (for each l_1 parity), then we expect the system to be “disordered” and this is true in regime I. Indeed, here the order parameter is independent from the boundary conditions, within a given parity of sub-lattices. When there is more than one ground state (per parity), the system is in an “ordered” phase.

The critical points can be understood and identified with an appropriate conformal point²¹:

- I \leftrightarrow II critical point: the system passes from a disordered to an ordered phase, where p acts like a temperature; the critical point has the conformal structure of parafermion.
- III \leftrightarrow IV critical point: here both phases are ordered and the groundstate degeneracy passes from $r-2$ to $r-3$; the critical point has the conformal structure of $(r-1)$ -unitary minimal model.

D. Thermodynamic limit

It is easier to approach the critical points starting from region III and I, thus, from now on, we will focus just on these regimes. We are interested in the thermodynamic limit of the replicated partition function, introduced in (7). The details of the calculation for $\alpha = 1$ can be found in the original work¹⁹, thus here we can concentrate only on the main points and the few modifications needed. For convenience, we collect some definitions and the relevant properties of elliptic functions in A.

1. Regime III

The finite-size partition function for α replicas is easily obtained from (13), by summing over the value of the central height

$$\mathcal{Z}_\alpha = \sum_{1 \leq a < r} [E(x^a, x^r)]^\alpha X_m(a, b, c; x^{2\alpha}), \quad (16)$$

where we singled out the boundary conditions as $a = l_1, b = l_{m+1}, c = l_{m+2}$ and

$$X_m(a, b, c; q) \equiv \sum_{l_2, \dots, l_m} q^{\phi[l]}.$$

The thermodynamic limit can be computed exactly once this expression is rewritten in terms of gaussian polynomials¹⁹, resulting in

$$X(a, b, c; q) \equiv \lim_{m \rightarrow \infty} X_m(a, b, c; q) = (q)_\infty^{-1} q^{bc/4} \Gamma\left(a, \frac{b+c-1}{2}; q\right), \quad (17)$$

where the q -Pochhammer symbol $(q)_\infty$ is defined in (A3) and

$$\Gamma(a, d; q) \equiv q^{\frac{a(a-1)}{4}} \left\{ q^{-\frac{ad}{2}} E[-q^{(r-a)(r-1)+rd}, q^{2r(r-1)}] - q^{\frac{ad}{2}} E[-q^{(r+a)(r-1)+rd}, q^{2r(r-1)}] \right\}. \quad (18)$$

The partition function can thus be written as

$$\begin{aligned}\mathcal{Z}_\alpha &= \lim_{m \rightarrow \infty} \sum_{1 \leq a < r} [E(x^a, x^r)]^\alpha X_m(a, b, c; x^{2\alpha}) \\ &= x^{\frac{\alpha bc}{2}} (x^{2\alpha})_\infty^{-1} \sum_{1 \leq a < r} [E(x^a, x^r)]^\alpha \Gamma\left(a, \frac{b+c-1}{2}; x^{2\alpha}\right).\end{aligned}\quad (19)$$

2. Regime I

In this regime, using the table in section II B, the finite-size, α -replicated partition function is given by

$$\mathcal{Z}_\alpha = \sum_{1 \leq a < r} x^{a\alpha(1+a-r)/2} \left[E(x^a, -x^{r/2}) \right]^\alpha Y_m(a, b, c; x^{\alpha(r-2)}) \quad (20)$$

where we introduced

$$Y_m(a, b, c; q) \equiv \sum_{l_2, \dots, l_m} q^{\psi(l)} \quad (21)$$

As before the thermodynamic limit is computed taking the limit $m \rightarrow \infty$ in (21). Unlike the regime III, here the sum is not convergent, due to the non-zero energy density of the groundstate. Thus, we factor out the diverging contribution (which amounts to an irrelevant redefinition of the partition function normalization), obtaining

$$\lim_{m \rightarrow \infty} q^{-m} Y_m(a, b, c; q) = (q)_\infty^{-1} f_{b,c}(q) E(q^a, q^r), \quad (22)$$

where the boundary conditions enter only in the function $f_{b,c}(q)$, defined as

$$f_{b,b+1}(q) = \begin{cases} \varepsilon_b(q) & 1 \leq b < n, \\ 1 & n \leq b \leq r-2 \end{cases} \quad (23)$$

$$f_{b,b-1}(q) = \begin{cases} 1 & 2 \leq b \leq n+1 \\ \varepsilon_{r-b}(q) & n+1 < b \leq r-1 \end{cases} \quad (24)$$

where n is the integer part of $r/2$ and we defined

$$\varepsilon_b(q) \equiv \frac{q^{1-b}(1-q^b)}{1-q}.$$

III. ENTROPY AND PARTITION FUNCTION: REGIME III

Now that we have introduced the model and the replicated partition functions, it is straightforward to proceed with the calculation of the Renyi entropy, using (8). However, before we take on the full computation, following¹¹, we would like to exploit the known relation between the partition function with fixed boundary conditions (both at infinity and at the origin) and the characters of primary fields in minimal models³¹. This link will drive the expansion of the entropy around the critical point, as we will show in section III B. Let us discuss this point in some detail.

A. Characters of the minimal models

It was noticed in³²⁻³⁶ that the quantity in (17) can be identified with a minimal model character. This can be shown by simply rewriting (17), using the sum expansion in (A6), yielding

$$X(a, b, c; q) = q^{\frac{1}{4}(a-d)(a-d-1) + \frac{c_r}{24} - \Delta_{d,a}} \chi_{d,a}^{(r-1)}(q), \quad (25)$$

which is the character in the minimal model $(r, r-1)$ of the primary with conformal dimension

$$\Delta_{d,a} = \frac{[dr - a(r-1)]^2 - 1}{4r(r-1)}, \quad (26)$$

and central charge

$$c_r = 1 - \frac{6}{r(r-1)}. \quad (27)$$

Here the boundary conditions at infinity are accounted just by the combination

$$d \equiv \frac{b+c-1}{2},$$

Therefore, $X(a, b, c; q)$ is equivalent to a generating function of the Verma module degeneracy for the representation fixed by the boundary conditions a, b, c . We stress here that this equality is valid only at a formal level: indeed, in the one-dimensional configurational sums, the elliptic nome q is a measure of the departure from criticality while in the conformal characters q is the modular nome related to the geometry on the torus at criticality. Even though they are both usually denoted by q , these are two very different objects.

Similarly, one should not confuse the formal identification of fields in the Renyi entropy expansion, with the operator actually responsible for opening the gap. We recall that regime III can be described as (6), that is as a lattice deformation of a minimal model by means of a perturbation given by $\epsilon(x) \simeq \phi_{1,3}(x)$, which is known to be both thermal and integrable³⁷. As we shall see, this operator does not appear among the most relevant ones in the Renyi expansion in regime III.

B. Fixed central height

We can now compute the Renyi entropy in a sector where the height at the origin is kept fixed. As already pointed out in¹¹, we stress that this is done at the level of the corner transfer matrix, and so, of the reduced density matrix: if such degree of freedom was fixed at the level of the Hamiltonian, it would indeed affect the interaction between the two parts of the system. Rather, fixing the height at the origin of the CTM corresponds to selecting a sector out of the whole Hilbert space of the model, and taking the groundstate within this projection. Thus, we are measuring the entanglement of the lowest energy state within this subspace. In general, these states will be a superposition of high energy states and thus the calculation of their Renyi entropy can shed some light on their properties.

In approaching the gapless point, the elliptic nome $q = x^{2\alpha}$, in (19), tends to unity. As this is not the best parameterization to extract the leading contributions, we perform the customary dual transformation of the elliptic nome, granting us an expansion in the original parameter p , which tends to zero at criticality. Using (A8) and (A10) for one term in the sum of Eq. (16), we obtain

$$\mathcal{Z}_\alpha^{(a)} \equiv \frac{\theta_3\left(\frac{\pi d}{2r-2} - \frac{\pi a}{2r}, p^{\frac{1}{8\alpha(r-1)}}\right) - \theta_3\left(\frac{\pi d}{2r-2} + \frac{\pi a}{2r}, p^{\frac{1}{8\alpha(r-1)}}\right)}{\theta_4\left(\frac{-ir \ln p}{8\alpha}, p^{\frac{3r}{4\alpha}}\right) p^{\frac{r}{48\alpha}} \sqrt{2r(r-1)}}, \quad (28)$$

where the index a refers to the fixed value of the height at the origin. As stated in¹¹, the correspondence with a conformal character, allows to reinterpret this duality, at the very end grounded on the Poisson resummation formula, as the invariance of the torus under the modular group. In a CFT, every character can then be expressed as a linear combinations of characters of the dual theory³¹

$$\chi_{t,s}^{(r-1)}(\tilde{q} \equiv e^{-i\pi/\tau}) = \sum_{t',s'} S_{t,s}^{t',s'} \chi_{t',s'}^{(r-1)}(q = e^{i\pi\tau}), \quad (29)$$

where

$$S_{t,s}^{t',s'} = 2\sqrt{\frac{2}{r(r-1)}} (-1)^{(t+s)(t'+s')} \sin\left(\pi \frac{tt'}{r-1}\right) \sin\left(\pi \frac{ss'}{r}\right) \quad (30)$$

is the so-called *modular matrix*.

To reproduce this result in our setting, we can expand (28) using (A1) and

$$\ln(q)_\infty^{-1} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{nk}}{k} = \sum_{n=1}^{\infty} \sigma_{-1}(n) q^n = q + \frac{3}{2}q^2 + \frac{4}{3}q^3 + \frac{7}{4}q^4 + \mathcal{O}(q^{10}), \quad (31)$$

where $\sigma_\kappa(n)$ is the sum of the κ -th powers of the divisors of n

$$\sigma_\kappa(n) \equiv \sum_{\substack{j < i=1 \\ j \cdot i = n}}^{\infty} (j^\kappa + i^\kappa) + \sum_{\substack{j=1 \\ j^2 = n}}^{\infty} i^\kappa. \quad (32)$$

To compare the expansion of (28) and (29), it is convenient to use the parameter truly dual to the one used in (25), that is

$$\tilde{q} \equiv p^{\frac{r}{2}} = p^{2\nu} \quad (33)$$

where $\nu = (2 - 2\Delta_{1,3})^{-1} = r/4$ is the correlation length critical exponent in Regime III³⁸. Collecting everything we obtain

$$\ln \mathcal{Z}_\alpha^{(a)} = -\frac{c_r}{24\alpha} \ln \tilde{q} + C'_{adr} + 4\gamma_{adr} \tilde{q}^{\frac{3}{4\alpha r(r-1)}} - 8\gamma_{adr}^2 \tilde{q}^{\frac{3}{2\alpha r(r-1)}} + \mathcal{O}\left(\tilde{q}^{\frac{2}{\alpha r(r-1)}}\right), \quad (34)$$

where

$$\gamma_{adr} \equiv \cos\left(\frac{\pi d}{r-1}\right) \cos\left(\frac{\pi a}{r}\right) \quad (35)$$

and

$$C'_{adr} \equiv \ln\left(\frac{4}{\sqrt{2r(r-1)}} \sin\frac{\pi d}{r-1} \sin\frac{\pi a}{r}\right). \quad (36)$$

is the zeroth-order correction, corresponding to the boundary entropy of³⁹.

Using (8), we can obtain the expansion for the Renyi entropy in \tilde{q} , while still keeping the central height fixed:

$$\begin{aligned} \mathcal{S}_\alpha^{(a)} = & -\frac{c_r}{24} \left(1 + \frac{1}{\alpha}\right) \ln \tilde{q} + C'_{adr} + \frac{4\gamma_{adr}}{1-\alpha} \left(\tilde{q}^{\frac{3}{4\alpha r(r-1)}} - \alpha \tilde{q}^{\frac{3}{4r(r-1)}}\right) \\ & - \frac{8\gamma_{adr}^2}{1-\alpha} \left(\tilde{q}^{\frac{3}{2\alpha r(r-1)}} - \alpha \tilde{q}^{\frac{3}{2r(r-1)}}\right) + \mathcal{O}\left(\tilde{q}^{\frac{2}{r(r-1)}}\right). \end{aligned} \quad (37)$$

It is well known⁹ that one can read off the central charge of the model from the coefficient of leading term of the entropy, as in (37). Let us remark, however, that it is a pleasant check to notice that the standard conformal result, obtained using the replica trick, remains valid also for the minimal models, where the twist operator introduced in the computation does not belong to the Kac table of the CFT.

As suggested in¹¹, the sub-leading corrections contain information on the operatorial content of the theory and their characters. In fact, from (37) and comparing (36) with (30) we recognize, consistently with¹¹,

$$C'_{adr} = \ln S_{1,1}^{d,a}. \quad (38)$$

Indeed, the zero-order term is related to the modular matrix between the primary field chosen by the boundary condition and the identity, which is giving the dominant contribution. The first correction in (37) is coming, as expected, from the most relevant field. Indeed, we see that:

$$\gamma_{adr} = \frac{S_{2,2}^{d,a}}{4S_{1,1}^{d,a}}$$

and, coherently, from (26) we recognize that the exponent of the correction is $\Delta_{2,2} = \frac{3}{4r(r-1)}$. The identification with the operators of a Virasoro algebra can continue to higher orders, but one should notice that the expansion of the logarithm generates additional terms which do not appear in the Kac table, such as the second sub-dominant correction in (37), which is just a $2\Delta_{2,2}$. This correction is always dominant over the $\Delta_{3,3} = \frac{2}{r(r-1)}$.

It should be noted here, that the parameter \tilde{q} is microscopical in nature and the entropy is usually measured as a function of a thermodynamical parameter, such as the correlation length ξ . From⁴⁰ we know that

$$\xi = -\frac{1}{\ln k'(|p|^\nu)} = -\frac{1}{\ln k'(|\tilde{q}|^{\frac{1}{2}})}, \quad (39)$$

where

$$k'(q) = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^4 = \frac{\theta_4^2(0)}{\theta_3^2(0)}, \quad (40)$$

From these expressions we get the expansion:

$$\tilde{q} = \frac{1}{64\xi^2} - \frac{1}{1536\xi^4} + \frac{113}{2949120\xi^6} + \mathcal{O}(\xi^{-8}), \quad (41)$$

which should be substituted order by order in (37). At the leading order, this substitution correctly fixes the usual normalization in front of the leading logarithm in terms of the central charge c_r and the exponents of the corrections as $h = 2\Delta$ in (5). The rest of the terms, however, which strictly vanish in the scaling limit, spoil the possibility of reading and reconstructing the operator content of the characters appearing in the entropy in any study at finite lattice spacing. This effect is completely analogous to the one discussed in¹³ for the XYZ chain.

C. Full entropy

To calculate the bipartite Renyi entropy of the model in its true ground state we should sum over the central height. Using the dual transformation in the full partition function (16), we have

$$\mathcal{Z}_\alpha = \sum_{a=1}^{r-1} \theta_1^\alpha \left(\frac{a\pi}{r}, \sqrt{p} \right) \mathcal{Z}_\alpha^{(a)}, \quad (42)$$

where $\mathcal{Z}_\alpha^{(a)}$ is given by (28). We remark that, while $\mathcal{Z}_\alpha^{(a)}$ has a simple interpretation in terms of a character, the coefficients in the sum over the central height in (42) do not. In the previous section, since a was kept fixed, the value of the coefficient could be absorbed in the normalization of the partition function, but now we cannot ignore these contributions anymore. The infinite sum representation of θ_1 in (A1b) allows an exact computation of this expression, at least for small integer values of $\alpha = n^{54}$:

$$\begin{aligned} \ln \mathcal{Z}_n = & -\frac{r}{48n} c_r \ln p + \frac{n}{8} \ln p + \ln \frac{2^{2+n}}{\sqrt{2r(r-1)}} - \sum_{j=1}^{\infty} \ln \left(1 - p^{\frac{rj}{2n}} \right) \\ & + \ln \sum_{j=1}^{\infty} p^{\frac{j^2-1}{8n(r-1)}} \sin \left(\frac{\pi dj}{r-1} \right) \sum_{a=1}^{r-1} \sin \frac{\pi a j}{r} \left[\sum_{k=0}^{\infty} (-1)^k p^{\frac{k(k+1)}{2}} \sin \left((2k+1) \frac{a\pi}{r} \right) \right]^n. \end{aligned} \quad (43)$$

Take, for instance, $\alpha = 1$: the sum over a can be computed immediately using the orthogonality condition

$$\sum_{a=1}^{r-1} \sin \frac{\pi a n}{r} \sin \frac{a m \pi}{r} = \frac{r}{2} \delta_{n,m} \quad (44)$$

recovering the partition function of the RSOS model with fixed boundary conditions at infinity¹⁹ already in the dual formulation, that once expressed in $\tilde{q} = p^{\frac{r}{2}}$ gives

$$\mathcal{Z}_1 = \sqrt{\frac{2r}{r-1}} \frac{\theta_1 \left(\frac{\pi d}{r-1}, \tilde{q}^{\frac{1}{(r-1)}} \right)}{\tilde{q}^{\frac{1}{24}} \prod_{j=1}^{\infty} (1 - \tilde{q}^j)}, \quad (45)$$

For general values of α , at our knowledge the sum in (42) can not be computed analytically. However, it is possible to obtain its expansion order by order close to the critical point introducing the coefficients

$$s_\alpha(n, k) \equiv \sum_{a=1}^{r-1} \sin^\alpha \frac{\pi a}{r} \sin \frac{\pi a n}{r} \cos^k \frac{2\pi a}{r},$$

The first few terms give

$$\begin{aligned} \mathcal{S}_\alpha = & \frac{c_r}{24} \frac{1+\alpha}{\alpha} \ln \tilde{q} + \ln \left[\sqrt{\frac{2r}{r-1}} \sin \frac{\pi d}{r-1} \right] + \frac{1}{1-\alpha} \ln \left[\frac{2}{r} s_\alpha(1, 0) \right] \\ & - \frac{1}{1-\alpha} \frac{s_\alpha(3, 0)}{s_\alpha(1, 0)} \left(4 \cos^2 \frac{\pi d}{r-1} - 1 \right) \tilde{q}^{\frac{2}{\alpha r(r-1)}} + \mathcal{O} \left(\tilde{q}^{\frac{4}{\alpha r(r-1)}} \right), \end{aligned} \quad (46)$$

Some observations about this expression are in order

- it remains finite as it should, in the $\alpha \rightarrow 1$ limit due to the properties of $s_{\alpha \rightarrow 1}(n, k)$: e.g. $s_1(3, 0) = 0$, $s_1(1, 0) = \frac{r}{2}$; note that this is a different mechanism w.r.t. (37), where terms with and without α at the exponent appear in pairs and together render the Von Neumann limit finite;
- the leading term remains the same as (37) being dictated by the CFT central charge;
- since $s_\alpha(2n, k) = 0$ for all integers n, k , every correction coming from the operator $\Delta_{2,2}$ and its descendants disappear and the first sub-leading term is now related to the primary field of dimension $\Delta_{3,3}$. We interpret this cancellation as due to the \mathbb{Z}_2 symmetry

$$l \rightarrow r - l$$

under which the full partition function (42) is invariant, while the most relevant field, being identifiable with the order parameter⁴¹, is indeed odd. Of course, this implies that all odd-operators identically vanish in the expansion of the entropy. In any case, as we already pointed out, these corrections are not directly ascribable to the operator opening the gap in (6), since in general $\Delta_{1,3} \neq \Delta_{2,2}, \Delta_{3,3}$. This will not be the case in regime I: as we are going to show in the next section, in the disordered phase the leading correction seems to be given by the same operator opening the gap.

IV. REGIME I

We can now turn back to regime I and its bipartite entanglement entropy. As we saw in section II C, this regime corresponds to a disordered phase, where local expectation values are independent from the boundary conditions. Indeed, the structure of the entanglement entropy is different from before. As can be seen from (20, 22), the contribution to the partition functions of boundary conditions at infinity factorizes out in the term $f_{b,c}(q)$. However, this contribution does not cancel out in the entanglement entropy (8) and can bring a finite and interesting contribution. In the analysis, we will separate the bulk and boundary contribution and consider them separately:

$$S_\alpha = S_\alpha^{(\text{bulk})} + S_\alpha^{(bc)} . \quad (47)$$

Moreover, looking at (22), we notice that, due to the \mathbb{Z}_2 symmetry, the fixed central height partitions function for a and $r - a$ are equal and additional relations can be established for certain values of r and a for their coefficients in (20).

In approaching the transition toward regime IV the system undergoes a second order phase transition described by the parafermionic conformal field theory²³. Here we summarize the main features of these conformal points.

A. Conformal content of parafermions

The critical point can be described as the coset

$$\frac{\hat{sl}(2)_{r-2}}{\hat{u}(1)} ,$$

with central charge

$$c_r^{\text{pf}} = \frac{2(r-3)}{r} . \quad (48)$$

Beyond the conformal one, these theories enjoy an additional $\mathbb{Z}_{r-2} \times \tilde{\mathbb{Z}}_{r-2}$ symmetry (which is actually enlarged to a W_{r-2}). This structure allows to reduce the number of allowed anomalous dimensions to a finite set, even for $r > 6$, i.e. $c > 1$. These dimensions are determined by the charges (Q, \tilde{Q}) under the two \mathbb{Z}_{r-2} symmetries, since each of them is defined modulus $r - 2$. Following⁴², we introduce the two indexes

$$\begin{aligned} l &= Q - \tilde{Q} , \\ m &= Q + \tilde{Q} , \end{aligned} \quad (49)$$

in terms of which the conformal dimension of the most relevant field in each sector can be parameterized as

$$\Delta_{l,m}^{\text{pf}} = \frac{l(l+2)}{4r} - \frac{m^2}{4(r-2)} , \quad \begin{aligned} 0 \leq l \leq r-2 , \\ 0 \leq |m| \leq l , \\ l - m = 0 \bmod 2 . \end{aligned} \quad (50)$$

As a matter of fact, each combination $(\Delta_{l,m}^{\text{pf}}, \bar{\Delta}_{l',m'}^{\text{pf}})$ of dimensions for the holomorphic and anti-holomorphic part can correspond to more than one primary field. To resolve this degeneracy, one needs to look into their representation under the W -algebra⁴³. In particular, within the sector neutral under the two \mathbb{Z}_{r-2} 's, i.e. with $(Q, \tilde{Q}) = (0, 0)$, we have the following allowed dimensions

$$\Delta_k^{(\epsilon)} = \frac{k(k+1)}{r}. \quad (51)$$

These fields ϵ_k , often called *energy* or *thermal fields*, are spin-less, that is $\Delta_k^{(\epsilon)} = \bar{\Delta}_k^{(\epsilon)}$ and the identity is ϵ_0 . We recognize that they are degenerate with the parafermionic operators with $(l, m) = (2k, 0)$, i.e. $\Delta_{l=2k,0}^{\text{pf}} = \Delta_k^{(\epsilon)}$.

In regime III we showed the exact mapping existing between the partition function at fixed boundary conditions and a conformal character, see (25). A similar relation can be established in regime I as well, but it is less explicit since the mapping is no more one-to-one: we refer to⁴² for the precise construction.

In passing, let us point out that the transition between regime I and II can be described as (6) where the gap-opening perturbation is due to the most relevant thermal field $\epsilon_1(x)$.

B. Fixed central height

In order to extract the behavior around criticality, we express each term of the sum in (20) using the parameter p , through a duality transformation, as we did in (28) and (42). We recall that in this regime $-1 < p < 0$. However, following¹⁹, the formulae for the partition function and the entropy are to be understood as depending only on the absolute value of p . Thus, in the following we will intend the substitution

$$p \longrightarrow |p| = -p.$$

The partition function at fixed height at the origin can be written as

$$\mathcal{Z}_\alpha^{(a)} = \frac{\theta_1\left(\frac{a\pi}{r}, p^{\frac{1}{\alpha(r-2)}}\right)}{\theta_4\left(\frac{ir \log p}{2\alpha(r-2)}, p^{\frac{3r}{\alpha(r-2)}}\right) p^{\frac{r}{12\alpha(r-2)}} \sqrt{r}}, \quad (52)$$

where, for the moment, we dropped the term $f_{bc}(q)$, as discussed.

The computation of the Renyi entropy is quite similar to what we did in section III B. Introducing again \tilde{q} following (33), with $\nu^{\text{pf}} = \left(2 - 2\Delta_1^{(\epsilon)}\right)^{-1} = \frac{r}{2(r-2)}$ in this regime, we obtain

$$S_\alpha^{(a)} = -\frac{c_r^{\text{pf}}}{24} \left(1 + \frac{1}{\alpha}\right) \log \tilde{q} + C'_a - \frac{\gamma_a}{1-\alpha} \left(\tilde{q}^{\frac{2}{\alpha r}} - \alpha \tilde{q}^{\frac{2}{r}}\right), \quad (53)$$

where we defined

$$\gamma_a = 1 + \cos \frac{2\pi a}{r}, \quad (54)$$

$$C'_a = \log \left(\frac{2 \sin \frac{\pi a}{r}}{\sqrt{r}} \right). \quad (55)$$

We recognize that, regardless of the boundary conditions, the exponent of the leading correction corresponds to $\Delta_{2,0}^{\text{pf}}$ in (50), which is not the smallest one. Since this phase is disordered and we do not break it explicitly in the computation of the entanglement entropy, we expect only neutral fields under the \mathbb{Z}_{r-2} symmetries to enter in (53). Thus, we find it natural to interpret the leading correction in (53) as due to the most relevant thermal operator ϵ_1 , see (51), which, coincidentally, is also the gap-opening operator. This interpretation is further corroborated by the observation that, expressing (53) in terms of the correlation length $\xi \simeq \tilde{q}^{-1/2}$, the dimension of the leading correction becomes $2\Delta_1^{(\epsilon)}$, which seems to be due to a spin-less operator. Moreover, as we observed in the introduction of section IV, by changing a we can generate only $[r/2]$ independent combinations of primary fields and their characters, which coincides with the number of allowed thermal operators in (51).

C. Full entropy

We consider the full partition function, that is obtained with the dual transformation of the full sum in (20). One gets

$$\mathcal{Z}_\alpha = \sum_{a=1}^{r-1} \left[e^{-\frac{i\pi}{8}} \theta_1 \left(\frac{a\pi}{r}, i\sqrt{p} \right) \right]^\alpha \mathcal{Z}_\alpha^{(a)}. \quad (56)$$

Also in this case, the series expansion of the θ_1 function (A1) is useful for integer $\alpha = n$

$$\begin{aligned} \ln \mathcal{Z}_n = & -\frac{\nu^{\text{pf}}}{12n} c_r^{\text{pf}} \ln p + \frac{n}{8} \ln p + \ln \frac{2^{1+n}}{\sqrt{r}} - \sum_{j=1}^{\infty} \ln \left(1 - p^{\frac{2rj}{(r-2)n}} \right) \\ & + \ln \sum_{j=0}^{\infty} p^{\frac{j(j+1)}{n(r-2)}} (-1)^j \sum_{a=1}^{r-1} \sin \frac{\pi a(2j+1)}{r} \left[\sum_{k=0}^{\infty} (-1)^k (-p)^{\frac{k(k+1)}{2}} \sin \left((2k+1) \frac{a\pi}{r} \right) \right]^n, \end{aligned}$$

and again the partition function can be reproduced exactly with (44)

$$\mathcal{Z}_1 = \sqrt{r} \frac{e^{-\frac{i\pi}{8}} \theta_2(0, i\sqrt{q})}{\tilde{q}^{\frac{1}{12}} \prod_{j=1}^{\infty} (1 - \tilde{q}^{2j})}. \quad (57)$$

For general α we can expand the Renyi entropy at desired order

$$\begin{aligned} S_\alpha^{(\text{bulk})} = & -\frac{c_r^{\text{pf}}}{24} \left(1 + \frac{1}{\alpha} \right) \log \tilde{q} + \frac{\log r}{2} + \frac{1}{1-\alpha} \log \frac{2s_\alpha(1,0)}{r} \\ & - \frac{1}{1-\alpha} \frac{s_\alpha(3,0)}{s_\alpha(1,0)} \tilde{q}^{\frac{2}{r\alpha}} + \mathcal{O} \left(p^{\frac{4}{(r-2)\alpha}} \right), \end{aligned} \quad (58)$$

where we see that the leading correction comes from the same $\Delta_1^{(\epsilon)}$ operator as in (53).

Thus, we see that, unlike for regime III, in the disordered phase the leading correction is less sensitive to the boundary condition at the origin and coincide with the scaling dimension of the gap-opening field.

D. The boundary contribution

Now we turn to the term related to the boundary heights $f_{bc}(q)$: from its definition in (23), we see that for the set of values of b, c that makes it non-trivial, it gives rise to a peculiar set of terms appearing in the expansion of the Renyi entropy

$$S_\alpha^{(bc)} = \frac{1}{1-\alpha} \ln \left[\frac{\epsilon_b(x^{\alpha(r-2)})}{\epsilon_b(x^{r-2})^\alpha} \right] = \ln b + \frac{(b^2-1)\pi^4\alpha}{6(\ln \tilde{q})^2} + \mathcal{O} \left(\frac{1}{\ln \tilde{q}} \right)^4. \quad (59)$$

We see that the boundary contribution modifies the constant term (boundary term³⁹) and generates sub-leading logarithmic corrections. This result may appear surprising: when local quantities are computed as in (12), such term cancels out, as expected since we are in a disordered phase. But in the Renyi entropies, being a highly non-local object, also the boundary appears. Of course, in considering the entropy of an actual state, one might need to sum over different boundary conditions, possibly with different weights, and the boundary contribution to the entropy might change significantly. Thus, it might be pointless to try to provide a CFT interpretation of (59), as these boundary conditions might not have any conformal counterpart. However, it should be noted that, once expressed in terms of the correlation length, the logarithmic corrections in (59) have the same form as those predicted in a CFT with a (bulk) marginal field^{10,29}. This is peculiar, since \mathbb{Z}_{r-2} parafermions present such a marginal field only for certain given values of r . One can check that a massless flow in the theory, essentially due to a free boson, is present, for example, for the series

$$r = m(m+1) \quad \Rightarrow \quad c_r^{\text{pf}} = 1 + \left(1 - \frac{6}{m(m+1)} \right) \quad (60)$$

where the central charge takes the form of a free boson plus a unitary minimal model. Since, the logarithmic corrections in the entanglement entropy (59) typically appear for every $r > 5$, as we will see in the next section, these terms must have a different origin.

V. SOME EXAMPLES

To better elucidate our results, it is instructive to specialize and consider two particular examples: the Ising model ($c = 1/2$) and the 3-state Potts model ($c = 4/5$). In fact, within the RSOS, we have two possible realizations of these models: one in Regime III (respectively $r = 4$ and $r = 6$) as unitary minimal models and one in Regime I ($r = 4$ and $r = 5$) as parafermions. The comparison between the two realizations of the same theory can shed some light on the two phases and the nature of the corrections.

A. Ising model

The Ising model is arguably the simplest CFT, since it consists of only three operators: $1, \sigma, \epsilon$. Characters of the Ising model are known to arise in the study of the CTM's of eight-vertex models⁴⁴, which has the same Yang-Baxter algebra as the RSOS. Depending on the choice of parameters, the 1-D quantum system corresponding to the eight-vertex model is either an anisotropic XY model in a transverse magnetic field, or an XYZ chain in zero field. The entanglement entropies of both models have been calculated analytically in the thermodynamic, bipartite limit we are also considering here. The first has both an ordered and a disordered phase^{45–47}, while the XYZ model presents only the ordered phase^{12,13,27}. In the direct parameter x of section II B, the generalized partition function in the ordered phase has been observed to be proportional to the character of the spin operator, while in the disordered it is a combination of the identity and energy:

$$\mathcal{Z}_\alpha^{\text{Ord}} \propto \prod_{n=1}^{\infty} (1 + x^{2n}) = 2 x^{-1/12} \chi_\sigma(x^2), \quad (61)$$

$$\mathcal{Z}_\alpha^{\text{Dis}} \propto \prod_{n=1}^{\infty} (1 + x^{2n-1}) = x^{1/24} [\chi_1(x^2) + \chi_\epsilon(x^2)]. \quad (62)$$

The Kac table for the Ising minimal and parafermionic model can be summarized as

$$\begin{cases} \Delta_1 \equiv \Delta_{1,1} = \Delta_{0,0}^{\text{pf}} = 1, \\ \Delta_\sigma \equiv \Delta_{2,2} = \Delta_{1,1}^{\text{pf}} = \frac{1}{16}, \\ \Delta_\epsilon \equiv \Delta_{1,3} = \Delta_{2,0}^{\text{pf}} = \frac{1}{2}. \end{cases} \quad (63)$$

In regime III, as explained in section III A, tuning the boundary conditions at the origin and at infinity, we generate each individual character. The entropy at fixed origin height then reflects the operator content of the theory under the duality transformation. Thus, looking at the modular matrix (30), we see that if we start with the identity or the energy field, the first correction to the entropy (37) comes from the most relevant operator, i.e. the spin operator σ . However, if the height at the origin is set to $a = 2$, the coefficients of the σ contributions vanishes (as the modular matrix has zero element for the (σ, σ) entry) and the most relevant correction is given by the energy, as in^{13,47}.

As we explained in section III C for the general case, the $\Delta_{2,2}$ field is odd under \mathbb{Z}_2 and therefore disappears in the full entropy (46), and only the energy and identity appear. Of course, the field $\Delta_{3,3}$, which in general would give most relevant correction in (46), does not appear in the Kac table of the Ising model and indeed its coefficients are vanishing.

In regime I, things are a bit different. By direct inspection of (22), we see that for $a = 2$ (62) is realized and both $a = 1$ and $a = 3$ give (61). Thus, as we conjectured at the end of section IV B, not every combination of operators and their characters appear. In this case it seems that fields with the same parity under \mathbb{Z}_2 appear together. It is then straightforward to see that after the modular transformation the character of the spin operator is never generated and the leading correction to the entropy is always given by the energy ϵ , both when fixing the central height at any value as in (53) and by summing over it (58). Indeed, the identity and the energy are the fields in (51).

Moreover, it is easy to check that the entropy contribution (59) due to the boundary condition at infinity is always vanishing since for all the allowed values of b, c in (23, 24) we have $f_{bc}(q) = 1$.

As a final remark, we notice that for the Ising model, through the identities collected in the appendix and some manipulations, the partition functions (19, 20) can be written in a relatively explicit way. One simplification arises because, due to the \mathbb{Z}_2 symmetry, it is sufficient to fix the boundary conditions to $b = 1, c = 2$. For regime III we have

$$\begin{aligned} \mathcal{Z}_\alpha^{(\text{III})} &= x^{\frac{\alpha}{24}} \left[\chi_1(x^{2\alpha}) + \chi_\epsilon(x^{2\alpha}) \right] \prod_{n=1}^{\infty} (1 - x^n)^\alpha (1 + x^{2n})^\alpha \\ &\quad + x^{-\frac{\alpha}{12}} \chi_\sigma(x^{2\alpha}) \prod_{n=1}^{\infty} (1 - x^n)^\alpha (1 + x^{2n-1})^\alpha, \end{aligned} \quad (64)$$

and for regime I

$$\begin{aligned} \mathcal{Z}_\alpha^{(1)} = & x^{\frac{1}{12}\alpha} \left[\chi_1(x^{4\alpha}) + \chi_\epsilon(x^{4\alpha}) \right] x^{-\alpha} \prod_{n=1}^{\infty} (1 - x^{2n})^\alpha (1 + x^{4n})^\alpha \\ & + x^{-\frac{\alpha}{6}} \chi_\sigma(x^{4\alpha}) [1 + (-1)^\alpha] x^{-\alpha} \prod_{n=1}^{\infty} (1 - x^{2n})^\alpha (1 + x^{4n-2})^\alpha. \end{aligned} \quad (65)$$

In regime I, the coefficients of the $a = 1$ and $a = 3$ terms are equal and opposite and we see that for $\alpha = 2m - 1$ the partition function is simply proportional to the one found in the disordered phase of the XY model (62), consistently with the fact that this regime is also disordered. We also notice that for $\alpha = 1$, the coefficients in (64, 65) have the same form as (61, 62). Thus, the partition function can be formally written as a bilinear in the characters of the model. This reminds us of what observed in¹³ and we take it as further indication that the character structure of the CTM in integrable models is mostly due to the analytical structure that permeates this beautiful construction, and not on some underlying Virasoro algebra.

B. 3-state Potts model

The operator content of the minimal model in this case is given by:

$$\Delta = \left\{ 0, \frac{1}{40}, \frac{1}{15}, \frac{1}{8}, \frac{2}{5}, \frac{21}{40}, \frac{2}{3}, \frac{7}{5}, \frac{13}{8}, 3 \right\}. \quad (66)$$

The most relevant is $\Delta_{2,2} = \frac{1}{40}$, which is the one appearing in (37), except for $a = 3$. As usual, this field cannot enter in the full entropy obtained summing over the central height, and the leading correction comes from the next relevant operator with $\Delta_{3,3} = \frac{1}{15}$.

In the parafermionic realization, we only have four allowed conformal anomalies

$$\Delta^{\text{pf}} = \left\{ 0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3} \right\}. \quad (67)$$

The leading correction in the entropies (53, 58) is coming from $\Delta_1^{(\epsilon)} = \Delta_{2,0}^{\text{pf}} = \frac{2}{5}$, as we found in (53). We notice that in this case, as it was for the Ising model, only two thermal operators (51) exist ($k = 0, 1$) and the boundary corrections again disappears for any allowed choice of b, c .

We see that, contrary to the Ising example, here the two different (minimal ordered and parafermionic disordered) realizations of the 3-state Potts model have different corrections in the entanglement entropy.

The CTM's spectra of the 3- and 5-state Potts model have been calculated numerically in⁴⁸ with a DMRG approach and an impressive agreement with the analytical expectations was found, also in the presence of integrability breaking terms, sufficiently close to criticality. This indicates that our results for the entanglement entropy should also remain valid under the same conditions.

It is worth to recall here that this parafermionic realization of the 3-state Potts model appears as the ferromagnetic phase in the Fibonacci chain⁴⁹. As stressed in²⁵, due to the topological symmetry present in the quantum realization, all the relevant perturbations are forced to vanish, and the critical point is topologically protected. It means that, in the RSOS, the topological symmetry is restored only at the gapless points. Therefore it would be interesting to compare our predictions with the numerical data for the entanglement entropy coming from the anyonic chain, once a perturbation breaking the topological charge is turned on.

VI. CONCLUSIONS

Following the suggestion put forward in¹¹, we show how to calculate the bipartite Renyi entropy in the thermodynamic limit of a set of models known as RSOS. The method we employed is quite general and powerful and requires just the knowledge of the structure of the Corner Transfer Matrix eigenvalues of the system under consideration. In our case, the model being exactly solvable, the CTM spectrum is fully known analytically, thanks to¹⁹. However, generic systems close to criticality are expected to organize their CTM eigenvalues according to the CFT reached at criticality, as seen, for instance, in⁴⁸. If one was able to determine the coefficients in the expansion of the CTM in terms of characters of the CFT, the approach we used in this work would apply directly.

Beside its feasibility, this study of the RSOS was motivated by the fact that this model provides a lattice realization of all minimal and parafermionic conformal models. It is remarkable that a single system can realize such a variety of phase transitions

and thus its entropy provides a unique case study for the approach to criticality in $1 + 1$ dimensions. The different CFT's are realized by varying an integer parameter r , while the continuous parameter p (or its dual x) measures the departure from criticality. Furthermore, the boundary conditions play an important role in fixing the phase under consideration.

We were thus able to compute the dependence of the Renyi entropy on p and to study its behavior. The expansion of the entropy in regime III is given by (37), if we project the Hilbert space on a subset specified by fixing the central height in the CTM, and by (46) for the absolute ground state. In regime I, we have (53) for the projected case and (58) for the whole case, with the addition of the boundary term (59), when present.

Our results confirm the expectation in (5), according to which, approaching the critical points, we have a leading logarithmic term with a universal prefactor (set by the conformal anomaly), a non universal constant term, and power-law corrections with non-universal coefficients. We related the exponents of the corrections to the conformal dimensions of one of the critical fields. The leading correction always has the form of an *unusual correction*, using the terminology of¹⁰, and its dimension is that of the most relevant field allowed. By changing the boundary conditions on the RSOS, we can select different states for which we calculate the bipartite Renyi entropy, and we noticed that certain corrections can be suppressed and thus the leading term can be determined by different operators. In particular, we found that symmetry considerations prevent the appearance of the most relevant field in the Renyi entropy of the absolute ground state. In the case of the minimal models, where the most relevant field $\Delta_{2,2}$ is the order parameter, the leading contribution is given by the next most relevant operator, that is $\Delta_{3,3}$. For parafermionic model, the effect is even more dramatic, because the \mathbb{Z}_N symmetry seems to select only certain fields and the first correction generally comes from the most relevant operator neutral under the symmetry, that is $\Delta_1^{(\epsilon)}$ in (51).

In our opinion, these sort of effects due to the boundary conditions could represent an interesting possibility for numerical studies in this and other models, where the operator content of the theory can be in principle read out, by a proper turning of the boundary conditions.

In the parafermionic phase, we also observed the emergence of non power-law corrections, of the same logarithmic form $(\log \xi)^{-2n}$ expected in the presence of a primary marginal field in the theory. These types of terms were already found in¹³ and would be in agreement with a naïve scaling argument applied to the $(\log \ell)^{-2}$ terms of¹⁰, where the expansion is computed for a finite interval of length ℓ . However, we already pointed out that these logarithmic corrections are present even when the parafermionic theory does not support a marginal field, and thus we should conclude that the origin of these terms is not so simple and might be a lattice effect due to non-conformal boundary conditions.

Another possible interpretation is that in general the relation between the corner transfer matrices and the Virasoro characters is “accidental”, in the sense that is purely due to the analytical structure of both quantities. Both are elliptic functions: the latter bi-periodic in real space, while the former in parameter space (we remind that the elliptic nome q has a different physical interpretation in the two cases). When expanded close to the critical point, for consistency the CTM has to give the correct central charge of the gapless CFT, and this constraints the structure of the elliptic series defining the CTM. Since the same constraint applies to the Virasoro characters, this might explain why in general one can write the CTM as a sum of characters and why in the RSOS we did not find any connection between the dimension of the operator opening the gap and the dimension of the most relevant correction in the Renyi entropy. And it might explain why, playing with the boundary condition, one can turn on logarithmic corrections with no counterpart in the CFT.

Finally, let us remark that the original work¹⁹ on the RSOS spent a considerable effort in developing advanced mathematical identities (known as generalized Rogers-Ramanujan identities) to access the partition functions of the model. In our calculations, we overcome some difficulties involved with summing up Gaussian polynomials, by performing first a duality transformation that, in our cases, turned a product of Gaussian polynomials into a sum over exponential one, which are easy to handle. We do not know how general and applicable this approach is, but it revealed to be quite powerful for us.

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Appendix A: Elliptic functions and q -series

In this appendix we recall standard definitions and useful identities for elliptic functions and q -series, used in the derivations in the text. For a more detailed treatment and for the derivations of the various equalities, we refer the reader to one of the standard textbooks on the topic, e.g.^{50,51}.

First of all, the Jacobi Elliptic θ functions are defined as

$$\theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin[(2n+1)z] , \quad (\text{A1a})$$

$$\theta_2(z, q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos[(2n+1)z] , \quad (\text{A1b})$$

$$\theta_3(z, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz) , \quad (\text{A1c})$$

$$\theta_4(z, q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz) . \quad (\text{A1d})$$

Employing the Jacobi triple product identity

$$(x^2; x^2)_{\infty} (xy^2; x^2)_{\infty} (xy^{-2}; x^2)_{\infty} = \sum_{n=-\infty}^{\infty} x^{n^2} y^{2n} , \quad (\text{A2})$$

where we introduced the q -Pochhammer symbol

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) , \quad (\text{A3})$$

$$(q)_{\infty} = (q; q)_{\infty} , \quad (\text{A4})$$

one can derive the product representations for the θ functions:

$$\theta_1(z, q) = 2 (q^2; q^2)_{\infty} q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} [1 - 2q^{2n} \cos(2z) + q^{4n}] , \quad (\text{A5a})$$

$$\theta_2(z, q) = 2 (q^2; q^2)_{\infty} q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} [1 + 2q^{2n} \cos(2z) + q^{4n}] , \quad (\text{A5b})$$

$$\theta_3(z, q) = (q^2; q^2)_{\infty} \prod_{n=1}^{\infty} [1 + 2q^{2n-1} \cos(2z) + q^{4n-2}] , \quad (\text{A5c})$$

$$\theta_4(z, q) = (q^2; q^2)_{\infty} \prod_{n=1}^{\infty} [1 - 2q^{2n-1} \cos(2z) + q^{4n-2}] . \quad (\text{A5d})$$

In the text we also used the function

$$E(z, x) \equiv (z; x)_{\infty} (xz^{-1}; x)_{\infty} (x; x)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(n-1)}{2}} z^n , \quad (\text{A6})$$

where the second equality follows again from (A2).

The duality transformation for θ functions can be derived using the Poisson summation formula, obtaining the so called *Jacobi identities*. Once we define q and \tilde{q} such that

$$q = e^{i\pi\tau} , \quad \tilde{q} = e^{-\frac{i\pi}{\tau}} , \quad \Im\tau > 0 , \quad (\text{A7})$$

they take the form

$$\theta_1(z, \tilde{q}) = -i(i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_1(\tau z, q) \quad (\text{A8a})$$

$$\theta_2(z, \tilde{q}) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_4(\tau z, q) \quad (\text{A8b})$$

$$\theta_3(z, \tilde{q}) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_3(\tau z, q) \quad (\text{A8c})$$

$$\theta_4(z, \tilde{q}) = (-i\tau)^{\frac{1}{2}} e^{\frac{i\tau z^2}{\pi}} \theta_2(\tau z, q) \quad (\text{A8d})$$

Finally, it is possible to repress the function $E(z, x)$ in (A6) by means of the θ functions

$$E(e^{2iz}, q^2) = iq^{-1/4} e^{iz} \theta_1(z, q), \quad (\text{A9})$$

$$E(-e^{2iz}, q^2) = q^{-1/4} e^{iz} \theta_2(z, q), \quad (\text{A10})$$

Combining these expression with (A8), it is possible to obtain the expression of the partition functions in the dual variables.

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- ⁵³ To avoid additional spurious degeneracies, in this classification we will assume r to be odd for regimes I and IV.
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