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Two Wave Mixing in Nonlinear Magnetization Dynamics: A Perturbation Expansion of The Landau-Lifschitz-Gilbert Equation

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Recent experiments have shown that two electromagnetic waves can be mixed together by a nonlinear process in magnetic materials and can produce a wide variety of output waves, each with a different frequency. A perturbation expansion of the Landau-Lifschitz-Gilbert equation is presented which provides qualitative and quantitative understanding of this process. The results of this expansion are compared to both experiment and direct numerical solutions.

I. INTRODUCTION

Nonlinear behavior in magnetic systems has been a topic of continuing interest.^{1–3} Recently there has been a renewal of this interest because it has become possible to generate large amplitude precessional motion in metals through a variety of techniques.^{4–11} There have been a number of experiments^{8,12,13} which have shown an interesting nonlinear mixing of two signals in magnetic materials. These experiments were done in ultra-small geometries where the microwave magnetic field could be varied over several orders of magnitude, from less than one Oe up to several hundred Oe. These experiments have demonstrated several interesting features such as the generation of new frequencies not present in the input and the amplification of these new frequencies by the input frequencies. In addition, the amplification of one input frequency by another has been seen.

In this paper we develop, in detail, a perturbation theory that can account for many of the features found in the experiments. Carrying this calculation to third order in a small expansion parameter we find, for inputs consisting of two distinct frequencies, expressions for the frequencies and amplitudes of six new signals not present in the input. The values of these frequencies agree with those seen in the experiments. The expressions for the amplitudes properly predict the relative sizes of the outputs, as well as their dependence on the amplitudes of the input signals. The output waves which are at the same frequencies as the input waves are particularly interesting. These signals are the sum of two terms, the original input wave and a generated wave. When the magnitudes of these two terms are similar, constructive or destructive interference may occur resulting in increased or decreased output amplitude.

In addition we also investigate a numerical macrospin model which shows good agreement with the perturbation theory for microwave fields up to about 20 Oe. At higher input powers more energy goes into the creation of frequencies not included in the third-order perturbation theory, so agreement in this regime is not expected.

The remainder of the paper is organized as follows: section II describes the outline of the perturbation calculation, section II A presents a general solution to the k -th order equation, section II B presents a specific solution to k -th order equation, section II C describes the solutions for each distinct output frequency, in sections III and IV we compare the calculated perturbation theory values to experiment and to a simple macrospin model, and finally section V contains a summary of the results.

II. OUTLINE OF THE PERTURBATION EXPANSION

In this section we develop the theory of the time development of the magnetization in the nonlinear regime when two input driving frequencies are present. For simplicity we use a macrospin assumption where the magnetization is spatially uniform over a significant region and may be represented by a single quantity. This is a reasonable approximation for the recent experiments where the magnetic system is driven by an electromagnetic wave with a wavelength on the order of several millimeters.

The Landau-Lifschitz-Gilbert (LLG) equation is

$$\frac{\partial \mathbf{M}}{\partial t} = -|\gamma| \mathbf{M} \times \mathbf{H}_{\text{eff}} + \frac{\alpha}{M_s} \mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t}$$

where $\mathbf{H}_{\text{eff}}$ is the effective field acting on the magnetization $\mathbf{M}$, α is the damping parameter, and γ is the gyromagnetic ratio. We replace the effective field $\mathbf{H}_{\text{eff}}$ with $\mathbf{H} - \mathbf{N}\mathbf{M}$, where $\mathbf{H}$ represents the applied fields, both static and dynamic. $\mathbf{N}$ is a diagonal demagnetizing tensor having the form

$$\mathbf{N} = \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \end{bmatrix}$$

We also use the non-dimensional variables $\mathbf{m} = \mathbf{M}/M_s$, $\mathbf{h} = \mathbf{H}/M_s$, and $\tau = |\gamma|M_s t$. The LLG equation becomes

$$\frac{\partial \mathbf{m}}{\partial \tau} = -\mathbf{m} \times (\mathbf{h} - \mathbf{N}\mathbf{m}) + \alpha \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial \tau} \quad (1)$$

Here we consider the z component of the applied field to be a positive constant, but the x and y components may depend on time. Explicitly,

$$\mathbf{m} = \begin{bmatrix} m_x(\tau) \\ m_y(\tau) \\ m_z(\tau) \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_x(\tau) \\ h_y(\tau) \\ h_z \end{bmatrix}$$

Assuming $|\mathbf{m}| = 1$, the variable $m_z(\tau)$ may be eliminated by the substitution

$$m_z(\tau) \rightarrow \sqrt{1 - m_x^2(\tau) - m_y^2(\tau)}$$

We also introduce a small perturbation expansion parameter δ and make the following substitutions

$$\begin{aligned} h_x(\tau) &\rightarrow \delta h_x(\tau), \\ h_y(\tau) &\rightarrow \delta h_y(\tau), \\ m_x(\tau) &\rightarrow \delta m_{1,x}(\tau) + \delta^2 m_{2,x}(\tau) + \delta^3 m_{3,x}(\tau), \\ m_y(\tau) &\rightarrow \delta m_{1,y}(\tau) + \delta^2 m_{2,y}(\tau) + \delta^3 m_{3,y}(\tau) \end{aligned}$$

The square roots in the resulting LLG equation are then expanded in a Taylor series in terms of δ . Equating all terms containing a factor of δ^k but no other power yields a two-component, k -th order equation in the variables $m_{k,x}(\tau)$ and $m_{k,y}(\tau)$. In all cases considered here, the equation corresponding to the z component becomes identically zero.

Remarkably, the first, second, and third order equations can all be written in the following simple forms:

$$\frac{\partial \mathbf{m}_1}{\partial \tau} = \mathbf{A} \mathbf{m}_1 + \mathbf{B} \mathbf{h}_1 \quad (2)$$

$$\frac{\partial \mathbf{m}_2}{\partial \tau} = \mathbf{A} \mathbf{m}_2 \quad (3)$$

$$\frac{\partial \mathbf{m}_3}{\partial \tau} = \mathbf{A} \mathbf{m}_3 + \mathbf{B} \mathbf{h}_3 \quad (4)$$

where

$$\mathbf{m}_k = \begin{bmatrix} m_{k,x}(\tau) \\ m_{k,y}(\tau) \end{bmatrix}, \quad k = 1, 2, 3$$

and the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\begin{aligned} \mathbf{A} &= \frac{1}{\alpha^2 + 1} \\ &\times \begin{bmatrix} -\alpha(h_z + n_x - n_z) & -h_z - n_y + n_z \\ h_z + n_x - n_z & -\alpha(h_z + n_y - n_z) \end{bmatrix}, \end{aligned} \quad (5)$$

$$\mathbf{B} = \frac{1}{\alpha^2 + 1} \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} \quad (6)$$

The Pauli matrices

$$\begin{aligned} \sigma_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

may be used to write $\mathbf{A}$ and $\mathbf{B}$ in a more compact form:

$$\begin{aligned} \mathbf{A} &= -\mathbf{B}(\sigma_0 h_z + \mathbf{n}), \\ \mathbf{B} &= \frac{1}{\alpha^2 + 1} (\sigma_3 \sigma_1 + \alpha \sigma_0) \end{aligned}$$

where $\mathbf{n}$ is a “reduced” demagnetizing tensor:

$$\mathbf{n} = \begin{bmatrix} n_x - n_z & 0 \\ 0 & n_y - n_z \end{bmatrix}$$

The expression $\mathbf{h}_1$ is simply

$$\mathbf{h}_1 = \begin{bmatrix} h_x(\tau) \\ h_y(\tau) \end{bmatrix}$$

but $\mathbf{h}_3$ is quite a bit more complicated; it is

$$\mathbf{h}_3 = \mathbf{C} \frac{\partial \mathbf{m}_1}{\partial \tau} - \mathbf{D} \mathbf{h}_1 + \mathbf{E} \mathbf{m}_1 \quad (7)$$

where

$$\begin{aligned} \mathbf{C} &= \frac{\alpha}{2} \begin{bmatrix} m_{1,y}^2(\tau) - m_{1,x}^2(\tau) & -2m_{1,x}(\tau)m_{1,y}(\tau) \\ -2m_{1,x}(\tau)m_{1,y}(\tau) & m_{1,x}^2(\tau) - m_{1,y}^2(\tau) \end{bmatrix}, \\ \mathbf{D} &= \frac{1}{2}(\mathbf{m}_1 \cdot \mathbf{m}_1)\sigma_0, \\ \mathbf{E} &= \frac{1}{2}(\mathbf{m}_1 \cdot \mathbf{m}_1)\mathbf{n} \end{aligned}$$

Using the Kronecker product, $\mathbf{C}$ may be written as

$$\mathbf{C} = -(\sigma_0 \mathbf{m}_1) \otimes (\sigma_0 \mathbf{m}_1) - (\sigma_2 \mathbf{m}_1) \otimes (\sigma_2 \mathbf{m}_1)$$

Note that the matrices $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are all quadratic forms in terms of the components of $\mathbf{m}_1$. Using the first-order equation (2), the time derivative in Eq. (7) may be eliminated resulting in

$$\mathbf{h}_3 = (\mathbf{C}\mathbf{A} + \mathbf{E})\mathbf{m}_1 + (\mathbf{C}\mathbf{B} - \mathbf{D})\mathbf{h}_1$$

Written this way, it is clear that $\mathbf{h}_3$ is cubic in the components of $\mathbf{m}_1$.

A. General solution to k-th order equation

Equations (2) through (4) are linear, two-dimensional dynamical systems. Equation (3) is homogeneous and autonomous, but the others are nonhomogeneous and nonautonomous.

Solutions to the homogeneous system

$$\frac{\partial \mathbf{m}_k}{\partial \tau} = \mathbf{A} \mathbf{m}_k$$

have the form

$$\mathbf{m}_k(\tau) = e^{\mathbf{A}\tau} \mathbf{m}_k(0)$$

The nature of these solutions depend on the properties of the matrix $\mathbf{A}$. Here, the determinant and trace of $\mathbf{A}$ are given by

$$\det(\mathbf{A}) = \frac{1}{\alpha^2 + 1} (h_z + n_x - n_z)(h_z + n_y - n_z)$$

and

$$\text{tr}(\mathbf{A}) = -\frac{\alpha}{\alpha^2 + 1} (2h_z + n_x + n_y - 2n_z)$$

For $h_z > n_z$ and $\alpha > 0$, we have $\det(\mathbf{A}) > 0$ and $\text{tr}(\mathbf{A}) < 0$, therefore solutions are stable¹⁴. The discriminant of the system is

$$\begin{aligned} \Delta &= (\text{tr}(\mathbf{A}))^2 - 4 \det(\mathbf{A}) \\ &= \left(\frac{1}{\alpha^2 + 1} \right)^2 \\ &\quad \times (\alpha^2(n_x - n_y)^2 - 4(h_z + n_x - n_z)(h_z + n_y - n_z)) \end{aligned}$$

For α sufficiently small we have $\Delta < 0$, thus solutions are spiral sinks.

For τ sufficiently large, these solutions decay to the fixed point $(0, 0)$. The value of the time constant of this decay depends on the real parts of the eigenvalues of $\mathbf{A}$. Here, the real parts of both eigenvalues are the same, resulting in a time constant of

$$\tau_0 = -\frac{1}{\text{Re}(\lambda)} = \frac{2(\alpha^2 + 1)}{\alpha(2h_z + n_x + n_y - 2n_z)}$$

Solutions to the nonhomogeneous system

$$\frac{\partial \mathbf{m}_k}{\partial \tau} = \mathbf{A} \mathbf{m}_k + \mathbf{B} \mathbf{h}_k$$

are of the form $u(\tau) + v(\tau)$ where $u(\tau)$ is a particular solution of the nonhomogeneous equation and $v(\tau)$ is a solution of the homogenous equation.

We now find a particular solution for $\mathbf{h}$ of the form

$$\mathbf{h}(\tau) = \mathbf{a} e^{i\omega\tau}, \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

where a_x and a_y are arbitrary complex constants. Assuming a solution $\mathbf{m}$ of the same form, substitution into the nonhomogeneous equation yields

$$\begin{aligned} i\omega \mathbf{m}(\tau) &= \mathbf{A} \mathbf{m}(\tau) + \mathbf{B} \mathbf{h}(\tau), \\ \mathbf{m}(\tau) &= (i\omega \sigma_0 - \mathbf{A})^{-1} \mathbf{B} \mathbf{h}(\tau) \end{aligned}$$

Define the susceptibility tensor $\chi(\omega)$ as

$$\chi(\omega) = (i\omega \sigma - \mathbf{A})^{-1} \mathbf{B}$$

and this becomes

$$\mathbf{m}(\tau) = \chi(\omega) \mathbf{h}(\tau)$$

where

$$\begin{aligned} \chi(\omega) &= \frac{1}{s(\omega)} \\ &\times \begin{bmatrix} i\alpha\omega + h_z + n_y - n_z & i\omega \\ -i\omega & i\alpha\omega + h_z + n_x - n_z \end{bmatrix} \end{aligned}$$

with

$$s(\omega) = (i\alpha\omega + h_z + n_x - n_z)(i\alpha\omega + h_z + n_y - n_z) - \omega^2$$

Note that χ is antisymmetric and also has the property

$$\chi(-\omega) = \chi^*(\omega)$$

The complete solution to the nonhomogeneous k -th order equation can now be written as

$$\mathbf{m}_k(\tau) = e^{\mathbf{A}\tau} \mathbf{m}_k(0) + \chi(\omega) \mathbf{h}_k(\tau)$$

provided, of course, that $\mathbf{h}_k$ has the simple form mentioned above. In what follows we will only consider solutions with τ sufficiently large that the first term may be neglected. Note that this has the consequence that there is no contribution from the second-order equation.

B. Specific solution to k -th order equation

It is now straightforward, albeit tedious, to solve the system for a specific $\mathbf{h}_1$. The procedure is to use the first-order equation to solve for $\mathbf{m}_1$, then use $\mathbf{h}_1$ and $\mathbf{m}_1$ to calculate $\mathbf{h}_3$, and finally use the third-order equation to obtain $\mathbf{m}_3$.

We consider the solution for a specific $\mathbf{h}_1$ consisting of a sum of two distinct frequencies ω_1 and ω_2 . Although the k -th order equations are linear in $\mathbf{m}_k$, the expression for $\mathbf{h}_3$ is not, so the usual trick of representing a solution as the real part of a complex exponential does not work. To ensure the correct evaluation of the solution we require $\mathbf{h}_1$ to be strictly real, which guarantees that $\mathbf{m}_1$, $\mathbf{h}_3$, and finally $\mathbf{m}_3$ to be real as well.

To this end we let

$$\mathbf{h}_1(\tau) = \mathbf{a}_1 e^{i\omega_1\tau} + \mathbf{a}_1^* e^{-i\omega_1\tau} + \mathbf{a}_2 e^{i\omega_2\tau} + \mathbf{a}_2^* e^{-i\omega_2\tau}$$

where

$$\mathbf{a}_1 = \begin{bmatrix} a_{1,x} \\ a_{1,y} \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} a_{2,x} \\ a_{2,y} \end{bmatrix}$$

Then the solution to the first-order equation is simply

$$\begin{aligned} \mathbf{m}_1(\tau) &= \chi(\omega_1)\mathbf{a}_1 e^{i\omega_1\tau} + \chi(-\omega_1)\mathbf{a}_1^* e^{-i\omega_1\tau} \\ &\quad + \chi(\omega_2)\mathbf{a}_2 e^{i\omega_2\tau} + \chi(-\omega_2)\mathbf{a}_2^* e^{-i\omega_2\tau}, \\ &= \chi(\omega_1)\mathbf{a}_1 e^{i\omega_1\tau} + \chi^*(\omega_1)\mathbf{a}_1^* e^{-i\omega_1\tau} \\ &\quad + \chi(\omega_2)\mathbf{a}_2 e^{i\omega_2\tau} + \chi^*(\omega_2)\mathbf{a}_2^* e^{-i\omega_2\tau} \end{aligned}$$

The nonlinear character of the expression for $\mathbf{h}_3$ results in the creation of new frequencies not present in $\mathbf{h}_1$. Each term in $\mathbf{h}_3$ will contain three factors, each of which is a term in $\mathbf{h}_1$ or $\mathbf{m}_1$. The exponentials in these factors may be combined into a single exponential representing a single frequency. Grouping terms of the same frequency results in an expression of the form

$$\mathbf{h}_3(\tau) = \sum_{n=0}^7 \mathbf{u}_n e^{i\Omega_n\tau} + \mathbf{u}_n^* e^{-i\Omega_n\tau}$$

where

$$\begin{aligned} \Omega_0 &= 2\omega_1 - \omega_2, & \Omega_1 &= \omega_1, \\ \Omega_2 &= \omega_2, & \Omega_3 &= 2\omega_2 - \omega_1, \\ \Omega_4 &= 3\omega_1, & \Omega_5 &= 2\omega_1 + \omega_2, \\ \Omega_6 &= 2\omega_2 + \omega_1, & \Omega_7 &= 3\omega_2 \end{aligned}$$

If $0 < \omega_1 < \omega_2$ and $\omega_2/\omega_1 < 2$ then the Ω_n are distinct and ordered from smallest to largest.

The solution to the third-order equation is then

$$\mathbf{m}_3(\tau) = \sum_{n=0}^7 \chi(\Omega_n)\mathbf{u}_n e^{i\Omega_n\tau} + \chi^*(\Omega_n)\mathbf{u}_n^* e^{-i\Omega_n\tau} \quad (8)$$

and the complete solution is

$$\mathbf{m}(\tau) = \mathbf{m}_1(\tau) + \mathbf{m}_3(\tau) \quad (9)$$

The complexity of the solution resides in the expression for $\mathbf{h}_3$, which is cubic in the components of $\mathbf{m}_1$ and $\mathbf{h}_1$. Each component of $\mathbf{h}_3$ consists of over 5000 individual terms, each containing one or more of the factors α , h_z , n_x , n_y , n_z , as well as the components of $\mathbf{a}_1$, $\mathbf{a}_2$, $\chi(\omega_1)$, $\chi(\omega_2)$ and their complex conjugates. Symmetry properties of the χ tensor may be used to reduce the number of unique terms to less than 2000.

To reduce the number of terms further, we restrict the varying applied field to be in the y direction only, and require the sample to have infinite extent in the z direction; that is, we set

$$a_{1,x} \rightarrow 0, \quad a_{2,x} \rightarrow 0, \quad n_z \rightarrow 0$$

This reduces the number of unique terms to a manageable 399.

We now assign symbols to certain recurring combinations of factors and then list $\mathbf{h}_3$ in terms of these symbols and grouped by frequency. To that end, we define a set of amplitude factors, A_m . Each amplitude factor is a third-degree

product of the components of $\mathbf{a}_1$ and $\mathbf{a}_2$ and their complex conjugates. These factors are defined as follows

$$\begin{aligned}
A_0 &= a_{1,y}^2 a_{2,y}^* \\
A_{1,1} &= a_{1,y}^2 a_{1,y}^* \\
A_{1,2} &= a_{1,y} a_{2,y} a_{2,y}^* \\
A_{2,1} &= a_{1,y} a_{1,y}^* a_{2,y} \\
A_{2,2} &= a_{2,y}^2 a_{2,y}^* \\
A_3 &= a_{1,y}^* a_{2,y}^2 \\
A_4 &= a_{1,y}^3 \\
A_5 &= a_{1,y}^2 a_{2,y} \\
A_6 &= a_{1,y} a_{2,y}^2 \\
A_7 &= a_{2,y}^3
\end{aligned}$$

We also define a set of coefficients, c_j , which are products of the other relevant parameters¹⁵. Coefficients c_1 through c_{69} do not contain a factor of α , c_{70} through c_{234} do contain a factor of α , and c_{235} through c_{399} contain a factor of α^2 .

Note that the amplitudes of the perturbation solutions at a given frequency are cubic polynomials in the driving amplitudes, $\mathbf{a}_1$ and $\mathbf{a}_2$; the geometric and physical properties of the system are contained in the coefficients, c_n .

In the next section we write the explicit expressions for the third-order driving fields which occur in Eq. (8). These allow one to calculate the strength of the response at any of the eight possible output frequencies.

C. Solutions for the output frequencies

The expressions for the $\mathbf{u}_n$, each respectively corresponding to an output frequency of Ω_n , are

$$\mathbf{u}_0 = \frac{A_0}{2(1+\alpha^2)} \begin{bmatrix} u_{0,x} \\ u_{0,y} \end{bmatrix} \quad (10)$$

$$\mathbf{u}_1 = \frac{1}{2(1+\alpha^2)} \left(A_{1,1} \begin{bmatrix} u_{1,1,x} \\ u_{1,1,y} \end{bmatrix} + A_{1,2} \begin{bmatrix} u_{1,2,x} \\ u_{1,2,y} \end{bmatrix} \right) \quad (11)$$

$$\mathbf{u}_2 = \frac{1}{2(1+\alpha^2)} \left(A_{2,1} \begin{bmatrix} u_{2,1,x} \\ u_{2,1,y} \end{bmatrix} + A_{2,2} \begin{bmatrix} u_{2,2,x} \\ u_{2,2,y} \end{bmatrix} \right) \quad (12)$$

$$\mathbf{u}_3 = \frac{A_3}{2(1+\alpha^2)} \begin{bmatrix} u_{3,x} \\ u_{3,y} \end{bmatrix} \quad (13)$$

$$\mathbf{u}_4 = \frac{A_4}{2(1+\alpha^2)} \begin{bmatrix} u_{4,x} \\ u_{4,y} \end{bmatrix} \quad (14)$$

$$\mathbf{u}_5 = \frac{A_5}{2(1+\alpha^2)} \begin{bmatrix} u_{5,x} \\ u_{5,y} \end{bmatrix} \quad (15)$$

$$\mathbf{u}_6 = \frac{A_6}{2(1+\alpha^2)} \begin{bmatrix} u_{6,x} \\ u_{6,y} \end{bmatrix} \quad (16)$$

$$\mathbf{u}_7 = \frac{A_7}{2(1+\alpha^2)} \begin{bmatrix} u_{7,x} \\ u_{7,y} \end{bmatrix} \quad (17)$$

The individual components of the $\mathbf{u}_n$ are defined in the supplemental material¹⁵. However here we present, as an example, the complete solution for $u_{4,x}$

$$\begin{aligned} u_{4,x} = & c_2 + c_{12} - c_{70} - c_{82} - 2c_{83} + c_{84} \\ & + c_{92} - c_{99} - c_{100} + c_{236} + c_{237} - 2c_{246} \\ & + c_{258} - c_{259} + 2c_{260} \end{aligned}$$

where

$$\begin{aligned} c_2 &= n_x \chi_{xy}^3(\omega_1) \\ c_{12} &= n_x \chi_{xy}(\omega_1) \chi_{yy}^2(\omega_1) \\ c_{70} &= \alpha \chi_{xy}^2(\omega_1) \\ c_{82} &= \alpha h_z \chi_{xy}^2(\omega_1) \chi_{yy}(\omega_1) \\ c_{83} &= \alpha n_x \chi_{xy}^2(\omega_1) \chi_{yy}(\omega_1) \\ c_{84} &= \alpha n_y \chi_{xy}^2(\omega_1) \chi_{yy}(\omega_1) \\ c_{92} &= \alpha \chi_{yy}^2(\omega_1) \\ c_{99} &= \alpha h_z \chi_{yy}^3(\omega_1) \\ c_{100} &= \alpha n_y \chi_{yy}^3(\omega_1) \\ c_{236} &= \alpha^2 h_z \chi_{xy}^3(\omega_1) \\ c_{237} &= \alpha^2 n_x \chi_{xy}^3(\omega_1) \\ c_{246} &= \alpha^2 \chi_{xy}(\omega_1) \chi_{yy}(\omega_1) \\ c_{258} &= \alpha^2 h_z \chi_{xy}(\omega_1) \chi_{yy}^2(\omega_1) \\ c_{259} &= \alpha^2 n_x \chi_{xy}(\omega_1) \chi_{yy}^2(\omega_1) \\ c_{260} &= \alpha^2 n_y \chi_{xy}(\omega_1) \chi_{yy}^2(\omega_1) \end{aligned}$$

III. COMPARISON TO EXPERIMENT

As has been previously reported⁸, this analysis explains features seen in experiments utilizing a microstrip waveguide with a thin Permalloy ribbon under the signal line. The waveguide was driven with two independent continuous wave frequencies; the input amplitude of the lower frequency, ω_1 , was kept fixed, while the amplitude of the higher frequency, ω_2 , was varied.

As the input power is increased, a new output is seen at a frequency of $\Omega_3 = 2\omega_2 - \omega_1$ as predicted by the perturbation expansion. At still higher input powers an output wave at frequency $\Omega_0 = 2\omega_1 - \omega_2$ appears.

Why does the output at Ω_3 appear before the output at Ω_0 in this experiment? Equations (8) and (13) tell us the amplitude of the Ω_3 signal, $\mathcal{A}_3$, will be proportional to $|A_3|$, since for fixed material geometry and parameters, as well as static field h_z , all other factors are constant. Similarly, equations (8) and (10) indicate that the amplitude of the Ω_0 signal, $\mathcal{A}_0$, is proportional to $|A_0|$ (of course, the constant of proportionality is in general different from that of the Ω_3 signal). The expressions for the amplitude factors yield

$$\frac{|A_3|}{|A_0|} = \frac{|a_{2,y}|}{|a_{1,y}|}$$

thus

$$\frac{\mathcal{A}_3}{\mathcal{A}_0} \sim \frac{|a_{2,y}|}{|a_{1,y}|}$$

For $|a_{2,y}|$ sufficiently larger than $|a_{1,y}|$, the Ω_3 signal will be larger than the Ω_0 signal, and will be seen first as the input power increases.

If $|a_{1,y}|$ is held fixed we have

$$\mathcal{A}_0 \sim |a_{2,y}|, \quad \mathcal{A}_3 \sim |a_{2,y}|^2 \quad (18)$$

FIG. 1. (color online) Log-log plot illustrating the qualitative features of the response at the lowest four frequencies as predicted by the perturbation expansion result. *Note:* the responses at frequencies Ω_1 and Ω_2 have been multiplied by 10 for clarity.

When these amplitudes, as a function of $a_{2,y}$, are plotted using a log-log scale, the results are simple straight lines with slopes proportional to the exponents. These features are illustrated in Fig. 1. To compute the data plotted in this figure we used values of the physical parameters as follows: $\alpha = 0.02$, $\gamma = 2.93$ GHz/kOe, $M_s = 0.7958$ kOe, $H_z = 0.6$ kOe, $n_x = (0.903)4\pi$, $n_y = (0.097)4\pi$, $\omega_1 = 9.95$ GHz, $\omega_2 = 10.05$ GHz and $|a_{1,y}| = 1.0$ Oe.

The response at frequencies Ω_1 and Ω_2 is more complicated since these signals are a sum of the first-order and third-order terms. The *first-order* response at frequencies Ω_1 and Ω_2 is

$$|\mathcal{A}_{1,1}| \sim \text{constant}, \quad |\mathcal{A}_{2,1}| \sim |a_{2,y}|$$

and the *third-order* response at these frequencies is

$$|\mathcal{A}_{1,2}| \sim |a_{2,y}|^2, \quad |\mathcal{A}_{2,2}| \sim |a_{2,y}|^3$$

The consequence of this is that as the driving amplitude $|a_{2,y}|$ is increased, the Ω_1 response will be initially constant since the first-order term dominates. But for sufficiently large driving amplitude, the third-order term will dominate and the response will increase. This is an *amplification* of the Ω_1 signal in response to the increased Ω_2 signal.

Similarly, the Ω_2 response will initially be proportional to the driving amplitude, but will then be amplified as the third-order contribution becomes dominant (see Fig. 1). These amplification effects were also seen in⁸.

Although outputs at frequencies Ω_4 through Ω_7 were not discussed in⁸, a similar analysis shows

$$\begin{aligned} |\mathcal{A}_4| &\sim \text{constant}, & |\mathcal{A}_5| &\sim |a_{2,y}|, \\ |\mathcal{A}_6| &\sim |a_{2,y}|^2, & |\mathcal{A}_7| &\sim |a_{2,y}|^3 \end{aligned}$$

Each frequency has a distinct functional dependence on $|a_{2,y}|$ and each has a distinct slope when plotted using log-log scales. This is illustrated in Fig. 2, which was produced using identical values of the physical parameters used for Fig. 1.

IV. COMPARISON TO NUMERICAL SOLUTIONS

The LLG equation (1) can also be solved numerically and the results compared with the perturbation expressions (9). To do so we have used adaptive time-step ODE solvers found in the popular MATLAB computing environment¹⁶. The response at the specific frequencies predicted by the perturbation result was then computed using a fast Fourier transform, taking care that the number of points in the transform, as well as the time-domain resolution, ensure that the computed points in the frequency domain fall exactly on the eight frequencies of interest, Ω_0 – Ω_7 . For these computations we used values of the physical parameters as follows: $\alpha = 0.02$, $\gamma = 2.93$ GHz/kOe, $M_s = 0.796$ kOe, $n_x = (0.9)4\pi$, $n_y = (0.1)4\pi$, $\omega_1 = 9.95$ GHz, $\omega_2 = 10.05$ GHz and $|a_{1,y}| = 1.0$ Oe.

In Fig. 3 we compare one such solution, for frequencies Ω_0 through Ω_3 , with the perturbation result using an applied static field of $H_z = 0.41$ kOe. For the values used here this results in an undamped resonant frequency of 10.67 GHz,

FIG. 2. (color online) Log-log plot illustrating the qualitative features of the response at the four highest frequencies, as predicted by the perturbation expansion result.

FIG. 3. (color online) Comparison of a numerical solution of the LLG equation to the perturbation expansion result for the lowest four frequencies at an applied static field of $H_z = 0.41$ kOe. The perturbation result is plotted using solid lines, and the numerical solutions are plotted using the discrete markers indicated in the legend.

so we have a sample being driven below resonance. As can be seen, there is good agreement up to a driving field of 20 Oe or so.

We note that the amplification of the Ω_2 mode seen in the experiment and seen in Fig. 1 does occur in both the perturbation theory and in the numerical solution. However as the driving field is further increased, the response in M_y actually decreases in the numerical solution while there is an increase found using perturbation theory.

Agreement at larger driving amplitudes is not expected as more energy goes into the creation of frequencies not considered here. Furthermore, the macrospin approximation also becomes less appropriate as it is possible to generate spin waves with short wavelengths. In this case true micromagnetic calculations would be more appropriate.

Fig. 4 is the same comparison for the higher frequencies Ω_4 through Ω_7 . Again, there is good agreement up to a driving field of 20 Oe or so.

In Fig. 5 and Fig. 6 we present a comparison computed at a static field of $H_z = 0.14$ kOe. This corresponds to a undamped resonant frequency of 9.45 GHz so we have the case of a sample being driven above the resonance. Again, we get good agreement up to a driving field of 20 Oe or so.

Fig. 5 also illustrates an interesting consequence of the fact that the perturbation result for frequency Ω_1 is the sum of a first-order term and a third-order term. The relative phase between these two terms need not be zero, and this has the consequence that there may be constructive or destructive interference between the two terms. For the particular case shown in Fig. 5, we see the perturbation result for the Ω_1 signal undergo a pronounced dip in the region around 60 Oe, followed by a rise at higher driving field. This region corresponds to where the two terms are similar

FIG. 4. (color online) Comparison of a numerical solution of the LLG equation to the perturbation expansion result for the highest four frequencies at an applied static field of $H_z = 0.41$ kOe. The perturbation result is plotted using solid lines, and the numerical solutions are plotted using the discrete markers indicated in the legend.

FIG. 5. (color online) Comparison of a numerical solution of the LLG equation to the perturbation expansion result for the lowest four frequencies at an applied static field of $H_z = 0.14$ kOe. The perturbation result is plotted using solid lines, and the numerical solutions are plotted using the discrete markers indicated in the legend.

in magnitude but opposite in phase, and the dip corresponds to destructive interference between the two terms.

It is not surprising that the perturbation result is not as good for the case of driving frequencies close to resonance. Close to the resonance frequency, the precession can be driven to large amplitudes at smaller values of the driving field. As a result the perturbation theory also deviates from the numerical results for smaller driving fields. A static field of $H_z = 0.2593$ kOe gives an undamped resonant frequency of 10.0 GHz, half-way between the driving frequencies used here. For this case we have found reasonable agreement only for driving fields up to 10 Oe or so. Nonetheless, the perturbation method gives a very good account (at low powers) of which waves can be produced by the nonlinear mixing and their relative amplitudes.

V. SUMMARY

In this paper we have found a series solution to the Landau-Lifschitz-Gilbert equation up to the third-order in terms of a small perturbation expansion parameter. The first, second and third-order equations in this expansion turn out to be simple linear two-dimensional dynamical systems which differ only in the nonhomogeneous terms. The solutions to the corresponding homogeneous equations are stable spiral sinks which vanish in the limit of large time values. For harmonic driving terms, and in the limit of large time values, solutions to the nonhomogeneous

FIG. 6. (color online) Comparison of a numerical solution of the LLG equation to the perturbation expansion result for the highest four frequencies at an applied static field of $H_z = 0.14$ kOe. The perturbation result is plotted using solid lines, and the numerical solutions are plotted using the discrete markers indicated in the legend.

equations are frequency-dependent linear combinations of the frequencies present in the driving terms. The solution to the first-order equation is straightforward and, in the limit of large time values, the second-order equation does not contribute. The complexity of the solution to the third-order equation lies in the complexity of the third-order driving term, which contains six new frequencies not present in the first-order driving term. For reasonable restrictions on the driving fields, we present analytic expressions for the solutions to the first and third order equations.

These expressions provide a ready qualitative understanding of the amplitudes of the frequencies present in the solution, as functions of the amplitudes of the driving fields, and agree quite well with features seen in actual experiments. For reasonable values of the parameters, quantitative agreement of the perturbation solution to direct numerical solution of the Landau-Lifschitz-Gilbert equation is also quite good, up to driving fields of 10 to 20 Oe.

In addition, it is clear from this paper how the solutions could be calculated in other geometries, e.g. systems where there are static or dynamic fields in different directions than those considered here. For example, it has been pointed out recently that the nonlinear generation of harmonics strongly depends on the orientation of the static field¹⁷.

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