Exotic continuous quantum phase transition between Z_{2} topological spin liquid and Néel order

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Exotic continuous quantum phase transition between Z_2 topological spin liquid and Néel order

Eun-Gook Moon[1] and Cenke Xu[1]
[1]Department of Physics, University of California, Santa Barbara, CA 93106
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Recent numerical simulations with different techniques have all suggested the existence of a continuous quantum phase transition between the Z_2 topological spin liquid phase and a conventional Néel order. Motivated by these numerical progress, we propose a candidate theory for such Z_2-Néel transition. We first argue on general grounds that, for a SU(2) invariant system, this transition *cannot* be interpreted as the condensation of spinons in the Z_2 spin liquid phase. Then we propose that such Z_2-Néel transition is driven by proliferating the bound state of the bosonic spinon and vison excitation of the Z_2 spin liquid, *i.e.* the so called $(e, m)-$type excitation. Universal critical exponents associated with this exotic transition are computed using $1/N$ expansion. This theory predicts that at the Z_2-Néel transition, there is an emergent quasi long range power law correlation of columnar valence bond solid order parameter.

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I. INTRODUCTION

Spin liquid is an exotic quantum many-body ground state of bosonic spin systems that does not break any global symmetry of the system, thus it should correspond to a disordered phase in the classic Ginzburg-Landau paradigm. However, besides being disordered, spin liquids usually also have many properties beyond the GL paradigm. For example, fully gapped spin liquids usually have exotic long range quantum entanglement, which is usually interpreted as topological order. Algebraic spin liquids have emergent stable gapless excitations even though the systems break no continuous symmetry (hence these gapless excitations are not ordinary Goldstone modes). In the last decade, candidates of gapless spin liquid states have been discovered in experimental systems such as $\kappa-(ET)_2Cu_2(CN)_3$[1–4], $EtMe_3Sb[Pd(dmit)_2]_2$[5–9], $Ba_3CuSb_2O_9$[10], $Ba_3NiSb_2O_9$[11], $ZnCu_3(OH)_6Cl_2$[12], etc. In all these materials, no evidence of spin order was found at temperature much lower compared with the spin interaction energy scale, while a large density of gapless excitations was discovered in thermodynamics or thermal conductivity measurements[3,4,8,9] , thus these systems cannot be understood using standard semiclassical physics of spin orders.

In addition to all the experimental studies, thanks to the rapid development of numerical techniques, more and more candidates of exotic liquid states have been identified in frustrated spin models[13–17], hard-core quantum boson model[19,20], or even Hubbard model[23]. All these phases that are identified numerically are fully gapped liquid phases with short range correlation between both spin order parameters and also valence bond solid (VBS) order parameters. The simplest fully gapped spin liquid state is the Z_2 topological liquid state, which has the same topological order as the toric code model[24]. In addition to the fully gapped spectrum, the computation of critical exponents at the order-disorder transition of these models[19,20,22], and the computation of topological entanglement entropy[15,21] both convincingly proved that the spin liquid states of some of these models (such as the $J_1 - J_2$ model on the square lattice, and the extended Bose-Hubbard model on the Kagome lattice) are indeed the Z_2 topological liquid. In some other models (such as the spin-1/2 Heisenberg model on the Kagome lattice[13,14], and the Hubbard model on the honeycomb lattice[23]), although an accurate topological entanglement entropy computation is still demanded, it is broadly believed that the spin liquid state is indeed the Z_2 liquid state, or a similar $(Z_2)^n$ liquid state.

Besides the spin liquid state itself, the quantum phase transitions of these models are equally interesting. For example, continuous quantum phase transitions between Néel order and a fully gapped spin liquid phase have been found in the honeycomb lattice Hubbard model[23], and the $J_1 - J_2$ spin-1/2 Heisenberg model on the square lattice[15,16]. : phases (plural) In terms of the Landau-Ginzburg (LG) theory, this transition should be an ordinary O(3) transition, and the Z_2 liquid phase is identified as the disordered phase, while the Néel phase is the ordered phase. However, because the Z_2 liquid phase has a nontrivial topological order and topological degeneracy[24], it cannot be adiabatically connected to the trivial direct product state, thus it should *not* be identified as the trivial disordered phase in the classical case. Thus if the Z_2-Néel transition exists, it means that the quantum disordering of the Néel order and the emergence of the Z_2 topological order happen simultaneously at one point, this unusual fact implies that this Z_2-Néel quantum critical point (QCP) must be an unconventional one that is beyond the LG paradigm. The goal of this paper is to understand this unconventional QCP.

II. FAILURE OF THE SPINON THEORIES

We first argue on general grounds that such a continuous Z_2-Néel transition *cannot* be understood using an ordinary spinon theory. We stress that we will only consider SU(2) invariant systems here.

First of all, if this Z_2 spin liquid phase has a gapped fermionic spinon excitation f_α, then a Néel order parameter can in principle be represented as $\vec{N} \sim (-1)^i f^\dagger_{i,\alpha}\vec{\sigma}_{\alpha\beta} f_{i,\beta}$. Thus it appears that we can interpret this Z_2-Néel transition as the disorder-order transition of the vector $\vec{N}$ using an ordinary Landau-Ginzburg theory. However, this theory is incorrect because the vector $\vec{N}$ does not carry any gauge charge, thus the order parameter does not immediately suppress the Z_2 topological order. This implies that between the Z_2 spin liquid and the Néel order with nonzero $\langle \vec{N} \rangle$, there must be an intermediate state with the coexistence of both Néel order and Z_2 topological order, and it is usually called the Néel* state. Thus a direct continuous transition between Z_2 and Néel order cannot be obtained this way without fine-tuning.

In order to suppress the Z_2 topological order, the usual wisdom is to condense a topological excitation that carries the Z_2 gauge charge. Then after the topological excitations are condensed, the Z_2 gauge field is gapped out due to Higgs mechanism, and the topological order disappears. Along with suppressing the topological order, if we want to induce spin order simultaneously, then the excitation that condenses must also carry certain representation of the spin SU(2) symmetry group, in addition to the Z_2 gauge charge. Let us call this gauge-charged spin excitation a *spinon* in general. Then the nature of the spin order and the universality class of this transition both depend on the particular spin representation of spinon.

The smallest representation of SU(2) is spin-1/2 representation, and there is no consistent "fractional" representation of SU(2) group that is smaller than spin-1/2. Thus let us first assume the spinon is a spin-1/2 boson, which is described by a two component complex boson field $z_\alpha = (z_1, z_2)^t$, and z_α is subject to the constraint $|z_1|^2 + |z_2|^2 = 1$. Then z_α is coupled to a Z_2 gauge field in the following way:

$$H = \sum_{i,\mu}\sum_{\alpha} -t\sigma^z_{i,\mu} z^*_{\alpha,i} z_{\alpha,i+\mu} + H.c. + \cdots \quad (1)$$

where the ellipsis stands for higher order interaction terms. $\sigma^z_{i,\mu}$ is the Z_2 gauge field that is defined on the link (i,μ) of the lattice, and Eq. 1 is invariant under the gauge transformation

$$z_{i,\alpha} \to \eta_i z_{i,\alpha}, \quad \sigma^z_{i,\mu} \to \eta_i \sigma^z_{i,\mu} \eta_{i+\mu}, \quad (2)$$

where $\eta_i = \pm 1$ is an arbitrary Ising function defined on the sites of the lattice. The condensed phase of z_α is the spin ordered phase, and because z_α is coupled to the Z_2 gauge field, the Z_2 topological order is automatically destroyed due to the Higgs mechanism in the condensate of z_α. The gapped phase of z_α is the deconfined Z_2 topological phase.

Since z_α has in total two complex bosonic fields, *i.e.* four real fields, then with the constraint $|z_1|^2 + |z_2|^2 = 1$, the entire configuration of z_α is equivalent to a three dimensional sphere S^3. Since the spinon field z_α is coupled to a Z_2 gauge field, then the physical configuration of the condensate of z_α is S^3/Z_2, which is mathematically equivalent to the group manifold SO(3). Since z_α itself is not a physical observable, inside the condensate of z_α the physical observables are the three following vectors:

$$\vec{N}_1 = \mathrm{Re}[z^t i\sigma^y \vec{\sigma} z], \quad \vec{N}_2 = \mathrm{Im}[z^t i\sigma^y \vec{\sigma} z], \quad \vec{N}_3 = z^\dagger \vec{\sigma} z. \quad (3)$$

A simple application of the Fierz identity $\sum_a \sigma^a_{\alpha\beta}\sigma^a_{\gamma\rho} = 2\delta_{\alpha\rho}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\rho}$ proves that these three vectors are orthogonal to each other. Since the first homotopy group of SO(3) is $\pi_1[\mathrm{SO}(3)] = Z_2$, inside this spin ordered phase there are vortex-like topological defects. Two of these vortices can annihilate each other.

The spin-1/2 boson field z_α can be viewed as the low energy mode of the usual Schwingber boson b_α, but our argument is more general, and it is independent of the microscopic origin of z_α. If one of the three vectors $\vec{N}_i$ is identified as the Néel vector, then this phase must have two other spin vector orders that are perpendicular to the Néel vector. The condensation transition of z_α while coupled to a Z_2 gauge field is usually called the O(4)* transition[25].

Now let us assume the spinon of the Z_2 topological phase carries a spin-1 representation. A spin-1 representation is a vector representation of SU(2), *i.e.* it can be parametrized as a unit real vector $\vec{n}$, $|\vec{n}|^2 = 1$. Now the coupling between the spinon and Z_2 gauge theory reads

$$H = \sum_{i,\mu}\sum_{a} -t\sigma^z_{i,\mu} n^a_i n^a_{i+\mu} + \cdots \quad (4)$$

Again, since $\vec{n}$ couples to a Z_2 gauge field, it is not a physical observable: $\vec{n}$ and $-\vec{n}$ are physically equivalent. If vector $\vec{n}$ condenses, the condensate is in fact a spin nematic, or quadrupole order, with physical order parameter

$$Q^{ab} = n^a n^b - \frac{1}{3}\delta_{ab}. \quad (5)$$

This spin order has manifold S^2/Z_2, which also supports vortex excitation since $\pi_1[S^2/Z_2] = Z_2$. One example state of this type is the spin quadrupolar state that has been observed in the spin-1 material $NiGa_2S_4$[26–28].

We have discussed two types of unconventional QCPs between Z_2 liquid phase and spin orders. In either case, the spin ordered phase is different from the ordinary collinear Néel order, because a Néel order should have ground state manifold (GSM) S^2. In particular, in both cases we have considered, the spin ordered phase must have a nontrivial homotopy group π_1, which corresponds to the vison excitation of the Z_2 gauge field. Generalization of our analysis to higher spin representations is straightforward, but the conclusion is unchanged. As we already discussed, in Ref.[15] and Ref.[23], a *continuous* quantum phase transition between a fully gapped spin liquid phase and a Néel order was reported. If the fully gapped spin liquid discovered in these numerical works is indeed a Z_2 spin liquid as we expected, then such continuous quantum phase transition is beyond the spinon theory discussed in this section. In order to understand the continuous transition between the gapped spin liquid and Néel order reported in the phase diagram of the Hubbard model on the honeycomb lattice[23], in Ref.[29–31] the authors had to introduce extra "hidden" order parameters in the Néel phase, which change the GSM of the Néel phase completely.

In this section we argued on general grounds that the Z_2-Néel transition cannot be interpreted as the condensation of an ordinary spinon. Our argument is independent of specific spin model or lattice structure. However, this argument can only be applied to SU(2) invariant systems. For a system with U(1) symmetry, for example the hard-core Boson model on the Kagome lattice discussed in Ref.[18,21,22], the transition between Z_2 topological phase and the superfluid phase can be understood as the condensation of a fractionalized "half-boson" that couples to the Z_2 gauge field, and this transition is the so-called $3d$ XY* transition.

III. EXOTIC Z_2-NÉEL QUANTUM CRITICAL POINT

A. Phase diagram around Z_2 spin liquid driven by e and m excitations

In order to understand the direct continuous transition between the Z_2 spin liquid and the Néel phase, we should first put these two phases in the same phase diagram. One candidate theory that contains both phases was proposed in Ref.[34]. Let us first write down a minimal unified field theory proposed in Ref.[34]:

$$\mathcal{L} = \sum_{\alpha=1}^{N_z} |(\partial_\mu - i a_\mu) z_\alpha|^2 + \sum_{\alpha=1}^{N_v} |(\partial_\mu - i b_\mu) v_\alpha|^2 + s_z |z_\alpha|^2 + s_v |v_\alpha|^2 + \frac{i}{\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu b_\rho + \cdots \quad (6)$$

In this field theory, there are two types of matter fields, z_α and v_α, and they are interacting with each other through mutual Chern-Simons (CS) fields a_μ and b_μ, which grant them a mutual semionic statistics *i.e.* when v_α adiabatically encircles z_α through a closed loop, the system wavefunction acquires a minus sign. This is one of the key properties of the Z_2 topological phase. Here z_α corresponds to the electric ($e-$type) excitation of the Z_2 liquid, and v_α corresponds to the magnetic ($m-$type) excitation. v_α is usually called the vison excitation.

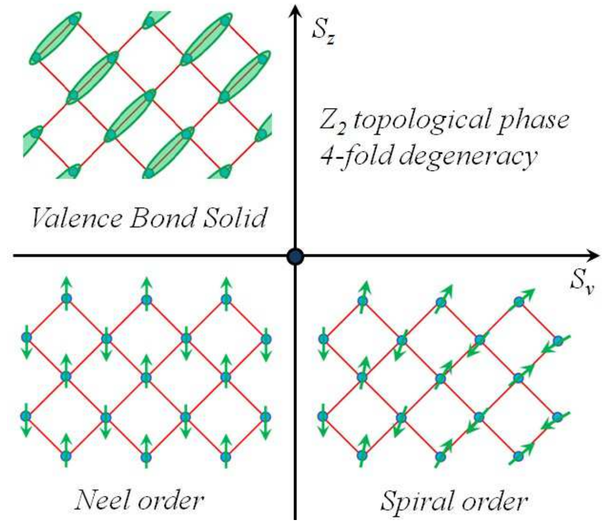

FIG. 1: The global phase diagram of Eq. 6, which describes four different states on a distorted triangular lattice, or a square lattice. Eq. 6 assumes that the $e-$type excitation z_α and $m-$type excitation v condense separately.

The minimal field theory Eq. 6 has symmetry SU(N_z)×SU(N_v). However, depending on the details of the microscopic model, the higher order interactions between matter fields can break this symmetry down to its subgroups. We will first ignore this higher order symmetry breaking effects, and focus on the case with $N_z = 2$, and $N_v = 1$. In Ref.[34], the authors used the model Eq. 6 with $N_z = 2$, $N_v = 1$ to describe the global phase diagram of spin-1/2 quantum magnets on a distorted triangular lattice, which is a very common structure in many materials. The same theory can be applied to the square and honeycomb lattice as well, and in this paper we will take the square lattice as an example. Here z_α is a bosonic spin-1/2 spinon, and v is the low energy mode of a vison, and it corresponds to the expansion of the vison at two opposite momenta $\pm\vec{Q}$:

$$\tau \sim v e^{i\vec{Q}\cdot\vec{r}} + v^* e^{-i\vec{Q}\cdot\vec{r}}, \quad (7)$$

thus v is a complex scalar field. On the square lattice or distorted triangular lattice, there is a Z_8 anisotropy on v, that is allowed by the symmetry of the lattice[34,35]. This anisotropy is highly irrelevant in the quantum critical region, and it will be ignore throughout the paper.

The phase diagram of this model is tuned by two parameters: s_z and s_v, and depending on the sign of these two parameters, there are in total four different phases (Fig. 1):

Phase 1. This is the phase with $s_z > 0$, $s_v > 0$. In this phase, both matter fields z_α and v are gapped, and they acquire a topological statistic interaction through the mutual CS fields. Since all the matter fields are gapped, the low energy properties of phase 1 is described by the mutual CS theory only. The mutual CS theory defined on a torus has a four-fold degenerate ground state, thus this phase is precisely the gapped Z_2 topological phase[36].

Phase 2. $s_v > 0$, $s_z < 0$. When vison v is gapped, integrating out v induces a Maxwell term for gauge field b_μ, which implies that the flux of b_μ is condensed. In other words the flux-creation operator (denoted as $\mathcal{M}_b$) acquires a nonzero expectation value. $\mathcal{M}_b$ corresponds to a Dirac monopole configuration of b_μ in the space-time. Due to the mutual CS coupling between gauge fields a_μ and b_μ, the condensate of $\mathcal{M}_b$ breaks a_μ to a Z_2 gauge field. Thus after we integrate out v and b_μ, the spinon z_α is only coupled to a Z_2 gauge fields. Thus when $N_z = 2$, the condensate of z_α has GSM SO(3) as was discussed in the previous section. An example of this phase is the spiral spin density wave phase. Once we assume $s_v > 0$, Eq. 6 precisely reduces to the previously studied O(4)* theory for the transition between Z_2 spin liquid and spiral spin order[25].

Phase 3. $s_v < 0$, $s_z > 0$. This is a phase where v condenses while z_α is gapped out. This phase is the four fold degenerate columnar VBS phase that breaks the reflection and translation symmetry of the lattice. The columnar VBS order parameter can be written as $v^2\mathcal{M}_a$, where $\mathcal{M}_a$ is the monopole operator of gauge field a_μ, which creates a 2π flux of a_μ. When $s_z > 0$, spinon z_α is gapped, and it leads to a Maxwell term for a_μ. This implies that $\mathcal{M}_a$ is condensed, and it breaks b_μ to a Z_2 gauge field. In this case the low energy effective theory that describes phase 3 is a complex field v that couples to a Z_2 gauge field, thus our theory reduces to the pure vison theory that was thoroughly discussed in Ref.[35].

Phase 4. $s_v < 0$, $s_z < 0$. This is a phase where both z_α and v condense. Because in this phase the only gauge invariant order parameter that condenses is $\vec{N} \sim z^\dagger \vec{\sigma} z$, this is precisely the collinear Néel phase with GSM S^2. In fact, when v is condensed, the gauge field b_μ acquires a mass term b_μ^2 due to the Higgs mechanism. Then integrating out v and b_μ leads to a Maxwell term for gauge field a_μ, due to the mutual CS coupling. Thus the spinon z_α is coupled to a dynamical gapless U(1) gauge field a_μ. Then the GSM of the condensate of z_α is $S^3/U(1) = S^2$, which is equivalent to the collinear Néel order. Thus under the assumption $s_v < 0$, Eq. 6 reduces to the CP(1) model that describes the deconfined QCP between Néel and VBS order[37,38].

We have shown that the mutual CS formalism Eq. 6 unifies many previously discussed exotic states and exotic phase transitions. A more detailed discussion of the phase diagram can be found in Ref.[34].

B. Z_2-Néel transition driven by (e, m) excitation

Now we are ready to discuss our theory for the direct continuous transition between Z_2 liquid phase and Néel phase. In a Z_2 topological phase, using the standard notation, there are three types of topological excitations: the electric excitation e, the magnetic excitation m, and

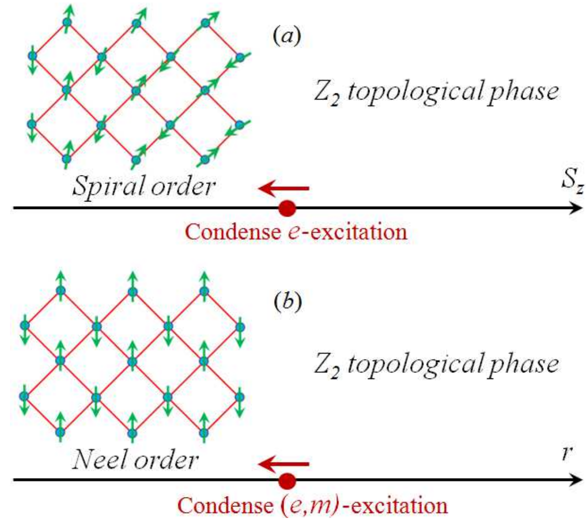

FIG. 2: (a) For a SU(2) invariant system, if one condenses the ordinary spinon of the Z_2 spin liquid phase, the spin ordered state must have ground state manifold SO(3). One example state of this kind is the spiral spin density wave. (b) If the (e, m) type of excitation of the Z_2 spin liquid condenses, the spin order can be the ordinary Néel order. At the Z_2-Néel QCP, both Néel and columnar VBS order parameters have power-law correlation.

their bound state (e, m). In Eq. 6, the spinon field z_α is the $e-$type excitation, while the vison field v is the $m-$type excitation. Eq. 6 is based on the assumption that inside the Z_2 liquid phase the $e-$type and $m-$type excitations have lower energy than (e, m), thus in the global phase diagram Fig. 1, the Néel and Z_2 topological phases are separated by a multicritical point $s_z = s_v = 0$. However, if we consider the opposite possibility, namely the $(e, m)-$type excitation has the lowest energy in the Z_2 spin liquid, then a different quantum phase transition can occur by condensing the $(e, m)-$type excitation.

Let us first take the simplest Toric code model[24] as an example: $H = \sum_i -\sigma^x_{i,-x}\sigma^x_{i,x}\sigma^x_{i,-y}\sigma^x_{i,y} - \sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+x,y}\sigma^z_{i+y,x}$. The "condensation" of an excitation simply means that the system enters a phase were the kinetic energy of this excitation dominates. It is well-known that in the Toric code model the condensation of the $e-$excitation is driven by a magnetic field $h_z\sigma^z_{i,\mu}$, while the condensation of $m-$excitation is driven by field $h_x\sigma^x_{i,\mu}$, because these two fields enable the hopping of e and m excitations respectively. In order to "condense" the (e, m) excitation, we simply need to turn on field $h_y\sigma^y_{i,\mu}$, which hops the (e, m) excitation along the diagonal directions of the square lattice. When any of the three excitations is condensed, the system enters a trivial polarized state without any topological degeneracy. Generally speaking, in the topological phase, starting from one of the topological sectors on the torus, the other sectors can be generated by locally creating a pair of topological excitations, and annihilating them after adiabatically moving one excitation of the pair around the torus. Because all three types of topological excitations are mutual semions, condensing one of the three excitations will lead to a strong local flux fluctuation for the other two excitations, thus the other two excitations are confined in this condensate, *i.e.* the system no longer has topological degeneracy.

In our current case, both e and m excitations carry extra global symmetries besides their gauge charges. In order to describe the (e, m) excitation in our situation, let us define new complex bosonic fields ϕ_α and ψ_α:

$$\phi_\alpha = z_\alpha v, \qquad \psi_\alpha = z_\alpha v^*. \tag{8}$$

ϕ_α and ψ_α carry the quantum number of (e, m) excitation. Because v is a complex variable, fields ϕ_α and ψ_α are independent of each other, and they interact with each other as follows:

$$\begin{aligned}\mathcal{L} \; &= \; \sum_\alpha |(\partial_\mu - ia_\mu - ib_\mu)\phi_\alpha|^2 + |(\partial_\mu - ia_\mu + ib_\mu)\psi_\alpha|^2 \\ &+ \; r(|\phi_\alpha|^2 + |\psi_\alpha|^2) + \frac{i}{\pi}\epsilon_{\mu\nu\rho}a_\mu\partial_\nu b_\rho \\ &+ \; g(|\phi|^2)^2 + g(|\psi|^2)^2 + u|\phi|^2|\psi|^2 - w\phi^\dagger\vec{\sigma}\phi\cdot\psi^\dagger\vec{\sigma}\psi\end{aligned} \tag{9}$$

Notice that ϕ_α and ψ_α carry gauge charges of both gauge fields a_μ and b_μ. In order to understand the QCP at $r = 0$ more quantitatively, it is more convenient to define new gauge field $A^\pm_\mu = a_\mu \pm b_\mu$, then the Lagrangian reads :

$$\begin{aligned}\mathcal{L} \; &= \; \sum_\alpha |(\partial_\mu - iA^+_\mu)\phi_\alpha|^2 + |(\partial_\mu - iA^-_\mu)\psi_\alpha|^2 \\ &+ \; r(|\phi_\alpha|^2 + |\psi_\alpha|^2) + \frac{i}{4\pi}\epsilon_{\mu\nu\rho}A^+_\mu\partial_\nu A^+_\rho - \frac{i}{4\pi}\epsilon_{\mu\nu\rho}A^-_\mu\partial_\nu A^-_\rho \\ &+ \; g(|\phi|^2)^2 + g(|\psi|^2)^2 + u|\phi|^2|\psi|^2 - w\phi^\dagger\vec{\sigma}\phi\cdot\psi^\dagger\vec{\sigma}\psi.\end{aligned} \tag{10}$$

In this field theory, ϕ_α and ψ_α are almost decoupled from each other, *i.e.* they are only coupled through the quartic terms u and w. The mass gaps r for ϕ_α and ψ_α are equal, because the vison modes v and v^* are guaranteed to be degenerate by the symmetry of the square lattice. ϕ_α and ψ_α are introduced as bosonic fields, but gauge fields A^+_μ and A^-_μ make them fermionic fields after the standard flux attachment, due to the existence of the Chern-Simons terms in this Lagrangian. In our formulation, fields ϕ_α and ψ_α can still condense by tuning parameter r in Eq. 10. After ϕ_α and ψ_α both condense simultaneously, A^+_μ and A^-_μ are both gapped out based on the Higgs mechanism, and in the Higgs phase the only gauge invariant operators are

$$\phi^\dagger\vec{\sigma}\phi, \qquad \psi^\dagger\vec{\sigma}\psi. \tag{11}$$

Since these two vectors both carry the same quantum number as the Néel order parameter $z^\dagger\vec{\sigma}z$, in Eq. 10 w is naturally positive, thus these two vectors are aligned parallel with each other, so the condensate of ϕ_α and ψ_α has a manifold S^2, *i.e.* it is the standard collinear Néel order. The difference between this new transition and the ordinary spinon theory is illustrated in Fig. 2.

What is the universality class of this transition? The simplest possibility is that, both u and w are irrelevant at the transition, although they are relevant in the condensate of ϕ_α and ψ_α. Under this assumption ϕ_α and ψ_α are completely decoupled at the transition $r = 0$. Then in this case this transition is described by the simple Chern-Simons-Higgs model:

$$\begin{aligned}\mathcal{L} \; &= \; \sum_{\alpha=1}^{N} |(\partial_\mu - iA_\mu)\phi_\alpha|^2 + r|\phi_\alpha|^2 + g(\sum_\beta|\phi_\beta|^2)^2 \\ &+ \; \frac{iN}{8\theta}\epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho.\end{aligned} \tag{12}$$

Here we have generalized the equation to have N flavors of matter fields ϕ_α, and introduced a statistical angle θ. Our physical situation corresponds to $N = 2$ and $\theta = \pi$. Notice that Eq. 12 explicitly breaks the time-reversal symmetry due to the Chern-Simons term. But the complete theory Eq. 10 is time-reversal invariant, because under time-reversal transformation ϕ_α and ψ_α are exchanged, the two gauge fields A^+_μ and A^-_μ are also exchanged.

The critical exponents of this transition can be computed using a systematic $1/N$ expansion. Ref.[33] has computed the critical exponent ν defined as $\xi \sim |r|^{-\nu}$, here we will focus on the scaling dimension of $\phi^\dagger T^a\phi$ at the QCP, where T^a is the SU(N) generator. To the first order $1/N$ expansion, this scaling dimension reads

$$\Delta[\phi^\dagger T^a\phi] = 1 + \frac{4}{3\pi^2}\left(\frac{4}{N} - \frac{1}{N}\frac{\theta^2/4}{1+\theta^2/64}\right). \tag{13}$$

Thus the so-called anomalous dimension of the Neel order parameter reads

$$\eta = 1 + \frac{8}{3\pi^2}\left(\frac{4}{N} - \frac{1}{N}\frac{\theta^2/4}{1+\theta^2/64}\right). \tag{14}$$

This number can be checked numerically. If such quantum phase transition between Néel order and topological order is discovered in experimental systems, then this critical exponent η can be detected by measuring the spin-lattice relaxation rate $1/T_1 \sim T^\eta$ in nuclear magnetic resonance (NMR) experiments at the quantum critical region of the phase diagram[43].

Let us briefly comment how we obtain this result in Eq. 13. Similar calculation without the θ term was obtained before. See Fig.3 and Fig.4 of the previous work[39] for necessary Feynman diagrams. First, we need to evaluate wave function renormalization of ϕ from both gauge fluctuation and the density fluctuation, which contain the factor$(1/N)$. Then, using the standard operator insertion method, one can calculate renoramlization function of the corresponding vertex. We note that the traceless condition, $(\mathrm{Tr}(T^a) = 0)$, reduces one diagram compared with the calculation of the scaling dimension of $|\vec{\phi}|^2$ and simplify our calculation.

In the limit of $\theta \to \infty$, *i.e.* the CS term is effectively zero, these results converges to the ordinary CP$(N-1)$ results computed in Ref.[39]. In the limit of $\theta \to 0$, the gauge fluctuation is totally frozen by the CS term, and the universality class of this quantum critical point only acquires corrections from the short range self-interaction between field ϕ_α, thus it is equivalent to an O$(2N)$ transition of the O$(2N)$ bosonic vector field $(\mathrm{Re}[\phi_1], \cdots \mathrm{Re}[\phi_N], \mathrm{Im}[\phi_1], \cdots \mathrm{Im}[\phi_N])$. Scaling dimension of the "Néel" type operator $(\Delta[\phi^\dagger T^a \phi])$ in our theory is larger than that in the CP$(N-1)$ theory with large N, *i.e.* at the Z_2-Néel QCP, the anomalous dimension of the Néel order parameter is predicted to be larger than that of the deconfined QCP between the Néel and VBS order. This prediction can be tested in the future by a careful comparison between the critical exponents of the $J_1 - J_2$ model and the $J - Q$ model[40–42].

It is pretty clear that at least in the large$-N$ limit, the perturbation of u in Eq. 10 is irrelevant, because in this limit the scaling dimension $\Delta[|\vec{\phi}|^2] = \Delta[|\vec{\psi}|^2] = 2$, *i.e.* $\Delta[u] = -1$. Higher order $1/N$ or ϵ expansion is demanded to determine whether w is relevant or not at this transition.

Assuming at the QCP $r = 0$ both u and w are irrelevant, then besides the Néel order parameter, some other physical order parameters also have power-law correlation. For example, the columnar VBS order parameter can be written as

$$\mathrm{VBS} \sim \psi^\dagger_\alpha \phi_\alpha \mathcal{M}_a \sim v^2 \mathcal{M}_a, \tag{15}$$

where $\mathcal{M}_a$ is the monopole operator for gauge field a_μ. When ϕ_α and ψ_α both have a large N component, the scaling dimension of $\mathcal{M}_a$ is proportional to N. Thus with large N the VBS order parameter is expected to have a much larger scaling dimension compared with the Néel order parameter at the Z_2-Néel QCP. We stress that the VBS order parameter has short-range correlation in the Z_2 spin liquid and the Néel phase, its emergent quasi long range correlation occurs *only* at the QCP. This result has already been confirmed numerically in Ref.[16], and it was demonstrated that the scaling dimension of the VBS order parameter is indeed larger than that of Néel order at the QCP[16].

In 2+1 dimension, the entanglement entropy of a conformal field theory can in general be written as $S = cL - \beta$, where the first term is the nonuniversal area law contribution, while the second term is a universal constant. In Ref.[44], it was argued that at a QCP where a bosonic field condenses while coupling to a discrete gauge field, the universal entanglement entropy is a direct sum of the contribution from the bosonic matter field and the contribution from the discrete gauge field: $\beta = \beta_b + \beta_{gauge}$. This conclusion is based on the assumption that the matter field dynamics is not affected by the discrete gauge field in the infrared limit, and this is indeed true for the XY* transition observed in Ref.[22]. However, at the exotic Z_2-Néel transition discussed here where the $(e, m)-$type excitations condense, the bosonic matter fields ϕ_α and ψ_α are indeed strongly affected by the gauge field, thus at this QCP the universal entanglement entropy β is no longer a direct sum of the two different degrees of freedom of the system. The universal entanglement entropy of field theory Eq. 12 in the large$-N$ limit can be found in Ref.[45].

C. A Toy model with $N = 1$

Now let us discuss a toy model with $N = 1$. This is actually the case where the critical theories can be all understood exactly. This field theory with $N = 1$ can be applied to the following extended Toric-code model:

$$\begin{aligned} H &= \sum_i K_x \sigma^x_{i,-x}\sigma^x_{i,x}\sigma^x_{i,-y}\sigma^x_{i,y} + K_z \sigma^z_{i,x}\sigma^z_{i,y}\sigma^z_{i+x,y}\sigma^z_{i+y,x} \\ &+ \sum_{i,\mu} h_x \sigma^x_{i,\mu} + h_z \sigma^z_{i,\mu} + \cdots . \end{aligned} \tag{16}$$

Here the $e-$type ($m-$type) excitation is the end of a string product of σ^x (σ^z). The e and $m-$type excitations view σ^z and σ^x as Z_2 gauge fields respectively, and the h_x and h_z terms enable the hopping of these excitations. Unlike the standard toric-code model[24], here we keep $K_x,\ K_z > 0$. When $K_x,\ K_z > 0$, both σ^z and σ^x have a $\pi-$flux in the ground state. Then the dynamics of both e and m type of excitations are frustrated, and both excitations have two different minima $\pm\vec{Q}$ in their band structure. As a result, the low energy dynamics of e and m excitations are described by complex scalar fields z and v expanded at momentum $\vec{Q}$. If one of these two fields condenses while the other one remains gapped, the Z_2 topological order is destroyed, and the system must spontaneously break the lattice translation symmetry as well. The condensates of e and $m-$type excitations physically correspond to the valence bond solid phases of σ^z and σ^x respectively.

When $N = 1$, if e and m type of excitations condense separately, then the phases in Fig. 1 would be Z_2 liquid, VBS order of σ^x, trivial phase, and VBS order of σ^z (counted counterclockwise around the multicritical point $s_z = s_v = 0$). On the other hand, if the bound state (e, m) has the lowest energy in the Z_2 liquid phase, then again we can introduce two independent complex fields ϕ and ψ as $\phi = zv$, $\psi = zv^*$. Then the transition driven by the condensation of ϕ and ψ is described by Eq. 12 with $N = 1$ and $\theta = \pi/2$.

What kind of transition is this? If in Eq. 12 the complex field ϕ is also coupled to an external U(1) gauge field A^{ext}_μ, then we can see that in the disordered phase of ϕ, after integrating out the massive ϕ and dynamical gauge field A_μ, the lowest order contribution to the effective Lagrangian of A^{ext}_μ is still a Maxwell term: $\mathcal{L}_{eff} \sim (\partial A^{ext})^2 + c\theta(\partial^2 A^{ext})(\partial A^{ext}) + \cdots$. While in the condensate of ϕ, the effective Lagrangian of A^{ext}_μ acquires a Chern-Simons term at level 1. This analysis implies that this transition is equivalent to a topological transition between a trivial insulator and a Chern insulator with Chern number 1. The universality class of this type of topological transition of Chern insulator is very well-understood, it can be simply described by a 2+1d Dirac fermion:

$$\mathcal{L} = \bar{\psi}\gamma_\mu\partial_\mu\psi + m\bar{\psi}\psi, \tag{17}$$

here the trivial insulator and Chern insulator correspond to $m > 0$ and $m < 0$ respectively, and $m = 0$ corresponds to the quantum critical point at $r = 0$ in Eq. 12 with $N = 1$. Thus we conjecture that when $N = 1$ and $\theta = \pi/2$, the critical point in Eq. 12 is dual to a massless free Dirac fermion. In Ref.[46] a similar conjecture was made that the 3D XY transition is dual to a massless Dirac fermion coupled to a noncompact U(1) gauge field.

IV. SUMMARY AND DISCUSSION

In this work we have discussed a possible theory for the direct continuous transition between the Z_2 liquid phase and the Néel order, and this is a candidate theory for the liquid-Néel transition observed in Ref.[15,23]. We have taken the square lattice as an example, but results discussed in this paper can also be applied to the honeycomb lattice after straightforward generalization.

Evidences of a quantum transition between spin liquid and magnetic ordered phase were discovered in the material $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$[47]. It was also proposed that the ground state of $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$ is a Z_2 spin liquid with Bosonic spinons[43]. Thus the theory proposed in our current paper may be applied to the material $\kappa-(\mathrm{ET})_2\mathrm{Cu}_2(\mathrm{CN})_3$, after the nature of its ground state is further clarified. The critical exponent η calculated in Eq. 14 can be measured by NMR relaxation rate $1/T_1 \sim T^\eta$ at the quantum critical region.

In our theory, we exploited the fact that in two spatial dimensions, the e and $m-$type excitations are both point like defects, thus their nontrivial statistics can be described well with a mutual Chern-Simons theory. By contrast, in a three dimensional Z_2 liquid phase, there is a mutual semion statistics between the point particle like $e-$excitation and loop like $m-$type excitation. Thus the effective field theory for the three dimensional Z_2 liquid phase is the so-called BF theory $\mathcal{L}_{eff} \sim \frac{i}{\pi}\epsilon_{\mu\nu\rho\tau}a_\mu\partial_\nu b_{\rho\tau}$, where a_μ is the U(1) gauge field that couples to the $e-$type point particle, and $b_{\mu\nu}$ is an antisymmetric rank-2 antisymmetric tensor gauge field that couples to the $m-$type loop excitation. The global phase diagram around the three dimensional Z_2 liquid phase is another interesting subject, and we will leave it to future studies.

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