# Edge states for topological insulators in two dimensions and their Luttinger-like liquids

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# Edge states for topological insulators in two dimensions and their Luttinger-like liquids

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Topological insulators in three spatial dimensions are known to possess a precise bulk/boundary correspondence, in that there is a one-to-one correspondence between the 5 classes characterized by bulk topological invariants and Dirac hamiltonians on the boundary with symmetry protected zero modes. This holographic characterization of topological insulators is studied in two dimensions. Dirac hamiltonians on the one dimensional edge are classified according to the discrete symmetries of time-reversal, particle-hole, and chirality, extending a previous classification in two dimensions. We find 17 inequivalent classes, of which 11 have protected zero modes. Although bulk topological invariants are thus far known for only 5 of these classes, we conjecture that the additional 6 describe edge states of new classes of topological insulators. The effects of interactions in two dimensions are also studied. We show that all interactions that preserve the symmetry are exactly marginal, i.e. preserve the gaplessness. This leads to a description of the distinct variations of Luttinger liquids that can be realized on the edge.

## I. INTRODUCTION

Topological insulators are characterized by bulk wave-functions in $d$ spatial dimensions with special topological properties characterized by certain topological invariants, such as the Chern number[1–8]. These physical systems possess a kind of holography, or bulk/boundary correspondence, in that they necessarily have protected gapless excitations on the $\overline{d} = d - 1$ dimensional surface. These surface modes are typically described by Dirac hamiltonians. For example in the integer quantum Hall effect (QHE) in $d = 2$, the Chern number is the same integer as in the quantized Hall conductivity, and the edge states are chiral Dirac fermions.

Schnyder *et al.*[9], Ryu *et al.*[10] and Kitaev[11] classified topological insulators in any dimension according to the discrete symmetries of time reversal $\mathbf{T}$, particle-hole symmetry $\mathbf{C}$ and chirality $\mathbf{P}$ and found 5 classes of topological insulators in any dimension. See also[12]. These classifications relied on generic properties in any dimension, namely the homotopy groups of replica sigma models for Anderson localization[9,10], or the 8-fold periodicity property of spinor representations of $so(n)$ based on their Clifford algebras, which is a mild form of Bott-periodicity in K-theory[11].

The bulk/boundary correspondence was described explicitly in[9] for $d = 3$ spatial dimensions: using the classification of $\overline{d} = 2$ dimensional Dirac hamiltonians in[13], it was found that precisely 5 of the 13 Dirac classes had protected surface states with the predicted discrete symmetries. In that analysis, it was crucial that the classification in[13] contained 3 additional classes beyond the 10 Altland-Zirnbauer (AZ) classes[14], since it was precisely these additional classes that corresponded to some of the topological insulators. The reason that there are more classes of Dirac hamiltonians is that AZ classify finite dimensional hermitian matrices (hamiltonians) without assuming any Dirac structure.

In this paper we explore this 'holographic classification' of topological insulators (TI's) and topological superconductors (TS's) in $d = 2$ spatial dimensions, in order to ascertain whether it works out as nicely as for $d = 3$. The general $d$ dimensional case will be presented elsewhere[15]. It is not obvious from the beginning that this holographic approach should reproduce precisely the classifications based on topological invariants. For instance, Anderson localization properties are generally different in $d < 2$ verses $d > 2$. Also, we assume that the surface states can be realized as Dirac fermions, which is an additional constraint on top of the discrete symmetries under consideration. More importantly, there is no guarantee that there exists a microscopic 2D model with topological wave-functions with the edge modes we classify. However the subsequent holographic classification by two of us[15] in arbitrary dimensions strengthens the case for the holographic approach as it was found using only generic properties of Clifford algebras that this approach gives precisely the known TI's and nothing more in odd dimensions. In even dimensions with $d \neq 2$, only one additional class with protected surface Dirac fermions were found. The $d = 2$ case turned out to be special and it is the focus of this paper. Also, it is important to examine this holographic classification since the edge states are the most experimentally accessible properties.

This study requires a classification of Dirac hamiltonians in $\overline{d} = 1$, which is carried out for the first time below. We identify 17 unitarily-inequivalent classes. Since the classifications in[9–11] were based on *generic* properties in any dimension, it is possible that there exist more classes of topological insulators in $d = 2$ due to this richer structure specific to $\overline{d} = 1$. Indeed, based on our classification, we find 11 classes of Dirac hamiltonians with protected zero modes on the 1 dimensional edge. In addition to the previously predicted topological insulators in classes A, C, D, DIII, and AII, we find the classes AIII, BDI, two versions of CII, an additional version of DIII, and a $\mathbb{Z}_2$ version of D (the definition of these classes will be reviewed below; the notation goes back to Cartan). One interpretation is that, unlike in $d = 3$, for $d = 2$ there are classes of $\overline{d} = 1$ Dirac hamiltonians that are protected for reasons other than the existence of a topological invariant for the $d = 2$ band structure. On the other hand, our new classes could in principle be characterized by some as yet unknown bulk topological invariants. Although this distinction needs to be kept in mind, henceforth, for simplicity, we will refer to all classes with protected zero modes on the boundary as TI's.

For the QHE, bulk interactions lead to the fractional QHE, and the effect of these interactions is that the edge states become Luttinger liquids[16]. This is unique to $d = 2$ since only in this dimension are quartic interactions on the boundary marginal, which is not unrelated to the fact that anyons only exist in 2 dimensions. Thus a criterion for the possible effects of bulk interactions is the existence of *exactly* marginal perturbations of the free boundary Dirac hamiltonian that are consistent with the discrete symmetries, since an exactly marginal perturbation deforms the theory but keeps it gapless. This leads us to also classify quartic, exactly marginal perturbations that are consistent with the discrete symmetries. In addition to the ordinary, chiral and helical Luttinger liquids, we find the possibility of 3 additional varieties in the classes DIII and CII.

The sections below cover the following. In section II we review the definitions of the 10 AZ classes. Section III reviews the holographic classification of TI in $d = 3$. One-dimensional Dirac hamiltonians are classified in section IV. This classification is completely general, and could have applications in other areas, such as disordered systems. In section V, we identify the Dirac theories with protected zero modes, and section VI describes their consequent Luttinger liquids.

## II. DISCRETE SYMMETRIES

The 10 Altland-Zirnbauer (AZ) classes of random hamiltonians arise when one considers time reversal symmetry (**T**), particle-hole symmetry (**C**), and parity or chirality (**P**). These discrete symmetries are defined to act as follows on a first-quantized hamiltonian $\mathcal{H}$:

$$\begin{aligned} \mathbf{T}: \qquad & T\mathcal{H}^*T^\dagger = \mathcal{H} \\ \mathbf{C}: \qquad & C\mathcal{H}^TC^\dagger = -\mathcal{H} \\ \mathbf{P}: \qquad & P\mathcal{H}P^\dagger = -\mathcal{H} \end{aligned} \tag{1}$$

with $TT^\dagger = CC^\dagger = PP^\dagger = \mathbf{1}$. We consider two hamiltonians $\mathcal{H}, \mathcal{H}'$ related by a unitary transformation $\mathcal{H}' = U\mathcal{H}U^\dagger$ to be in the same class, since they have the same eigenvalues. For $C$ and $T$, this translates to $C \to C' = UCU^T$ and $T \to T' = UTU^T$. For $P$, it amounts to $P \to P' = UPU^\dagger$. It is thus important to identify s these unitary equivalences in order not to overcount classes. We will sometimes refer to these unitary transformations as gauge transformations.

| AZ-classes | $T^2$ | $C^2$ | $P^2$ |
|---|---|---|---|
| A | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| AIII | $\emptyset$ | $\emptyset$ | 1 |
| AII | $-1$ | $\emptyset$ | $\emptyset$ |
| AI | $+1$ | $\emptyset$ | $\emptyset$ |
| C | $\emptyset$ | $-1$ | $\emptyset$ |
| D | $\emptyset$ | $+1$ | $\emptyset$ |
| BDI | $+1$ | $+1$ | 1 |
| DIII | $-1$ | $+1$ | 1 |
| CII | $-1$ | $-1$ | 1 |
| CI | $+1$ | $-1$ | 1 |

TABLE I. The 10 Altland-Zirnbauer (AZ) hamiltonian classes. $\emptyset$ denotes the absence of respective symmetry.

For hermitian hamiltonians, $\mathcal{H}^T = \mathcal{H}^*$, thus, up to a sign, **C** and **T** symmetries are the same. We focus then on these symmetries involving the transpose: $T\mathcal{H}^TT^\dagger = \mathcal{H}$ and $C\mathcal{H}^TC^\dagger = -\mathcal{H}$. Taking the transpose of this relation, one finds there are two consistent possibilities: $T^T = \pm T, C^T = \pm C$, which are unitarily-invariant relations. It turns out that unitary transformations allow us to choose $T, C$ to be real; unitarity of $T, C$ then implies $C^2 = \pm 1, T^2 = \pm 1$. The various classes are thus distinguished by $T^2 = \pm 1, \emptyset$ and $C^2 = \pm 1, \emptyset$, where $\emptyset$ indicates that the hamiltonian does not have the symmetry, and the sign is equivalent to the sign in the relation between $T, C$ and their transpose. One obtains $9 = 3 \times 3$ classes just by considering the 3 cases for **T** and **C**. If the hamiltonian has both **T** and **C** symmetry, then it automatically has a **P** symmetry, with $P = TC^\dagger$ up to a phase. If there is neither **T** nor **C** symmetry, then there are two choices $P = \emptyset, 1$, and this gives the additional class AIII, leading to a total of 10. Their properties are shown in Table I. We also mention that one normally requires $P^2 = 1$. Below, we will require **T** and **C** to commute, thus $P^2 = T^2{C^\dagger}^2 = \pm 1$. However one has the freedom $P \to iP$ to restore $P^2 = 1$. In the sequel, in the cases with both $\mathbf{T}, \mathbf{C}$ symmetry, we simply define $P = TC^\dagger$, up to a phase.

## III. REVIEW OF THE $\overline{d} = 2$ DIMENSIONAL CASE

The connection between the bulk topological properties and the existence of protected zero modes on the boundary was first pointed out for $d = 3$ by Schnyder et. al.[9]. This relied on the classification of $\overline{d} = 2$ dimensional Dirac hamiltonians found by two of us[13]. In this section we review this holographic classification of $d = 3$ TI's since this illustrates what we are attempting to accomplish in $d = 2$.

If one requires a Dirac structure of the hamiltonian, then the AZ classification can be more refined. The most general hamiltonian in $\overline{d} = 2$ dimensions is of the form:

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\partial_{\overline{z}} + A_{\overline{z}} \\ -i\partial_z + A_z & V_+ - V_- \end{pmatrix} \quad (2)$$

where $\partial_z = \partial_x - i\partial_y, \partial_{\overline{z}} = \partial_x + i\partial_y$ with $x, y$ the spatial coordinates and $V_\pm, A_{z,\overline{z}}$ are matrices. The above $\mathcal{H}$ is just a relabeling of $\mathcal{H} = -i\sigma_x\partial_x - i\sigma_y\partial_y + \vec{\sigma}\cdot\vec{V} + V_0$, i.e. the block structure comes from the Pauli matrices $\sigma$.

One then finds the most general form of the $T, C, P$ matrices that preserve the Dirac structure. Thirteen inequivalent classes were found[13]. In particular, there exist two inequivalent versions of the chiral classes AIII, DIII, and CI, simply because the discrete symmetries can take different forms. In was shown in[9] that precisely 5 of the 13 classes corresponded to the surface states of TI's, with discrete symmetries consistent with the predictions from bulk topology. As argued there, the criterion for a TI is that $V_-$ has a zero mode, i.e. det $V_- = 0$. This led to the following identification of TI's, where the nomenclature of[13] is given in parentheses. As far as the bulk properties, the are two types of topological invariants, $\mathbb{Z}$ and $\mathbb{Z}_2$, which are also indicated. In the holographic approach, $\mathbb{Z}$ verses $\mathbb{Z}_2$ corresponds to the two ways of obtaining a zero mode, namely $V_- = 0$ or $\det V_- = -\det V_-$ for $V_-$ odd dimensional, and the exceptional case CII, which is also $\mathbb{Z}_2$. (See section VA for a more detailed discussion of these topological identifications.).

- **AIII (1) , DIII (5) , CI (6)** . These are the three classes that are doubled in comparison with AZ. For one of the two in each these classes, the discrete symmetry forces $V_- = 0$. These are all of type $\mathbb{Z}$ or $2\mathbb{Z}$.[35]

- **AII** ($3_+$). Here the discrete symmetries require $V_-^T = -V_-$, which implies that if $V_-$ is odd-dimensional, det $V_- = 0$. Type $\mathbb{Z}_2$.

- **CII** ($9_-$). In this case, the discrete symmetries constrain $V_- = \begin{pmatrix} 0 & v_- \\ w_- & 0 \end{pmatrix}$ with $v_-^T = -v_-, w_-^T = -w_-$. Thus if $v_-, w_-$ are odd-dimensional, then up to a sign, det $V_- =$ det $v_-$ det $w_- = 0$. Type $\mathbb{Z}_2$.

## IV. THE $\overline{d} = 1$ DIMENSIONAL CLASSIFICATION OF DIRAC HAMILTONIANS

In this section, we present the complete classification of $\overline{d} = 1$ dimensional Dirac hamiltonians. Although the identification of TI's and TS's will be the subject of the next section, it is useful to motivate what follows by discussing chiral Dirac Hamiltonians with only right moving or left moving fermions[36]. Since a mass term necessarily couples left and right movers (see section V), these classes have a protected zero mode for somewhat trivial reasons. Such Hamiltonians cannot be realized on a 1d lattice and they necessarily break **T** and **P**. However they can appear as a $\overline{d} = 1$-edge state of a 2d TI or TS in classes A, C, and D which break both **T** and **P**. An example of class A is the quantum Hall effect. Depending on the number of filled Landau levels there are $\mathbb{Z}$ number of edge states[1]. An example of class C is the spin quantum Hall effect in a singlet time-reversal breaking superconductor. The spin quantum Hall conductivity will be proportional to the Cooper pair angular momentum, hence this is a $\mathbb{Z}$ TS. Although there is no known experimental realization, $d_{x^2-y^2} + id_{xy}$ superconductor (SC) was extensively discussed theoretically[18,19]. A realization of class D would be the thermal Hall effect of a time-reversal breaking superfluid of spinless (fully spin polarized) fermions. The $\nu = 5/2$ quantum Hall state could be a $p_x + ip_y$ paired superfluid of composite fermions[20].

All non-"chiral" non-interacting 1d Dirac hamiltonians with equal number of right-movers and left-movers can be written as $\mathcal{H} = -i\sigma_x\partial_x + \vec{\sigma}\cdot\vec{A} + V_+$, where $\vec{\sigma}$ are the Pauli matrices acting on a space of right/left-movers $|\sigma_x = \pm\rangle$. Redefining $A_z = V_-$, these hamiltonians can be expressed as

$$\mathcal{H} = \begin{pmatrix} V_+ + V_- & -i\partial_x + A \\ -i\partial_x + A^\dagger & V_+ - V_- \end{pmatrix}. \quad (3)$$

The potentials $V_\pm$ are hermitian matrices and $A = A_x + iA_y$ where $A_{x,y}$ are also hermitian matrices in general. The dimension of $V_\pm$ and $A$ is the number of edge mode species for each chirality. When $V_\pm$ and $A$ are even dimensional we use $\vec{\tau}$ to denote a set of Pauli matrices acting on the even dimensional flavor space. **1** will denote the identity in either the $\sigma$ or $\tau$ space. Note that $\vec{\sigma}$ and $\vec{\tau}$ have distinct physical meaning: $\vec{\sigma}$ acts on the space of "chirality" as we show explicitly in sectionV B, and it is responsible for the block structure of Eq.(3), whereas $\vec{\tau}$ acts on the space of flavors which could be spin or pseudo-spin. If there is spin-momentum locking (see sectionV B) $\vec{\sigma}$ will act on the spin space as well as on the space of "chirality".

The Dirac derivative structure of $\mathcal{H}$ constrains the form of $T, C,$ and $P$ in terms of $\vec{\sigma}$ and $\vec{\tau}$. Furthermore, we can specify the conditions $V_\pm$ and $A$ have to satisfy in order for $\mathcal{H}$ to have discrete symmetries under specific $T, C,$ or $P$. Hence the specific forms of symmetry transformations can be used to classify hamiltonians of form Eq.(3). Since, as described below, there are multiple sets of matrices $T, C, P$ with the same $T^2$, $C^2$, $P^2$, this scheme refines the AZ classification of Table I. Here we find even more classes of Dirac hamiltonians in $\overline{d} = 1$ than in $\overline{d} = 2$, and more classes with symmetry protected zero modes (see sectionV).

In the rest of this section, we first specify the forms of $T, C$ and $P$ symmetry that preserve the Dirac structure, and describe the resulting conditions on $V_\pm$ and $A$ in a fixed $\vec{\sigma}$ basis and arrive at 25 classes as summarized in

Table II. We then check for unitary equivalences. The unitary transform is

$$\mathcal{H} \to U_\theta \mathcal{H} U_\theta^\dagger \tag{4}$$

with $U_\theta$ a rotation about the $x$-axis in $\sigma$-space by an angle $\theta$:

$$U_\theta = u \cdot e^{i\theta\sigma_x/2} = u \cdot (\mathbf{1}\cos(\theta/2) + i\sigma_x \sin(\theta/2)) \tag{5}$$

where $u$ is unitary and commutes with $\sigma_x$. We find 17 unitarily-inequivalent classes, each forming a row separated by a horizontal line in Table II.

Consider first the $\mathbf{T}$ symmetry. In order to preserve the derivative structure of the hamiltonian Eq.(3), using $(-i\partial_x)^T = i\partial_x$, one finds that $T$ must anti-commute with $\sigma_x$. Since $T$ is (anti)-symmetric and unitary, it is then either proportional to $\sigma_z$ or $i\sigma_y$. This leads to 2 ways of implementing of $\mathbf{T}$-symmetry transformations: using either

$$T = \eta_t \otimes i\sigma_y = \begin{pmatrix} 0 & \eta_t \\ -\eta_t & 0 \end{pmatrix} \tag{6}$$

$$\widetilde{T} = \widetilde{\eta}_t \otimes \sigma_z = \begin{pmatrix} \widetilde{\eta}_t & 0 \\ 0 & -\widetilde{\eta}_t \end{pmatrix}, \tag{7}$$

where $\eta_t$ or $\widetilde{\eta}_t$ are unitary matrices in general. Then, for a hamiltonian of form Eq.(3) to have $\mathbf{T}$ symmetry the potentials have to satisfy either

$$\eta_t\, V_\pm^T = \pm V_\pm\, \eta_t, \quad \eta_t\, A^T = -A\, \eta_t \tag{8}$$

or

$$\widetilde{\eta}_t\, V_\pm^T = V_\pm\, \widetilde{\eta}_t, \quad \widetilde{\eta}_t\, A^* = -A\, \widetilde{\eta}_t \tag{9}$$

Now the condition $T^T = \pm T$ ($T^2 = \pm 1$) which distinguishes AI from AII for instance, implies either $\eta_t^T = \pm\eta_t$ or $\widetilde{\eta}_t^T = \pm\widetilde{\eta}_t$. Hence all AZ classes with $\mathbf{T}$-symmetry are further refined depending on whether $T$ (Eq.(6)) or $\widetilde{T}$ (Eq.(7)) is used to implement $\mathbf{T}$. This distinction has a physical significance: the use of $T \propto i\sigma_y$ leads to spin-momentum locking (see section V B).

Finally we can choose representations of $\eta_t$ in terms of $\vec{\tau}$ up to the unitary transformations: $\eta_t = \mathbf{1}$ if $\eta_t^T = \eta_t$, and $\eta_t = i\tau_y$ if $\eta_t^T = -\eta_t$[21]. We can do the same for $\widetilde{\eta}_t$. The unitary transform $T \to UTU^T$ corresponds to $\eta \to u\eta u^T$ with $u$ unitary, for all $\eta$'s. The unitary transformation affects the choice of $\mathbf{1}$ v.s. $\tau_x$ for $\eta_t$'s. However the unitary transform cannot affect the distinction between $T$ and $\widetilde{T}$. In particular when $\mathbf{T}$ is the only available discrete symmetry, $T^2, \widetilde{T}^2 = \pm 1$ completely classifies $\overline{d} = 1$ Dirac Hamiltonians into $\mathrm{AI}_{(1)}$, $\mathrm{AI}_{(2)}$ and $\mathrm{AII}_{(1)}$, $\mathrm{AII}_{(2)}$. See Table II.

We can specify $C$, following steps analogous to those for specifying $T$. As $C$ must commute with $\sigma_x$ for Dirac hamiltonian Eq.(3), it is in the linear span of $\mathbf{1}$ and $\sigma_x$. Hence there are two possibilities:

$$\begin{aligned} C &= \eta_c \otimes \sigma_x, \quad \eta_c\, V_\pm^T = \mp V_\pm\, \eta_c,\ \eta_c\, A^T = -A\, \eta_c \\ \widetilde{C} &= \widetilde{\eta}_c \otimes \mathbf{1}, \quad \widetilde{\eta}_c\, V_\pm^T = -V_\pm\, \widetilde{\eta}_c,\ \widetilde{\eta}_c\, A^* = -A\, \widetilde{\eta}_c \end{aligned} \tag{10}$$

with $\eta_c$ and $\widetilde{\eta}_c$ unitary. The condition $C^T = \pm C$ that distinguishes AZ class C from D for instance, implies that $\eta_c^T = \pm\eta_c$ or $\widetilde{\eta}_c^T = \pm\widetilde{\eta}_c$. One can again represent up to unitary transformations $\eta_c = \mathbf{1}$ if $\eta_c^T = \eta_c$, and $\eta_c = i\tau_y$ if $\eta_c^T = -\eta_c$. This again refines the AZ classes with $\mathbf{C}$ symmetry. However unlike $T$ and $\widetilde{T}$ which are unitarily-inequivalent, $C$ and $\widetilde{C}$ are unitarily-equivalent for non-zero $A_y$ (see the end of this section). We denote such unitarily-equivalent refinements using primed notation within the same row in Table II. In particular, this completes our classification of $\overline{d} = 1$ Dirac hamiltonians with only $\mathbf{C}$ symmetry into C, C', D, D'.

Consider now $\mathbf{P}$ symmetry. $P$ must anti-commute with $\sigma_x$ for the Dirac hamiltonian Eq.(3), so $P$ is in the linear span of $\sigma_y$ and $\sigma_z$. For $P$ unitary, this implies that $P = \eta_p \cdot (\cos b\ \sigma_y + \sin b\ \sigma_z)$ for some real $b$. All these choices are unitarily-equivalent by rotations around the $x$-axis in the sigma space. However, in order to accommodate $P = TC^\dagger$ in all cases, we define two unitarily-equivalent types:

$$\begin{aligned} P &= \eta_p \otimes \sigma_z \quad \eta_p\, V_\pm = -V_\pm\, \eta_p,\ \eta_p\, A = A\, \eta_p \\ \widetilde{P} &= \widetilde{\eta}_p \otimes i\sigma_y \quad \widetilde{\eta}_p\, V_\pm = \mp V_\pm\, \widetilde{\eta}_p,\ \widetilde{\eta}_p\, A^\dagger = A\, \widetilde{\eta}_p \end{aligned} \tag{11}$$

where $\eta_p$ and $\widetilde{\eta}_p$ are unitary. The unitary freedom reduces to $\eta_p \to u\eta_p u^\dagger$ and the same for $\widetilde{\eta}$. Up to unitary transformations there are two choices: $\eta_p, \widetilde{\eta}_p = 1$ or $\tau_z$. This gives 4 AIII classes.

Finally for the classes with both $\mathbf{T}, \mathbf{C}$ symmetries, $\mathbf{T}$ and $\mathbf{C}$ must either commute or anti-commute[12]. The argument is simple. Given both $T$ and $C$, a $P$ symmetry is provided by $P = TC^\dagger$ or $P = C^\dagger T$. These two $P$'s must be equivalent up to a sign since $P^2 = 1$, thus $TC^\dagger = \pm C^\dagger T$, which is a gauge-invariant condition. Thus $T, C$ commute or anti-commute, since in all cases, $C^\dagger = \pm C$.

Now the AZ classes BDI, CI, DIII, and CII refines into 12 classes; among these 8 are gauge inequivalent. We label the three subclasses associated with the BDI class by $\mathrm{BDI}_{(1)}$, $\mathrm{BDI}_{(2)}$, $\mathrm{BDI}'_{(2)}$, and similarly for CI, DIII, and CII. Table II shows this classification with respective representations of $\mathbf{T}$, $\mathbf{C}$ and $\mathbf{P}$. In some cases $\eta_t$ or $\eta_c$ had to be taken to be $\tau_x$ which is unitarily-equivalent to $\mathbf{1}$, in order for $\mathbf{T}$ and $\mathbf{C}$ to anti-commute. When there are both $\mathbf{T}, \mathbf{C}$ symmetries, then there is automatically a $P = TC^\dagger$ symmetry (up to a phase). Depending on the type of $C, T$, one finds the $\mathbb{Z}_2$ graded multiplication: $P = TC^\dagger, P = \widetilde{T}\widetilde{C}^\dagger, \widetilde{P} = T\widetilde{C}^\dagger, \widetilde{P} = \widetilde{T}C^\dagger$. This gives $\eta_p = \eta_t\eta_c^\dagger$ or $\widetilde{\eta}_t\widetilde{\eta}_c^\dagger$ and $\widetilde{\eta}_p = \eta_t\widetilde{\eta}_c^\dagger$ or $\widetilde{\eta}_t\eta_c^\dagger$.

Let us finally return to the issue of unitary equivalence. The unitary transform of Eq. (4) preserves the Dirac structure for $U_\theta$ of Eq. (5). The two possibilities $T$ and $\widetilde{T}$

| 1d-classes | **T** | **C** | **P** | $V_\pm$ | $A$ | **zero-mode** |
|---|---|---|---|---|---|---|
| A | $\emptyset$ | $\emptyset$ | $\emptyset$ | $V_\pm^\dagger = V_\pm$ | | $\mathbb{Z}$ |
| $\mathrm{AIII}_{(1)}$ | $\emptyset$ | $\emptyset$ | $\mathbf{1}\otimes\sigma_z$ | $V_\pm = 0$ | | $\mathbb{Z}$ |
| $\mathrm{AIII}'_{(1)}$ | $\emptyset$ | $\emptyset$ | $\mathbf{1}\otimes i\sigma_y$ | $V_+ = 0$ | | |
| $\mathrm{AIII}_{(2)}$ | $\emptyset$ | $\emptyset$ | $\tau_z\otimes\sigma_z$ | $\tau_z V_\pm = -V_\pm\tau_z$ | $\tau_z A = A\tau_z$ | |
| $\mathrm{AIII}'_{(2)}$ | $\emptyset$ | $\emptyset$ | $\tau_z\otimes i\sigma_y$ | $\tau_z V_\pm = \mp V_\pm\tau_z$ | | |
| $\mathrm{AII}_{(1)}$ | $\mathbf{1}\otimes i\sigma_y$ | $\emptyset$ | $\emptyset$ | $V_\pm = \pm V_\pm^T$ | $A^T = -A$ | $\mathbb{Z}_2$ |
| $\mathrm{AII}_{(2)}$ | $i\tau_y\otimes\sigma_z$ | $\emptyset$ | $\emptyset$ | $\tau_y V_\pm^T = V_\pm\tau_y$ | $\tau_y A^* = -A\tau_y$ | |
| $\mathrm{AI}_{(1)}$ | $i\tau_y\otimes i\sigma_y$ | $\emptyset$ | $\emptyset$ | $\tau_y V_\pm^T = \pm V_\pm\tau_y$ | $\tau_y A^T = -A\tau_y$ | |
| $\mathrm{AI}_{(2)}$ | $\mathbf{1}\otimes\sigma_z$ | $\emptyset$ | $\emptyset$ | $V_\pm^T = V_\pm$ | $A^* = -A$ | |
| C | $\emptyset$ | $i\tau_y\otimes\mathbf{1}$ | $\emptyset$ | $\tau_y V_\pm^T = -V_\pm\tau_y$ | $\tau_y A^* = -A\tau_y$ | $2\mathbb{Z}$ |
| $\mathrm{C}'$ | $\emptyset$ | $i\tau_y\otimes\sigma_x$ | $\emptyset$ | $\tau_y V_\pm^T = \mp V_\pm\tau_y$ | $\tau_y A^T = -A\tau_y$ | |
| D | $\emptyset$ | $\mathbf{1}\otimes\mathbf{1}$ | $\emptyset$ | $V_\pm = -V_\pm^T$ | $A^* = -A$ | $\mathbb{Z},\ \mathbb{Z}_2$ |
| $\mathrm{D}'$ | $\emptyset$ | $\mathbf{1}\otimes\sigma_x$ | $\emptyset$ | $V_\pm = \mp V_\pm^T$ | $A^T = -A$ | |
| $\mathrm{BDI}_{(1)}$ | $i\tau_y\otimes i\sigma_y$ | $\mathbf{1}\otimes\mathbf{1}$ | $i\tau_y\otimes i\sigma_y$ | $V_\pm = -V_\pm^T = \mp\tau_y V_\pm\tau_y$ | $A = -A^* = -\tau_y A^T\tau_y$ | |
| $\mathrm{BDI}'_{(1)}$ | $i\tau_y\otimes i\sigma_y$ | $\tau_x\otimes\sigma_x$ | $\tau_z\otimes\sigma_z$ | $V_\pm = \pm\tau_y V_\pm^T\tau_y = \mp\tau_x V_\pm^T\tau_x$ | $\tau_{x,y}A^T = -A\tau_{x,y}$ | |
| $\mathrm{BDI}_{(2)}$ | $\mathbf{1}\otimes\sigma_z$ | $\mathbf{1}\otimes\mathbf{1}$ | $1\otimes\sigma_z$ | $V_\pm = 0$ | $A^* = -A$ | $\mathbb{Z}$ |
| $\mathrm{DIII}_{(1)}$ | $\mathbf{1}\otimes i\sigma_y$ | $\mathbf{1}\otimes\mathbf{1}$ | $1\otimes i\sigma_y$ | $V_+ = 0,\ V_-^T = -V_-$ | $A = -A^* = -A^T$ | $\mathbb{Z}_2$ |
| $\mathrm{DIII}_{(2)}$ | $i\tau_y\otimes\sigma_z$ | $\mathbf{1}\otimes\mathbf{1}$ | $i\tau_y\otimes\sigma_z$ | $V_\pm = -V_\pm^T = -\tau_y V_\pm\tau_y$ | $A = -A^* = -\tau_y A^T\tau_y$ | $\mathbb{Z}_2$ |
| $\mathrm{DIII}'_{(2)}$ | $i\tau_y\otimes\sigma_z$ | $\tau_x\otimes\sigma_x$ | $\tau_z\otimes i\sigma_y$ | $V_\pm = \tau_y V_\pm^T\tau_y = \mp\tau_x V_\pm^T\tau_x$ | $A = -\tau_y A^*\tau^y = -\tau_x A^T\tau_x$ | |
| $\mathrm{CII}_{(1)}$ | $\mathbf{1}\otimes i\sigma_y$ | $i\tau_y\otimes\mathbf{1}$ | $i\tau_y\otimes i\sigma_y$ | $V_\pm = \pm V_\pm^T = \mp\tau_y V_\pm\tau_y$ | $A = -A^T = -\tau_y A^*\tau_y$ | $\mathbb{Z}_2$ |
| $\mathrm{CII}'_{(1)}$ | $\tau_x\otimes i\sigma_y$ | $i\tau_y\otimes\sigma_x$ | $\tau_z\otimes\sigma_z$ | $V_\pm = \pm\tau_x V_\pm^T\tau_x = \mp\tau_y V_\pm^T\tau_y$ | $\tau_{x,y}A^T = -A\tau_{x,y}$ | |
| $\mathrm{CII}_{(2)}$ | $i\tau_y\otimes\sigma_z$ | $i\tau_y\otimes\mathbf{1}$ | $1\otimes\sigma_z$ | $V_\pm = 0$ | $A = -\tau_y A^*\tau_y$ | $2\mathbb{Z}$ |
| $\mathrm{CI}_{(1)}$ | $i\tau_y\otimes i\sigma_y$ | $i\tau_y\otimes\mathbf{1}$ | $1\otimes i\sigma_y$ | $V_+ = 0,\ \tau_y V_-^T = -V_-\tau_y$ | $A = -\tau_y A^T\tau_y = -\tau_y A^*\tau_y$ | |
| $\mathrm{CI}_{(2)}$ | $\mathbf{1}\otimes\sigma_z$ | $i\tau_y\otimes\mathbf{1}$ | $i\tau_y\otimes\sigma_z$ | $V_\pm = V_\pm^T = -\tau_y V_\pm\tau_y$ | $A = -A^* = -\tau_y A^*\tau_y$ | |
| $\mathrm{CI}'_{(2)}$ | $\tau_x\otimes\sigma_z$ | $i\tau_y\otimes\sigma_x$ | $\tau_z\otimes i\sigma_y$ | $V_\pm = \tau_x V_\pm^T\tau_x = \mp\tau_y V_\pm^T\tau_y$ | $A = -\tau_x A^*\tau_x = -\tau_y A^T\tau_y$ | |

TABLE II. The properties of the 25 non-chiral $\overline{d} = 1$ Dirac classes. 17 unitarily-inequivalent classes separated from each other by a horizontal line. The first column lists the $\overline{d} = 1$ Dirac classes. Columns **T**, **C** and **P** show representations of symmetry transformations for each class. The columns $V_\pm$ and $A$ show symmetry constraints on the potentials. A blank cell denotes absence thereof. The symmetry constraints guarantee zero modes in some classes (see section V). The last column shows classes with symmetry protected zero modes and the type of zero modes.

| $\overline{d} = 1$ classes | zero modes | topological invariant | examples |
|---|---|---|---|
| A | $\mathbb{Z}$ | $\mathbb{Z}$ | QH edge states |
| C | $2\mathbb{Z}$ | $2\mathbb{Z}$ | spin QH edge states in $d + id$-wave SC[18,19] |
| D | $\mathbb{Z}$ | $\mathbb{Z}$ | thermal QH edge states in spinless chiral $p$-wave SC[18] |

TABLE III. $\overline{d} = 1$ chiral Dirac hamiltonian classes.

for **T** are unitarily-inequivalent, because unitary transformations preserve the relation $T^T = \pm T$, or equivalently, $U_\theta\sigma_{y,z}U_\theta^T = \sigma_{y,z}$. However $C$ and $\widetilde{C}$ are unitarily-equivalent for non-zero $A_y$, since $U_{\pi/2}\sigma_x U_{\pi/2}^T = i$. In Table II, we listed all 25 classes separating 17 unitarily-inequivalent classes by horizontal lines. It is important to note however that all of the 25 classes should be viewed as inequivalent once $U_\theta$ is used to set $A_y = 0$ since $C, \widetilde{C}$ are inequivalent under the residual symmetry. (If $A_y = 0$, $A^* = A^T$.) We will take this route in the next section where we investigate the symmetry protection of zero modes.

## V. "TOPOLOGICAL INSULATORS" IN TWO DIMENSIONS

We conjecture a 'holographic' classification of 2D TI-TS based on the classification of $\overline{d} = 1$ Dirac hamiltonians that are symmetry protected to be gapless, i.e. have a protected zero mode. We list such $\overline{d} = 1$ Dirac hamiltonian classes in Tables III and IV. For a *subset* of these classes, there exists a $d = 2$ gapped hamiltonian in the same class and a known topological invariant which one can calculate from the ground state wave function which takes on $\mathbb{Z}$-values or $\mathbb{Z}_2$-values[9,10]; these are indicated

| $\overline{d}=1$ classes | **T** | **C** | **P** | zero modes | top. inv. | locking | examples |
|---|---|---|---|---|---|---|---|
| **AIII$_{(1)}$** | $\emptyset$ | $\emptyset$ | $\sigma_z$ | $\mathbb{Z}$ | | | |
| AII$_{(1)}$ | $i\sigma_y$ | $\emptyset$ | $\emptyset$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | Y | HgTe/(Hg,Cd)Te |
| **D** | $\emptyset$ | $\mathbf{1}$ | $\emptyset$ | $\mathbb{Z}_2$ | | | |
| **BDI$_{(2)}$** | $\sigma_z$ | $\mathbf{1}$ | $\sigma_z$ | $\mathbb{Z}$ | | | |
| DIII$_{(1)}$ | $i\sigma_y$ | $\mathbf{1}$ | $i\sigma_y$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | Y | $(p+ip)\times(p-ip)$-wave SC |
| **DIII$_{(2)}$** | $i\tau_y\otimes\sigma_z$ | $\mathbf{1}$ | $i\tau_y\otimes\sigma_z$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | N | particle-hole symmetric KM model |
| **CII$_{(1)}$** | $\mathbf{1}\otimes i\sigma_y$ | $i\tau_y\otimes\mathbf{1}$ | $i\tau_y\otimes i\sigma_y$ | $\mathbb{Z}_2$ | | Y | doubled KM |
| **CII$_{(2)}$** | $i\tau_y\otimes\sigma_z$ | $i\tau_y\otimes\mathbf{1}$ | $\mathbf{1}\otimes\sigma_z$ | $2\mathbb{Z}$ | | N | trigonally strained graphene[30] |

TABLE IV. $\overline{d}=1$ non-chiral Dirac hamiltonian classes with symmetry protected zero modes. The spin-momentum locking column is left blank when spins cannot be assigned because the time-reversal operator do not involve either $i\sigma_y$ or $i\tau_y$. New classes are shown in boldface (red online).

in the columns denoted "topological invariant". Surprisingly, for a class with a known bulk topological invariant, there is a correspondence between the values it can take and the number of gapless Dirac edge branches (dimension of the block matrices Eq(3) for the non-chiral case). Namely, classes with $\mathbb{Z}$-invariants are gapless for any number of Dirac edge branches; classes with $\mathbb{Z}_2$-invariants are gapless only when there are odd-number of branches for each chirality. The main point of this paper is that there are additional classes with protected edge zero modes beyond the 5 predicted on the basis of the known topological invariants.

In the rest of this section we enumerate the classes of $\overline{d}=1$ Dirac hamiltonians that have a protected zero mode as a consequence of the discrete symmtries. We then comment on the microscopic 2d models corresponding to a subset of our new classes. We finally discuss physical properties of these classes such as spin-momentum locking through a second quantized description.

### A. First quantized description

First we discuss the chiral (only right or left moving) Dirac fermion classes we mentioned at the beginning of section IV. These are massless for a "trivial" reason since a mass term necessarily couples left to right. As **T** and **P** transform left to right movers (see below), hamiltonians with these symmetries cannot be chiral. On the other hand, AZ classes A, C, D have at most a **C** symmetry and can be chiral. For chiral hamiltonians in classes A, C, D, any $\mathbb{Z}$ number of branches will be gapless. For chiral class C, since the auxiliary $\tau$ space is doubled, as explained above this is of type $2\mathbb{Z}$. See Table III for the summary.

Now consider non-chiral hamiltonians of the form Eq. (3) whose block diagonal structure implies that the second quantized theory has both right movers $\psi_R\equiv\langle x|\sigma_x=+\rangle$ and left movers $\psi_L\equiv\langle x|\sigma_x=-\rangle$ (see below). The hamiltonian $\mathcal{H}$ is gapless if it has a zero eigenvalue at $\mathbf{k}=0$, i.e. $\det\mathcal{H}(\mathbf{k}=0)=0$. Below we simplify this into a condition on $V_-$.

The potential $A_x$ can be removed by redefining the fields in the second quantized theory: $\psi_{L,R}\to e^{-i\int^x A_x(x)dx}\psi_{L,R}$ (see subsection V B). A constant $V_+$ is a chemical potential which shifts the overall energy levels. Hence we set this to zero. Now the condition for existence of a zero mode and hence a gapless spectrum is

$$\det\begin{pmatrix} V_- & iA_y \\ -iA_y & -V_- \end{pmatrix}=0 \tag{12}$$

However Eq. (12) is difficult to use in general[37]. Hence we use the freedom of unitary transform $U_\theta$ to set $A_y=0$. The criterion for a TI is now simply $\det V_-=0$ for fixed $A_y=0$.

Now we test if the conditions on $V_-$ imposed by symmetry listed in Table II guarantee $\det V_-=0$. As the choice of $A_y=0$ makes $C$ and $\widetilde{C}$ inequivalent we consider all 25 entries. Once we identify symmetry protected gapless Dirac classes, we check for unitary equivalence among those by consulting the Table II. In Table IV we list unitarily inequivalent protected classes.

There are two generic types of constraints on $V_-$ that protect a gapless spectrum. First, $V_-=0$ guarantees $\det V_-=0$ independent of the dimension of $V_-$ nor the $\mathbb{Z}$ number of edge modes. This is identified with a type $\mathbb{Z}$ TI. If the **T** or **C** symmetry involves a doubling of the auxiliary $\tau$ space, then this doubling is the signature of a type $2\mathbb{Z}$ TI.[15] Second, $V_-^T=-V_-$ implies $\det V_-=-\det V_-$ when $V_-$ is *odd dimensional*, and hence $\det V_-=0$. By analogy with the 3d case, those that rely on $V_-^T=-V_-$ with $V_-$ odd-dimensional should be of $\mathbb{Z}_2$ type because of the even/odd aspect.

There are also two exceptional cases: **DIII$_{(1)}$** Here $\widetilde{\eta}_t=i\tau_y$, $\widetilde{\eta}_c=\mathbf{1}$, $\eta_p=i\tau_y$. Here $V_-^T=-V_-$, however it is even dimensional, and constrained to be of the form $V_-=\left(\begin{smallmatrix} a_- & b_- \\ b_- & -a_- \end{smallmatrix}\right)$ with $a_-^T=-a_-, b_-^T=-b_-$. Thus, if $a_-, b_-$ are one dimensional, then $V_-=0$. Type $\mathbb{Z}_2$.

**CII$_{(1)}$** Here $\eta_t=\tau_x, \eta_c=i\tau_y, \eta_p=-\tau_z$. $V_-=\left(\begin{smallmatrix} 0 & b_- \\ c_- & 0 \end{smallmatrix}\right)$

with $b_-^T = -b_-, c_-^T = -c_-$. If $b_-, c_-$ are odd dimensional, then, up to a sign, $\det V_- = \det b_- \det c_- = 0$. Type $\mathbb{Z}_2$.

The table IV lists new classes with protected Dirac edge modes in boldface(red online). An immediate question is whether these classes can be realized in a microscopic 2D model and if so, why they were missed in previous classifications. First we point out that by considering an additional reflection symmetry, Yao and Ryu[25] recently found topological invariants for all of our new classes except $\mathrm{CII}_(1)$. As first noticed by Fu[26], when considering microscopic realizations of topological insulators, point-group symmetry can play an important role. While we required our non-chiral edge state to be described by a Dirac hamiltonian, it is plausible that the latter assumption automatically implies a reflection symmetry for some of the classes for $\overline{d} = 1$. This is a topic to be investigated further in the future. Nevertheless, what is clear from the work[25] is that indeed there are microscopic 2d theories whose edge states are described by our new classes.

Turning to physical realizations of the new classes of edge states so far we have found two examples: $\mathrm{DIII}_{(2)}$ and $\mathrm{CII}_{(2)}$. An example of $\mathrm{DIII}_{(2)}$ is the Kane-Mele model in the presence of particle hole symmetry[28,29]. This can be viewed aa special case of $\mathrm{AII}_{(1)}$-type TI with additional particle-hole symmetry. The additional symmetry enables quantum Montecarlo simulations without sign-problems. But it also means absence of spin or charge edge current as we will discuss further in the next section. Of particular interest is the zero field QHE in trigonally strained graphene[30,31] as an example of $\mathrm{CII}_{(2)}$. The details of this identification will be presented elsewhere[27]. However, the underlying reasoning is rather simple. The observation of Landau levels in[30] in the absence of magnetic field calls for a $\mathbb{Z}$ type TI among time-reversal symmetric classes. In the original classification by Schnyder *et al.*[9] $\mathbb{Z}$ type TI are found only among $\mathbf{T}$ breaking classes. Since trigonal strain introduces pseudo-magnetic fields of opposite direction for two valleys, there are $2\mathbb{Z}$ edge modes when the system is subject to a confining potential.

### B. Second quantized description and spin-momentum locking

One can define a second-quantized hamiltonian:

$$H = \int dx \sum_{a,b} \psi_a^\dagger(x) \mathcal{H}_{ab} \psi_b(x) \tag{13}$$

from $\mathcal{H}$ of Eq. (3). Now let $\mathbf{T}, \mathbf{C}$ be time-reversal and particle-hole transformation operators in the field theory and define

$$\mathbf{T}\psi_a\mathbf{T}^\dagger = T_{ab}\psi_b, \qquad \mathbf{C}\psi_a\mathbf{C}^\dagger = C_{ab}\psi_b^\dagger. \tag{14}$$

This and the $T, C$ properties of $\mathcal{H}$ (Eq. (1)) implies the invariance: $\mathbf{T}H\mathbf{T}^\dagger = H$, $\mathbf{C}H\mathbf{C}^\dagger = H$.

Since right movers and left movers are $\psi_R \equiv \langle x|\sigma_x = +\rangle$ and left movers $\psi_L \equiv \langle x|\sigma_x = -\rangle$, the spinor field $\psi$ has the block structure:

$$\psi = \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} \tag{15}$$

in the eigenbasis of $\sigma_z$. Upon passing to Euclidean space by $t \to -i\tau$, the Schrodinger equation for $\mathcal{H}$ in Eq. (3), $i\partial_t\psi = \mathcal{H}\psi$, becomes $\partial_z\psi_R = \partial_{\overline{z}}\psi_L = 0$, where $\partial_{\overline{z}} = \partial_\tau + i\partial_x, \partial_z = \partial_\tau - i\partial_x$. This confirms the anticipated chirality of $\psi_R$ and $\psi_L$.

The $\mathbf{T}$ and $\mathbf{P}$ transformations exchange left and right movers:

$$\begin{aligned} T: &\quad \psi_R \to -\eta_t\psi_L, \quad \psi_L \to \eta_t\psi_R \\ \widetilde{T}: &\quad \psi_R \to \widetilde{\eta}_t\psi_L, \quad \psi_L \to \widetilde{\eta}_t\psi_R \end{aligned} \tag{16}$$

and

$$\begin{aligned} P: &\quad \psi_R \to \eta_p\psi_L, \qquad \psi_L \to \eta_p\psi_R \\ \widetilde{P}: &\quad \psi_R \to -\widetilde{\eta}_p\psi_L, \qquad \psi_L \to \widetilde{\eta}_p\psi_R \end{aligned} \tag{17}$$

On the other hand, $C$ transforms fields into their conjugates:

$$\begin{aligned} C: &\quad \psi_R \to \eta_c\psi_R^\dagger, \qquad \psi_L \to -\eta_c\psi_L^\dagger \\ \widetilde{C}: &\quad \psi_R \to \widetilde{\eta}_c\psi_R^\dagger, \qquad \psi_L \to \widetilde{\eta}_c\psi_L^\dagger. \end{aligned} \tag{18}$$

Hence for the AZ classes A,C,D which do not have $\mathbf{T}$ or $\mathbf{P}$ symmetry, chiral states with only $\psi_R$ or $\psi_L$ can be realized as edge states and are protected from a mass gap since mass term couples left and right.

We now use the $\mathbf{T}$ symmetry to assign (pseudo-) spins and check for spin-momentum locking. On physical grounds, we consider the smallest number of components in each class, i.e. either 1 or 2. It is well-known that $\mathbf{T}$ has the representation $\mathbf{T} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on spin 1/2 particles and $\mathbf{T}^2 = -1$. Hence when the representation of $\mathbf{T}$ involves $i\sigma_y$ or $i\tau_y$ and $\mathbf{T}^2 = -1$ in Table II, $\vec{\sigma}$ or $\vec{\tau}$ should act on the spin space. This is particularly interesting since $|\sigma_x = +\rangle$ and $|\sigma_x = -\rangle$ are right- and left-moving states by definition of the hamiltonian Eq. (3): this, as we mentioned earlier, is a manifestation of spin-momentum locking.

The classes with spin-momentum locking are $\mathrm{AII}_{(1)}$, $\mathrm{DIII}_{(1)}$, $\mathrm{CII}_{(1)}$. These are all TI-TS edge states of type $\mathbb{Z}_2$ within our scheme. For these, we can label the fields $\psi_R = \psi_{R\uparrow}$, $\psi_L = \psi_{L\downarrow}$. $\mathrm{AII}_{(1)}$ and $\mathrm{DIII}_{(1)}$ have well known examples. QSH edge states[4,5,23] *in the absence of particle hole symmetry* are examples of $\mathrm{AII}_{(1)}$ class. Note that we derived here the spin-momentum locking, which arises from the spin-orbit coupling in QSH systems, on very general grounds. A 2d version of a $\mathrm{He}_3\mathrm{B}$ superfluid phase where up-spin pairs and down-spin pairs have opposite angular momentum, would be an example of the $\mathrm{DIII}_{(1)}$ class.[38]. Such a state has not been realized yet, but perhaps could be in a film geometry with control over the boundary conditions. $\mathrm{CII}_{(1)}$ can be realized[27] as a particle-hole doubled version of $\mathrm{AII}_{(1)}$ much the same way as how in 3d a CII TI was constructed out of two copies of 3d Dirac Hamiltonian in Schnyder *et al.*[9].

$\mathrm{DIII}_{(2)}$ and $\mathrm{CII}_{(2)}$ classes have both spin components for right-movers and left-movers each. The Kane-Mele (KM) model[4] at zero chemical potential has particle-hole symmetry and hence does not strictly speaking belong to class AII. Moreover the spin or charge edge current is absent as the current operators are odd under charge conjugation[28]. Nevertheless, there is a charge neutral gapless edge mode[28,29]. This is an example of $\mathrm{DIII}_{(2)}$ class[27]. $\mathrm{CII}_{(2)}$ is unique in that spin is tied to charge, i.e. particle-hole transformations flip spin: $(\psi_{R\uparrow}, \psi_{R\downarrow}) \to (-\psi^\dagger_{R\downarrow}, \psi^\dagger_{R\uparrow})$. Note that these spin-momentum locking properties offer concrete distinctions between classes ($\mathrm{DIII}_{(1)}$, $\mathrm{CII}_{(1)}$) and ($\mathrm{DIII}_{(2)}$, $\mathrm{CII}_{(2)}$).

$\mathrm{AIII}_{(1)}$, non-chiral D, and $\mathrm{BDI}_{(2)}$ are spinless fermions. Note that we find the non-chiral D TI to be of $\mathbb{Z}_2$ type and distinct from the chiral $D$ which is of $\mathbb{Z}$ type.

## VI. VARIATIONS OF LUTTINGER LIQUIDS

We are now in the position to consider how interactions consistent with the $\mathbf{T}, \mathbf{C}, \mathbf{P}$ symmetries could affect the $\overline{d} = 1$ edge states. In general, bulk interactions should lead to interactions on the edge. If the bulk stays gapped, one can focus on the edge states even in the presence of interactions. While the topological invariants based on single particle wave functions cannot be applied to interacting systems, the edge state theory can incorporate the effects of interactions.

The fractional quantum Hall effect (FQH) is the prime example. The FQH edge state resulting from Coulomb interaction in the bulk has no topological invariant associated with it, while the integer QHE is associated with the Chern number[2]. However the fractional quantum Hall edge states are chiral Luttinger liquids which are related to the integer quantum Hall edge states (chiral Fermi liquid) by the addition of an exactly marginal perturbation to the Dirac action[16]. An exactly marginal perturbation on a non-interacting edge state preserves the gaplessness, but deforms it into an interacting theory with non-trivial exponents, fractional charges, etc.

Motivated by the FQH case, we classify the exactly marginal perturbations for each proposed TI-TS's in Table IV, as a way of characterizing the effect of bulk interactions.

The starting point is the action for the generic free Dirac Hamiltonian Eq. (13):

$$\begin{aligned} S = \int dxdt \ \Big[ &\psi^\dagger_R(\partial_z + A_x + V_+)\psi_R \\ &+ \psi^\dagger_L(\partial_{\overline{z}} - A_x + V_+)\psi_L \\ &+ \Big(\psi^\dagger_L(V_- + iA_y)\psi_R + h.c.\Big)\Big] . \end{aligned} \tag{19}$$

Recall that $\psi_R$ and $\psi_L$ are vectors in the space represented by $\tau$. $V_+$ can be interpreted as a chemical potential, or equivalently the time component of a gauge field as it couples to currents $\psi^\dagger_R V_+ \psi_R + \psi^\dagger_L V_+ \psi_L$. We set it to zero. If $V_- + iA_y$ is one dimensional, it simply corresponds to a complex mass. Hence removing $A_y$ through a unitary transform $U_\theta$ is equivalent to removing the phase of the mass by redefining $\psi_L$. After removing $A_y$, and absorbing the physical gauge field $A_x$ to the definition of the $\psi$ fields, the action for the massless zero mode simplifies to

$$S = \int dxdt \left(\psi^\dagger_R \partial_z \psi_R + \psi^\dagger_L \partial_{\overline{z}} \psi_L\right). \tag{20}$$

We consider left-right current-current perturbations in analogy with Luttinger liquids and single out those preserving the $\mathbf{T}$, $\mathbf{C}$, $\mathbf{P}$ of the free theory. Consider the currents $J^a_L = \psi^\dagger_L t^a \psi_L$, $J^a_R = \psi^\dagger_R t^a \psi_R$, where $t^a$ is a hermitian matrix acting on the $\tau$ space, and define the operator $\mathcal{O}^a = J^a_L J^a_R$ (no sum on $a$). Since $\psi$ has scaling dimension 1/2, the operator $\mathcal{O}^a$ has dimension two, i.e. it is marginal, and a term $g\,\mathcal{O}^a$ can be added to the lagrangian. For the $T, \widetilde{T}, P, \widetilde{P}$ symmetries, $\mathcal{O}^a$ is invariant if the appropriate $\eta$ commutes with $t^a$. For the $C, \widetilde{C}$ symmetries which transform fields into their conjugates, invariance of the operator additionally requires $(t^a)^T = \pm t^a$. The renormalization group beta function for $\mathcal{O}^a$ is in general proportional to the quadratic Casimir for the Lie algebra generated by the $t^a$. If this beta function vanishes for a symmetry invariant $\mathcal{O}^a$, it is an exactly marginal perturbation.

For all TI-TS's, the marginal perturbation $\mathcal{O}^a$ is invariant for $t^a = \mathbf{1}$, and we can consider the action

$$S = \int dxdt \ \left(\psi^\dagger_R \partial_z \psi_R + \psi^\dagger_L \partial_{\overline{z}} \psi_L + g J_L J_R\right). \tag{21}$$

Since the currents $J_{L,R}$ are then $U(1)$ currents, the beta function vanishes making this perturbation exactly marginal. Eq. (21) describes different versions of Luttinger liquids for different classes.

The choice $t^a = \tau_y$, which requires at least 2 components for each chirality, also yields an invariant $\mathcal{O}^a$ for the classes $\mathrm{DIII}_{(2)}$ and $\mathrm{CII}_{(1,2)}$. Since this involves a single $t^a$, it again generates a U(1) current and the associated $\mathcal{O}^a$ is again exactly marginal.

We list each exactly marginal perturbation for the above TI-TS's:

- **$\mathrm{AII}_{(1)}$ and $\mathrm{DIII}_{(1)}$.** Both are one-component spin-momentum locked classes. The only allowed perturbation is with $t^a = \mathbf{1}$:

$$\mathcal{O}^a = \left(\psi^\dagger_{L\downarrow}\psi_{L\downarrow}\right)\left(\psi^\dagger_{R\uparrow}\psi_{R\uparrow}\right). \tag{22}$$

  The so-called helical liquid for interacting QSH edge state[32] requires such a perturbation. Interestingly such a bulk interaction effect on the edge states has been recently confirmed[28,29,33].

- **DIII$_{(2)}$ and CII$_{(2)}$**. Both are two-component classes which can be perturbed with $t^a = \mathbf{1}$ and $t^a = \tau_y$. $t^a = \mathbf{1}$ yields the spin-full Luttinger liquid with

$$\mathcal{O}^a = \left(\psi^\dagger_{L\uparrow}\psi_{L\uparrow} + \psi^\dagger_{L\downarrow}\psi_{L\downarrow}\right)\left(\psi^\dagger_{R\uparrow}\psi_{R\uparrow} + \psi^\dagger_{R\downarrow}\psi_{R\downarrow}\right). \quad (23)$$

Whereas $t^a = \tau_y$ turn $J^a_L$ and $J^a_R$ into a spin-singlet currents and

$$\mathcal{O}^a = -\left(\psi^\dagger_{L\uparrow}\psi_{L\downarrow} - \psi^\dagger_{L\downarrow}\psi_{L\uparrow}\right)\left(\psi^\dagger_{R\uparrow}\psi_{R\downarrow} - \psi^\dagger_{R\downarrow}\psi_{R\uparrow}\right). \quad (24)$$

These are new types of Luttinger liquids which we refer to as the "spin-singlet liquid".

- **AIII$_{(1)}$**, non-chiral **D** and **BDI$_{(2)}$**. These are spinless fermion classes which can be single component. They can only be perturbed with $t^a = \mathbf{1}$.

- **CII$_{(1)}$**. This has both particle and hole components with spin-momentum locking for each component. It is a different kind of Luttinger liquid, which we refer to as the "double helix", since the free part is essentially a doubled KM model.

$$\mathcal{O}^a = \left(\psi^\dagger_{L\downarrow}\psi_{L\downarrow} + \psi'^\dagger_{L\downarrow}\psi'_{L\downarrow}\right)\left(\psi^\dagger_{R\uparrow}\psi_{R\uparrow} + \psi'^\dagger_{R\uparrow}\psi'_{R\uparrow}\right) \quad (25)$$

Next consider adding more than one perturbation, i.e. $\sum_a g_a \mathcal{O}^a$. In general, the operator product expansion of $\mathcal{O}^a$ with $\mathcal{O}^b$ generates another $\mathcal{O}$ operator associated with the current corresponding to $[t^a, t^b]$, and this gives rise to a renormalization group beta function proportional to the quadratic casimir of the Lie algebra generated by the $t^a$. Only classes DIII$_{(1)}$ and CII$_{(2)}$ have two allowed $\mathcal{O}^a$ listed above: $t^a = \mathbf{1}$ or $\tau_y$. However since these $t^a$ commute, this two parameter perturbation is also exactly marginal. In summary, we find all possible symmetry preserving quartic interactions to be exactly marginal, deforming the free Dirac edge theory into an interacting one that preserves the gaplessness .

## VII. CONCLUSIONS

We classified Dirac hamiltonians in one dimension according to the discrete symmetries of time-reversal, particle-hole and chiral symmetry, and found 17 inequivalent ones. Assuming that two-dimensional topological insulators (or superconductors) are realized on their one dimensional boundary as Dirac fermions, we found 11 of these classes that possessed a zero mode which was protected by the symmetries. This should be compared with the classifications based on bulk topological or boundary localization properties in[9–11], which predict 5 classes in any dimension. The classes we find beyond the standard 5 are in classes AIII, BDI, two versions of CII, a distinct version of DIII and a $\mathbb{Z}_2$ version of D. We suggested that physical realizations for the new TI's in classes CII$_{(1)}$ and CII$_{(2)}$ could perhaps be a doubled Kane-Mele model and trigonally strained graphene respectively.

The simplest interpretation of the existence of these new classes of TI in two spatial dimensions is that there are theories with boundary zero modes that are not necessarily protected by topology, and this is attributed to the richer structure of the classification of Dirac hamiltonians in 1 dimension. On the other hand, it remains a possibility that the new classes are characterized by some as yet unknown topological invariants.

We also studied possible manifestations of bulk interactions as quartic interactions on the boundary in two dimensions. For all classes of potential TI's, we found that all such interactions that preserve the discrete symmetries are exactly marginal. The exact marginality preserves the gaplessness, but deforms the theory into distinct variations of Luttinger liquids.

## VIII. ACKNOWLEDGMENTS

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