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Fractional Quantum Hall Effect in Topological Flat Bands with Chern Number Two

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Recent theoretical works have demonstrated various robust Abelian and non-Abelian fractional topological phases in lattice models with topological flat bands carrying Chern number $C = 1$. Here we study hard-core bosons and interacting fermions in a three-band triangular-lattice model with the lowest topological flat band of Chern number $C = 2$. We find convincing numerical evidence of bosonic fractional quantum Hall effect at the $\nu = 1/3$ filling characterized by three-fold quasi-degeneracy of ground states on a torus, a fractional Chern number for each ground state, a robust spectrum gap, and a gap in quasihole excitation spectrum. We also observe numerical evidence of a robust fermionic fractional quantum Hall effect for spinless fermions at the $\nu = 1/5$ filling with short-range interactions.

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Introduction.— Topological states of matter have been the focus of intensive studies since the discovery of the integer quantum Hall effect (QHE) [1] and the fractional QHE (FQHE) [2]. The latter, occurring at fractional filling of Landau levels (LLs), provides the first example of fractionalization in two dimensions. The precise quantization of Hall conductance was found to be directly connected to a topological invariant Chern number [3, 4] soon after its experimental discovery. The FQHE is further characterized by quasi-particles with fractional charge [5] and fractional statistics [6, 7] as well as topological ground-state degeneracy [8], which are manifestations of its topological order [9]. The idea of flux attachment and composite-particle theory has provided a simple but profound picture of the FQHE [10, 11].

Recently, a series of numerical works have demonstrated convincing evidence of the Abelian [12–14] and non-Abelian FQHEs [15–17] in topological flat band (TFB) models [18] without an external magnetic field. These TFB models, belonging to the topological class of the well-known Haldane model [19], have at least one topologically nontrivial nearly flat band with a Chern number $C = 1$, which is separated from the other bands by large gaps [18, 20–22]. This intriguing fractionalization effect in TFBs without LLs, defines a new class of fractional topological phases (also known as fractional Chern insulators), and has stimulated a lot of recent research activities [23–32].

In contrast to the continuum model in a magnetic field where the Chern number of a LL is always one, higher Chern numbers are possible for nearly flat bands in lattice models [33]. The Abelian and non-Abelian FQHE states found in $C = 1$ TFBs [12–17] generally have analogy with ones in the continuum LL, while FQHE in TFBs with high Chern numbers might do not have such sim-

ple analogy; thus new exotic topological states of matter might occur in these TFBs [34]. Nonetheless, exotic FQHE in TFBs with high Chern numbers has not been studied in microscopic models. In this Letter, we aim to fill in the gap by demonstrating that exotic FQHE for both hard-core bosons and interacting fermions can indeed be realized in TFBs with $C = 2$.

We first introduce a three-band triangular lattice tight-binding model whose lowest TFB has Chern number $C = 2$. Through extensive exact diagonalization (ED) studies of hard-core bosons in the TFB, we find convincing numerical evidence of the bosonic FQHE at the filling $\nu = 1/3$ in the $C = 2$ TFB. This $1/3$ bosonic FQHE is characterized by three-fold quasi-degeneracy ($d = 3$) of ground states on a torus, a fractional quantized Chern number for each ground state, a robust spectrum gap, and a gap in quasihole excitation spectrum. For the $1/5$ filling of spinless fermions in the $C = 2$ TFB, clear FQHE features are also observed. We discuss the plausibility of understanding the exotic $\nu = 1/3$ bosonic and $\nu = 1/5$ fermionic FQHEs in the $C = 2$ TFB in terms of effective two-component (or bilayer) FQHEs.

Formulation.—We introduce a three-band triangular-lattice model of interacting hard-core bosons:

$$\begin{aligned}
 H = & \pm t \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \left[b_{\mathbf{r}'}^\dagger b_{\mathbf{r}} \exp(i\phi_{\mathbf{r}'\mathbf{r}}) + \text{H.c.} \right] \\
 & \pm t' \sum_{\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle} \left[b_{\mathbf{r}'}^\dagger b_{\mathbf{r}} \exp(i\phi_{\mathbf{r}'\mathbf{r}}) + \text{H.c.} \right] \\
 & + V_1 \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} n_{\mathbf{r}} n_{\mathbf{r}'} + V_2 \sum_{\langle\langle \mathbf{r}\mathbf{r}' \rangle\rangle} n_{\mathbf{r}} n_{\mathbf{r}'} \quad (1)
 \end{aligned}$$

where $b_{\mathbf{r}}^\dagger$ creates a hard-core boson at site \mathbf{r} , $\langle \dots \rangle$ and $\langle\langle \dots \rangle\rangle$ denote the nearest-neighbor (NN) and the next-nearest-neighbor (NNN) pairs of sites, respectively

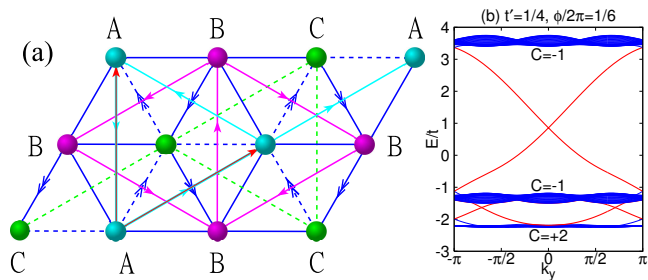


FIG. 1: (color online). (a) The three-band triangular-lattice model: The NN and NNN hopping amplitudes are positive (negative) along the solid (dashed) lines; The arrows represent the phases $\pm 2\phi$ (signs are represented by arrow directions) in the NN hoppings and $\pm\phi$ in the NNN hoppings. (b) Edge states of the triangular-lattice model in (a), and the lower TFB owns the Chern number $C = +2$.

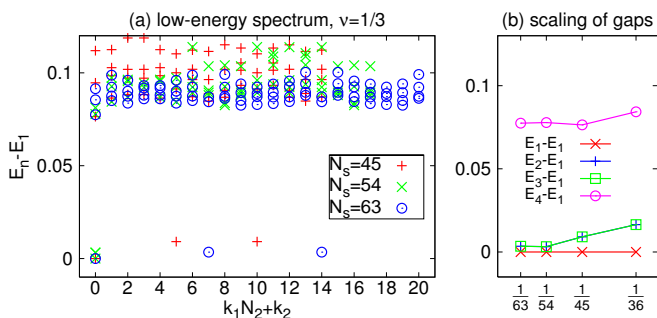


FIG. 2: (color online). The $1/3$ bosonic FQHE. (a) Low energy spectrum $E_n - E_1$ versus the momentum $k_1 N_2 + k_2$ of the $1/3$ bosonic FQHE phase for three lattice sizes $N_s = 45, 54$ and 63 at $\nu = 1/3$ filling with $V_1 = V_2 = 0.0$. (b) Spectrum gaps versus $1/N_s$ for four lattice sizes.

[Fig. 1(a)], and V_1 and V_2 are the NN and the NNN repulsions.

The triangular-lattice model has a unit cell of three sites, and therefore has three single-particle bands. Here, we adopt the parameters $t = 1$, $t' = 1/4$ [the signs of hoppings are described in Fig. 1(a)] and $\phi/2\pi = 1/6$, such that a lowest TFB of $C = 2$ is formed with a flatness ratio (of the band gap over bandwidth) of about 15 [Fig. 1(b)]. In our ED study, we consider a finite system of $N_1 \times N_2$ unit cells (total number of sites $N_s = 3 \times N_1 \times N_2$ and total number of single-particle orbitals $N_{\text{orb}} = N_1 N_2$ in each band) with basis vectors shown in Fig. 1(a) and we use periodic boundary conditions. We denote the boson numbers as N_b , and the filling factor of the flat band is $\nu = N_b/N_{\text{orb}}$. The momentum vector $\mathbf{q} = (2\pi k_1/N_1, 2\pi k_2/N_2)$ will be denoted by a pair of integer quantum numbers (k_1, k_2) . The amplitude of the NN hopping t is set as the unit of energy.

The $1/3$ bosonic FQHE. (a) Low energy spectrum.— We first look at the low-energy spectrum for a finite lattice with $N_s = 45$ ($3 \times 3 \times 5$) sites at filling $\nu = 1/3$ with

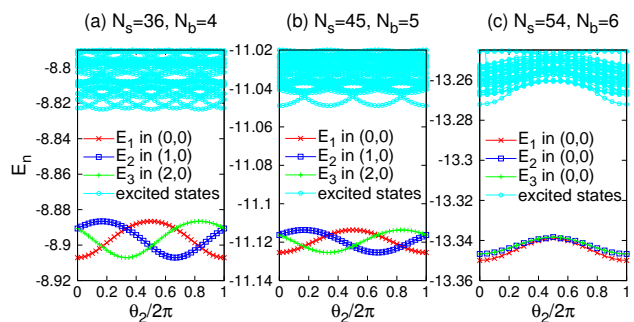


FIG. 3: (color online). The $1/3$ bosonic FQHE. Low energy spectra versus θ_2 at a fixed $\theta_1 = 0$ for three lattice sizes at $\nu = 1/3$ filling with $V_1 = V_2 = 0.0$: (a) $N_s = 36$; (b) $N_s = 45$; (c) $N_s = 54$.

$V_1 = V_2 = 0.0$ as shown by Fig. 2(a). We denote E_i as the energy of the i -th lowest many-body eigenstate. The ground state manifold (GSM) is defined as a set of lowest states with close energies well separated from other excited states by a finite spectrum gap. For the $\nu = 1/3$ bosonic FQHE phase, two necessary conditions are satisfied: a GSM with three quasi-degenerate ($d = 3$) lowest eigenstates ($E_3 - E_1 \sim 0$); and the $d = 3$ GSM being separated from the higher eigenstates by a finite spectrum gap $E_4 - E_3 \gg E_3 - E_1$.

We have also obtained numerical results from other lattice sizes of $N_s = 36$ ($3 \times 3 \times 4$), 54 ($3 \times 3 \times 6$) and 63 ($3 \times 3 \times 7$) around $V_1 = V_2 = 0.0$. Similar to the FQHE in the $C = 1$ TFBs [12–14], if (k_1, k_2) is the momentum sector for one of the states in the GSM, we find that other state in the GSM can be found in the sector $(k_1 + N_b, k_2 + N_b)$ [module (N_1, N_2)] demonstrating the momentum space translation invariant as an emerging symmetry of the system. Indeed, for $N_s = 36, 45, 63$, the three GSs are in the $(0,0)$, $(1,0)$, and $(2,0)$ sectors, respectively; while for $N_s = 54$, both N_b/N_1 and N_b/N_2 are integers, and all three GSs are in the same $(0,0)$ sector [with very close energies as shown in Fig. 2(a)]. Therefore, for each system size, there is an obvious GSM with three-fold quasi-degenerate states, which is well separated from the higher energy spectrum by a large spectrum gap. We have also attempted a scaling plot of spectrum gaps for the four lattice sizes [as shown in Fig. 2(b)], which indicates that the spectrum gap of the $1/3$ bosonic FQHE phase should survive in the thermodynamic limit. We further find that this $\nu = 1/3$ FQHE is stable (with large spectrum gap and well-defined $d = 3$ GSM) in the presence of relatively weak repulsions ($V_1 < 0.5$ and $V_2 < 0.5$).

(b) Berry curvature and Chern number.—The Chern number [3] (which is the Berry phase in units of 2π) of a many-body state is an integral invariant in the boundary phase space [4, 35]: $C = \frac{1}{2\pi} \int d\theta_1 d\theta_2 F(\theta_1, \theta_2)$, where two boundary phases θ_1 and θ_2 are introduced as the generalized boundary conditions in both direc-

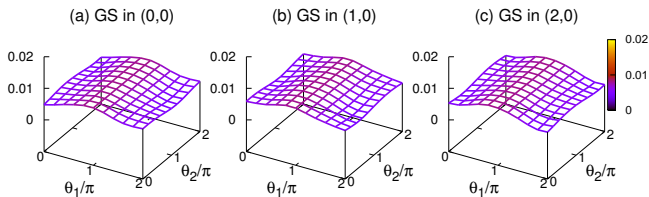


FIG. 4: (color online). The $1/3$ bosonic FQHE. Berry curvatures $F(\theta_1, \theta_2)\Delta\theta_1\Delta\theta_2/2\pi$ at 10×10 mesh points for the GSM of the $N_s = 45$ lattice at $\nu = 1/3$ filling with $V_1 = V_2 = 0.0$: (a) the 1st GS in $(0,0)$ sector; (b) the 2nd GS in $(1,0)$ sector; (c) the 3rd GS in $(2,0)$ sector.

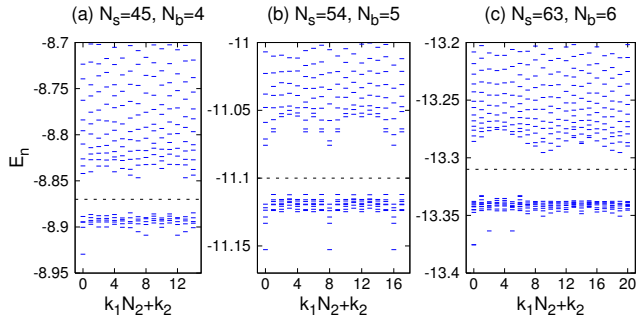


FIG. 5: (color online). The $1/3$ bosonic FQHE. Quasihole excitations for three lattice sizes with $V_1 = V_2 = 0.0$: (a) $N_s = 45$ and $N_b = 4$; (b) $N_s = 54$ and $N_b = 5$; (c) $N_s = 63$ and $N_b = 6$;

tions, respectively. The Berry curvature is given by $F(\theta_1, \theta_2) = \text{Im} \left(\left\langle \frac{\partial \Psi}{\partial \theta_2} \left| \frac{\partial \Psi}{\partial \theta_1} \right\rangle - \left\langle \frac{\partial \Psi}{\partial \theta_1} \left| \frac{\partial \Psi}{\partial \theta_2} \right\rangle \right) \right)$. For the GSM of $1/3$ bosonic FQHE phase, the three GSs maintain their quasi-degeneracy and are well separated from the other low-energy excitation spectrum upon tuning the boundary phases, which indicates the robustness of this FQHE phase (Fig. 3). For each GSM of $N_s = 36, 45, 63$, the three states are found to evolve into each other with level crossings when boundary phases are changed. [Fig. 3(a) and 3(b)]. While for $N_s = 54$, with all three states of the GSM in the $(0,0)$ sector, through tuning the boundary phases, each state evolves into itself without level crossing [Fig. 3(c)], consistent with the level repulsion principle.

Furthermore, for the three GSs of the GSM in the $(0,0)$, $(1,0)$ and $(2,0)$ sectors of the $N_s = 45$ case, the Berry curvatures in boundary phase space (with 10×10 mesh points) are shown in Fig. 4(a)-4(c). The summation of Berry curvatures in three sectors gives the integral Berry phase 4π (each GS contributes an almost precisely quantized Berry phase $4\pi/3$ with 6-digit high accuracy), and thus the total Chern number of the $d = 3$ GSM is $C_{\text{tot}} = 2$ which corresponds to a fractional quantized Hall conductance of $2e^2/3h$ per GS.

(c) *Quasihole excitation spectrum.*—In order to investigate the possible fractional statistics of the $1/3$ bosonic FQHE state, we study the quasihole spectrum by re-

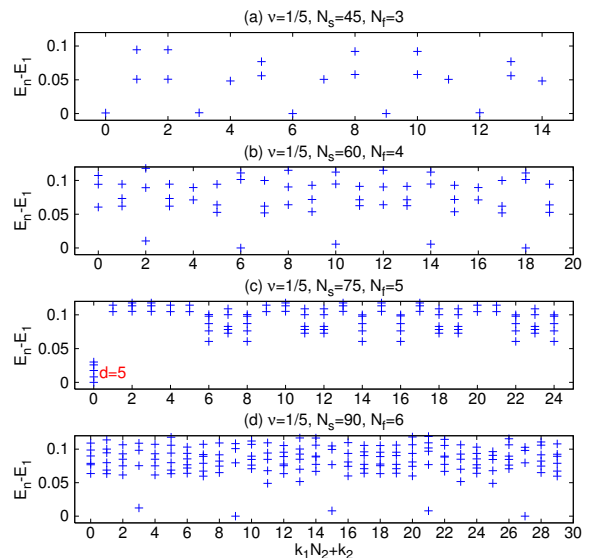


FIG. 6: (color online). The $1/5$ fermionic FQHE. Low energy spectrum $E_n - E_1$ versus the momentum $k_1 N_2 + k_2$ of the $1/5$ fermionic FQHE for four lattice sizes with $V_1 = 8.0$ and $V_2 = V_3 = 1.0$: (a) $N_s = 45$; (b) $N_s = 60$; (c) $N_s = 75$; (d) $N_s = 90$. The quasi-degeneracy has been labeled for the $(0,0)$ sector of the $N_s = 75$ case in (c).

moving one boson from the $\nu = 1/3$ filling. As shown in Fig. 5(a), for the case of $N_s = 45$ and $N_b = 4$, the quasihole spectrum exhibits a distinguishable gap which separates 5 lowest states in each momentum sector from the other higher-energy states, and there are 75 low-energy quasihole states in total. This number of low-energy quasihole states is consistent with the counting rule of splitting one hole into three quasiholes (each with fractional charge $1/3$), i.e. the quasihole-counting in Laughlin's $1/3$ fermionic FQHE state, based upon the generalized Pauli principle [14, 16]. Similarly, for the $N_s = 54$ and $N_b = 5$ case [Fig. 5(b)], and the $N_s = 63$ and $N_b = 6$ case [Fig. 5(c)], there are also distinguishable spectrum gaps and well-separated lower-energy quasihole manifolds.

The $1/5$ fermionic FQHE.—For the case of interacting spinless fermions [where the bosonic operators in Eq. (1) are replaced by the fermionic ones] in the $C = 2$ TFB, we have also observed FQHE features at the $\nu = 1/5$ filling [36] for four different lattice sizes of $N_s = 45$ ($3 \times 5 \times 3$), 60 ($3 \times 5 \times 4$), 75 ($3 \times 5 \times 5$) and 90 ($3 \times 5 \times 6$). We denote the fermion numbers as N_f , and the filling factor of the TFB is $\nu = N_f/N_{\text{orb}}$. In contrast to the $\nu = 1/3$ bosonic FQHE around $V_1 = V_2 = 0.0$, the onset of fermionic FQHE features at $\nu = 1/5$ needs finite values of short-range repulsions (V_1 , V_2 and a third-neighbor interaction V_3), similar to the fermionic $1/5$ FQHE [12] or the bosonic $1/4$ FQHE [13] in the $C = 1$ TFBs. At $\nu = 1/5$, the topological ground-state degeneracy is $d = 5$: for $N_s = 45, 60, 90$, the five GSs are in five different sectors, e.g. the $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$ and $(4,0)$ sectors for

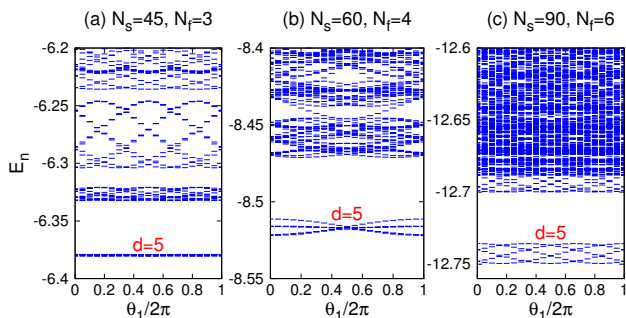


FIG. 7: (color online). The $1/5$ fermionic FQHE. Low energy spectra versus θ_1 at a fixed $\theta_2 = 0$ for three lattice sizes at $\nu = 1/5$ filling with $V_1 = 8.0$ and $V_2 = V_3 = 1.0$: (a) $N_s = 45$; (b) $N_s = 60$; (c) $N_s = 90$. The quasi-degeneracy of the GSM has been labeled.

$N_s = 45$ [Fig. 6(a), 6(b) and 6(d)]; while for $N_s = 75$, both N_f/N_1 and N_f/N_2 are integers, and all five GSs are in the same $(0,0)$ sector [with very close energies as shown in Fig. 6(c)]. Therefore, for each system size, there is an obvious GSM with five-fold quasi-degenerate states, which is well separated from the higher energy spectrum by a distinguishable spectrum gap.

For the $1/5$ fermionic FQHE, the five GSs also maintain their quasi-degeneracy and are well separated from the other low-energy excitation spectrum when we tune the boundary phases (Fig. 7), indicating a possible robust topological phase. Moreover, the $d = 5$ GSM in the $1/5$ fermionic FQHE is found to share a total Chern number $C_{\text{tot}} = 2$: e.g. for the $N_s = 60$ case [Fig. 6(b)], the summation of Berry curvatures in 10×10 mesh points gives the Chern numbers 0.39629, 0.40975, 0.38792, 0.40975 and 0.39629 for the five GSs in $(0,2)$, $(1,2)$, $(2,2)$, $(3,2)$ and $(4,2)$ sectors, respectively; and thus the total Chern number is found to be almost precisely $C_{\text{tot}} = 2$ for this $d = 5$ GSM, which implies that each GS supports a fractional quantized Hall conductance of $2e^2/5h$.

Concluding discussions.— We find convincing numerical evidences of $\nu = 1/3$ bosonic FQHE in the $C = 2$ TFB near $V_1 = V_2 = 0$. This odd-denominator bosonic FQHE phase is in stark contrast to the $C = 1$ TFB where the most robust bosonic FQHE occurs at the $\nu = 1/2$ filling for hard-core bosons with $V_1 = V_2 = 0$ [13]. It is desired to physically understand the nature of this odd-denominator bosonic FQHE. Due to the absence of higher topological ground-state degeneracy and the same quasihole counting as Laughlin’s $1/3$ fermionic FQHE, we believe that the $1/3$ bosonic FQHE phase is of Abelian nature. Moreover, when we treat the $C = 2$ band as an effective two-component (or bilayer) system [34] with the number of orbitals reduced to half for each component (which doubles the effective total filling to $2/3$), the $1/3$ bosonic FQHE would be consistent, in terms of fractional quasihole charge and ground state degeneracy, with Halperin’s mmn state (with $m = 2$ and $n = 1$)

at the $2/3$ total filling. For the $1/5$ filling of interacting spinless fermions in the $C = 2$ TFB, clear FQHE features have also been observed with a five-fold degenerate GSM and a fractional quantized Hall conductance of $2e^2/5h$ per GS, which could also be consistent with the Halperin 332 state. Nonetheless, the two “components” in a $C = 2$ TFB are mutually entangled, one may speculate that our states may be different from the conventional bilayer mmn states where the two separated layers are only coupled by interaction. We believe that more direct evidence is needed to verify the nature of the obtained FQHE states, which calls for further studies in the future.

Other topological phases are also possible in other fractional fillings. For the $\nu = 1/4$ hard-core bosons, we have observed a bosonic state with some topological features. However we can not determine the nature of this $1/4$ bosonic state as its ground-state degeneracy and the total Chern number varies with the particle numbers which will be presented in the Supplementary Material [37].

Although our results are obtained in a specific three-band triangular-lattice model, one may speculate that such results, especially the very robust $1/3$ bosonic state, might be universal for generic $C = 2$ TFB models [38]. In the future, it would be also highly interesting to demonstrate more exotic fractional topological phases in microscopic TFB models with high Chern numbers.

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Note added.—After the submission of the present Letter, a few related works appeared very recently [39–43]. Two works have shown that flat bands with arbitrary high Chern numbers can be systematically constructed using multi-layer lattice models [39, 40]. Another two works reported some numerical evidence for a series of fermionic and bosonic fractional incompressible states in a multi-layer kagome-lattice model with high Chern numbers [41, 43]. These results together with ours suggest some universality of FQHE in TFBs with high Chern numbers.

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- [37] See the Supplementary Material for details of this $1/4$ bosonic state.
- [38] Indeed, very recently, a similar $1/3$ bosonic FQHE has also been observed for a bilayer kagome-lattice model in Refs. [41, 43] and a two-orbital square-lattice model (see an updated version of Ref. [40]), supporting this universality conjecture.
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