Magnetic translation algebra with or without magnetic field in the continuum or on arbitrary Bravais lattices in any dimension

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Magnetic translation algebra with or without magnetic field in the continuum or on arbitrary Bravais lattices in any dimension

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The magnetic translation algebra plays an important role in the quantum Hall effect. Murthy and Shankar have shown how to realize this algebra using fermionic bilinears defined on a two-dimensional square lattice. We show that, in any dimension d, it is always possible to close the magnetic translation algebra using fermionic bilinears, be it in the continuum or on the lattice. We also show that these generators are complete in even, but not odd, dimensions, in the sense that any fermionic Hamiltonian in even dimensions that conserves particle number can be represented in terms of the generators of this algebra, whether or not time-reversal symmetry is broken. As an example, we reproduce the f-sum rule of interacting electrons at vanishing magnetic field using this representation. We also show that interactions can significantly change the bare band width of lattice Hamiltonians when represented in terms of the generators of the magnetic translation algebra.

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I. INTRODUCTION

The two coordinates of an electron in the plane orthogonal to an uniform magnetic field $B\,\boldsymbol{e}$ pointing along the direction $\boldsymbol{e}$ do not commute. There follows the $U(1)$ algebra

$$[t(\boldsymbol{a}),t(\boldsymbol{b})]=\mathrm{i}\,\sin\left(\frac{(\boldsymbol{a}\wedge\boldsymbol{b})\cdot\boldsymbol{e}}{2}\right)t(\boldsymbol{a}+\boldsymbol{b}) \tag{1.1}$$

obeyed by the triplet of generators $t(\boldsymbol{a})$, $t(\boldsymbol{b})$, and $t(\boldsymbol{a}+\boldsymbol{b})$ for any pair $\boldsymbol{a}$ and $\boldsymbol{b}$ of vectors orthogonal to $\boldsymbol{e}$, which is called the magnetic translation algebra in this context.[1] The magnetic translation algebra can be used to derive the transverse conductivity of the integer quantum Hall effect (IQHE). It has also been used by Girvin, MacDonald, and Platzman in Ref. 2 to place a variational estimate on the excitation gap for the fractional quantum Hall effect (FQHE), following closely the approach of Feynman and Bijl in their study of excitations in ^{4}He.[3]

Hamiltonians defined on two-dimensional lattices with topologically non-trivial bands can also display quantum Hall physics. The IQHE can occur in band insulators when the Bloch bands have a nonvanishing Chern number, as shown by Haldane.[4] The FQHE effect requires strong electronic correlations. This is possible if the Chern bands are sufficiently narrow (or even flat).[5–7] Whether flat Chern bands can sustain or not a FQHE is a matter of energetics. Exact diagonalization studies of fractionally filled Chern bands with added short-range interactions are consistent with a correlated liquid ground state supporting a FQHE for certain filling fractions.[7–21] Such topological correlated states on the lattice are now known as fractional Chern insulators (FCI).

In an effort to draw a bridge between the case when the FQHE is realized in the continuum or in a FCI, Parameswaran, Roy, and Sondhi in Ref. 22 have pioneered an algebraic approach to FCIs by deriving the algebra obeyed by the density operators projected to the partially filled band.[12,23–25] They found that the algebra (1.1) emerges to leading order in a gradient expansion. Remarkably, Murthy and Shankar have (i) constructed in Ref. 26 a coherent superposition of the projected density operator that closes the $U(1)$ algebra (1.1) on the square lattice and (ii) represented any Hamiltonian that commutes with the number operator and describes the competition between the electronic hopping and the electronic interaction in terms of these generators.[27]

In this paper, we are going to generalize the results by Murthy and Shankar as follows. We shall represent the $U(1)$ algebra (1.1) in terms of coherent superpositions of electron-hole pairs in arbitrary dimensions both in the continuum and for Bravais lattices. We shall then show that these generators provide a complete basis for the linear space of operators spanned by charge neutral fermion bilinears provided the Bravais lattice, or its embedding space in the continuum limit, is even dimensional. For odd dimensions, the generators of the $U(1)$ algebra (1.1) form an incomplete basis of the space of operators spanned by charge neutral fermion bilinears.

We first treat the case of Hamiltonians acting on wavefunctions supported in the continuum for pedagogical reasons in Sec. II. After this warm up, we turn our attention to Hamiltonians acting on wavefunctions supported on Bravais lattices in Sec. III. Sections II and III constitute the main results of this paper.

As a sanity check, we verify that the f-sum rule is obeyed if one represents the electronic density operator in terms of the particle-hole generators of the algebra (1.1) in Sec. IV A. This exercise also suggests caution when performing uncontrolled approximations using the magnetic algebra, for such uncontrolled approximations could predict effects associated to a spurious breaking of time-reversal symmetry.

In Sec. IV B, we explain how, when represented in terms of these generators of the algebra (1.1), interactions induce one-body terms that can significantly change the bare band width of lattice Hamiltonians. The same effect in the FQHE requires the addition of a strong one-body perturbation to a Landau band, one that is of the order of the FQHE gap. Thus, whereas the FQH is a strong coupling problem, the FCI in a flat band is more like a problem at intermediate coupling. This result explains why in Ref. 19 a FCI with Chern number two was more stable if the bare dispersion was not flat rather than flat, for the bare and induced one-body terms can conspire to neutralize each other.

II. THE CASE OF THE CONTINUUM

We define the fermionic Fock space $\mathfrak{F}$ with the help of the algebra

$$\begin{aligned}\big\{\hat{c}(\boldsymbol{k}),\hat{c}^\dagger(\boldsymbol{k}')\big\}&=\delta(\boldsymbol{k}-\boldsymbol{k}'),\\ \{\hat{c}(\boldsymbol{k}),\hat{c}(\boldsymbol{k}')\}&=\big\{\hat{c}^\dagger(\boldsymbol{k}),\hat{c}^\dagger(\boldsymbol{k}')\big\}=0,\end{aligned}\tag{2.1}$$

for any pair of momenta $\boldsymbol{k},\boldsymbol{k}'\in\mathbb{R}^d$. Without loss of generality, we ignore any internal degrees of freedom such as the spin quantum numbers since we are after the $U(1)$ algebra (1.1).

The linear space of fermionic bilinears that we study is spanned by the basis

$$\hat{T}(\boldsymbol{q}_1,\boldsymbol{q}_2):=\hat{c}^\dagger(\boldsymbol{q}_1)\,\hat{c}(\boldsymbol{q}_2)\tag{2.2a}$$

that obeys the algebra

$$\begin{aligned}\Big[\hat{T}(\boldsymbol{q}_1,\boldsymbol{q}_2),\hat{T}(\boldsymbol{q}_1',\boldsymbol{q}_2')\Big]=&\,\delta(\boldsymbol{q}_2-\boldsymbol{q}_1')\,\hat{T}(\boldsymbol{q}_1,\boldsymbol{q}_2')\\ &-\delta(\boldsymbol{q}_1-\boldsymbol{q}_2')\,\hat{T}(\boldsymbol{q}_1',\boldsymbol{q}_2)\end{aligned}\tag{2.2b}$$

for any quadruple $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_1'$, and $\boldsymbol{q}_2'$ from $\mathbb{R}^d$.

For any momentum $\boldsymbol{q}\in\mathbb{R}^d$ and for any function $f:\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{C}$, define the coherent superposition

$$\hat{\varrho}^f(\boldsymbol{q}):=\int\limits_{\boldsymbol{p}} f(\boldsymbol{q},\boldsymbol{p})\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{p})\,\hat{c}(\boldsymbol{p}).\tag{2.3a}$$

There follows the algebra

$$\Big[\hat{\varrho}^f(\boldsymbol{q}),\hat{\varrho}^{f'}(\boldsymbol{q}')\Big]=\int\limits_{\boldsymbol{p}}\Big[f(\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{p})\,f'(\boldsymbol{q}',\boldsymbol{p})-(\boldsymbol{q}\leftrightarrow\boldsymbol{q}'\text{ and }f\leftrightarrow f')\Big]\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p})\,\hat{c}\,(\boldsymbol{p})\tag{2.3b}$$

for any pair of momenta $\boldsymbol{q}$ and $\boldsymbol{q}'$ and for any pair of functions f and f'.

The choice $f(\boldsymbol{q},\boldsymbol{p})=1$ for any pair of momenta $\boldsymbol{q}$ and $\boldsymbol{p}$ from $\mathbb{R}^d$ defines the momentum representation of the local density operator

$$\hat{\rho}(\boldsymbol{q}):=\int\limits_{\boldsymbol{p}}\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{p})\,\hat{c}(\boldsymbol{p}).\tag{2.4a}$$

Any pair thereof commutes

$$[\hat{\rho}(\boldsymbol{q}),\hat{\rho}(\boldsymbol{q}')]=0.\tag{2.4b}$$

Another choice of the function f is made with the family

$$\hat{\varrho}(\boldsymbol{q};\boldsymbol{G}):=\int\limits_{\boldsymbol{p}} e^{+\mathrm{i}\,\Phi(\boldsymbol{q},\boldsymbol{p};\boldsymbol{G})}\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{p})\,\hat{c}\,(\boldsymbol{p})\tag{2.5a}$$

for any pair $\boldsymbol{q}$ and $\boldsymbol{G}$ from $\mathbb{R}^d$ where

$$\Phi(\boldsymbol{q},\boldsymbol{p};\boldsymbol{G}):=(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{p}-\frac{1}{2}\boldsymbol{q}*\boldsymbol{G}\tag{2.5b}$$

and the $*$-product

$$\boldsymbol{a}*\boldsymbol{b}=-\boldsymbol{b}*\boldsymbol{a}\equiv\sum_{i,j=1}^{d}a_i\,M_{ij}^{(*)}b_j\tag{2.5c}$$

defines a real antisymmetric bilinear form specified by the real-valued $d\times d$ antisymmetric matrix $M^{(*)}$. When d is even, we assume that $M^{(*)}$ is invertible. When d is odd, $M^{(*)}$ has at least one vanishing eigenvalue and is thus not invertible. Observe that

$$\hat{\rho}(\boldsymbol{q})=\hat{\varrho}(\boldsymbol{q};-\boldsymbol{q}).\tag{2.6}$$

We are going to prove that

(1) The family $\hat{\varrho}(\boldsymbol{q};\boldsymbol{G})$ labeled by the pair $\boldsymbol{q}$ and $\boldsymbol{G}$ from $\mathbb{R}^d$ obeys the $U(1)$ algebra (1.1).

(2) In even-dimensional space, the family $\hat{\varrho}(\boldsymbol{q};\boldsymbol{G})$ labeled by the pair $\boldsymbol{q}$ and $\boldsymbol{G}$ from $\mathbb{R}^d$ is complete.

Proof of closure: We define

$$\Gamma(\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G},\boldsymbol{G}') := \Phi(\boldsymbol{q},\boldsymbol{q}'+\boldsymbol{p};\boldsymbol{G}) + \Phi(\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G}') - \Phi(\boldsymbol{q}+\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G}+\boldsymbol{G}') \tag{2.7a}$$

in terms of which Eq. (2.3b) can be rewritten as

$$[\hat{\varrho}(\boldsymbol{q};\boldsymbol{G}),\hat{\varrho}(\boldsymbol{q}';\boldsymbol{G}')] = \int\limits_{\boldsymbol{p}} \left[e^{\mathrm{i}\,\Gamma(\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G},\boldsymbol{G}')} - (\boldsymbol{q}\leftrightarrow\boldsymbol{q}' \text{ and } \boldsymbol{G}\leftrightarrow\boldsymbol{G}')\right] e^{+\mathrm{i}\,\Phi(\boldsymbol{q}+\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G}+\boldsymbol{G}')}\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p})\,\hat{c}(\boldsymbol{p}). \tag{2.7b}$$

Since

$$\Gamma(\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p};\boldsymbol{G},\boldsymbol{G}') = \left(\boldsymbol{q}+\frac{1}{2}\boldsymbol{G}\right) * \left(\boldsymbol{q}'+\frac{1}{2}\boldsymbol{G}'\right) - \frac{1}{4}\boldsymbol{G}*\boldsymbol{G}' \equiv \Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}') \tag{2.8a}$$

is independent of $\boldsymbol{p}$ and antisymmetric under $\boldsymbol{q}\leftrightarrow\boldsymbol{q}'$ and $\boldsymbol{G}\leftrightarrow\boldsymbol{G}'$,

$$\Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}') = -\Upsilon(\boldsymbol{q}',\boldsymbol{q};\boldsymbol{G}',\boldsymbol{G}), \tag{2.8b}$$

the algebra (2.7b) closes to

$$[\hat{\varrho}(\boldsymbol{q};\boldsymbol{G}),\hat{\varrho}(\boldsymbol{q}';\boldsymbol{G}')] = F(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}')\,\hat{\varrho}(\boldsymbol{q}+\boldsymbol{q}';\boldsymbol{G}+\boldsymbol{G}') \tag{2.9a}$$

with the structure constant

$$F(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}') = e^{\mathrm{i}\,\Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}')} - (\boldsymbol{q}\leftrightarrow\boldsymbol{q}' \text{ and } \boldsymbol{G}\leftrightarrow\boldsymbol{G}') = 2\mathrm{i}\,\sin\,\Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}'). \tag{2.9b}$$

Proof of completeness: Choose any function $f:\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{C}$ such that the Fourier transform

$$f(\boldsymbol{q},\boldsymbol{p}) =: \bar{f}(\boldsymbol{q},\boldsymbol{p})\,e^{\mathrm{i}\,\boldsymbol{q}*\boldsymbol{p}} =: \left(\int\limits_{\boldsymbol{G}} e^{+\mathrm{i}\,\boldsymbol{G}*\boldsymbol{p}}\,\tilde{f}(\boldsymbol{q},\boldsymbol{G})\right) e^{\mathrm{i}\,\boldsymbol{q}*\boldsymbol{p}} \tag{2.10}$$

is well defined. For the second equality to be true for arbitrary functions $\bar{f}(\boldsymbol{q},\cdot):\mathbb{R}^d\to\mathbb{C}$ with the well-defined Fourier transform $\tilde{f}(\boldsymbol{q},\cdot):\mathbb{R}^d\to\mathbb{C}$, the square matrix $M^{(*)}$ that defines the $*$-product must be invertible and thus have an even number d of rows (columns). Indeed, the rank of an antisymmetric matrix $M^{(*)}$ is necessarily even. Hence, in odd dimensional space, $M^{(*)}$ is never invertible as it has at least one vanishing eigenvalue. This means that the $*$ Fourier transform $\int\limits_{\boldsymbol{G}} e^{+\mathrm{i}\,\boldsymbol{G}*\boldsymbol{p}}\,\tilde{h}(\boldsymbol{G})$ is at best a function of $d-1$ coordinates of $\boldsymbol{p}$ if d is odd. For completeness to hold, it is thus necessary that d be even, which we now assume. A sufficient condition for completeness to hold is that the linear space spanned by the operators (2.2a) is limited to the coherent superpositions of the form (2.3a) such that the function $\bar{f}(\boldsymbol{q},\cdot):\mathbb{R}^d\to\mathbb{C}$ has a Fourier transform for any given momentum $\boldsymbol{q}$. With the help of Eq. (2.5b), we can then write

$$f(\boldsymbol{q},\boldsymbol{p}) = \int\limits_{\boldsymbol{G}} \tilde{f}(\boldsymbol{q},\boldsymbol{G})\,e^{\mathrm{i}\,\boldsymbol{q}*\boldsymbol{G}/2}\,e^{\mathrm{i}\Phi(\boldsymbol{q},\boldsymbol{p};\boldsymbol{G})}. \tag{2.11}$$

In turn and with the help of Eq. (2.3a), we conclude with

$$\hat{\varrho}^f(\boldsymbol{q}) = \int\limits_{\boldsymbol{p}} f(\boldsymbol{q},\boldsymbol{p})\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{p})\,\hat{c}\,(\boldsymbol{p}) = \int\limits_{\boldsymbol{G}} \tilde{f}(\boldsymbol{q},\boldsymbol{G}) e^{\mathrm{i}\,\boldsymbol{q}*\boldsymbol{G}/2} \int\limits_{\boldsymbol{p}} e^{\mathrm{i}\Phi(\boldsymbol{q},\boldsymbol{p};\boldsymbol{G})}\,\hat{c}^\dagger(\boldsymbol{q}+\boldsymbol{p})\,\hat{c}\,(\boldsymbol{p}) = \int\limits_{\boldsymbol{G}} \tilde{f}(\boldsymbol{q},\boldsymbol{G})\,e^{\mathrm{i}\,\boldsymbol{q}*\boldsymbol{G}/2}\,\hat{\varrho}(\boldsymbol{q};\boldsymbol{G}). \tag{2.12}$$

III. THE CASE OF THE LATTICE

We begin with some notation. Let Λ be a Bravais lattice and $\Lambda^\star$ its dual. Sites in Λ are denoted by $\boldsymbol{r}$, sites in $\Lambda^\star$ by $\boldsymbol{G}$. The first Brillouin zone is denoted Ω_{BZ}, it contains the origin of $\mathbb{R}^d$. We shall decompose $\mathbb{R}^d$ into a set of shifted Brillouin zones $\Omega_{\mathrm{BZ}}^{\boldsymbol{G}}$ obtained by translation of Ω_{BZ} by $\boldsymbol{G}\in\Lambda^\star$,

$$\mathbb{R}^d = \bigcup_{\boldsymbol{G}\in\Lambda^\star} \Omega_{\mathrm{BZ}}^{\boldsymbol{G}}. \tag{3.1}$$

Sites in $\Omega_{\rm BZ}$ are denoted $\boldsymbol{k}$, $\boldsymbol{q}$, and $\boldsymbol{p}$. If $\boldsymbol{q}$ and $\boldsymbol{p}$ belong to the Brillouin zone $\Omega_{\rm BZ}$, this might not be the case for $\boldsymbol{q}+\boldsymbol{p}$. There is a unique $\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Lambda^\star$ such that $\boldsymbol{q}+\boldsymbol{p} \in \Omega_{\rm BZ}^{\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}}}$. Correspondingly, $\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Omega_{\rm BZ}$. We shall use the notation

$$[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ} \equiv \boldsymbol{q}+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Omega_{\rm BZ}. \tag{3.2a}$$

Two observations are pertinent to what follows. First, the bracketing (3.2a) obeys the nesting rule

$$\Big[\boldsymbol{q}'+[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ}\Big]_{\rm BZ} = [\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}]_{\rm BZ} \tag{3.2b}$$

for any triplet $\boldsymbol{q}$, $\boldsymbol{q}'$, and $\boldsymbol{p}$ from the first Brillouin zone. Second, if we hold $\boldsymbol{q} \in \Omega_{\rm BZ}$ fixed and vary $\boldsymbol{p}$ across the Brillouin zone $\Omega_{\rm BZ}$, the unique reciprocal wave vector $\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Lambda^\star$ such that $\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Omega_{\rm BZ}$ defines an implicit function of $\boldsymbol{q}$ that is piecewise constant with discontinuous jumps each time $\boldsymbol{q}+\boldsymbol{p}$ crosses the boundary separating neighboring Brillouin zones.

We define the fermionic Fock space $\mathfrak{F}$ with the help of the algebra

$$\begin{aligned} &\Big\{\hat{c}_{\boldsymbol{k}+\boldsymbol{G}}, \hat{c}^\dagger_{\boldsymbol{k}'+\boldsymbol{G}'}\Big\} = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \\ &\big\{\hat{c}_{\boldsymbol{k}+\boldsymbol{G}}, \hat{c}_{\boldsymbol{k}'+\boldsymbol{G}'}\big\} = \Big\{\hat{c}^\dagger_{\boldsymbol{k}+\boldsymbol{G}}, \hat{c}^\dagger_{\boldsymbol{k}'+\boldsymbol{G}'}\Big\} = 0, \end{aligned} \tag{3.3}$$

for any pair $\boldsymbol{k}$ and $\boldsymbol{k}'$ from the Brillouin zone $\Omega_{\rm BZ}$ and any pair $\boldsymbol{G}$ and $\boldsymbol{G}'$ from the dual lattice $\Lambda^\star$.

The linear space of fermionic bilinears that we study is spanned by the basis

$$\hat{T}_{\boldsymbol{q}_1,\boldsymbol{q}_2} := \hat{c}^\dagger_{\boldsymbol{q}_1}\, \hat{c}_{\boldsymbol{q}_2} \tag{3.4a}$$

that obeys the algebra

$$\Big[\hat{T}_{\boldsymbol{q}_1,\boldsymbol{q}_2}, \hat{T}_{\boldsymbol{q}_1',\boldsymbol{q}_2'}\Big] = \delta_{\boldsymbol{q}_2,\boldsymbol{q}_1'}\, \hat{T}_{\boldsymbol{q}_1,\boldsymbol{q}_2'} - \delta_{\boldsymbol{q}_1,\boldsymbol{q}_2'}\, \hat{T}_{\boldsymbol{q}_1',\boldsymbol{q}_2} \tag{3.4b}$$

for any quadruple $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_1'$, and $\boldsymbol{q}_2'$ from the Brillouin zone.

For any $\boldsymbol{q}$ from the Brillouin zone $\Omega_{\rm BZ}$ and for any function $f : \Omega_{\rm BZ} \times \Omega_{\rm BZ} \longrightarrow \mathbb{C}$, define

$$\hat{\varrho}^f_{\boldsymbol{q}} := \sum_{\boldsymbol{p} \in \Omega_{\rm BZ}} f_{\boldsymbol{q},\boldsymbol{p}}\, \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ}}\, \hat{c}_{\boldsymbol{p}}. \tag{3.5}$$

There follows the algebra [with the help of Eq. (3.2b)]

$$\begin{aligned} \Big[\hat{\varrho}^f_{\boldsymbol{q}}, \hat{\varrho}^{f'}_{\boldsymbol{q}'}\Big] = \sum_{\boldsymbol{p} \in \Omega_{\rm BZ}} &\Big[f_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\rm BZ}}\, f'_{\boldsymbol{q}',\boldsymbol{p}} - (\boldsymbol{q}, f \leftrightarrow \boldsymbol{q}', f')\Big] \\ &\times \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}]_{\rm BZ}}\, \hat{c}_{\boldsymbol{p}} \end{aligned} \tag{3.6}$$

for any pair of momenta $\boldsymbol{q}$ and $\boldsymbol{q}'$ from the Brillouin zone and for any pair of functions f and f'.

The choice $f_{\boldsymbol{q},\boldsymbol{p}} = 1$ for any pair $\boldsymbol{q}$ and $\boldsymbol{p}$ from the Brillouin zone defines the momentum representation of the local density operator

$$\hat{\rho}_{\boldsymbol{q}} := \sum_{\boldsymbol{p} \in \Omega_{\rm BZ}} \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ}}\, \hat{c}_{\boldsymbol{p}}. \tag{3.7a}$$

Any pair thereof commutes

$$\big[\hat{\rho}_{\boldsymbol{q}}, \hat{\rho}_{\boldsymbol{q}'}\big] = 0. \tag{3.7b}$$

Another choice of the function f is made with the family

$$\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}} := \sum_{\boldsymbol{p} \in \Omega_{\rm BZ}} e^{+{\rm i}\,\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}}}\, \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ}}\, \hat{c}_{\boldsymbol{p}} \tag{3.8a}$$

for any $\boldsymbol{G}$ from the dual lattice and $\boldsymbol{q}$ from the Brillouin zone where

$$\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}} := \frac{1}{2\pi}\,\big[(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{p} - (\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G})*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G}}\big] \tag{3.8b}$$

and the $d \times d$ matrix $M^{(*)}_\Lambda$ that defines the $*$-product is antisymmetric, as was the case in the continuum, but with the restriction that

$$\frac{1}{2\pi}\,\boldsymbol{G}*\boldsymbol{G}' = 0 \bmod 2\pi, \qquad \forall \boldsymbol{G}, \boldsymbol{G}' \in \Lambda^\star, \tag{3.9}$$

to accommodate the d-dimensional Bravais lattice Λ. When d is even, $M^{(*)}_\Lambda$ has a nonvanishing determinant by assumption. As advertised, the $U(1)$ algebra

$$\Big[\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}, \hat{\varrho}^{\boldsymbol{G}'}_{\boldsymbol{q}'}\Big] = 2{\rm i}\,\sin\left(\frac{(\boldsymbol{q}+\boldsymbol{G})*(\boldsymbol{q}'+\boldsymbol{G}')}{2\pi}\right)\,\hat{\varrho}^{\boldsymbol{G}+\boldsymbol{G}'}_{\boldsymbol{q}+\boldsymbol{q}'} \tag{3.10}$$

follows for any quadruple $\boldsymbol{q}$, $\boldsymbol{q}'$, $\boldsymbol{G}$, and $\boldsymbol{G}'$. The proof of Eq. (3.10) is technically more involved than that of Eq. (2.9) as one needs to account for the restriction on momenta to the first Brillouin zone. For this reason, we refer the reader to Appendix A for the details of the proof.

To prove completeness, we assume that the dimensionality d is even for the same reasons as given below Eq. (2.5c). One verifies that

$$\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}} = \Theta_{\boldsymbol{q},\boldsymbol{p}} + \frac{\boldsymbol{G}*\boldsymbol{p} - \boldsymbol{p}*\boldsymbol{G}}{2\pi} + \Theta^{\boldsymbol{G}}_{\boldsymbol{q}} + \bmod 2\pi \tag{3.11a}$$

where the function

$$2\pi\,\Theta_{\boldsymbol{q},\boldsymbol{p}} := \boldsymbol{q}*\boldsymbol{p} - (\boldsymbol{q}+\boldsymbol{p})*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \tag{3.11b}$$

is independent of $\boldsymbol{G}$, while the function

$$2\pi\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}} := -\boldsymbol{q}*\boldsymbol{G} \tag{3.11c}$$

is independent of $\boldsymbol{p}$. We will use the fact that

$$\Theta_{\boldsymbol{q}=\boldsymbol{0},\boldsymbol{p}} = \Theta^{\boldsymbol{G}}_{\boldsymbol{q}=\boldsymbol{0}} = 0 \tag{3.12}$$

in Sec. IV B and Appendix B. We define the function $\bar{f} : \Omega_{\rm BZ} \times \Omega_{\rm BZ} \longrightarrow \mathbb{C}$ by

$$f_{\boldsymbol{q},\boldsymbol{p}} =: \bar{f}_{\boldsymbol{q},\boldsymbol{p}}\, e^{+\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}}. \tag{3.13}$$

We then use the Fourier expansion

$$\bar{f}_{\boldsymbol{q},\boldsymbol{p}} =: \sum_{\boldsymbol{G}\,\in\Lambda^\star} \tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}}\, e^{+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)} \tag{3.14}$$

to do the following manipulations

$$\begin{aligned} f_{\boldsymbol{q},\boldsymbol{p}} &= e^{+\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}}\,\bar{f}_{\boldsymbol{q},\boldsymbol{p}} \\ &= e^{+\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}} \left[\sum_{\boldsymbol{G}\,\in\Lambda^\star} \tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} e^{+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}\right] \\ &= \sum_{\boldsymbol{G}\,\in\Lambda^\star} \tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} e^{+\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)+\mathrm{i}\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}-\mathrm{i}\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}} \\ &= \sum_{\boldsymbol{G}\,\in\Lambda^\star} \underbrace{\tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} e^{-\mathrm{i}\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}}}_{\text{independent of } \boldsymbol{p}} \times e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}}}. \end{aligned} \tag{3.15}$$

Insertion of Eq. (3.15) into Eq. (3.5) gives

$$\begin{aligned} \hat{\varrho}^f_{\boldsymbol{q}} &= \sum_{\boldsymbol{G}\,\in\Lambda^\star} \tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} e^{-\mathrm{i}\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}} \left(\sum_{\boldsymbol{p}\,\in\Omega_{\rm BZ}} e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}}}\, \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ}}\, \hat{c}_{\boldsymbol{p}}\right) \\ &= \sum_{\boldsymbol{G}\,\in\Lambda^\star} \tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} e^{-\mathrm{i}\,\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}}\, \hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}} \end{aligned} \tag{3.16}$$

where we made use of the definition (3.8a) to reach the last equality. Completeness has thus been proved if the space of functions f is restricted to those for which the Fourier transform (3.14) exists.

IV. DISCUSSION

As pointed out by Murthy and Shankar, the magnetic translation algebra is not limited to situations in which time-reversal symmetry is broken. From the point of view of many-body physics, the generators of the magnetic translation algebra can also be thought of as special coherent superpositions of particle-hole excitations. As such they are always present in the many-body Fock space.

If time-reversal symmetry is either explicitly or spontaneously broken, it is plausible that these excitations might be selected by the many-body interactions to play an important role at low energies and long distances. However, the breaking of time-reversal symmetry alone is no guarantee for the FQHE. The selection of a ground state supporting the FQHE is a subtle compromise between the kinetic energy and the interactions.

If time-reversal symmetry is neither explicitly nor spontaneously broken, it is harder to imagine that these excitations are of relevance to the low-energy and long-distance properties of interacting electrons.

With this motivation in mind, we are going to discuss the following two cases.

(a) f-sum rule. We begin in Sec. IV A with the case of interacting electrons in the continuum limit without explicit breaking of time-reversal symmetry and for which spontaneous symmetry breaking of time-reversal symmetry is not anticipated. This situation is the one expected if electrons interact through sufficiently weak density-density interactions. We are going to show how to recover the f-sum rule when we choose to represent the many-body Hamiltonian in terms of the generators (2.5) of the magnetic translation algebra for any even dimension d of space.

This exercise serves two purposes. First, it gives us the confidence that we can solve an interacting problem devoid of any magnetic field using the magnetic translation algebra, i.e., using a technology that is geared to the presence of a magnetic field. We find this result remarkable. Second, it is a warning against blindly performing a mean-field approximation of the Hamiltonian when represented in terms of the generators (2.5) that delivers the FQHE. In other words, one should be cautious when using the magnetic translation algebra in an approximate fashion to predict a FQHE, for such treatments can predict a FQHE when none is known to occur.

(b) FCIs at intermediate rather than strong couplings. To illustrate the delicate competition between the kinetic energy and the interactions, we consider in Sec. IV B a Hamiltonian describing a band insulator to which we add density-density interactions that preserve translation invariance. We represent the projection of this Hamiltonian onto a single band in terms of the generators (3.8) for any even dimension d of the Bravais lattice. In doing so, we are going to show that normal ordering can change the bare band width by a value comparable to the characteristic energy for the interactions. Hence, if the bare band width is smaller than the characteristic energy for the interactions, as is usually believed to be necessary to stabilize a FCI, normal ordering can be an effect of order one!

As an application of this result, we consider any projected and normal-ordered Hamiltonian $\hat{H}$ describing itinerant fermions in a flat band carrying a nonvanishing Chern number and interacting though a density-density interaction that preserves translation invariance. We assume that $\hat{H}$ supports a FCI as the ground state at the partial filling $0 < \nu < 1$ of the flat band. A particle-hole transformation turns the normal-ordered $\hat{H}$ into $\widetilde{H}$, whereby $\widetilde{H}$ must support a FCI made of holes as the ground state at the partial filling $\widetilde{\nu} = 1 - \nu$. What is remarkable is that the projected Hamiltonian $\widetilde{H}$, when decomposed into a one-body term and a normal-ordered interaction, can be thought of as describing holes with a genuine dispersion and interacting through a normal-ordered density-density interaction sharing the same functional form as $\hat{H}$. The dispersion of the holes is genuine for its width is generically nonvanishing and of the order of the characteristic interaction strength times

a numerical factor of geometrical origin. Indeed, this numerical factor arises because of the geometry induced by the overlaps between pairs of Bloch states from the original flat band.[28] When these overlaps are constant, as is the case in the FQHE, this numerical factor vanishes so that $\widetilde{H}$ can also be assigned a flat band. When these overlaps are functions of both the relative and center of mass momenta of the pair of Bloch states, then this numerical factor can be nonvanishing.

That this numerical factor can be of order unity, and thus matters in a crucial way in order to stabilize the FCI at the filling fraction $\widetilde{\nu}$, can be inferred from the following numerical results.

In Ref. 7, a band insulator with two flat bands supporting the Chern numbers ± 1 was shown to support a FCI phase at the filling fraction 1/3 in the presence of a repulsive nearest-neighbor density-density interaction projected onto the lower flat band. In Ref. 10, the *same* band insulator was shown to support the *same* FCI phase at the *same* filling fraction 1/3 in the presence of a *different* interaction, namely the repulsive nearest-neighbor density-density interaction projected onto the lower flat band and then *normal-ordered*. Hence, at the filling fraction 1/3, the FCI phase is robust to weather the projected interaction is normal-ordered or not. In Ref. 11, the same model as in Ref. 7 was also shown to support a FCI phase at the filling fraction 2/3. However, no evidences for a topological phase were found at the filling fraction 2/3 using the *normal-ordered* projected interaction in Ref. 10. Hence, at the filling fraction 2/3, the FCI phase is either not selected as the ground state or very close to a phase transition to a phase without topological order when the projected interaction is normal ordered, while the FCI phase is selected as the ground state when the projected interaction is not normal ordered. We conclude that the characteristic band width of the one-body term that is generated by normal ordering the repulsive nearest-neighbor density-density interaction must be of the same order as the characteristic energy scale of the interaction.

Both quantitative examples are consistent with the fact that interactions projected onto a single Chern band can induce one-body terms that can significantly alter the band width of lattice Hamiltonians for itinerant fermions.

A. f-sum rule

The f-sum rule holds for electrons with the mass m and the quadratic dispersion

$$\varepsilon(\boldsymbol{p}) := \frac{\boldsymbol{p}^2}{2m} \tag{4.1a}$$

subjected to any one-body potential V and interacting with any translation-invariant density-density interaction U in any dimension d. Pines and Nozières presented a derivation thereof in Ref. 29 that hinges on the fact that the operator identity

$$\Big[\hat{\rho}(\boldsymbol{q}), [\hat{H}, \hat{\rho}(-\boldsymbol{q})]\Big] = \frac{\boldsymbol{q}^2}{m}\,\hat{N} \tag{4.1b}$$

holds for any momentum $\boldsymbol{q} \in \mathbb{R}^d$. Here,

$$\hat{N} := \int_{\boldsymbol{p}} \hat{c}^\dagger(\boldsymbol{p})\,\hat{c}(\boldsymbol{p}) \tag{4.1c}$$

is the conserved particle number operator, and the many-body Hamiltonian $\hat{H} := \hat{H}_0 + \hat{H}_V + \hat{H}_U$ is the sum of the dispersion

$$\hat{H}_0 := \int_{\boldsymbol{p}} \varepsilon(\boldsymbol{p})\,\hat{c}^\dagger(\boldsymbol{p})\,\hat{c}(\boldsymbol{p}), \tag{4.1d}$$

the one-body potential

$$\hat{H}_V := \int_{\boldsymbol{q}} V(\boldsymbol{q})\,\hat{\rho}(-\boldsymbol{q}), \tag{4.1e}$$

and the two-body potential

$$\hat{H}_U := \int_{\boldsymbol{q}} V(\boldsymbol{q})\;\hat{\rho}(\boldsymbol{q})\,\hat{\rho}(-\boldsymbol{q}). \tag{4.1f}$$

The only nonvanishing contribution to the nested commutator in Eq. (4.1b) arises from the quadratic dispersion $\hat{H}_0$ in view of the Abelian algebra (2.4b). Equation (4.1b) follows from the algebra (2.1).

As a sanity check, we are going to verify Eq. (4.1b) for any even dimension d with the help of the magnetic translation algebra

$$\begin{aligned}[\hat{\varrho}(\boldsymbol{q};\boldsymbol{G}), \hat{\varrho}(\boldsymbol{q}';\boldsymbol{G}')] = {}& 2\mathrm{i}\,\sin\Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}') \\ & \times\;\hat{\varrho}(\boldsymbol{q}+\boldsymbol{q}';\boldsymbol{G}+\boldsymbol{G}')\end{aligned} \tag{4.2}$$

where $\Upsilon(\boldsymbol{q},\boldsymbol{q}';\boldsymbol{G},\boldsymbol{G}')$ is defined in Eq. (2.8). We shall only evaluate the contribution from the quadratic dispersion (4.1d).

First, we recall that $\hat{\rho}(\boldsymbol{q}) = \hat{\varrho}(\boldsymbol{q};-\boldsymbol{q})$ according to Eq. (2.6). Second, we expand $\hat{H}_0$ in terms of the magnetic translation densities $\hat{\varrho}(\boldsymbol{q};\boldsymbol{G})$,

$$\hat{H}_0 \;=\; \int_{\boldsymbol{G}} \tilde{\varepsilon}(\boldsymbol{G})\;\hat{\varrho}(\boldsymbol{q}=0;\boldsymbol{G}) \tag{4.3a}$$

where

$$\tilde{\varepsilon}(\boldsymbol{G}) \;=\; \int_{\boldsymbol{p}} e^{-\mathrm{i}\,\boldsymbol{G}*\boldsymbol{p}}\,\varepsilon(\boldsymbol{p}). \tag{4.3b}$$

It is with Eq. (4.3b) that we made use of d being even.

Third, we make a first use of Eq. (4.2) to evaluate the internal commutator

$$\begin{aligned}\Big[\hat\rho(\boldsymbol{q}),[\hat H,\hat\rho(-\boldsymbol{q})]\Big] &= \int\limits_{\boldsymbol{G}} \tilde\varepsilon(\boldsymbol{G})\,\Big[\hat\varrho(\boldsymbol{q};-\boldsymbol{q}),[\hat\varrho(0;\boldsymbol{G}),\hat\varrho(-\boldsymbol{q};\boldsymbol{q})]\Big]\\ &= \int\limits_{\boldsymbol{G}} \tilde\varepsilon(\boldsymbol{G})\;2\mathrm{i}\,\sin\,\Upsilon(0,-\boldsymbol{q};\boldsymbol{G},\boldsymbol{q})\;[\hat\varrho(\boldsymbol{q};-\boldsymbol{q}),\hat\varrho(-\boldsymbol{q};\boldsymbol{G}+\boldsymbol{q})].\end{aligned} \tag{4.4}$$

We make a second use of Eq. (4.2) to evaluate the external commutator

$$\begin{aligned}\Big[\hat\rho(\boldsymbol{q}),[\hat H,\hat\rho(-\boldsymbol{q})]\Big] &= \int\limits_{\boldsymbol{G}} \tilde\varepsilon(\boldsymbol{G})\;2\mathrm{i}\,\sin\,\Upsilon(0,-\boldsymbol{q};\boldsymbol{G},\boldsymbol{q})\;2\mathrm{i}\,\sin\,\Upsilon(\boldsymbol{q},-\boldsymbol{q};-\boldsymbol{q},\boldsymbol{G}+\boldsymbol{q})\;\hat\varrho(0;\boldsymbol{G})\\ &= \int\limits_{\boldsymbol{G}} \tilde\varepsilon(\boldsymbol{G})\;\left[2\mathrm{i}\,\sin\left(\frac{\boldsymbol{q}*\boldsymbol{G}}{2}\right)\right]^2\;\hat\varrho(0;\boldsymbol{G}).\end{aligned} \tag{4.5}$$

The integral over $\boldsymbol{G}$ can now be performed,

$$\begin{aligned}\Big[\hat\rho(\boldsymbol{q}),[\hat H,\hat\rho(-\boldsymbol{q})]\Big] &= \int\limits_{\boldsymbol{G}} \tilde\varepsilon(\boldsymbol{G})\;\left(e^{+\mathrm{i}\,\boldsymbol{q}*\boldsymbol{G}}+e^{-\mathrm{i}\,\boldsymbol{q}*\boldsymbol{G}}-2\right)\int\limits_{\boldsymbol{p}} e^{+\mathrm{i}\,\boldsymbol{G}*\boldsymbol{p}}\;\hat c^\dagger(\boldsymbol{p})\,\hat c(\boldsymbol{p})\\ &= \int\limits_{\boldsymbol{p}} \frac{1}{2m}\left(|\boldsymbol{p}+\boldsymbol{q}|^2+|\boldsymbol{p}-\boldsymbol{q}|^2-2|\boldsymbol{p}|^2\right)\;\hat c^\dagger(\boldsymbol{p})\,\hat c(\boldsymbol{p})\\ &= \frac{\boldsymbol{q}^2}{m}\int\limits_{\boldsymbol{p}} \hat c^\dagger(\boldsymbol{p})\,\hat c(\boldsymbol{p}).\end{aligned} \tag{4.6}$$

Equation (4.1b) follows from the definition (4.1c).

B. Projected Hamiltonians and the importance of induced one-body terms

We begin with the generic lattice Hamiltonian

$$\hat H := \hat H_0 + \hat H_U \tag{4.7}$$

whereby the dimensionality d of the lattice is assumed even. Our goal is to understand how does normal ordering of the interaction $\hat H_U$ changes the band width of the kinetic Hamiltonian $\hat H_0$. To this end, we need to choose the representation in which we define $\hat H_0$ and $\hat H_U$. We will see that the choice of the representation of $\hat H$ can change the effects on $\hat H_0$ of normal ordering on $\hat H_U$.

The kinetic Hamiltonian is defined by

$$\hat H_0 := \frac{1}{2}\sum_{\boldsymbol{r},\boldsymbol{r}'\in\Lambda}\sum_{\alpha,\alpha'}\Big(\hat\psi^\dagger_{\boldsymbol{r},\alpha}\,t^{\alpha,\alpha'}_{\boldsymbol{r}-\boldsymbol{r}'}\,\hat\psi_{\boldsymbol{r}',\alpha'}+\mathrm{H.c.}\Big) \tag{4.8a}$$

where the hopping amplitudes

$$t^{\alpha,\alpha'}_{\boldsymbol{r}-\boldsymbol{r}'} = \Big(t^{\alpha',\alpha}_{\boldsymbol{r}'-\boldsymbol{r}}\Big)^* \tag{4.8b}$$

decay exponentially fast with the separation between any pair of sites $\boldsymbol{r}$ and $\boldsymbol{r}'$ from the lattice Λ and we have reinstated a finite number of internal degrees of freedom labeled by the *orbital index* α. If N denotes the number of sites in Λ, we can perform the Fourier transformation to the band basis in two steps. First, we do the Fourier transformation

$$\hat\psi^\dagger_{\boldsymbol{r},\alpha} =: \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\frac{e^{-\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{r}}}{\sqrt{N}}\hat\psi^\dagger_{\boldsymbol{p},\alpha},\quad \hat\psi_{\boldsymbol{r},\alpha} =: \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\frac{e^{+\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{r}}}{\sqrt{N}}\hat\psi_{\boldsymbol{p},\alpha}, \tag{4.9a}$$

in terms of which

$$\begin{aligned}\hat H_0 &= \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\sum_{\alpha,\alpha'}\hat\psi^\dagger_{\boldsymbol{p},\alpha}\,\mathcal{H}^{\alpha,\alpha'}_{\boldsymbol{p}}\,\hat\psi_{\boldsymbol{p},\alpha'},\\ \mathcal{H}^{\alpha,\alpha'}_{\boldsymbol{p}} &:= \sum_{\boldsymbol{r}\in\Lambda} e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\,t^{\alpha,\alpha'}_{\boldsymbol{r}}.\end{aligned} \tag{4.9b}$$

Second, for any given $\boldsymbol{p}$ from the Brillouin zone, we do the unitary transformation

$$\hat\psi^\dagger_{\boldsymbol{p},\alpha} =: \sum_a \hat c^\dagger_{\boldsymbol{p},a}u^{\alpha\,*}_{\boldsymbol{p},a},\qquad \hat\psi_{\boldsymbol{p},\alpha} =: \sum_a u^{\alpha}_{\boldsymbol{p},a}\,\hat c_{\boldsymbol{p},a}, \tag{4.10a}$$

in terms of which

$$\hat H_0 = \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\sum_a \hat c^\dagger_{\boldsymbol{p},a}\,\varepsilon_{\boldsymbol{p},a}\,\hat c_{\boldsymbol{p},a}. \tag{4.10b}$$

The algebra (3.3) applies to the band operators labeled by the *band index* a if one multiplies the Kroenecker symbol $\delta_{\boldsymbol{p},\boldsymbol{p}'}$ in the Brillouin zone by the Kroenecker symbol

$\delta_{a,a'}$ among the bands. The algebra (3.3) thus endows the orbital creation and annihilation operators with the canonical fermion algebra.

The interacting Hamiltonian is defined by

$$\begin{aligned}\hat{H}_U &:= \sum_{\boldsymbol{r},\boldsymbol{r}'\in\Lambda}\sum_{\alpha,\alpha'}\hat{\rho}^{\psi}_{\boldsymbol{r},\alpha}\,U^{\alpha,\alpha'}_{\boldsymbol{r}-\boldsymbol{r}'}\,\hat{\rho}^{\psi}_{\boldsymbol{r}',\alpha'}\\ &= \sum_{\boldsymbol{q}\in\Omega_{\mathrm{BZ}}}\sum_{\alpha,\alpha'}\hat{\rho}^{\psi}_{+\boldsymbol{q},\alpha}\,\tilde{U}^{\alpha,\alpha'}_{\boldsymbol{q}}\,\hat{\rho}^{\psi}_{-\boldsymbol{q},\alpha'}\end{aligned} \tag{4.11a}$$

with

$$\hat{\rho}^{\psi}_{\boldsymbol{r},\alpha} := \hat{\psi}^{\dagger}_{\boldsymbol{r},\alpha}\,\hat{\psi}_{\boldsymbol{r},\alpha} \tag{4.11b}$$

the local density at the site $\boldsymbol{r}\in\Lambda$ and for the orbital α. The corresponding Fourier transforms are

$$\begin{aligned}\hat{\rho}^{\psi}_{\boldsymbol{q},\alpha} &:= \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\hat{\psi}^{\dagger}_{[\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\alpha}\,\hat{\psi}_{\boldsymbol{p},\alpha},\\ \tilde{U}^{\alpha,\alpha'}_{\boldsymbol{q}} &:= \frac{1}{N}\sum_{\boldsymbol{r}\in\Lambda}e^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{r}}\,U^{\alpha,\alpha'}_{\boldsymbol{r}}.\end{aligned} \tag{4.11c}$$

For simplicity, we shall focus on orbital independent (density-density) interactions, in which case

$$U^{\alpha,\alpha'}_{\boldsymbol{r}} = U_{\boldsymbol{r}},\quad \forall\alpha,\alpha'\,. \tag{4.12}$$

Normal ordering is the operation by which all creation operators are to be moved to the left of the annihilation operators. In the orbital basis, normal ordering results in

$$\hat{H}_U = \hat{H}'^{\psi}_U + \hat{H}''^{\psi}_U\,. \tag{4.13a}$$

The one-body Hamiltonian H'^{ψ}_U, a consequence of the fermion algebra, is proportional to the conserved number operator,

$$\hat{H}'^{\psi}_U := \sum_{\boldsymbol{r}\in\Lambda}\sum_{\alpha}U_{\boldsymbol{0}}\,\hat{\psi}^{\dagger}_{\boldsymbol{r},\alpha}\,\hat{\psi}_{\boldsymbol{r},\alpha} \equiv U\,\hat{N}, \tag{4.13b}$$

where we defined $U\equiv U_{\boldsymbol{r}=\boldsymbol{0}}$. The normal-ordered interaction $\hat{H}''^{\psi}_U$ is

$$\hat{H}''^{\psi}_U \equiv \sum_{\boldsymbol{r},\boldsymbol{r}'\in\Lambda}\sum_{\alpha,\alpha'}U_{\boldsymbol{r}-\boldsymbol{r}'}\,\hat{\psi}^{\dagger}_{\boldsymbol{r},\alpha}\,\hat{\psi}^{\dagger}_{\boldsymbol{r}',\alpha'}\,\hat{\psi}_{\boldsymbol{r}',\alpha'}\,\hat{\psi}_{\boldsymbol{r},\alpha}. \tag{4.13c}$$

The one-body term induced by normal ordering is, in the band basis,

$$\hat{H}'^{c}_U := U\sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\sum_{a}\hat{c}^{\dagger}_{\boldsymbol{p},a}\,\hat{c}_{\boldsymbol{p},a} \equiv U\,\hat{N}. \tag{4.14a}$$

The normal-ordered interaction is, in the band basis,

$$\begin{aligned}\hat{H}''^{c}_U &:= \sum_{\boldsymbol{q}\in\Omega_{\mathrm{BZ}}}\sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\sum_{\boldsymbol{p}'\in\Omega_{\mathrm{BZ}}}\sum_{a}\sum_{b}\sum_{a'}\sum_{b'}V^{a,b;a',b'}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}'}\,\hat{c}^{\dagger}_{[+\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},a}\,\hat{c}^{\dagger}_{[-\boldsymbol{q}+\boldsymbol{p}']_{\mathrm{BZ}},a'}\,\hat{c}_{\boldsymbol{p}',b'}\,\hat{c}_{\boldsymbol{p},b},\\ V^{a,b;a',b'}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}'} &:= \tilde{U}_{\boldsymbol{q}}\sum_{\alpha,\alpha'}u^{\alpha\,*}_{[+\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},a}\,u^{\alpha}_{\boldsymbol{p},b}\,u^{\alpha'\,*}_{[-\boldsymbol{q}+\boldsymbol{p}']_{\mathrm{BZ}},a'}\,u^{\alpha'}_{\boldsymbol{p}',b'}.\end{aligned} \tag{4.14b}$$

In any subspace of the Fock space with a fixed number of particles, normal ordering thus produces a rigid shift of all single-particle energy eigenvalues of $\hat{H}_0$. For any band a, the width of the single-particle dispersion ε_a is not affected by the normal ordering $-U\,\hat{N}$. We are going to show that this needs not be true anymore if we *first* project Hamiltonian (4.7) onto the band $\bar{a}$ and *then* express the resulting projected Hamiltonian in terms of the generators (3.8).

The projection of Hamiltonian (4.7) onto the band $\bar{a}$ is

$$\hat{H}^{\bar{a}} = \hat{H}^{\bar{a}}_0 + \hat{H}^{\bar{a}}_U \tag{4.15a}$$

where the projected kinetic Hamiltonian is

$$\hat{H}^{\bar{a}}_0 = \sum_{\boldsymbol{p}\in\Omega_{\mathrm{BZ}}}\hat{c}^{\dagger}_{\boldsymbol{p},\bar{a}}\,\left(\varepsilon_{\boldsymbol{p},\bar{a}}+U\right)\hat{c}_{\boldsymbol{p},\bar{a}} \tag{4.15b}$$

while the projected interacting Hamiltonian is

$$\begin{aligned}\hat{H}_U^{\bar{a}} &= \sum_{\boldsymbol{q}\in\Omega_{\rm BZ}}\sum_{\boldsymbol{p}\in\Omega_{\rm BZ}}\sum_{\boldsymbol{p}'\in\Omega_{\rm BZ}} V^{\bar{a}}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}'}\,\hat{c}^\dagger_{[+\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ},\bar{a}}\,\hat{c}^\dagger_{[-\boldsymbol{q}+\boldsymbol{p}']_{\rm BZ},\bar{a}}\,\hat{c}_{\boldsymbol{p}',\bar{a}}\,\hat{c}_{\boldsymbol{p},\bar{a}},\\ V^{\bar{a}}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}'} &= \tilde{U}_{\boldsymbol{q}}\sum_{\alpha,\alpha'} u^{\alpha\,*}_{[+\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ},\bar{a}}\,u^{\alpha}_{\boldsymbol{p},\bar{a}}\,u^{\alpha'\,*}_{[-\boldsymbol{q}+\boldsymbol{p}']_{\rm BZ},\bar{a}}\,u^{\alpha'}_{\boldsymbol{p}',\bar{a}}.\end{aligned} \tag{4.15c}$$

For the purpose of representing the projection of Hamiltonian (4.7) onto the band $\bar{a}$ by the magnetic density operators (3.8), it is necessary to undo the normal ordering in Eq. (4.15c). In doing so, a second one-body term is produced,

$$\hat{H}^{\bar{a}} = \hat{H}_0'^{\bar{a}} + \hat{H}_U'^{\bar{a}} \tag{4.16a}$$

where the projected kinetic Hamiltonian is

$$\hat{H}_0'^{\bar{a}} = \sum_{\boldsymbol{p}\in\Omega_{\rm BZ}} \left(\varepsilon_{\boldsymbol{p},\bar{a}} + U\right)\,\hat{c}^\dagger_{\boldsymbol{p},\bar{a}}\,\hat{c}_{\boldsymbol{p},\bar{a}} - \sum_{\boldsymbol{p}\in\Omega_{\rm BZ}}\left(\sum_{\boldsymbol{q}\in\Omega_{\rm BZ}} V^{\bar{a}}_{\boldsymbol{q},[-\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ},\boldsymbol{p}}\right)\,\hat{c}^\dagger_{\boldsymbol{p},\bar{a}}\,\hat{c}_{\boldsymbol{p},\bar{a}}, \tag{4.16b}$$

while the projected interacting Hamiltonian is

$$\hat{H}_U'^{\bar{a}} = \sum_{\boldsymbol{q}\in\Omega_{\rm BZ}}\sum_{\boldsymbol{p}\in\Omega_{\rm BZ}}\sum_{\boldsymbol{p}'\in\Omega_{\rm BZ}} V^{\bar{a}}_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{p}'}\,\hat{c}^\dagger_{[+\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ},\bar{a}}\,\hat{c}_{\boldsymbol{p},\bar{a}}\,\hat{c}^\dagger_{[-\boldsymbol{q}+\boldsymbol{p}']_{\rm BZ},\bar{a}}\,\hat{c}_{\boldsymbol{p}',\bar{a}}. \tag{4.16c}$$

Observe that had we first represented Eq. (4.11) in the band basis, followed by the projection consisting in restricting all the band indices to $\bar{a}$ prior to normal-ordering, then we would have obtained Eq. (4.16) upon normal ordering without the second term on the right-hand side of Eq. (4.16b). The correct implementation of projection is to normal order first and then to project, leading to Eq. (4.16b). Indeed, the order by which normal ordering is followed by restricting all band indices to the projected ones corresponds to sandwiching the Hamiltonian by the projection operator onto a subset of bands. The reverse order by which the density operators is projected onto a subset of bands followed by normal ordering corresponds to sandwiching first all density operators by the projection operator onto a subset of bands and then assembling a Hamiltonian out of these projected density operators. As the projection operators do not commute with the density operators, the order in which the operations of normal ordering and projection are performed matters.

We can now express the Hamiltonian in terms of the magnetic density operators (the details are provided in Appendix B)

$$\begin{aligned}\hat{H}_0'^{\bar{a}} &= \sum_{\boldsymbol{G}\in\Lambda^\star}\left(\tilde{\varepsilon}_{\boldsymbol{G}} + U\,\delta_{\boldsymbol{G},0} - \sum_{\boldsymbol{q}}\,\tilde{U}_{\boldsymbol{q}}\,\tilde{h}^{\boldsymbol{G}}_{-\boldsymbol{q}}\right)\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{0}},\\ \hat{H}_U'^{\bar{a}} &= \sum_{\boldsymbol{q}\in\Omega_{\rm BZ}}\tilde{U}_{\boldsymbol{q}}\sum_{\boldsymbol{G},\boldsymbol{G}'\in\Lambda^\star}\tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}}\,\tilde{f}^{-\boldsymbol{G}'}_{-\boldsymbol{q}}\,e^{-\mathrm{i}\,(\Theta^{\boldsymbol{G}}_{+\boldsymbol{q}}+\Theta^{-\boldsymbol{G}'}_{-\boldsymbol{q}})}\,\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}\,\hat{\varrho}^{-\boldsymbol{G}'}_{-\boldsymbol{q}},\end{aligned} \tag{4.17a}$$

where

$$\tilde{\varepsilon}_{\boldsymbol{G}} = \frac{1}{N}\sum_{\boldsymbol{p}} \varepsilon_{\boldsymbol{p}}\, e^{-\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}, \tag{4.17b}$$

$$\tilde{h}^{\boldsymbol{G}}_{\boldsymbol{q}} = \frac{1}{N}\sum_{\boldsymbol{p}} \left|\sum_{\alpha} u^{\alpha\,*}_{[\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\bar{a}}\, u^{\alpha}_{\boldsymbol{p},\bar{a}}\right|^2 e^{-\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}, \tag{4.17c}$$

$$\tilde{f}^{\boldsymbol{G}}_{\boldsymbol{q}} = \frac{1}{N}\sum_{\boldsymbol{p}} \left(\sum_{\alpha} u^{\alpha\,*}_{[\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\bar{a}}\, u^{\alpha}_{\boldsymbol{p},\bar{a}}\right) e^{-\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}}\, e^{-\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}, \tag{4.17d}$$

with $\Theta_{\boldsymbol{q},\boldsymbol{p}}$ and $\Theta^{\boldsymbol{G}}_{\boldsymbol{q}}$ defined in Eqs. (3.11b) and (3.11c), respectively.

Equation (4.17) is the main result of Sec. IV B. Applied to a Chern insulator to which density-density interactions have been added, Eq. (4.17) suggests that there will always be linear in $\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}=\boldsymbol{0}}$ contributions to the Hamiltonian even if the bare band is flat to begin with, i.e., even if $\varepsilon_{\boldsymbol{p},\bar{a}} = 0$. Because of the topological attributes of the Bloch spinors as they wrap around the Brillouin zone, we expect a nonvanishing $\tilde{h}^{\boldsymbol{G}}_{\boldsymbol{q}}$. (An extreme case of a topologically trivial band insulator has Bloch spinors that are constant across the Brillouin zone, in which case only $\tilde{h}^{\boldsymbol{G}=\boldsymbol{0}}_{\boldsymbol{q}} \neq 0$ and the additional one-body contribution is just proportional to the total particle number. This would also be the case in the context of the quantum Hall effect.) This effect on the bare dispersion is controlled by the bare interaction $U_{\boldsymbol{q}}$. Hence, it could be as large as the effects of the density-density interaction.

It is far from evident that a FCI is selected by the competition between the one-body and two body terms in Eq. (4.17) since they are both controlled by one characteristic energy scale in the limit of a flat bare band width. On the other hand, if a ground state supporting a FCI is selected for some range of parameters, then the effective quantum field theory describing the low-energy and long-distance properties of this phase should belong to one of the universality class associated to the FQHE.

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Appendix A: Proof of Eq. (3.10)

For any $\boldsymbol{q}$ and $\boldsymbol{q}'$ from the first Brillouin zone Ω_{BZ}, and for any $\boldsymbol{G}$ and $\boldsymbol{G}'$ from the dual lattice $\Lambda^\star$, Eq. (3.6) dictates that

$$\begin{aligned}
\left[\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}, \hat{\varrho}^{\boldsymbol{G}'}_{\boldsymbol{q}'}\right] &= \sum_{\boldsymbol{p}\,\in\Omega_{\mathrm{BZ}}} \left[e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}}_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}} + \mathrm{i}\,\Phi^{\boldsymbol{G}'}_{\boldsymbol{q}',\boldsymbol{p}}} - (\boldsymbol{q}\leftrightarrow\boldsymbol{q}' \text{ and } \boldsymbol{G}\leftrightarrow\boldsymbol{G}')\right] \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}}\,\hat{c}_{\boldsymbol{p}} \\
&= \sum_{\boldsymbol{p}\,\in\Omega_{\mathrm{BZ}}} e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}},\boldsymbol{p}}} \left[e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}}_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}} + \mathrm{i}\,\Phi^{\boldsymbol{G}'}_{\boldsymbol{q}',\boldsymbol{p}} - \mathrm{i}\,\Phi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}},\boldsymbol{p}}} - (\boldsymbol{q}\leftrightarrow\boldsymbol{q}' \text{ and } \boldsymbol{G}\leftrightarrow\boldsymbol{G}')\right] \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}}\,\hat{c}_{\boldsymbol{p}} \qquad \text{(A1a)} \\
&\equiv \sum_{\boldsymbol{p}\,\in\Omega_{\mathrm{BZ}}} \mathcal{F}^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}\; e^{+\mathrm{i}\,\Phi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}',\boldsymbol{p}]_{\mathrm{BZ}}}}\; \hat{c}^\dagger_{[\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}}\,\hat{c}_{\boldsymbol{p}}.
\end{aligned}$$

Here, we have introduced the short-hand notations

$$[\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}} \equiv \boldsymbol{q}+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}} \in \Omega_{\mathrm{BZ}}, \tag{A1b}$$

$$\mathcal{F}^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} := e^{+\mathrm{i}\,\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}} - (\boldsymbol{q}\leftrightarrow\boldsymbol{q}' \text{ and } \boldsymbol{G}\leftrightarrow\boldsymbol{G}'), \tag{A1c}$$

and

$$\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} := \Phi^{\boldsymbol{G}}_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}} + \Phi^{\boldsymbol{G}'}_{\boldsymbol{q}',\boldsymbol{p}} - \Phi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}},\boldsymbol{p}} \tag{A1d}$$

for any triplet $\boldsymbol{q}$, $\boldsymbol{q}'$, $\boldsymbol{p}$ from the first Brillouin zone and for any triplet $\boldsymbol{G}$, $\boldsymbol{G}'$, $\boldsymbol{G}''$ from the dual lattice.

First, we are going to show that the algebra (A1a) closes to

$$\left[\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}, \hat{\varrho}^{\boldsymbol{G}'}_{\boldsymbol{q}'}\right] = F^{\boldsymbol{G},\boldsymbol{G}'}_{\boldsymbol{q},\boldsymbol{q}'}\; \hat{\varrho}^{\boldsymbol{G}+\boldsymbol{G}'}_{\boldsymbol{q}+\boldsymbol{q}'} \tag{A2a}$$

as a consequence of the fact that the kernel $\mathcal{F}^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}$ is independent of $\boldsymbol{p}$ and $\boldsymbol{G}''$ on the right-hand side of

Eq. (A1a) (as we will show shortly), and for which reason we have introduced the notations

$$\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \equiv \Upsilon^{\boldsymbol{G},\boldsymbol{G}'}_{\boldsymbol{q},\boldsymbol{q}'} \qquad \mathcal{F}^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \equiv F^{\boldsymbol{G},\boldsymbol{G}'}_{\boldsymbol{q},\boldsymbol{q}'}. \tag{A2b}$$

Second, we are going to show that the algebra (A2a) simplifies to the algebra

$$\left[\hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}, \hat{\varrho}^{\boldsymbol{G}'}_{\boldsymbol{q}'}\right] = 2\mathrm{i}\,\sin\left(\frac{(\boldsymbol{q}+\boldsymbol{G})*(\boldsymbol{q}'+\boldsymbol{G}')}{2\pi}\right)\hat{\varrho}^{\boldsymbol{G}+\boldsymbol{G}'}_{\boldsymbol{q}+\boldsymbol{q}'} \tag{A3a}$$

as a consequence of the fact that (as we will show shortly)

$$\Upsilon^{\boldsymbol{G},\boldsymbol{G}'}_{\boldsymbol{q},\boldsymbol{q}'} = -\Upsilon^{\boldsymbol{G}',\boldsymbol{G}}_{\boldsymbol{q}',\boldsymbol{q}}. \tag{A3b}$$

Proof. Equations (A2b) and (A3b) follow at once from our choice

$$\Phi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}} := \frac{1}{2\pi}\left[(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{p} + \varphi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}}\right] \tag{A4a}$$

where

$$\varphi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}} := -(\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G})*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G}}. \tag{A4b}$$

To verify this claim, we start from

$$\begin{aligned} 2\pi\,\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} &= 2\pi\left(\Phi^{\boldsymbol{G}}_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}} + \Phi^{\boldsymbol{G}'}_{\boldsymbol{q}',\boldsymbol{p}} - \Phi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}},\boldsymbol{p}}\right) \\ &= (\boldsymbol{q}+\boldsymbol{G})*[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}} + (\boldsymbol{q}'+\boldsymbol{G}')*\boldsymbol{p} - ([\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}}+\boldsymbol{G}'')*\boldsymbol{p} + 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \end{aligned} \tag{A5a}$$

where

$$2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} := \varphi^{\boldsymbol{G}}_{\boldsymbol{q},[\boldsymbol{q}'+\boldsymbol{p}]_{\mathrm{BZ}}} + \varphi^{\boldsymbol{G}'}_{\boldsymbol{q}',\boldsymbol{p}} - \varphi^{\boldsymbol{G}''}_{[\boldsymbol{q}+\boldsymbol{q}']_{\mathrm{BZ}},\boldsymbol{p}}. \tag{A5b}$$

If it was not for the symbol $[\cdots]_{\mathrm{BZ}}$ and the dependence on $\boldsymbol{G}$, $\boldsymbol{G}'$, and $\boldsymbol{G}''$, all the *explicit* dependence on $\boldsymbol{p}$ would drop by linearity on the right-hand side very much as was the case in the continuum for Eq. (2.8) The condition for periodicity prevents this cancellation, however. Instead, according to Eq. (3.2a),

$$2\pi\,\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} = (\boldsymbol{q}+\boldsymbol{G})*(\underline{\boldsymbol{q}'} + \boldsymbol{p} - \underline{\underline{\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}}) + (\boldsymbol{q}'+\boldsymbol{G}')*\boldsymbol{p} - (\boldsymbol{q}+\boldsymbol{q}'-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}+\boldsymbol{G}'')*\boldsymbol{p} + 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \tag{A6}$$

simplifies to, if we collect all terms explicitly linear in $\boldsymbol{p}$,

$$2\pi\,\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} = (\boldsymbol{q}+\boldsymbol{G})*\underline{\boldsymbol{q}'} - (\boldsymbol{q}+\boldsymbol{G})*\underline{\underline{\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}} + (\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}-\boldsymbol{G}'')*\boldsymbol{p} + 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}. \tag{A7}$$

We choose

$$\boldsymbol{G}'' = \boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}. \tag{A8}$$

Then, all terms explicitly linear in $\boldsymbol{p}$ drop out and we are left with

$$2\pi\,\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} = +(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{q}' - \underline{\underline{(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}} + 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}. \tag{A9}$$

An implicit dependence on $\boldsymbol{p}$ remains through the underlined term on the one hand, and the functions of the form $\varphi^{\boldsymbol{G}}_{\boldsymbol{q},\boldsymbol{p}}$ on the other hand.

It is time to evaluate the contribution $2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}}$. Observe that

$$\begin{aligned} \boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G}} &= \boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G} - [\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G}]_{\mathrm{BZ}} \\ &= \boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G} - (\boldsymbol{q}+\boldsymbol{p}+\boldsymbol{G}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}}-\boldsymbol{G}) \\ &= \boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{p}}+\boldsymbol{G} \end{aligned} \tag{A10}$$

for any pair $\boldsymbol{q}$ and $\boldsymbol{p}$ from the Brillouin zone $\Omega_{\rm BZ}$ and for any $\boldsymbol{G}$ from the dual lattice $\Lambda^\star$. Hence, we can rewrite

$$\begin{aligned} 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \;=\; &-(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G})*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}} \\ &-(\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G}')*\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G}'} \\ &+(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}+\boldsymbol{G}'')*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}+\boldsymbol{G}''} \end{aligned} \tag{A11}$$

as

$$\begin{aligned} 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}''}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \;=\; &-(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G})*(\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}-\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}) \\ &-(\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G}')*(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}') \\ &+(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}+\boldsymbol{G}'')*(\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}-\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}+\boldsymbol{G}''). \end{aligned} \tag{A12}$$

If we use Eq. (A8) on the last line of Eq. (A12), we get

$$\begin{aligned} 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} \;=\; &-\big[(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p})-(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}-\boldsymbol{G}\qquad)\big]*(\underline{\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}}_{\#1}-\underline{\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}_{\#2}+\underline{\boldsymbol{G}}_{\#3}) \\ &-[(\qquad\boldsymbol{q}'+\boldsymbol{p})-(\qquad\qquad-\boldsymbol{G}')]*(\underline{\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}_{\#2}+\underline{\boldsymbol{G}'}_{\#4}) \\ &+[(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p})-(\qquad-\boldsymbol{G}-\boldsymbol{G}')]*(\underline{\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}}_{\#1}+\underline{\boldsymbol{G}}_{\#3}+\underline{\boldsymbol{G}'}_{\#4}). \end{aligned} \tag{A13}$$

The following terms cancel pairwise:

1. From lines 1 and 3, $(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G})*\underline{\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}}_{\#1}$ cancel. There is the left over $(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}')*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}$.
2. From lines 1 and 2, $(\qquad\boldsymbol{q}'+\boldsymbol{p})*\underline{\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}}_{\#2}$ cancel. There is the left over $(\underline{\underline{\boldsymbol{q}}}-\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\underline{\underline{\boldsymbol{G}}}+\boldsymbol{G}')*\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}$.
3. From lines 1 and 3, $(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G})*\underline{\boldsymbol{G}}_{\#3}$ cancel. There is the left over $(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}')*\boldsymbol{G}$.
4. From lines 2 and 3, $(\boldsymbol{q}'+\boldsymbol{p}+\boldsymbol{G}')*\underline{\boldsymbol{G}'}_{\#4}$ cancel. There is the left over $(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{G}'$.

Collecting all nonvanishing contributions to the right-hand side of Eq. (A13) yields

$$\begin{aligned} 2\pi\,\Xi^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} =\;&(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}')*\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p}}-(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}-\boldsymbol{G}')*\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+(\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+\boldsymbol{G}')*\boldsymbol{G} \\ &+\underline{\underline{(\boldsymbol{q}+\boldsymbol{G})}}*\boldsymbol{G}_{\boldsymbol{q}'+\boldsymbol{p}}+(\boldsymbol{q}+\boldsymbol{G})*\boldsymbol{G}'. \end{aligned} \tag{A14}$$

By assumption (3.9) the $*$-product between any pair from the dual lattice $\Lambda^\star$ is a multiple of $(2\pi)^2$. Hence, combining Eq. (A14) with Eq. (A9) delivers the desired expression

$$\Gamma^{\boldsymbol{G},\boldsymbol{G}',\boldsymbol{G}+\boldsymbol{G}'+\boldsymbol{G}_{\boldsymbol{q}+\boldsymbol{q}'}}_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{p}} = \frac{1}{2\pi}(\boldsymbol{q}+\boldsymbol{G})*(\boldsymbol{q}'+\boldsymbol{G}')+\mathrm{mod}\,2\pi \equiv \Upsilon^{\boldsymbol{G},\boldsymbol{G}'}_{\boldsymbol{q},\boldsymbol{q}'} = -\Upsilon^{\boldsymbol{G}'\boldsymbol{G}}_{\boldsymbol{q}',\boldsymbol{q}}. \tag{A15}$$

□

Appendix B: Details for reaching Eq. (4.17)

Equipped with Eq. (4.16), we are in position to take advantage of the fact that, for any even dimension d, the magnetic density operators (3.8) form a complete basis for the charge neutral fermion bilinears made out of the band creation and annihilation operators. Define the functions $f^i : \Omega_{\rm BZ}\times\Omega_{\rm BZ}\to\mathbb{C}$, with $i=1,2,3,4$ by

$$\begin{aligned} f^1_{\boldsymbol{q},\boldsymbol{p}} &:= \delta_{\boldsymbol{q},\boldsymbol{0}}\,\varepsilon_{\boldsymbol{p},\bar{a}}, \\ f^2_{\boldsymbol{q},\boldsymbol{p}} &:= \delta_{\boldsymbol{q},\boldsymbol{0}}\,U, \\ f^3_{\boldsymbol{q}',\boldsymbol{p}'} &:= \delta_{\boldsymbol{q}',\boldsymbol{0}}\,\delta_{\boldsymbol{p}',\boldsymbol{p}}, \\ f^4_{\boldsymbol{q},\boldsymbol{p}} &:= u^{\alpha\,*}_{[\boldsymbol{q}+\boldsymbol{p}]_{\rm BZ},\bar{a}}\,u^{\alpha}_{\boldsymbol{p},\bar{a}}. \end{aligned} \tag{B1a}$$

(Notice that $\boldsymbol{p}$ is a parameter in the definition of f^3.) In terms of the operators defined in Eq. (3.5), the projected

kinetic Hamiltonian is

$$\hat{H}_0'^{\bar{a}} = \sum_{\boldsymbol{q}\in\Omega_{\mathrm{BZ}}} \left(\varrho^{f^1}_{\boldsymbol{q}} + \varrho^{f^2}_{\boldsymbol{q}} - \sum_{\boldsymbol{p},\boldsymbol{q}'\in\Omega_{\mathrm{BZ}}} V^{\bar{a}}_{\boldsymbol{q},[-\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\boldsymbol{p}}\, \varrho^{f^3}_{\boldsymbol{q}'} \right), \tag{B1b}$$

while the projected interacting Hamiltonian is

$$\hat{H}_U'^{\bar{a}} = \sum_{\boldsymbol{q}\in\Omega_{\mathrm{BZ}}} \tilde{U}_{\boldsymbol{q}}\, \varrho^{f^4}_{\boldsymbol{q}}\, \varrho^{f^4}_{-\boldsymbol{q}}. \tag{B1c}$$

The bare kinetic energy is the first term on the right-hand side of Eq. (B1b). The correction to the bare kinetic energy from standard normal ordering is the second term on the right-hand side of Eq. (B1b). The function f^2 is a delta function with respect to its first argument and constant with respect to its second argument. This correction does not change the band width of the bare dispersion. The last correction to the bare kinetic energy is the third term on the right-hand side of Eq. (B1b). It is controlled by the interaction $\tilde{U}_{\boldsymbol{q}}$ dressed by the Bloch functions that diagonalize the bare kinetic energy,

$$V^{\bar{a}}_{\boldsymbol{q},[-\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\boldsymbol{p}} = \tilde{U}_{\boldsymbol{q}} \left| \sum_{\alpha} u^{\alpha\,*}_{[-\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\bar{a}}\, u^{\alpha}_{\boldsymbol{p},\bar{a}} \right|^2 . \tag{B2}$$

It can change the bare band width by an amount of the order of the characteristic energy of the interaction.

At last, Eq. (3.13) dictates that we define the functions $\bar{f}^i : \Omega_{\mathrm{BZ}} \times \Omega_{\mathrm{BZ}} \to \mathbb{C}$, with $i = 1, 2, 3, 4$ by

$$\bar{f}^i_{\boldsymbol{q},\boldsymbol{p}} := e^{-\mathrm{i}\,\Theta_{\boldsymbol{q},\boldsymbol{p}}}\, f^i_{\boldsymbol{q},\boldsymbol{p}} \tag{B3}$$

where the argument of the multiplicative exponential factor on the right-hand side was defined in Eq. (3.11c). We observe that any of the three functions f^1, f^2, and f^3 is proportional to $\delta_{\boldsymbol{q},\boldsymbol{0}}$ so that Eq. (3.12) teaches us that

$$\bar{f}^i_{\boldsymbol{q},\boldsymbol{p}} := \delta_{\boldsymbol{q},\boldsymbol{0}}\, f^i_{\boldsymbol{0},\boldsymbol{p}}, \qquad i = 1, 2, 3. \tag{B4}$$

In even dimensional space, we can safely use the $*$-Fourier expansion [see Eq. (3.14)]

$$\begin{aligned}
\bar{f}^i_{\boldsymbol{q},\boldsymbol{p}} &= \delta_{\boldsymbol{q},\boldsymbol{0}} \sum_{\boldsymbol{G}\in\Lambda^\star} \tilde{f}^{i;\boldsymbol{G}}_{\boldsymbol{0}}\, e^{+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}, \quad i = 1, 2, 3, \\
\bar{f}^4_{\boldsymbol{q},\boldsymbol{p}} &= \sum_{\boldsymbol{G}\in\Lambda^\star} \tilde{f}^{4;\boldsymbol{G}}_{\boldsymbol{q}}\, e^{+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)}, \\
\sum_{\alpha} \left| u^{\alpha\,*}_{[-\boldsymbol{q}+\boldsymbol{p}]_{\mathrm{BZ}},\bar{a}}\, u^{\alpha}_{\boldsymbol{p},\bar{a}} \right|^2 &= \sum_{\boldsymbol{G}\in\Lambda^\star} \tilde{h}^{-\boldsymbol{G}}_{-\boldsymbol{q}}\, e^{+\mathrm{i}\,(\boldsymbol{G}*\boldsymbol{p}-\boldsymbol{p}*\boldsymbol{G})/(2\pi)},
\end{aligned} \tag{B5}$$

to compute the Fourier coefficients $\tilde{f}^{i;\boldsymbol{G}}_{\boldsymbol{q}}$ with $i = 1, 2, 3$, $\tilde{f}^{4;\boldsymbol{G}}_{\boldsymbol{q}}$, and $\tilde{h}^{\boldsymbol{G}}_{\boldsymbol{q}}$. Application of Eq. (3.16) then delivers the desired representation of the projection of Hamiltonian (4.7) onto the band $\bar{a}$ by the magnetic density operators (3.8),

$$\begin{aligned}
\hat{H}^{\bar{a}} &= \hat{H}_0'^{\bar{a}} + \hat{H}_U'^{\bar{a}}, \\
\hat{H}_0'^{\bar{a}} &= \sum_{\boldsymbol{G}\in\Lambda^\star} \left(\tilde{f}^{1;\boldsymbol{G}}_{\boldsymbol{0}} + \tilde{f}^{2;\boldsymbol{G}}_{\boldsymbol{0}} - \sum_{\boldsymbol{q}} U_{\boldsymbol{q}}\, \tilde{h}^{\boldsymbol{G}}_{-\boldsymbol{q}} \right) \hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{0}}, \\
\hat{H}_U'^{\bar{a}} &= \sum_{\boldsymbol{q}\in\Omega_{\mathrm{BZ}}} \sum_{\boldsymbol{G},\boldsymbol{G}'\in\Lambda^\star} \tilde{f}^{4;\boldsymbol{G}}_{\boldsymbol{q}}\, \tilde{f}^{4;-\boldsymbol{G}'}_{-\boldsymbol{q}}\, e^{-\mathrm{i}\,(\Theta^{\boldsymbol{G}}_{+\boldsymbol{q}}+\Theta^{-\boldsymbol{G}'}_{-\boldsymbol{q}})}\, \hat{\varrho}^{\boldsymbol{G}}_{\boldsymbol{q}}\, \hat{\varrho}^{-\boldsymbol{G}'}_{-\boldsymbol{q}}.
\end{aligned} \tag{B6}$$

Equation (4.17) follows.

[1] J. Zak Phys. Rev. 134, A1602 (1964); *ibid*, A1607 (1964).
[2] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. Lett. **54**, 581 (1985).
[3] R. P. Feynman, Statistical Mechanics (Benjamin, Reading, Mass., 1972), Chap. 11.
[4] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[5] E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011).
[6] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. **106**, 236803 (2011).
[7] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).
[8] D. N. Sheng, Z. Gu, K. Sun, and L. Sheng, Nature Communications **2**, 389 (2011).
[9] Y.-F. Wang, Z.-C. Gu, C.-D. Gong, D. N. Sheng, Phys. Rev. Lett. **107**, 146803 (2011).
[10] N. Regnault and B. A. Bernevig, Phys. Rev. X **1**, 021014 (2011); B. A. Bernevig, *"Inversion Symmetric Topological Insulators and Fractional Chern Insulator,"* talk given at the Kavli Institute for Theoretical Physics China at the Chinese Academy of Sciences, 29 August 2011.
[11] T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165107 (2011).
[12] B. A. Bernevig and N. Regnault, Phys. Rev. B **85**, 075128 (2012).
[13] Jorn W. F. Venderbos, Stefanos Kourtis, Jeroen van den Brink, and Maria Daghofer, Phys. Rev. Lett. **108**, 126405 (2012).
[14] Y.-F. Wang, H. Yao, Z.-C. Gu, C.-D. Gong, D. N. Sheng, Phys. Rev. Lett. **108**, 126805 (2012)
[15] Y.-L. Wu, B. A. Bernevig, N. Regnault, Phys. Rev. B **85**, 075116 (2012).
[16] T. Liu, C. Repellin, B. A. Bernevig, and N. Regnault, arXiv:1206.2626 (unpublished).
[17] Y.-F. Wang, H. Yao, C.-D. Gong, and D. N. Sheng, arXiv:1204.1697 (unpublished).
[18] Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Lauchli, arXiv:1206.3759 (unpublished).
[19] Adolfo G. Grushin, Titus Neupert, Claudio Chamon, and Christopher Mudry, arXiv:1207.4097 (unpublished).
[20] A. M. Lauchli, Z. Liu, E.J. Bergholtz, and R. Moessner, arXiv:1207.6094 (unpublished).
[21] A. Sterdyniak, C. Repellin, B. Andrei Bernevig, and N. Regnault, arXiv:1207.6385 (unpublished).
[22] S. Parameswaran, R. Roy, and S. Sondhi, Phys. Rev. B **85**, 241308(R) (2012).
[23] M. O. Goerbig Eur. Phys. Journal B **85**, 14 (2012).
[24] T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **86**, 035125 (2012).
[25] B. Estienne, N. Regnault, and B. A. Bernevig, arXiv:1202.5543.
[26] Ganpathy Murthy and R. Shankar, arXiv:1207.2133.
[27] Ganpathy Murthy and R. Shankar, arXiv:1108.5501.
[28] D. Page, Phys. Rev. A **36**, 3479 (1987).
[29] David Pines and Philippe Nozières, The Theory of Quantum Liquids (Benjamin, New York, 1966).