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Elementary formula for the Hall conductivity of interacting systems

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A formula for the Hall conductivity of interacting electrons is given under the assumption that the ground state manifold is $N_{\rm gs}$-fold degenerate and discrete translation symmetry is neither explicitly nor spontaneously broken.

I. INTRODUCTION AND RESULTS

The Hall conductivity in the integer (IQHE) and fractional (FQHE) quantum Hall effect is observed to be quantized.[1] This quantization can be explained by expressing the Hall conductivity $\sigma_{\rm H}$ in terms of a topological invariant C that takes integer values through the relation

$$\sigma_{\rm H} = \frac{e^2}{h}\,\nu\,C \tag{1}$$

where ν is the filling fraction of the single-particle Landau levels.[2–6] For example, the relation between the quantized number C and the *non-degenerate* many-body ground state wave functions $|\Psi(\phi,\varphi)\rangle$ obeying twisted boundary conditions parametrized by the pair of angles $0 \leq \phi,\varphi \leq 2\pi$ for a gas of electrons confined in two-dimensional (position) space and subjected to a uniform magnetic field is

$$C = -\frac{\mathrm{i}}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\varphi\left[\left\langle\frac{\partial\Psi}{\partial\phi}\middle|\,\frac{\partial\Psi}{\partial\varphi}\right\rangle - \left\langle\frac{\partial\Psi}{\partial\varphi}\middle|\,\frac{\partial\Psi}{\partial\phi}\right\rangle\right], \tag{2}$$

when the cyclotron energy dominates over the characteristic electron-electron interactions while the characteristic spatial variations of any one-body potential felt by the electrons are much longer than the magnetic length.[5,6]

In this paper, we derive a formula for the Hall conductivity that applies to electrons moving in d-dimensional space as long as (i) there exists a gap between the $N_{\rm gs}$-fold degenerate ground state manifold and (ii) discrete translation invariance holds.[7] We show that the Hall conductivity averaged over the degenerate ground states is given by Eq. (50), which reduces to

$$\bar\sigma_{\rm H} = \frac{e^2}{h}\int_\Omega \mathrm{d}^2\boldsymbol{k}\;F(\boldsymbol{k})\;\bar{\rm n}(\boldsymbol{k}), \tag{3a}$$

$$F(\boldsymbol{k}) := \mathrm{i}\partial_{k_2}\langle\chi(\boldsymbol{k})|\partial_{k_1}\chi(\boldsymbol{k})\rangle - (1\leftrightarrow 2), \tag{3b}$$

in two-dimensional space if restricted to the case when the many-body ground states are exclusively built out of a single Bloch band with Bloch states $|\chi(\boldsymbol{k})\rangle$ and the single-particle Berry curvature $F(\boldsymbol{k})$. All the many-body correlations in Eq. (3b) are encoded by $\bar{\rm n}(\boldsymbol{k})$ defined in Eq. (47a), i.e., the expectation value of the occupation number operator of the Bloch momentum $\boldsymbol{k}$ from the Brillouin zone with volume Ω averaged over the $N_{\rm gs}$-fold degenerate many-body ground states.

Equation (3b) reproduces the following results.

Cases of the IQHE or the Chern band insulator. When $N_{\rm gs}=1$ and the many-body ground state is the Slater determinant made of all available Bloch states in the band, $\bar{\rm n}(\boldsymbol{k})=1$ for all $\boldsymbol{k}\in\Omega$ and

$$\bar\sigma_{\rm H} = \frac{e^2}{h}\int_\Omega \mathrm{d}^2\boldsymbol{k}\;F(\boldsymbol{k}) =: \frac{e^2}{h}\;C \tag{4}$$

with C the Chern number of the band (in particular, the Berry curvature is the constant $F(\boldsymbol{k})=1/\Omega$ and the Chern number is $C=1$ for the lowest Landau level).

Case of the FQHE. When $N_{\rm gs}>1$ is the degeneracy expected from a hierarchical ground state in the FQHE at the partial filling ν of the lowest Landau level, the Berry curvature is the constant $F(\boldsymbol{k})=C/\Omega$ with $C=1$ the Chern number of the lowest Landau level and

$$\bar\sigma_{\rm H} = \frac{e^2}{h}\int_\Omega \frac{\mathrm{d}^2\boldsymbol{k}}{\Omega}\;\bar{\rm n}(\boldsymbol{k}) =: \frac{e^2}{h}\;\nu. \tag{5}$$

Case of the anomalous Hall effect. When $N_{\rm gs}=1$ and the many-body ground state is a Fermi liquid, we can interpret Eq. (3b) as the anomalous Hall conductivity assuming the order of limits by which the gap has been taken to zero after all other limits have been taken. The anomalous Hall conductivity has been derived in the noninteracting limit,[8–10] but its derivation allows for interactions in this paper.

Equation (3b) predicts the following results for fractional Chern insulators.[11–21]

1. The integral over the Brillouin zone of $F(\boldsymbol{k})\times\bar{\rm n}(\boldsymbol{k})$ equals a rational number p/q, since Laughlin's gauge argument for quantization then applies.[2]

2. The integral over the Brillouin zone of $F(\boldsymbol{k})\times\bar{\rm n}(\boldsymbol{k})$ obeys a sum rule, for one cannot change the rational number p/q continuously.

3. The Hall conductivity $\bar\sigma_{\rm H}$ does not need to be equal to the filling fraction ν obtained by integrating $\bar{\rm n}(\boldsymbol{k})/\Omega$ over Ω whenever the Berry curvature $F(\boldsymbol{k})$ is not uniform over the Brillouin zone. In this case,

if $F_{\rm min}$ and $F_{\rm max}$ are the minimum and maximum of $F(\boldsymbol{k})$ over the Brillouin zone, respectively, then

$$\frac{e^2}{h}\,\nu\,(\Omega\,F_{\rm min}) \le \bar{\sigma}_{\rm H} \le \frac{e^2}{h}\,\nu\,(\Omega\,F_{\rm max})\,. \tag{6}$$

The remaining of the paper is devoted to deriving these results.

II. DEFINITIONS

We consider $N_{\rm e}$ electrons confined to a box of volume V in position space $\mathbb{R}^d$ where $d = 1, 2, \cdots$ whose dynamics is governed by the conserved many-body Hamiltonian H_0 acting on the many-body Hilbert space $\mathfrak{F}_{N_{\rm e}}$. Boundary conditions have been imposed such that H_0 has a countable number of eigenstates. The spectral decomposition of H_0 is written

$$H_0 = E_{\rm gs}\sum_{n=1}^{N_{\rm gs}} |n\rangle\langle n| + \sum_{m=N_{\rm gs}+1}^{\infty} E_m |m\rangle\langle m|. \tag{7a}$$

The ground state eigenenergy $E_{\rm gs}$ is assumed $N_{\rm gs}$-fold degenerate. We reserve the integer-valued index $n = 1, \cdots, N_{\rm gs}$ for its $N_{\rm gs}$ linearly independent ground states. A gap separating $E_{\rm gs}$ from the eigenenergies E_m of all excited states is assumed, whereby we reserve the integer-valued index $m = N_{\rm gs}+1, \cdots$ for the excited states. Eigenstates of H_0 are normalized according to

$$\langle n|n'\rangle = \delta_{n,n'}, \quad \langle m|m'\rangle = \delta_{m,m'}, \quad \langle m|n\rangle = 0. \tag{7b}$$

This means that the scalar product of any two states from the Hilbert space $\mathfrak{F}_{N_{\rm e}}$ is dimensionless. Finally, the total electric charge is assumed conserved by H_0 and this symmetry is not spontaneously broken.

Because of the boundary conditions, the set of center of mass momenta is countable, i.e., we can resolve the identity according to

$$1 = \sum_{\boldsymbol{Q}} P_{\boldsymbol{Q}}, \qquad P_{\boldsymbol{Q}}\,P_{\boldsymbol{Q}'} = \delta_{\boldsymbol{Q},\boldsymbol{Q}'}P_{\boldsymbol{Q}}, \tag{8}$$

where $\boldsymbol{Q} \in \mathbb{R}^d$ denotes a center of mass momentum and $P_{\boldsymbol{Q}}$ denotes the projector onto states with this center of mass momentum.

Finally, we shall assume that neither does H_0 break discrete translation invariance,[7] i.e.,

$$[H_0, P_{\boldsymbol{Q}}] = 0 \tag{9}$$

for any center of mass momenta $\boldsymbol{Q}$, nor is discrete translation invariance spontaneously broken.

We work in units where the electric charge e, the Planck constant $\hbar$, and the speed of light c are all unity. Therefore, only dimensions of energy appear in the theory.

III. LINEAR RESPONSE

We are after the linear response of one ground state $|n\rangle$ of the many-body Hamiltonian H_0 to adiabatically switching on a spatially homogeneous and static electric field $\boldsymbol{\mathcal{E}}$. The coupling to $\boldsymbol{\mathcal{E}}$ is described by the Hamiltonian

$$H := H_0 + H_1(t), \qquad H_1(t) := -\boldsymbol{X}\cdot\boldsymbol{\mathcal{E}}\,e^{\eta\,t}, \tag{10}$$

where $\boldsymbol{X} \equiv (X_i)$ is the many-body position operator in the Schrödinger picture,[23,24] $i = 1, \cdots, d$ labels the spatial coordinates, and η is a small positive number that implements the adiabatic turn-on of $\boldsymbol{\mathcal{E}}$.

We would like to make explicit the existence of a conserved electronic current that couples to the applied static and uniform electric field $\boldsymbol{\mathcal{E}}$. To this end, observe that we can always write

$$\boldsymbol{\mathcal{E}} = -\,(\partial_t\boldsymbol{\mathcal{A}})\,(t), \qquad \boldsymbol{\mathcal{A}}(t) := -t\,\boldsymbol{\mathcal{E}}. \tag{11a}$$

Insertion of Eq. (11a) into Eq. (10) gives

$$H = H_0 - \boldsymbol{J}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t} + \partial_t\left[\boldsymbol{X}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t}\right] \tag{11b}$$

with the current[25]

$$\boldsymbol{J} := \mathrm{i}\,[H_0, \boldsymbol{X}] \equiv \partial_t\boldsymbol{X}. \tag{11c}$$

In effect, we have performed the time-dependent gauge transformation

$$\begin{aligned}
&\widetilde{H}(t) := \widetilde{H}_0 - \widetilde{\boldsymbol{J}}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t},\\
&\widetilde{H}_0(t) := e^{+\mathrm{i}\boldsymbol{X}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t}}\,H_0\,e^{-\mathrm{i}\boldsymbol{X}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t}},\\
&\widetilde{\boldsymbol{J}}(t) := \mathrm{i}\left[\widetilde{H}_0(t), \boldsymbol{X}\right].
\end{aligned} \tag{12}$$

In anticipation of linear response theory, we are going to use the interaction picture that endows any operator O in the Schrödinger picture with the time dependence

$$O^{\rm I}(t) := e^{+\mathrm{i}\,H_0\,t}\,O\;\,e^{-\mathrm{i}\,H_0\,t}. \tag{13}$$

We do the same with the instantaneous ground states $|\widetilde{n}\rangle_t$ and the instantaneous excited states $|\widetilde{m}\rangle_t$ of $\widetilde{H}_0(t)$. In linear response theory, we approximate the time evolution of any state from $\mathfrak{F}_{N_{\rm e}}$ in the interaction picture by linearizing in the perturbation

$$H_1^{\rm I}(t) = -\boldsymbol{J}^{\rm I}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t} + \partial_t\left[\boldsymbol{X}^{\rm I}\cdot\boldsymbol{\mathcal{A}}(t)\,e^{\eta\,t}\right]. \tag{14}$$

For example, for one of the degenerate ground states,

$$|n(t)\rangle^{\rm I} := \left(1 - \mathrm{i}\int_{-\infty}^{t}\mathrm{d}t'\,H_1^{\rm I}(t')\right)|n\rangle + \cdots. \tag{15}$$

Hence, if we rule out any level crossing [see Eq. (21) and Ref. 6], then

$$\begin{aligned}
{}_t\langle\widetilde{n}|\widetilde{\boldsymbol{J}}(t)\,|\widetilde{n}\rangle_t &= {}^{\rm I}\langle n(t)\,|\boldsymbol{J}^{\rm I}(t)|\,n(t)\rangle^{\rm I}\\
&= -\,\mathrm{i}\int_{-\infty}^{t}\mathrm{d}t_1\,\langle n\,|\left[\boldsymbol{J}^{\rm I}(t), H_1^{\rm I}(t_1)\right]|\,n\rangle
\end{aligned} \tag{16}$$

to linear order in the time-dependent perturbation and after making use of the identity [following Eq. (11c)]

$$\left\langle n \left| \boldsymbol{J}^{\mathrm{I}}(t) \right| n \right\rangle = \left\langle n \left| \boldsymbol{J} \right| n \right\rangle = 0 \tag{17}$$

that disposes of the lowest order contribution to the expansion. With some intermediary steps involving the integral representation $\boldsymbol{J}^{\mathrm{I}}(t_1) = \frac{\mathrm{d}}{\mathrm{d}t_1} \int_{-\infty}^{t_1} \mathrm{d}t_2 \, \boldsymbol{J}^{\mathrm{I}}(t_2)$ and the use of partial integration, we arrive for any given $i = 1, \cdots, d$ at the leading order estimate (summation convention over the repeated index $j = 1, \cdots, d$)

$${}_t\!\left\langle \widetilde{n} \right| \widetilde{J}_i \left(t \right) \left| \widetilde{n} \right\rangle_t = \mathrm{i}\, \mathcal{E}_j \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \left\langle n \left| \left[J_i^{\mathrm{I}}(t), J_j^{\mathrm{I}}(t_2) \right] \right| n \right\rangle . \tag{18}$$

Finally, insertion of the resolution of the identity in $\mathfrak{F}_{N_\mathrm{e}}$

$$1 = \sum_n |n\rangle\langle n| + \sum_m |m\rangle\langle m| \tag{19}$$

delivers

$${}_t\!\left\langle \widetilde{n} \right| \widetilde{J}_i \left(t \right) \left| \widetilde{n} \right\rangle_t = -\mathrm{i}\, \mathcal{E}_j \sum_m \frac{\langle n|J_i|m\rangle\langle m|J_j|n\rangle - (i \leftrightarrow j)}{\left(E_m - E_{\mathrm{gs}} \right)^2}. \tag{20}$$

Here, we used that

$$\langle n|\boldsymbol{J}|n'\rangle = 0 \tag{21}$$

for any pair $n, n' = 1, \cdots, N_{\mathrm{gs}}$ in view of Eq. (11c). Equation (21) prevents level crossing among the degenerate ground states. For any $i, j = 1, \cdots, d$, we introduce the conductivity tensor $\sigma_{ij}^{(n)}$ through

$${}_t\!\left\langle \widetilde{n} \right| \widetilde{J}_i \left(t \right) \left| \widetilde{n} \right\rangle_t =: 2\pi\, V\, \sigma_{ij}^{(n)}\, \mathcal{E}_j. \tag{22}$$

For any pair $i \neq j = 1, \cdots, d$, the volume V on the right-hand side guarantees that the Hall conductivity tensor

$$\sigma_{ij}^{(n)} = -\frac{\mathrm{i}}{2\pi\, V} \sum_m \frac{\langle n|J_i|m\rangle\langle m|J_j|n\rangle - (i \leftrightarrow j)}{\left(E_m - E_{\mathrm{gs}} \right)^2} \tag{23}$$

is (i) intensive despite the fact that the left-hand side scales with the system size and (ii) dimensionless if $d = 2$. Equation (23) is the conventional representation of the Hall conductivity tensor. The right-hand side is an infinite series which is presumed convergent because of the energy denominators.

For our purpose, it is, however, more useful to trade the current for the position operator in the matrix elements of the right-hand side of Eq. (23) with the help of Eq. (11c) and, in turn, dispose of the energy denominator

$$\sigma_{ij}^{(n)} = -\frac{\mathrm{i}}{2\pi\, V} \sum_m \left[\langle n|X_i|m\rangle\langle m|X_j|n\rangle - (i \leftrightarrow j) \right]. \tag{24}$$

However, we must be very careful with the interpretation of the matrix element $\langle m|X_j|n\rangle$. In the thermodynamic limit $N_\mathrm{e}, V \to \infty$ holding the electronic density N_e/V fixed, the position operator X_j is unbounded, its expectation value in any momentum eigenstate is ill-defined, and so is its trace over the Hilbert space. Hence, the representation (24) of the Hall conductivity tensor is done in terms of the difference of two series, each of which is divergent in the thermodynamic limit $N_\mathrm{e}, V \to \infty$ holding the electronic density N_e/V fixed. Our goal is to properly regularize the subtraction of two infinities on the right-hand side by means of formal algebraic manipulations.

IV. ALGEBRAIC REGULARIZATION

For any $i \neq j = 1, \cdots, d$, instead of the Hall conductivity of a single degenerate ground state, we define the average

$$\bar{\sigma}_{ij} := \frac{1}{N_{\mathrm{gs}}} \sum_n \sigma_{ij}^{(n)}. \tag{25}$$

Because of the normalization (7b), we can introduce the projector on the ground states

$$P_{\mathrm{g}} := \sum_n |n\rangle\langle n| \tag{26a}$$

and the projector on the excited states

$$P_{\mathrm{e}} := \sum_m |m\rangle\langle m|. \tag{26b}$$

Evidently, the normalization (7b) implies that

$$\begin{aligned} &P_{\mathrm{g}}^2 = P_{\mathrm{g}} = P_{\mathrm{g}}^\dagger, \qquad P_{\mathrm{e}}^2 = P_{\mathrm{e}} = P_{\mathrm{e}}^\dagger, \\ &P_{\mathrm{g}}\, P_{\mathrm{e}} = P_{\mathrm{e}}\, P_{\mathrm{g}} = 0, \qquad P_{\mathrm{e}} + P_{\mathrm{g}} = 1. \end{aligned} \tag{26c}$$

In terms of the projectors (26a) and (26b), Eq. (25) becomes

$$\bar{\sigma}_{ij} = -\frac{\mathrm{i}}{2\pi\, V\, N_{\mathrm{gs}}} \mathrm{Tr}\, \left[P_{\mathrm{g}} X_i P_{\mathrm{e}}^2 X_j P_{\mathrm{g}} - (i \leftrightarrow j) \right] \tag{27}$$

where Tr denotes the trace over the Hilbert space $\mathfrak{F}_{N_\mathrm{e}}$.

The discrete translational invariance of the Hamiltonian H_0 guarantees that it commutes with $P_{\boldsymbol{Q}}$ for any center of mass momentum $\boldsymbol{Q}$, $P_{\boldsymbol{Q}}\, H_0 = H_0\, P_{\boldsymbol{Q}}$. Correspondingly, any eigenstate of H_0 can be labeled by a center of mass momentum and

$$P_{\boldsymbol{Q}}\, P_{\mathrm{g}} = P_{\mathrm{g}}\, P_{\boldsymbol{Q}}, \qquad P_{\boldsymbol{Q}}\, P_{\mathrm{e}} = P_{\mathrm{e}}\, P_{\boldsymbol{Q}}, \qquad \forall \boldsymbol{Q}. \tag{28}$$

Since we ruled out spontaneous symmetry breaking of discrete translation invariance by assumption, any ground state must be an eigenstate of the translation operator.[22] Indeed, if two ground states differ in their

center of mass momentum, then not all linear superpositions of them are eigenstates of the translation operator. We conclude that the absence of spontaneous breaking of discrete translation invariance implies that all states in the ground state manifold have the same center of mass momentum $\boldsymbol{Q}_0$. If so,

$$P_{\boldsymbol{Q}}\, P_{\rm g} = P_{\rm g}\, P_{\boldsymbol{Q}} = 0, \qquad \forall \boldsymbol{Q} \neq \boldsymbol{Q}_0, \tag{29}$$

so that, combined with the application

$$P_{\boldsymbol{Q}} = P_{\boldsymbol{Q}}\, \big(P_{\rm g} + P_{\rm e}\big) = \big(P_{\rm g} + P_{\rm e}\big)\, P_{\boldsymbol{Q}}, \tag{30}$$

of the resolution of the identity, we deduce that

$$P_{\boldsymbol{Q}} = P_{\boldsymbol{Q}}\, P_{\rm e} = P_{\rm e}\, P_{\boldsymbol{Q}}, \qquad \forall \boldsymbol{Q} \neq \boldsymbol{Q}_0. \tag{31}$$

Now, we use the fact (see Ref. 8) that the position operator $\boldsymbol{X} \equiv (X_i)$ can always be written as the sum of two operators $\boldsymbol{T} \equiv (T_i)$ and $\boldsymbol{A} \equiv (A_i)$, such that the former shifts momentum by an infinitesimal amount in the $i = 1, \cdots, d$ direction and the latter does not shift the momentum, i.e., $\boldsymbol{A}$ commutes with any $P_{\boldsymbol{Q}}$,

$$\boldsymbol{X} = \boldsymbol{T} + \boldsymbol{A}. \tag{32}$$

It is of crucial importance to note that the decomposition (32) is not unique, but basis dependent.[23,24] Indeed, under those basis transformations of the single-particle Hilbert space that are diagonal in momentum, i.e., those that commute with any $P_{\boldsymbol{Q}}$, the operator $\boldsymbol{A}$ transforms like an operator-valued gauge field. Explicit representations of the operators $\boldsymbol{T}$ and $\boldsymbol{A}$ will be given in Eqs. (44) and (45), respectivley.

We are now in position to do the following manipulations for any $i = 1, \cdots, d$. First, we do the decomposition

$$\begin{aligned} P_{\rm e}\, X_i\, P_{\rm g} &= P_{\rm e}\, (T_i + A_i)\, P_{\rm g} \\ &= P_{\rm e}\, \lim_{q\to 0} \left(\frac{e^{{\rm i}\, q\, T_i} - 1}{{\rm i}\, q} \right) P_{\rm g} + P_{\rm e}\, A_i\, P_{\rm g}. \end{aligned} \tag{33}$$

Second, we use the orthogonality $P_{\rm e}\, P_{\rm g} = 0$ to dispose of the term ${\rm i}/q$ that would blow up in the limit of $q \to 0$,

$$P_{\rm e}\, X_i\, P_{\rm g} = \lim_{q\to 0} P_{\rm e}\, \left(\frac{e^{{\rm i}\, q\, T_i}}{{\rm i}\, q} \right) P_{\rm g} + P_{\rm e}\, A_i\, P_{\rm g}. \tag{34}$$

Third, we make use of the resolution of the identity and the orthogonality from Eq. (8) together with Eqs. (29) and (31) to infer that

$$P_{\rm e}\, X_i\, P_{\rm g} = \lim_{q\to 0} P_{\boldsymbol{Q}_0 + q\, \boldsymbol{e}_i} \left(\frac{e^{{\rm i}\, q\, T_i}}{{\rm i}\, q} \right) P_{\rm g} + P_{\rm e}\, A_i\, P_{\rm g}\, P_{\boldsymbol{Q}_0}. \tag{35}$$

The first term on the right-hand side connects two sectors of the Hilbert space $\mathfrak{F}_{N_{\rm e}}$ with well-defined center of mass momenta differing by the momentum $q\boldsymbol{e}_i$. The second term on the right-hand side annihilates any many-body state with $\boldsymbol{Q} \neq \boldsymbol{Q}_0$. Henceforth, the product

$$P_{\rm g}\, X_i\, P_{\rm e}^2\, X_j\, P_{\rm g} = P_{\rm g}\, X_i\, P_{\rm e}\, X_j\, P_{\rm g} \tag{36}$$

on the right-hand side of Eq. (27) becomes

$$P_{\rm g} X_i\, P_{\rm e}\, X_j P_{\rm g} = P_{\rm g}\, A_i\, A_j\, P_{\rm g} - P_{\rm g}\, A_i\, P_{\rm g}\, A_j\, P_{\rm g} \tag{37}$$

if $i \neq j$ (so that $P_{q\,\boldsymbol{e}_i} \neq P_{q\,\boldsymbol{e}_j}$) and where we have assumed that we can freely interchange the limit with the evaluation of the products.

We now insert Eq. (37) into Eq. (27),

$$\begin{aligned} \bar{\sigma}_{ij} = &- \frac{\rm i}{2\pi\, V\, N_{\rm gs}}\, {\rm Tr}\, \Big[P_{\rm g} [A_i, A_j] P_{\rm g} \\ &- (P_{\rm g}\, A_i\, P_{\rm g})(P_{\rm g}\, A_j\, P_{\rm g}) + (P_{\rm g}\, A_j\, P_{\rm g})(P_{\rm g}\, A_i\, P_{\rm g}) \Big]. \end{aligned} \tag{38}$$

The full trace Tr over the Hilbert space $\mathfrak{F}_{N_{\rm e}}$ is thus reduced to a trace over the ground state manifold. This is a *finite sum* since we have assumed that the ground state manifold is a finite dimensional vector space. To dispose of the second contribution, we make use of the cyclicity of the trace restricted to the ground state manifold. We are then left with the *finite sum*

$$\begin{aligned} \bar{\sigma}_{ij} &= - \frac{\rm i}{2\pi\, V\, N_{\rm gs}}\, {\rm Tr}\, \big[P_{\rm g} [A_i, A_j] P_{\rm g} \big] \\ &= - \frac{\rm i}{2\pi\, V\, N_{\rm gs}} \sum_{n=1}^{N_{\rm gs}} \langle n | [A_i, A_j] | n \rangle. \end{aligned} \tag{39}$$

Equation (39) is the main result of this paper. It is an algebraic counterpart to the many-body presentation of the Hall conductance in terms of a many-body Berry phase defined by twisting boundary conditions.[6]

V. BLOCH REPRESENTATION

To proceed, we need to choose a basis of the Hilbert space $\mathfrak{F}_{N_{\rm e}}$. We choose the basis that follows from using the Fock space $\mathfrak{F}$ spanned by the creation and annihilation operators $\chi_a^\dagger(\boldsymbol{k})$ and $\chi_a(\boldsymbol{k})$, respectively, whereby any such pair labeled by the band index $a = 1, \cdots, N$ and the Bloch momentum $\boldsymbol{k}$ corresponds to a Bloch state $|\chi^{(a)}(\boldsymbol{k})\rangle$. We choose the normalization conventions

$$\{\chi_a(\boldsymbol{k}), \chi_b^\dagger(\boldsymbol{k}')\} = \Omega\, \delta(\boldsymbol{k} - \boldsymbol{k}')\, \delta_{a,b} \approx \Omega\, V \delta_{\boldsymbol{k},\boldsymbol{k}'}\, \delta_{a,b} \tag{40}$$

given the volume V in position space set by the infrared cutoff (the linear size $L \gg \mathfrak{a}$ say) and the volume Ω in momentum space set by the ultraviolet cutoff (the lattice spacing $\mathfrak{a} \ll L$ say). In other words, we have assumed that H_0 obeys the additive decomposition

$$H_0 = H_0^{\rm Blo} + H_0^{\rm Int}. \tag{41a}$$

Here

$$H_0^{\rm Blo} = \sum_{a=1}^{N} \int_\Omega \frac{{\rm d}^d \boldsymbol{k}}{\Omega}\, \varepsilon_a(\boldsymbol{k})\, \chi_a^\dagger(\boldsymbol{k})\, \chi_a(\boldsymbol{k}) \tag{41b}$$

where, for any band index $a = 1, \cdots, N$ and Bloch momentum $\boldsymbol{k}$, the single-particle eigenvalue $\varepsilon_a(\boldsymbol{k})$ and the single-particle Bloch state $|\chi^{(a)}(\boldsymbol{k})\rangle$ are the solution to the eigenvalue problem

$$\mathcal{H}_0^{\rm Blo}(\boldsymbol{k})\,|\chi^{(a)}(\boldsymbol{k})\rangle = \varepsilon_a(\boldsymbol{k})\,|\chi^{(a)}(\boldsymbol{k})\rangle. \tag{41c}$$

The interacting term $H_0^{\rm Int}$ is of higher order than two in the number of creation and annihilation operators.

The decomposition of the position operator on the right-hand side of Eq. (32) can now be understood as follows. Assume that the $N \times N$ Hermitean matrix $\mathcal{H}_0^{\rm Blo}(\boldsymbol{k})$ has been specified in the basis $|\psi^{(\mathtt{a})}(\boldsymbol{k})\rangle$ where $\mathtt{a} = 1, \cdots, N$. We shall call this basis the orbital basis. Diagonalization of $\mathcal{H}_0^{\rm Blo}(\boldsymbol{k})$ delivers the Bloch basis $|\chi^{(a)}(\boldsymbol{k})\rangle$ where $a = 1, \cdots, N$. The unitary transformation that brings the orbital to the Bloch basis has the N^2 matrix elements $u_{\mathtt{a}}^{(a)}(\boldsymbol{k})$ where $a, \mathtt{a} = 1, \cdots, N$. Any pair of columns or rows from this matrix must be orthogonal,

$$u_{\mathtt{a}}^{(a)*}(\boldsymbol{k})\,u_{\mathtt{a}}^{(b)}(\boldsymbol{k}) = \delta_{a,b}, \qquad u_{\mathtt{a}}^{(a)}(\boldsymbol{k})\,u_{\mathtt{b}}^{(a)*}(\boldsymbol{k}) = \delta_{\mathtt{a},\mathtt{b}}. \tag{42}$$

Here and in what follows, the summation convention over repeated band $a = 1, \cdots, N$ or orbital index $\mathtt{a} = 1, \cdots, N$ is implied. For any spatial coordinate $i = 1, \cdots, d$, any pair $a, b = 1, \cdots, N$ of bands, and any Bloch momentum $\boldsymbol{k}$ we define the non-Abelian Berry connection

$$A_i^{ab}(\boldsymbol{k}) := -\mathrm{i} u_{\mathtt{a}}^{(a)*}(\boldsymbol{k}) \left(\frac{\partial u_{\mathtt{a}}^{(b)}}{\partial k_i}\right)(\boldsymbol{k}). \tag{43}$$

The operators on the right-hand side of Eq. (32) are then represented by [24]

$$\boldsymbol{T} = \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{\Omega}\, \chi_a^\dagger(\boldsymbol{k}) \left(\mathrm{i}\frac{\partial \chi_a}{\partial \boldsymbol{k}}\right)(\boldsymbol{k}) \tag{44}$$

and

$$\boldsymbol{A} = \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{\Omega} \chi_a^\dagger(\boldsymbol{k})\, \boldsymbol{A}^{ab}(\boldsymbol{k})\, \chi_b(\boldsymbol{k}). \tag{45}$$

Consequently, for any $i \neq j = 1, \cdots, d$,

$$[A_i, A_j] = \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{\Omega} \chi_a^\dagger(\boldsymbol{k})\, [A_i(\boldsymbol{k}), A_j(\boldsymbol{k})]^{ab}\, \chi_b(\boldsymbol{k}). \tag{46}$$

Equation (46) suggests that we define the N^2 dimensionless intensive numbers

$$\bar{\mathrm{n}}^{ab}(\boldsymbol{k}) := \frac{1}{\Omega\, V\, N_{\rm gs}} \sum_{n=1}^{N_{\rm gs}} \langle n|\chi_a^\dagger(\boldsymbol{k})\chi_b(\boldsymbol{k})|n\rangle. \tag{47a}$$

With the normalization (40), one verifies that

$$0 \leq \bar{\mathrm{n}}^{aa}(\boldsymbol{k}) \leq 1 \tag{47b}$$

for any band a and any momentum $\boldsymbol{k}$. Insertion of Eq. (46) into Eq. (39) yields

$$\bar\sigma_{ij} = -\mathrm{i} \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{2\pi}\, [A_i(\boldsymbol{k}), A_j(\boldsymbol{k})]^{ab}\, \bar{\mathrm{n}}^{ab}(\boldsymbol{k}). \tag{48}$$

It remains to evaluate with the help of Eq. (43) the commutator $[A_i(\boldsymbol{k}), A_j(\boldsymbol{k})]^{ab}$ of the non-Abelian Berry connection. It is

$$[A_i(\boldsymbol{k}), A_j(\boldsymbol{k})]^{ab} = \mathrm{i} \left(\frac{\partial A_j^{ab}}{\partial k_i}\right)(\boldsymbol{k}) - (i \leftrightarrow j). \tag{49}$$

Hence, for any pair $i \neq j = 1, \cdots, d$, we conclude with the desired representation

$$\bar\sigma_{ij} = \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{2\pi} \left[\left(\frac{\partial A_j^{ab}}{\partial k_i}\right)(\boldsymbol{k}) - (i \leftrightarrow j)\right] \bar{\mathrm{n}}^{ab}(\boldsymbol{k}) \tag{50}$$

of the Hall conductivity tensor averaged over the degenerate ground states.

A. Noninteracting band insulator

For a noninteracting band insulator with the lowest $\widetilde{N} \leq N$ bands filled, we have $N_{\rm gs} = 1$ and

$$\bar{\mathrm{n}}^{ab}(\boldsymbol{k}) = \mathrm{n}_1^{ab}(\boldsymbol{k}) = \begin{cases} \delta_{a,b}, & 1 \leq a, b \leq \widetilde{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

Due to the presence of the gap, the Bloch Hamiltonian (41c) can be adiabatically deformed to

$$H_0^{\rm Blo}(\boldsymbol{k}) \longrightarrow 1 - 2\,\widetilde{P}(\boldsymbol{k}), \tag{52}$$

where $\widetilde{P}(\boldsymbol{k})$ is the projector on the single-particle states in the lower bands. The right-hand side of Eq. (52) is invariant under any unitary transformation of the filled bands, an element of the unitary group $U(\widetilde{N})$ which is a subgroup of the group of unitary transformations $U(N)$ that mixes all N bands. The Hall conductivity (50) can be written in a form for which this symmetry manifests itself as a local $U(\widetilde{N})$ gauge invariance,

$$\bar\sigma_{ij} = \int_\Omega \frac{\mathrm{d}^d\boldsymbol{k}}{2\pi}\, \widetilde{F}_{ij}^{\tilde{a}\tilde{a}}(\boldsymbol{k}). \tag{53}$$

Here, the non-Abelian Berry curvature of the *lower bands* is given by

$$\widetilde{F}_{ij}^{\tilde{a}\tilde{b}}(\boldsymbol{k}) := \partial_{k_i} A_j^{\tilde{a}\tilde{b}}(\boldsymbol{k}) + \mathrm{i}\, A_i^{\tilde{a}\tilde{c}}(\boldsymbol{k})\, A_j^{\tilde{c}\tilde{b}}(\boldsymbol{k}) - (i \leftrightarrow j) \tag{54}$$

and the indices with $\tilde{}$ are only summed over the lower bands. Note that

$$A_i^{\tilde{a}\tilde{c}}(\boldsymbol{k})\, A_j^{\tilde{c}\tilde{a}}(\boldsymbol{k}) - (i \leftrightarrow j) = A_j^{\tilde{a}\tilde{c}}(\boldsymbol{k})\, A_i^{\tilde{c}\tilde{a}}(\boldsymbol{k}) - (i \leftrightarrow j) \tag{55}$$

vanishes if both $\tilde{a}$ and $\tilde{c}$ are summed over. This allows us to recast the expression for the Hall conductivity (50) in terms of the manifestly $U(\widetilde{N})$ gauge invariant Berry curvature (53).

B. Noninteracting Fermi sea

Let us now consider a Fermi sea ground state for which $N_{\rm gs} = 1$. Even though our derivation relies on the existence of a gap from the outset, we can think of letting this gap go to zero at the end, as long as the ground state remains unique. We shall work with a single partially occupied band $\tilde{a} = 1$ for simplicity. With FS $\subset \Omega$ denoting the Fermi sea

$$\bar{\rm n}^{ab}(\boldsymbol{k}) \equiv {\rm n}^{ab}(\boldsymbol{k}) = \delta_{a,1} \times \delta_{b,1} \times \begin{cases} 1, & \boldsymbol{k} \in {\rm FS}, \\ 0, & \boldsymbol{k} \in \Omega \setminus {\rm FS}. \end{cases} \tag{56}$$

The Hall conductivity (50) becomes

$$\bar{\sigma}_{ij} = \int\limits_{{\rm FS}\subset\Omega} \frac{{\rm d}^d\boldsymbol{k}}{2\pi}\, \widetilde{F}^{11}_{ij}(\boldsymbol{k}). \tag{57}$$

This result agrees with the zero temperature limit of the Hall conductivity for the anomalous Hall effect.[9,10]

C. Interacting partially filled single Bloch band

Finally, we assume that the ground states $|n\rangle$ have nonvanishing amplitudes only with those Slater determinants that are made of single-particle Bloch states from the band $\tilde{a} = 1$. The Hall conductivity (50) becomes

$$\bar{\sigma}_{ij} = \int\limits_{\Omega} \frac{{\rm d}^d\boldsymbol{k}}{2\pi}\, \widetilde{F}^{11}_{ij}(\boldsymbol{k})\, \bar{\rm n}(\boldsymbol{k}), \tag{58}$$

where $\bar{\rm n}(\boldsymbol{k})$ is the occupation number of the Bloch momentum $\boldsymbol{k}$ in that single band $\tilde{a} = 1$ that contributes to any one of the $N_{\rm gs}$ ground states $|n\rangle$. The right-hand side is invariant under any $U(1)$ gauge transformation of the $U(1)$ Berry connection $\boldsymbol{A}^1(\boldsymbol{k})$ that defines the Berry curvature $\widetilde{F}^{11}_{ij}(\boldsymbol{k})$.

Equation (58) sets bounds to the Hall conductivity that can arise in an interacting system from a partially occupied single Bloch band. If we define the filling factor

$$\nu := \int\limits_{\Omega} \frac{{\rm d}^d\boldsymbol{k}}{\Omega}\, \bar{\rm n}(\boldsymbol{k}), \tag{59}$$

we conclude that

$$\frac{\Omega\,\nu}{2\pi} \times \inf_{\boldsymbol{k}\in\Omega} \widetilde{F}^{11}_{ij}(\boldsymbol{k}) \leq \bar{\sigma}_{ij} \leq \frac{\Omega\,\nu}{2\pi} \times \sup_{\boldsymbol{k}\in\Omega} \widetilde{F}^{11}_{ij}(\boldsymbol{k}) \tag{60}$$

for any pair $i, j = 1, \cdots, d$.[27]

When $d = 2$, we deduce from Laughlin's flux insertion argument[2] that

$$\bar{\sigma}_{ij} = \frac{p}{N_{\rm gs}}, \tag{61}$$

where $N_{\rm gs}$ is the topological ground state degeneracy on the 2-torus and p is any integer that does not need to be co-prime with $N_{\rm gs}$. If we combine Eq. (61) with Eq. (60), we conclude that

$$\frac{2\pi}{\Omega} \left[\sup_{\boldsymbol{k}\in\Omega} F_{ij}(\boldsymbol{k}) - \inf_{\boldsymbol{k}\in\Omega} F_{ij}(\boldsymbol{k}) \right] \geq \frac{1}{\nu} \times \frac{1}{N_{\rm gs}} \tag{62}$$

is a necessary condition for the Hall conductivity to deviate from

$$\bar{\sigma}_{12} = \nu \times C_{12}, \qquad C_{12} := \int\limits_{\Omega} \frac{{\rm d}^2\boldsymbol{k}}{2\pi}\, \widetilde{F}^{11}_{12}(\boldsymbol{k}), \tag{63}$$

in two dimensions. Such a deviation was discussed for the FQHE in a periodic potential[28] and could in principle also appear in two-dimensional fractional Chern insulators.[29]

We close by exploring another implication of Eq. (58) for fractional Chern insulators. It relates to the following question. Can topologically ordered many-body states arise from a topologically trivial single-particle band structure when interactions are added? Let us start by discussing two cases for which the answer is negative, before turning to cases where the answer might be positive.

First, according to Eq. (58), if the single-particle Berry curvature vanishes everywhere in the Brillouin zone, $\widetilde{F}^{11}_{12}(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \Omega$, then the many-body Hall conductivity $\bar{\sigma}_{12}$ has to vanish as well. Second, if $\bar{\rm n}(\boldsymbol{k})$ is independent of $\boldsymbol{k}$ and if the band $\tilde{a} = 1$ has the vanishing Chern number $C_{12} = 0$, then $\bar{\sigma}_{12} = 0$.

However, the condition that $\bar{\rm n}(\boldsymbol{k})$ is constant throughout the Brillouin zone of volume Ω is not required for a topologically ordered phase of matter. When $\bar{\rm n}(\boldsymbol{k})$ varies in the Brillouin zone, we can use the filling fraction ν defined by Eq. (59) and the inequality (62) to establish a necessary condition to be fulfilled by the variations of the Berry curvature across the Brillouin zone of volume Ω for the many-body Hall conductivity $\bar{\sigma}_{12}$ to acquire a nonvanishing value even though $C_{12} = 0$.

Even if the single-particle Berry curvature vanishes or the necessary condition encoded by the inequality (62) is not fulfilled, a FQHE might still be stabilized by interactions if the assumptions of Sec. V C are relaxed. These assumptions are that (i) only one isolated band is partially occupied and (ii) discrete translational symmetry is not spontaneously broken.

If degrees of freedom from more than one band are available or if discrete translational symmetry is spontaneously broken in such a way that the folding of the Brillouin zone results in new bands, interactions might then change the band structure from topologically trivial to non-trivial. The former case is known to occur for Kramers degenerate bands.[30] For the latter case, we have in mind the scenario by which a mean-field treatment of the interaction within a single topologically trivial band breaks spontaneously the discrete translation symmetry by reducing the Brillouin zone from the volume Ω to the volume $\Omega_{\rm MF}$.[31–34] In the process of folding the Brillouin zone from one with volume Ω to one with volume $\Omega_{\rm MF}$,

the original band might split into several sub-bands (separated by energy gaps), some of which carrying nonvanishing Chern numbers. The residual interactions that have been ignored by this mean-field treatment might then stabilize a FQHE characterized by Eq. (58) provided Ω is substituted by $\Omega_{\rm MF}$ and the original band $\tilde{a}$ is replaced by the relevant sub-band. The "spontaneous" formation of a fractional Chern insulator is thus allowed if more than one band is involved or discrete translational symmetry is spontaneously broken by the interaction.

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