Exact zero modes in closed systems of interacting fermions

G. Goldstein and C. Chamon

Phys. Rev. B **86**, 115122 — Published 14 September 2012
DOI: 10.1103/PhysRevB.86.115122

Exact Zero Modes in Closed Systems of Interacting Fermions

G. Goldstein
Physics Department, Harvard University, Cambridge MA 02138, USA

C. Chamon
Physics Department, Boston University, Boston MA 02215 USA

We show that for closed finite sized systems with an odd number of real fermionic modes, even in the presence of many-body interactions, there are always at least two fermionic operators that commute with the Hamiltonian. There is a zero mode corresponding to the total Majorana operator, as shown by Akhmerov[1], as well as additional linearly independent zero modes, one of which 1) is continuously connected to the Majorana mode solution in the non-interacting limit, and 2) is less prone to decoherence when the system is opened to contact with an infinite bath. We also show that in the idealized situation where there are two or more well separated zero modes each associated with a finite number of interacting fermions at a localized vortex, these modes have non-Abelian Ising statistics under braiding. Furthermore the algebra of the zero mode operators makes them useful for fermionic quantum computation[2].

I. INTRODUCTION

Zero modes in non-interacting systems, *i.e.* eigenstates annihilated by a single-particle Hamiltonian, have a long history in physics and in mathematics. Zero energy states are associated to certain types of topological defects in the background fields in which electrons or quasiparticles propagate. The first example of such modes in physics appeared in the seminal work of Jackiw and Rebbi[3] in one-dimensional and three-dimensional systems, where the topological defects were domain walls and hedgehogs, respectively. In both these examples the physical consequence of the zero modes is the fractionalization of electron charge. Fractional charges can also be bound to vortices in a Kékule dimerization pattern in two-dimensional graphene-like systems[4]. The zero mode solutions in two-dimensions were first found by Jackiw and Rossi[5] in the study of Dirac fermions in the background of scalar and vector gauge fields of the Abelian Higgs model. In the condensed matter context this corresponds to a superconductor (where charge cannot be fractionalized, since it is not conserved). The number of zero modes in such system of Dirac fermions in two-dimensions equals the magnitude of the net vorticity independent of the details of the profile of the Higgs fields, a result that was shown by Weinberg[6] to be tied to the index theorem.

A modern example of a physical realization of the model in Ref. 5 was presented by Fu and Kane[7], who showed that a Dirac-type matrix equation governs surface excitations in a topological insulator in contact with an s-wave superconductor. A vortex in the superconducting order parameter leads to a zero mode solution. Because of the reality conditions imposed by the symmetries of the Bogoliubov-de Gennes (BdG) equations describing the superconductor within the mean-field approximation, the zero energy solutions correspond to Majorana zero modes, which are the focus of our study. Majorana fermions are self-adjoint operators γ_i which can be written as a sum of an annihilation and creation operator for one fermion mode and which satisfy the algebra:

$$\{\gamma_i,\, \gamma_j\} = 2\delta_{ij},\, \gamma_i^\dagger = \gamma_i. \tag{1}$$

Because they are zero modes of some mean field Hamiltonian, $[H_{\rm MF},\, \gamma_i] = 0$, these modes are in principle protected from decoherence as the mean field Hamiltonian, when restricted to the subspace generated by these modes, is zero. Recently it has been argued that quantum and classical fluctuations in open infinite systems (for example when the system is in contact to a bath) lead to decoherence of information stored in such modes[8]. Below, instead, we shall focus on closed, finite systems, which have markedly different properties from those coupled to an infinite environment.

The purpose of this letter is to study zero modes of interacting many-body fermionic Hamiltonians, beyond mean-field approximations. We will assume that the relevant degrees of freedom may be described by an odd number of Majorana fermions $\{\gamma_1, \gamma_2, \dots, \gamma_{2N+1}\}$. This formalism also handles the case when complex fermions are present, as we may change basis from complex to Majorana fermions: $c_j = \frac{1}{2}\left(\gamma_{2j} + i\gamma_{2j+1}\right),\, c_j^\dagger = \frac{1}{2}\left(\gamma_{2j} - i\gamma_{2j+1}\right)$. For an interacting many-body Hamiltonian, a zero mode means a Hermitian fermionic operator

$$\mathcal{O} = \sum_i \alpha_i\ \gamma_i + i \sum_{i,j,k} \beta_{i,j,k}\ \gamma_i\gamma_j\gamma_k + \dots\ , \tag{2}$$

written as a multinomial with sums and products of γ_i's, that commutes with the Hamiltonian, $[H, \mathcal{O}] = 0$. For any such operator, $\mathcal{O}$, $\exp(itH)\,\mathcal{O}\exp(-itH) = \mathcal{O}$ for all times t. As such there is no decoherence of the information stored in the correlators of such operators.

We will find below, for systems of interacting fermions, 2^N linearly independent solutions of the form given in Eq. (2). We will also extend our results to the case when interactions include bosonic modes (with finite dimensional Hilbert space) coupled to the Majorana modes.

II. QUADRATIC HAMILTONIANS

Let us start, as a warm up, with the simplest case where $H^{\rm Gauss} = i\sum_{i,j} h_{i,j}\ \gamma_i\gamma_j$ with $h_{i,j} = -h_{j,i}$ and $h_{i,j}$ real. We note that any quadratic Hamiltonian may be written in this manner. Generic eigenoperator solutions satisfying

$\left[H^{\rm Gauss}, \mathcal{O}_\lambda\right] = \lambda\, \mathcal{O}_\lambda$ are obtained by computing the commutators for operators of the form $\mathcal{O} = \sum_i \alpha_i \gamma_i$ using the relations Eq. (1), and matching the coefficients multiplying each operator γ_i on both sides of the equation. One arrives in this manner at an eigenvalue equation for the matrix

$$\mathcal{H}^{\rm Gauss} = 4i \begin{pmatrix} 0 & h_{1,2} & h_{1,3} & \cdots & h_{1,2N+1} \\ h_{2,1} & 0 & \ddots & & \vdots \\ h_{3,1} & \ddots & 0 & & \vdots \\ \vdots & & & \ddots & h_{2N,2N+1} \\ h_{2N+1,1} & \cdots & \cdots & h_{2N+1,2N} & 0 \end{pmatrix} . \tag{3}$$

The elements of the matrices $\mathcal{H}^{\rm Gauss}$ and h are closely related because the theory is Gaussian – there will be modifications in the case of interacting systems. Note that $\mathcal{H}^{\rm Gauss}$ is an odd-dimensional Hermitian antisymmetric matrix so it has an eigenvector with zero eigenvalue and real components $(\alpha_1, \alpha_2, \ldots, \alpha_{2N+1})$ which corresponds to the zero mode $\mathcal{O} = \sum_i \alpha_i \gamma_i$. Notice that it follows from the relations in Eq. (1) that $\mathcal{O}^\dagger = \mathcal{O}$ and $\mathcal{O}^2 = \sum_i \alpha_i^2 \times \mathbb{1}$.

Let us now introduce notation so as to arrive at the same $\mathcal{H}^{\rm Gauss}$ in a way that will be similar to the calculations for interacting systems below. Matching the coefficients multiplying each operator γ_i on both sides of the equation $\left[H^{\rm Gauss}, \mathcal{O}_\lambda\right] = \lambda\, \mathcal{O}_\lambda$ can be achieved easily if we think of the γ_i as basis vectors and define an inner product for operators A and B as $(A, B) \equiv {\rm Coeff}_{\mathbb{1}}(A^\dagger B)$, where

$${\rm Coeff}_{\mathbb{1}} \left(z\, \mathbb{1} + \sum_i \alpha_i\, \gamma_i + \sum_{i,j} \beta_{i,j}\, \gamma_i \gamma_j + \ldots \right) \equiv z, \tag{4}$$

i.e., the function ${\rm Coeff}_{\mathbb{1}}(\mathcal{Q})$ returns the coefficient proportional to the identity in the multinomial expansion of the operator $\mathcal{Q}$. One can check that the inner product is Hermitian, $(A, B) = (B, A)^*$ and it follows from the algebra of the γ_i's that the inner product gives $(\gamma_i, \gamma_j) = \delta_{i,j}$.

Armed with this inner product we then compute the matrix

$$\begin{aligned} \mathcal{H}^{\rm Gauss}_{ij} &= \left(\gamma_i, \left[H^{\rm Gauss}, \gamma_j\right]\right) \\ &= -\left(\gamma_j, \left[H^{\rm Gauss}, \gamma_i\right]\right) = -\mathcal{H}^{\rm Gauss}_{ji} , \end{aligned} \tag{5}$$

where the last line follows by direct computation and the fact that $h_{i,j} = -h_{j,i} \in \mathbb{R}$. Once again $\mathcal{H}^{\rm Gauss}_{ji}$ is given by Eq. (3) above. We thus arrive once more at the result that zero modes can be determined from null vectors of a linear eigenvector equation for a Hermitian anti-symmetric matrix $\mathcal{H}^{\rm Gauss}_{ij}$ (of odd dimension).

III. INTERACTING HAMILTONIANS

A. Quartic Hamiltonian

We will consider a Hamiltonian given by:

$$H^{\rm Quart} = i \sum_{i,j} h_{i,j}\, \gamma_i \gamma_j + \sum_{i,j,k,l} V_{i,j,k,l}\, \gamma_i \gamma_j \gamma_k \gamma_l, \tag{6}$$

with $h_{i,j}$ a real and anti-symmetric matrix and $V_{i,j,k,l}$ real and antisymmetric under odd permutations of i, j, k, l (we have dropped an irrelevant constant that gives a state independent energy shift). We will look for operators that commute with $H^{\rm Quart}$. We will work with a vector space that is spanned by all linearly independent Hermitian modes obtained from products of individual Majorana fermions γ_i:

$$\begin{aligned} 0\,\gamma: &\quad \mathbb{1}, \\ 1\,\gamma: &\quad \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2N+1}, \\ 2\,\gamma'{\rm s}: &\quad i\gamma_1\gamma_2, i\gamma_1\gamma_3, \ldots, i\gamma_{2N}\gamma_{2N+1}, \\ 3\,\gamma'{\rm s}: &\quad -i\gamma_1\gamma_2\gamma_3, \ldots, -i\gamma_{2N-1}\gamma_{2N}\gamma_{2N+1}, \\ \ldots: &\quad \ldots \\ 2N+1\,\gamma'{\rm s}: &\quad i^{(2N+1)N}\gamma_1\gamma_2\ldots\gamma_{2N+1} . \end{aligned} \tag{7}$$

There are in total $\sum_{k=0}^{2N+1} \binom{2N+1}{k} = 2^{2N+1}$ such operators, which we will denote by Υ_a, for $a = 1, \ldots, 2^{2N+1}$. For each a we define n_a to be the number of γ's in the product Υ_a, and we let $L(a) \equiv \{i_1(a), \ldots, i_{n_a}(a)\}$ be the list of indices appearing in the product Υ_a. With this notation, one can write

$$\Upsilon_a \equiv i^{\,n_a(n_a-1)/2}\, \gamma_{i_1(a)}\gamma_{i_2(a)}\cdots\gamma_{i_{n_a}(a)}. \tag{8}$$

The choice of phase factor guarantees that $\Upsilon_a = \Upsilon_a^\dagger$ and $\Upsilon_a^2 = \mathbb{1}$. Using Eq. (8) one verifies that, up to a phase, the product of two Υ_a's gives a third: $\Upsilon_a\, \Upsilon_b = (i)^{s(a,b)}\, \Upsilon_c$, where c satisfies $L(c) = L(a) \cup L(b) \setminus L(a) \cap L(b)$ and $s\,(a, b) \in \mathbb{N}$. Without loss of generality, we shall reserve the labels $a = 1$ and $a = 2^{2N+1}$ for the identity and the total Majorana operators: $\Upsilon_1 = \mathbb{1}$ and $\Upsilon_{2^{2N+1}} = i^{(2N+1)N}\gamma_1\gamma_2\ldots\gamma_{2N+1} \equiv \Upsilon_{\rm Maj}$.

We can now rewrite the Hamiltonian Eq. (6) as

$$H^{\rm Quart} = \sum_{a|n(a)=2} h_a\, \Upsilon_a + \sum_{a|n(a)=4} V_a\, \Upsilon_a\ , \tag{9}$$

for some coefficients h_a , V_a defined when $n(a) = 2$ or 4, respectively, and h_a, $V_a \in \mathbb{R}$. Below we will convert $H^{\rm Quart}$ into an operator acting on the vector space spanned by the Υ_a's with the action being given by the linear transformation where $H^{\rm Quart}$ acts by commutation: $\mathcal{O} \to \left[H^{\rm Quart}, \mathcal{O}\right]$. As a first step we extend the inner product given in Eq. (4) above to the space spanned by Υ_a i.e. $(A, B) \equiv {\rm Coeff}_{\mathbb{1}}(A^\dagger B)$. One can check that the inner product is Hermitian, $(A, B) = (B, A)^*$ and the set Υ_a forms an orthonormal basis. Furthermore, up to a multiplicative constant, we see that it is also given by the usual trace inner product:

$$(A, B) = \frac{1}{2^{2N+1}} tr\left(A^\dagger B\right). \tag{10}$$

Here, tr is taken over the space spanned by Υ_a. Indeed this can be checked by noting that Eq. (10) is linear, so it is sufficient to consider only terms of the form $A = \Upsilon_a$, $B = \Upsilon_b$. There are two possibilities: 1) $\Upsilon_a = \Upsilon_b$ in which case $tr\left({\Upsilon_a}^\dagger \Upsilon_b\right) = 2^{2N+1}$ (the dimension of the vector space) 2) $\Upsilon_a \neq \Upsilon_b$, for which case $tr\left({\Upsilon_a}^\dagger \Upsilon_b\right) = 0$, and Eq. (10) holds. We now compute the matrix elements $\mathcal{H}^{\rm Quart}_{ab}$. Since

$\left[H^{\rm Quart}, \Upsilon_b\right]$ is an anti-Hermitian operator (or i times a Hermitian operator) all the matrix elements of $\mathcal{H}^{\rm Quart}_{ab}$ are imaginary. Now because $\{\Upsilon_b\}$ is an orthonormal set we may compute matrix elements by taking inner products:

$$\begin{aligned}\mathcal{H}^{\rm Quart}_{ab} &= \left(\Upsilon_a, \left[H^{\rm Quart}, \Upsilon_b\right]\right) \qquad (11)\\ &= \frac{1}{2^{2N+1}} tr\left(\Upsilon_a H^{\rm Quart}\Upsilon_b - \Upsilon_a\Upsilon_b H^{\rm Quart}\right)\\ &= -\left(\Upsilon_b, \left[H^{\rm Quart}, \Upsilon_a\right]\right) = -\mathcal{H}^{\rm Quart}_{ba} ,\end{aligned}$$

so $\mathcal{H}^{\rm Quart}_{ab}$ is antisymmetric. The equality in the last line of Eq. (11) comes from the cyclic property of trace. Therefore we arrive at a Hermitian anti-symmetric matrix $\mathcal{H}^{\rm Quart}$. So far, this matrix has dimension $2^{2N+1}\times 2^{2N+1}$, which is even. However, one can break this matrix into four block-diagonal pieces. First, because $H^{\rm Quart}$ contains only even Υ_c, that is with n_c even, sectors with opposite parity are not mixed by $\mathcal{H}^{\rm Quart}_{ab}$, so necessarily $n_a \equiv n_b$ mod 2. Therefore we break $\mathcal{H}^{\rm Quart}$ into blocks acting on the fermionic and bosonic $\{\Upsilon_a\}$, each block a $2^{2N}\times 2^{2N}$ matrix. Second, notice that both the identity and the total Majorana operator commute trivially with $H^{\rm Quart}$, so they each reside in a 1×1 block. The identity is in the even sector ($n_1 = 0$) and the total Majorana operator is in the odd sector ($n_{\rm Maj} = 2N+1$). Therefore we have broken down $\mathcal{H}^{\rm Quart}$ into four odd-dimensional Hermitian and anti-symmetric block matrices: there are four operators that commute with the Hamiltonian $H^{\rm Quart}$, or zero mode solutions. They are, in the even block, the trivial identity $\Upsilon_1 = \mathbb{1}$ and the Hamiltonian $H^{\rm Quart}$ proper, and in the odd sector the total Majorana operator $\Upsilon_{\rm Maj}$[1] and *another non-trivial solution* $\mathcal{O} = \sum_a \alpha_a\,\Upsilon_a$, with α_a solutions of $\sum_b \mathcal{H}^{\rm Quart}_{ab}\,\alpha_b = 0$.

B. Generic Fermionic Hamiltonians

Let us allow for arbitrarily high order interactions. That is we will consider Hamiltonians of the form $H^{\rm Gen} = i\sum h_{i,j}\,\gamma_i\gamma_j + \sum_{i,j,k,l} V_{i,j,k,l}\,\gamma_i\gamma_j\gamma_k\gamma_l + i\sum_{i,j,k,l,m,n} Q_{i,j,k,l,m,n}\,\gamma_i\gamma_j\gamma_k\gamma_l\gamma_m\gamma_n + \ldots$, which may also be expressed as

$$H^{\rm Gen} = \sum_{a|n(a)=2} h_a\,\Upsilon_a + \sum_{a|n(a)=4} V_a\,\Upsilon_a + \sum_{a|n(a)=6} Q_a\,\Upsilon_a + \ldots, \qquad (12)$$

where $h_a,\ V_a,\ Q_a, \ldots \in \mathbb{R}$. We can construct the matrix $\mathcal{H}^{\rm Gen}$ similarly to what we did above, it is still a Hermitian antisymmetric matrix. Nothing changes in the argument, and the essence is that the Hamiltonian contains only Υ_c with even n_c, and therefore one can break $\mathcal{H}^{\rm Gen}$ into four block diagonal pieces exactly the same way we did for quartic Hamiltonians and obtain zero modes.

C. Bosonic Modes

We now partially extend our ideas to the case of an odd number of Majorana fermions coupled to some bosonic modes. Our main limitation is that in order to insure convergence, to have finite dimensional matrices only – we will "truncate" the Hilbert space of the bosonic modes to a finite number of states. More precisely we will assume that the relevant Hilbert space for the bosons is M dimensional and labeled by the states $\{|1\rangle, |2\rangle \ldots |M\rangle\}$[9]. As such we may represent all boson operators by $M\times M$ Hermitian matrices. One can then write a Hamiltonian that generalizes Eq. (12):

$$\begin{aligned}H^{\rm Gen-Bose} &= \Theta^{M\times M} + \sum_{a|n(a)=2} h_a^{M\times M}\otimes\Upsilon_a + \\ &\quad + \sum_{a|n(a)=4} V_a^{M\times M}\otimes\Upsilon_a + \sum_{a|n(a)=6} Q_a^{M\times M}\otimes\Upsilon_a + \ldots\\ &= \sum_{a|n(a)\ \rm even}\ \sum_{p=1}^{M^2} W_{a,p}\,\Upsilon_a\otimes h_p , \qquad (13)\end{aligned}$$

with $\Theta^{M\times M}$, $h_a^{M\times M}$, $V_a^{M\times M}$, $Q_a^{M\times M}$ Hermitian matrices and we expanded the bosonic $M\times M$ Hermitian matrices into an orthonormal basis $\{h_1,\, h_2, \ldots h_{M^2}\}$, with $(h_p, h_q)_{\rm Bose} = \delta_{pq}$. The inner product is $(A,B)_{\rm Bose} \equiv \frac{1}{M} tr\left(A^\dagger B\right)$. It is not too hard to see that this is a positive definite symmetric form on the space of bosonic operators[10]. Without loss of generality, we take $h_1 = \mathbb{1}_{M\times M}$.

We can combine the operators in the fermionic and bosonic spaces and define $\Omega_{a,p} \equiv \Upsilon_a\otimes h_p$, with the usual tensor space inner product[10]. These states are orthonormal because $(\Omega_{a,q},\Omega_{b,q})_{\rm total} \equiv (\Upsilon_a,\Upsilon_b)\times(h_p,h_q)_{\rm Bose} = \delta_{a,b}\,\delta_{p,q}$. We can also check that this is expressible as a trace: $(A,B)_{\rm total} = \frac{1}{2^{2N+1}}\frac{1}{M} tr\left(A^\dagger B\right)$. Here the trace is over the total space spanned by $\Omega_{a,p}$.

Armed with these combined operators, we can show that there is an exact zero mode in exactly the same way we have done in the previous case. We need the matrix:

$$\begin{aligned}\mathcal{H}^{\rm Gen-Bose}_{a,p;b,q} &= \left(\Omega_{a,p}, \left[H^{\rm Gen-Bose}, \Omega_{b,q}\right]\right) \qquad (14)\\ &= -\left(\Omega_{b,q}, \left[H^{\rm Gen-Bose}, \Omega_{a,p}\right]\right) = -\mathcal{H}^{\rm Gen-Bose}_{b,q;a,p} ,\end{aligned}$$

which is Hermitian and anti-symmetric. The last equality in Eq. (14) can be checked similarly to Eq. (11). We then break $\mathcal{H}^{\rm Gen-Bose}_{a,p;b,q}$ into even and odd block diagonal spaces, as before. In this way, we find two zero modes in the even sector, $\Upsilon_1\otimes h_1 = \mathbb{1}\otimes\mathbb{1}_{M\times M}$, and $H^{\rm Gen-Bose}$ proper, and two zero modes in the odd sector, $\Upsilon_{\rm Maj}\otimes\mathbb{1}_{M\times M}$ and *another non-trivial solution* $\mathcal{O} = \sum_{a,p}\alpha_{a,p}\,\Upsilon_a\otimes h_p$, with $\alpha_{a,p}$ solutions of $\sum_{b,q}\mathcal{H}^{\rm Quart}_{a,p;b,q}\,\alpha_{b,q} = 0$.

IV. MODE COUNTING AND STRUCTURE

Let us count all zero modes in the system. We first start with the Gaussian part of the theory, including bosons, and then later we add the interactions. Consider a Hamiltonian given by:

$$H^{\rm Gauss} = \sum_{m=1}^{M} E_m\,|m\rangle\langle m| + \frac{1}{2}\sum_{j=1}^{N}\epsilon_j\ i\gamma_{2j}\gamma_{2j+1} . \qquad (15)$$

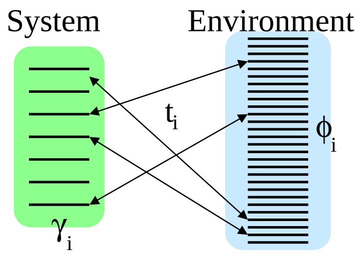

Figure 1: The system in tunneling contact with the environment. The system is composed of CdGM states[11], while the environment is everything else.

(Notice that $i\gamma_{2j}\gamma_{2j+1} = 2\,c_i^\dagger c_i - 1$.) By inspection, there are $M\times 2^N$ bosonic zero modes all given by operators of the form $\mathcal{O}^{\text{Bose}}_{m,\{\theta_j\}} \equiv |m\rangle\langle m| \otimes \prod_{j=1}^{N}\left(i\gamma_{2j}\gamma_{2j+1}\right)^{\theta_j}$ with $m = 1,\ldots,M$ and $\theta_j = 0, 1$ for $j = 1,\ldots,N$. There are similarly $M \times 2^N$ fermionic zero modes, simply given by $\mathcal{O}^{\text{Fermi}}_{n,\{\theta_j\}} \equiv \mathcal{O}^{\text{Bose}}_{n,\{\theta_j\}}\,\gamma_1$. These zero modes have a nice algebraic structure: 1) they are all Hermitian, 2) appropriate linear combinations of them square to one: $\left(\sum_m \mathcal{O}^{\text{Fermi/Bose}}_{m,\{\theta_j\}}\right)^2 = \mathbb{1}$, and 3) all zero modes commute: $\left[\mathcal{O}^{\text{Fermi/Bose}}_{m,\{\theta_j\}}, \mathcal{O}^{\text{Fermi/Bose}}_{m',\{\theta'_j\}}\right] = 0$. As such any one of the fermionic modes (which squares to one), and only one mode at a time, can be used for fermionic quantum computation[2].

Let us now show that the number of zero modes and their commutation relations do not change in the presence of weak interactions. To do so, as a first step, consider the following family of Hamiltonians $H^{\{\delta\}} \equiv H^{\text{Gauss}} + \sum_{m,\{\theta_j\}} \delta_{m,\{\theta_j\}}\,\mathcal{O}^{\text{Bose}}_{m,\{\theta_j\}}$ with $\delta_{m,\{\theta_j\}} \in \mathbb{R}$, and we note that $\left\{\delta_{m,\{\theta_j\}}\right\} \in \mathbb{R}^{M\times 2^N}$. It is not to hard to see that other then for points of accidental degeneracy all zero modes of all Hamiltonians of the form $H^{\{\delta\}}$ are given by $\mathcal{O}^{\text{Fermi/Bose}}_{m,\{\theta_j\}}$. As the next step, consider zero modes of Hamiltonians given by $H^{\{\delta\},U} \equiv U^\dagger H^{\{\delta\}}\,U$. All the zero modes are now given by $U^\dagger\,\mathcal{O}^{\text{Fermi/Bose}}_{m,\{\theta_j\}}\,U$, and as such also satisfy conditions 1), 2), and 3) of the previous paragraph. As before, exactly one appropriate mode from the fermionic set can be used for quantum computation[2]. To complete the discussion of the counting and structure of the zero modes for interacting systems, it remains for us to show that any Hamiltonian with weak interactions can be written as a $H^{\{\delta\},U}$.

To show this, we consider the map $\mathcal{F} : U\left(M^2 \times 2^{2N}\right) \oplus \mathbb{R}^{M\times 2^N} \to \mathbb{R}^{M^2\times 2^{2N}}$ given by $\mathcal{F}\left(U, \left\{\delta_{m,\{\theta_j\}}\right\}\right) = U^\dagger H^{\{\delta\}} U$. It is enough to show that the image of $U\left(M^2 \times 2^{2N}\right) \oplus \mathbb{R}^{M\times 2^N}$ contains a small open neighborhood of H^{Gauss}. Indeed, as any sufficiently weakly interacting Hamiltonian can be found in a small neighborhood of a non-interacting one this would show that $U^\dagger H^{\{\delta\}} U$ is a representation of all sufficiently weakly interacting Hamiltonians. By the implicit function theorem it is enough to show that $d\mathcal{F}$ is a surjective mapping onto $\mathbb{R}^{M^2\times 2^{2N}}$. Now writing $U = e^{-i\widetilde{H}}$ we get $d\mathcal{F}\left(\widetilde{H}, \left\{\delta_{m,\{\theta_j\}}\right\}\right) = i\left[\widetilde{H}, H^{\text{Gauss}}\right] + \sum_{m,\{\theta_j\}} \delta_{m,\{\theta_j\}}\,\mathcal{O}^{\text{Bose}}_{m,\{\theta_j\}}$. From this we see that all the zero modes are explicitly in the image of $d\mathcal{F}$. Since the transformation $* \to i\left[*, H^{Gauss}_{\{n\},\{\gamma_j\}}\right]$ is an invertible linear operator when restricted to the space of all non-zero modes, all non-zero modes are also in the image of $d\mathcal{F}$ as well. As such all of $\mathbb{R}^{M^2\times 2^{2N}}$ is in the image of $d\mathcal{F}$. This shows that up to conjugation by a unitary transformation the structure of the zero modes is the same as in the non-interacting case completing the proof.

V. COMPARISON WITH PREVIOUS WORK

In Ref.[1], the fermion parity operator Υ_{Maj} was discussed. This Majorana operator commutes with *any* Hamiltonian, since it is formed by the product of *all* the operators γ_i. This operator sits on its own 1×1 block of the matrix $\mathcal{H}$, for all cases studied, including in our generalization that includes bosons interacting with the fermionic modes.

In contrast, the other zero mode solutions found in the larger odd-dimensional block of $\mathcal{H}$ *do* depend on the form of the Hamiltonian. There are $M\times 2^N - 1$ of them. Furthermore one of the modes has a particularly simple structure $\mathcal{O} = e^{i\widetilde{H}}\sum_i \alpha_i\gamma_i\,e^{-i\widetilde{H}}$ which is continuously connected to the non interacting mode (consider $\mathcal{O}_t = e^{it\widetilde{H}}\sum_i \alpha_i\gamma_i\,e^{-it\widetilde{H}}$). This mode is different from the fermion parity mode[1] and, as we shall see below, for weak interactions (small $\widetilde{H}$) it is better protected from various forms of decoherence when the system is coupled to a generic bath.

VI. DECOHERENCE

Consider the setup shown in Fig. (1). We consider a simple perturbing tunneling Hamiltonian of the form: $\Delta H = i\sum_i t_i\gamma_i\eta_i$, with $t_i \in \mathbb{R}$. Here η_i refer to Hermitian fermionic modes relevant to the environment. In previous works it was demonstrated that $\langle \mathcal{O}(0)\,\mathcal{O}(T)\rangle$ is a good measure of the coherence of a qubit composed of localized Majorana modes[8]. Here $\mathcal{O}$ is an operator used to encode the qubit, and we will assume that the qubit and environment start uncorrelated. By Taylor expanding $e^{iT\Delta H}$ and keeping only leading order terms we obtain $\langle \mathcal{O}(0)\,\mathcal{O}(T)\rangle =$

$$1 - \tfrac{1}{2}T^2\sum_{i,j} t_i t_j \left\{\langle\eta_i\eta_j\rangle \times \left\{\langle\mathcal{O}\gamma_i\gamma_j\mathcal{O}\rangle + \langle\mathcal{O}\gamma_i\mathcal{O}\gamma_j\rangle\right\} + \langle\eta_j\eta_i\rangle \times \left\{\langle\mathcal{O}\gamma_j\mathcal{O}\gamma_i\rangle + \left\langle\mathcal{O}^2\gamma_j\gamma_i\right\rangle\right\}\right\}. \tag{16}$$

We can understand how this expression scales for various operators, in particular for $\mathcal{O} = \Upsilon_a$, n_a odd, we get that $\langle\Upsilon_a(0)\,\Upsilon_a(T)\rangle = 1 - 2T^2\sum_{i\in L(a)} t_i^2\left\langle\eta_i^2\right\rangle_{\text{Env}}$. Since $t_i^2\left\langle\eta_i^2\right\rangle_{\text{Env}} \geq 0$, operators with larger n_a decohere more quickly, at least for short times. This indicates enhanced stability for operators that are similar to single Majorana fermions, like the new zero modes presented here.

VII. BRAIDING

A. Quadratic Hamiltonian

As a warm up we will start with the case of quadratic Hamiltonians. We would focus on the holonomy under the exchange of vortices labeled by 1 and 2. We would like to consider the idealized case of several sets of fermionic zero modes $\left\{\mathcal{O}^{Fermi,l}_{m\{\theta_J\}}\right\}$, of the form $\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}} \equiv |m_l\rangle\langle m_l| \otimes \prod_{j=1}^{N}\left(i\gamma^l_{2j}\gamma^l_{2j+1}\right)^{\theta_j}\cdot\gamma^l_1$, each set corresponding to its own individual finite environment, vortex. The sets are labeled by ℓ. We further assume that the individual environments do not interact with the rest of the system. Since holomony is given by a unitary transformation it preserves product structure: $U^\dagger_{Hol}A\cdot BU_{Hol} = U^\dagger_{Hol}AU^\dagger_{Hol}\cdot U_{Hol}BU_{Hol}$. As such it is enough to consider the holomony of single particle modes $|m_l\rangle\langle m_l|$ and γ^l_i. We start with $|m_l\rangle\langle m_l|$. Since holomony preserves energy ordering, assuming no degeneracies, under braiding $|m_1\rangle \to e^{i\theta_m}|m_2\rangle$ and $|m_2\rangle \to e^{i\overline{\theta}_m}|m_1\rangle$, so overall $|m_1\rangle\langle m_1| \to |m_2\rangle\langle m_2|$ and $|m_2\rangle\langle m_2| \to |m_1\rangle\langle m_1|$. Similarly following Ivanov[12] we can work out the holomony for the Majorana modes. We know that under a change of superconducting phase by φ the Majorana modes transform as $\gamma^l_i = \begin{pmatrix} u^l_i \\ v^l_i\end{pmatrix} \to \begin{pmatrix} e^{i\varphi/2}u^l_i \\ e^{-i\varphi/2}v^l_i\end{pmatrix}$. Since there is a change by 2π of the superconducting phase when winding around a vortex and given that vortex two winds around vortex one under braiding, we see that $\gamma^1_i \to \gamma^2_i$ and $\gamma^2_i \to -\gamma^1_i$. Combining we get that[13]:

$$\begin{aligned}\mathcal{O}^{Fermi,1}_{m,\{\theta_j\}} &\to \mathcal{O}^{Fermi,2}_{m,\{\theta_j\}} \\ \mathcal{O}^{Fermi,2}_{m,\{\theta_j\}} &\to -\mathcal{O}^{Fermi,1}_{m,\{\theta_j\}}.\end{aligned} \tag{17}$$

We have reproduced Ising braiding statics.

B. Generic Hamiltonians

We would like to extend the derivation of Eq. (17) to the case of interacting modes. To do so we note that the many body holomony for interacting zero modes is the same as the one body holomony plus the effect of an additional Hamiltonian[14–16]. This Hamiltonian has matrix elements only between states of degenerate energy for the instanteneous Hamiltonian of the system. For example in the ground state manifold it is given by $H^{Hol}_{\Omega,\Omega'} = i\langle\Omega|\frac{d}{dt}|\Omega'\rangle$. Here $|\Omega\rangle$ and $|\Omega'\rangle$ are instantaneous zero energy eigenkets. Similarly for other instanteneous degenerate eigenkets. This Hamiltonian, which we shall not explicitly compute, corresponds within the Heisenberg picture to an effective evolution of the operators $U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l$. This evolution is given by a unitary transformation generated by the effective Hamiltonian $U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l \to P_U\left[H^{Hol}, U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right]$, where P_U is the projector onto the space of zero modes (operators in the manifold spanned by $U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l$). Now we claim that for any Hamiltonian, in particular the holomony Hamiltonian, $P_U\left[H, U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right] = 0$. We first note that: $P_U = \sum_{m,\{\theta_j\}}\left|U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right\rangle\left\langle U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right|$, so its enough to show that $\left(U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l, \left[H, U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right]\right) = 0$. Now:

$$\begin{aligned}&\left(U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l, \left[H, U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right]\right) \\ &= tr\left\{U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\left[H, U^\dagger_l\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}U_l\right]\right\} \\ &= tr\left\{\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\left[U_lHU^\dagger_l, \mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\right]\right\}.\end{aligned} \tag{18}$$

So its enough to prove $tr\left\{\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\left[U_lHU^\dagger_l, \mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\right]\right\} = 0$ for any Hamiltonian $U_lHU^\dagger_l$, e.g consider only the non-interacting case. However by inspection $tr\left\{\mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\left[|m'\rangle\langle m'|\Upsilon_a, \mathcal{O}^{Fermi,l}_{m,\{\theta_j\}}\right]\right\} = 0$. So by taking linear combinations of terms of the form $|m'\rangle\langle m'|\Upsilon_a$ we see that any Hamiltonian is zero when acting on the space of zero modes. As such the holomony reduces to the one given in Eq. (17).

VIII. CONCLUSIONS

We presented a systematic treatment of closed interacting systems with an odd number of real fermions. This formulation allowed us to find the zero mode solutions of interacting Hamiltonians, *i.e.*, operators that commute with the many-body Hamiltonian. In addition to the fermion parity operator that can be viewed as a constant of the motion for *any* Hamiltonian, we have found the solution that connects continuously to the Majorana mode for non-interacting systems as the interactions are switched off. These modes couple more weakly than the fermion parity mode to an environment once the system is opened up to an outside infinite bath[8]. Therefore, the solutions that are continuously connected to the non-interacting Majorana modes should lead to slower decay rates in the presence of a bath. We have also verified that, under idealized conditions when multiple such modes exist, they obey Ising like statistics under braiding.

This work was supported by NSF grant CCF-1116590.

[1] A. R. Akhmerov, Phys. Rev. B **82**, 020509 (2010).
[2] S. B. Bravii and A. Y. Kitaev, Ann. Phys. **298**, 210 (2002).
[3] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[4] C.-Y. Hou, C. Chamon, and C. Mudry, Phys. Rev. Lett. **98**, 186809 (2007).
[5] R. Jackiw and P. Rossi, Nucl. Phys. B **190**, 681 (1981).
[6] E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).
[7] L. Fu and C. L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008).

[8] G. Goldstein, and C. Chamon, arXiv 1107.0288.

[9] Spins are included in this formalism as a spin S system corresponds to a $2S+1$ dimensional bosonic subspace.

[10] S. Lang, *Linear Algebra*, Springer Science + Business Media Inc., (1987).

[11] C. Carroli, P. G. de Gennes, and J. Matricon, Phys. Lett. **9**, 307 (1964).

[12] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).

[13] For Bose modes the transformation is even simpler $\mathcal{O}^{Bose,1}_{m,\{\theta_j\}} \rightarrow \mathcal{O}^{Bose,2}_{m,\{\theta_j\}}$, $\mathcal{O}^{Bose,2}_{m,\{\theta_j\}} \rightarrow \mathcal{O}^{Bose,1}_{m,\{\theta_j\}}$.

[14] J. E. Avron, R. Seiler and L. G. Yaffe, Commun. Math. Phys. **110**, 33 (1987)

[15] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).

[16] R. R. Aldinger, A. Bohm, M. Loewe, Found. of Phys. Lett. **4**, 219 (1991).