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Continuous transitions between composite Fermi liquid and Landau Fermi liquid: a route to fractionalized Mott insulators

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One of the most successful theories of a non-Fermi liquid metallic state is the composite Fermi liquid (CFL) theory of the half-filled Landau level. In this paper, we study continuous quantum phase transitions out of the CFL state and into a Landau Fermi liquid, in the limit of no disorder and fixed particle number. This transition can be induced by tuning the bandwidth of the Landau level relative to the interaction energy, for instance through an externally applied periodic potential. We find a transition to the Landau Fermi liquid through a gapless Mott insulator with a Fermi surface of neutral fermionic excitations. In the presence of spatial symmetries, we also find a direct continuous transition between the CFL and the Landau Fermi liquid. The transitions have a number of characteristic observable signatures, including the presence of two crossover temperature scales, resistivity jumps, and vanishing compressibility. When the composite fermions are paired instead, our results imply quantum critical points between various non-Abelian topological states, including the $\nu = 1/2$ Moore-Read Pfaffian (Ising $\times$ U(1) topological order), a version of the Kitaev B phase (Ising topological order), and paired electronic superconductors. To study such transitions, we use a projective construction of the CFL, which goes beyond the conventional framework of flux attachment to include a broader set of quantum fluctuations. These considerations suggest a possible route to fractionalized Mott insulators by starting with FQH states and tuning the Landau level bandwidth.

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I. INTRODUCTION

Despite decades of work, the breakdown of Landau Fermi liquid theory in metallic states still poses some of the most challenging, unsolved problems in condensed matter physics. This breakdown often occurs in the vicinity of quantum phase transitions in Fermi liquid metals, although there are some situations, such as in the half-filled Landau level in two-dimensional electron gases (2DEGs), where entire non-Fermi liquid metallic phases have been found to exist. Perhaps the most experimentally successful theory of any non-Fermi liquid metal in more than one dimension is the composite Fermi liquid (CFL) theory of the half-filled Landau level, which has had a number of striking theoretical predictions that have been experimentally verified in GaAs 2DEGs.^{1–5} The CFL also provides structural insight: as the magnetic field is tuned, the conventional series of fractional quantum Hall (FQH) plateaus in the lowest Landau level in GaAs can be understood as integer quantum Hall (IQH) states of the composite fermions^{1–6}.

The crucial reason that the half-filled Landau level gives rise to this distinct non-Fermi liquid state of electrons is that the Landau level has essentially zero bandwidth, thus quenching the kinetic energy and leaving the interaction energy to dominate. As the bandwidth is increased to be on the order of the interaction strength, the system should pass through a quantum phase transition. In the limit that the bandwidth is large compared to the interaction strength, the resulting state will be well-described by Landau Fermi liquid theory. This raises the question of whether bandwidth-tuned transitions out of

quantum Hall states and into more conventional states can be continuous, and if so, what the possible critical theories are. For example, can there be a continuous quantum phase transition out of the CFL state and into a Landau Fermi liquid in a clean system? Such a continuous phase transition between a non-Fermi liquid metal and a Fermi liquid metal would be quite exotic; starting from the Landau Fermi liquid side, it would describe the continuous destruction of the electron Fermi surface and the emergence of a Fermi surface of composite fermions with singular gauge interactions. Previous work on the fate of *incompressible* (Abelian) FQH states includes Ref. 7–14. Our analysis below will allow us to consider also the fate of incompressible non-Abelian FQH states, such as the Moore-Read Pfaffian state¹⁵, as the bandwidth is increased.

One possible way to experimentally induce such bandwidth-tuned transitions is by imposing a periodic potential on a 2DEG subjected to an external magnetic field. When there is $2\pi p/q$ flux per unit cell of a weak periodic potential, each Landau level splits into p subbands, where the bandwidths and gaps between the subbands are on the order of the strength of the periodic potential.^{16,17} For 2π flux per plaquette, the periodic potential therefore does not split the bands, but only gives a bandwidth to the Landau levels¹⁶ and may be used to reach the regime in which the bandwidth is comparable to, or much larger than, the interaction strength. There are potentially other physical realizations as well. Recent attention has focussed on bands with non-zero Chern number in lattice systems without an external magnetic field^{18–27}. Partially filling such Chern bands can lead to

FIG. 1: (a) Schematic illustration of the effect of a periodic potential V_p with 2π flux per plaquette on a half-filled Landau level. The Landau level acquires a bandwidth (B) on the order of V_p ; in the limit that V_p is much larger than the interaction energy, the system should be well-described by Landau Fermi liquid theory. (b) As the bandwidth (B) of the flat band is tuned, the system will transition out of the CFL and ultimately into the Landau Fermi liquid. More generally, the system will transition out of a fractionalized QH state and ultimately into a more conventional state.

various fractional quantum Hall (FQH) states. The existence of incompressible FQH states has already been numerically established, and it is natural to expect that the CFL state can also be realized in such situations. The bandwidth of the Chern band can be increased by applying pressure, eventually resulting in a Landau Fermi liquid and providing another possible physical realization of the transitions discussed in this paper. In the context of the partially filled Chern bands, there is no background magnetic field, so the conventional framework of flux attachment and flux-smearing mean-field theory is inapplicable, raising a further fundamental conceptual question of how to understand the effective field theory of the CFL in such a situation.²⁵

In this paper, we address the questions discussed above. We begin by studying a lesser known, alternative formulation of the theory of the CFL state using the projective construction.⁷⁶ This construction has several advantages over the conventional flux attachment approach. Most importantly it allows us to include a broader range of quantum fluctuations, which provides access to nearby states, including ones in which the gauge fluctuations are destroyed to yield more conventional electronic states. We use this construction to develop the CFL theory for partially filled Chern bands without an external magnetic field, where the conventional notion of flux attachment may not be applicable in general. Subsequently, we study several possible states that can be described within the same low energy effective theory. For gapless states, these include the composite Fermi liquid, a gapless Mott insulator (GMI) with a Fermi surface of emergent neutral fermions, and the Landau Fermi liquid. When the fermions are paired instead, the CFL and GMI descend into either paired composite fermion states, such as the $\nu = 1/2$ Moore-Read (MR) Pfaffian, or gapped topological states with an emergent Z_2 gauge field coupled to the fermions. The Landau Fermi liquid is then replaced by a paired electronic superconductor. These considerations then allow us to study zero temperature phase

transitions out of the quantum Hall states and into other exotic fractionalized states, such as the GMI or its paired descendants, and ultimately into conventional electronic states, such as Landau Fermi liquids and superconductors (Fig. 1).

We find a transition between the CFL and the Landau FL through an intervening gapless Mott insulator state with emergent Fermi surface. In the presence of some spatial symmetry, such as inversion, we find a quantum critical point directly separating the CFL and the Landau FL. When we consider $p_x + ip_y$ pairing of the composite fermions, these considerations lead to a transition between the $\nu = 1/2$ MR Pfaffian¹⁵ and a $p_x + ip_y$ paired superconductor of electrons through an intervening non-Abelian topological state that is topologically equivalent to the non-Abelian B phase of Kitaev's honeycomb model²⁸. In the presence of some spatial symmetry, our results then imply a quantum critical point directly separating the $\nu = 1/2$ MR Pfaffian and the conventional $p_x + ip_y$ electronic superconductor; in the presence of a strong magnetic field this may be induced by tuning a periodic potential with 2π flux per plaquette.

Recently, building on a slave-rotor mean-field theory,²⁹ T. Senthil has developed a theory of a continuous Mott transition between a $U(1)$ spin liquid Mott insulator with a spinon Fermi surface and a Landau FL by including the effects of the $U(1)$ gauge fluctuations³⁰. That theory describes how the Fermi surface is continuously destroyed as the transition is approached, and it contains a number of striking, experimentally testable predictions. These include the existence of two crossover temperature scales and therefore two distinct quantum critical regimes, a universal resistivity jump at the transition, and diverging quasiparticle effective masses and Landau parameters as the transition is approached from the Fermi liquid side. It has been conjectured^{31,32} that such a gapless spin liquid state may be realized in a number of different compounds,^{33–35} which would then provide the possibility of observing such a bandwidth-tuned continuous Mott transition.

The transitions between the CFL, GMI, and Landau Fermi liquid display phenomenology similar to that found in the $U(1)$ spin liquid Mott transition of Ref. 30. We find that they can be continuous, and there are two crossover temperature scales and resistivity jumps at the transitions. Remarkably, we find that although the CFL and Landau FL are both compressible states, the quantum critical point between them is incompressible at zero temperature. On the composite Fermi liquid side of the transitions, the second crossover temperature scale does not exist in the presence of long-range Coulomb interactions, but can be made to appear by adding a metallic gate to screen the long-range interactions.

While the compounds studied in Refs. 33–35 provide experimentally promising venues for the observation of such a continuous Mott transition, our work suggests another experimentally promising venue for similar physics. If the CFL theory is taken seriously as describing the half-

filled Landau level, then our results predict that there is a GMI state nearby with a Fermi surface of neutral fermions and it may be realized by tuning appropriate periodic potentials with a wavelength on the order of the interparticle spacing; this state is a non-trivial spinless analog of the $U(1)$ spin liquid Mott insulator with spinon Fermi surface. We may refer to this as a $U(1)$ orbital liquid Mott insulator. The proximity to a Landau Fermi liquid by tuning the bandwidth would then also allow for another experimentally promising venue to study the continuous destruction of the Fermi liquid and the appearance of a fractionalized state with a Fermi surface coupled to an emergent $U(1)$ gauge field.

This paper is organized as follows. In Section II, we begin with a brief review of the construction of the CFL theory of the half-filled Landau level, and we review some of the results of the Halperin-Lee-Read theory of the gauge fluctuations of this state. In Section III, we develop a projective/parton construction for the CFL state, and we show how it can be applied to situations without an external magnetic field, and can be generalized to any filling fraction, including odd-denominator fillings. In Section IV, we discuss states proximate to the CFL at half-filling, including the GMI with emergent Fermi surface, and the Landau Fermi liquid. In Section V, we study the continuous transitions separating these states. In Section VI we discuss the consequences of this theory for various paired non-Abelian states.

II. BRIEF REVIEW OF THE CFL THEORY

The CFL theory of the compressible FQH state at $\nu = 1/2m$ begins by performing an exact transformation by which $2m$ units of flux quanta are attached to each electron. Similar flux transmutations have been used to derive effective field theories for a variety of QH states.^{1,2,36,37} In the Lagrangian formulation, in the imaginary time formalism, we have

$$Z = \int \mathcal{D}f^\dagger \mathcal{D}f \mathcal{D}a e^{-\int_0^\beta d\tau d^2r \mathcal{L}}, \quad (1)$$

with $\beta = 1/T$, T is the temperature, and the Lagrangian density is

$$\begin{aligned} \mathcal{L} = & f^\dagger (\partial_\tau - ia_0 - \mu) f - \frac{1}{2m_b} f^\dagger (\partial - iA - ia)^2 f \\ & + \frac{1}{2} \int d^2r' V(r - r') f^\dagger(r) f^\dagger(r') f(r') f(r) \\ & + \frac{i}{4\pi(2m)} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. \end{aligned} \quad (2)$$

f is the composite fermion field and A is the background electromagnetic field. The interaction may be chosen to be of the form $V(r) \sim 1/r^\eta$, where $\eta = 1$ corresponds to the case of Coulomb interactions. If the fluctuations of a are treated exactly, this is an exact transformation of the original theory. The value of this rewriting of the

original theory is that it allows for a mean-field approximation that was not available before. Since the filling fraction is $\nu = \frac{N_e}{N_\Phi} = \frac{1}{2m}$, there are $2m$ flux quanta for each electron. Since we have also added $2m$ units of a flux to each electron, we can consider a mean-field state where $\langle a \rangle = -A$. With this mean-field approximation, the composite fermions f on average do not feel any magnetic field.

It is found that the theory above describes a compressible,⁷⁷ metallic state, which describes well the phenomenology that is experimentally observed in the half-filled Landau level.¹

The single-particle properties of f exhibit various infrared singularities.³ For example, the self-energy $\Sigma_f(\omega)$ of f has a leading singularity that, for Coulomb interactions ($\eta = 1$), behaves as

$$\Sigma_f(\omega) \sim \omega \ln(i\omega). \quad (3)$$

For shorter-range interactions, where $1 < \eta \leq 2$, the self-energy gives rise to stronger singularities:

$$\Sigma_f(\omega) \sim (i\omega)^{\frac{2}{\eta+1}} \quad (4)$$

However, since f is not gauge-invariant, its single-particle correlation functions cannot be directly measured. Instead, the physical observables are the gauge-invariant response functions, which are found to be non-singular and Fermi liquid-like³⁸.

The electron operator, $c(r)$, in this theory is described as

$$c(r) = \hat{M}^{2m}(r) f(r), \quad (5)$$

where $\hat{M}(r)$ is an instanton operator for the gauge field a that annihilates 2π units of flux. The electron is simply the composite fermion f , together with $2m$ units of flux of the a gauge field. Using this electron operator, the equal-space electron Green's function was computed in Ref. 39 using a semi-classical approximation for the instanton action, with the result:

$$G_+(\tau) \equiv \langle c(0, \tau) c^\dagger(0, 0) \rangle \approx G_0(\tau) e^{-S_{MM}(\tau)}, \quad (6)$$

where $G_0(\tau)$ is an algebraically decaying function of τ and $S_{MM} \propto \tau^s$, where the exponent s depends on the form of the interactions between the composite fermions. It was found that the spectral function, $A_+(\omega)$, defined as the inverse Laplace transform of $G_+(\tau)$, behaves like

$$A_+(\omega) \sim e^{-\alpha/\omega^\beta}, \quad (7)$$

where α and β again depend on the interactions between the composite fermions. This indicates a strong exponential suppression of the tunneling density of states at low frequencies (see also Ref. 40 for a different derivation with the same conclusion).

In the presence of long-range Coulomb interactions ($\eta = 1$), the specific heat was calculated to scale as³

$$C_v \sim T \ln T. \quad (8)$$

With short-range interactions, the specific heat instead scales as

$$C_v \sim T^{2/3}. \quad (9)$$

The behavior of the specific heat and the exponentially decaying spectral function are both signatures of strongly non-Fermi liquid behavior.

A wave function for the CFL state that is expected to capture its long-wavelength properties is of the form⁴¹:

$$\Psi(\{r_i\}) = \mathcal{P}_{LLL}((z_i - z_j)^{2m} \text{Det}[e^{ik_i r_j}]), \quad (10)$$

where $\mathcal{P}_{LLL}$ indicates the projection to the lowest Landau level. The factor $(z_i - z_j)^{2m}$ can be thought of as attaching $2m$ units of flux quanta to the composite fermions, which are filling a Fermi sea.

The above theory has been quite successful in explaining many long-wavelength phenomena observed experimentally at $\nu = 1/2$. Nevertheless, the above formulation has three shortcomings that we address in this paper⁷⁸. First, it yields a limited framework for understanding continuous phase transitions out of the state. While this formulation is useful for studying transitions of the composite fermion Fermi surface, such as the pairing instability that leads to the Moore-Read Pfaffian state, more general continuous transitions, such as into a conventional Fermi liquid, cannot be understood through the above construction.

The second shortcoming is that while this flux attachment procedure and the associated flux-smearing mean-field theory can be defined in models where the Chern bands are induced by an external magnetic field, it is in general unclear how to extend this to lattice models without an external magnetic field (see Ref. 25 for a recent discussion). In fact, in many cases, the flux attachment procedure for a half-filled Chern band appears to fail entirely.

Thirdly, the above formulation of the composite particle theories on a compact space is problematic⁴²: because the Chern-Simons level is not quantized, the partition sum on a surface of genus ≥ 1 is not invariant under large gauge transformations^{43,44}.

In order to address these shortcomings, in the following section we develop a theory of the CFL through a projective construction.

III. PROJECTIVE CONSTRUCTION OF THE CFL STATE

Here we will study a derivation of the CFL through a totally different approach that does not begin from notions of flux attachment and flux smearing. Instead, we use a projective/parton construction, which provides several crucial advantages: most importantly, it incorporates a broader set of quantum fluctuations, which yields a path towards understanding transitions out of the CFL and into, for instance, Fermi liquids. Additionally, this

formulation can be extended to lattice models without an external magnetic field, where the conventional flux attachment picture fails; it yields insight into the CFL state and its topological properties; and finally it suggests generalizations of the CFL state to odd-denominator filling fractions or to states at the same filling fraction but which differ in their topological properties. Such projective constructions are more familiar in the study of quantum spin liquids⁴⁵, although they have been developed both for non-Abelian FQH states^{46–49}, and more conventional Abelian states^{22,26,27,50,51}.

For concreteness, let us consider a system of spinless interacting fermions on a square lattice in the presence of a background external magnetic field:

$$H = \sum_{ij} [t_{ij} c_i^\dagger c_j e^{iA_{ij}} - (\mu + A_i^0) \delta_{ij} n_j + V_{ij} n_i n_j], \quad (11)$$

with

$$n_i = c_i^\dagger c_i. \quad (12)$$

Let $\phi = 2\pi p/q$ be the amount of flux of A per plaquette. In the absence of interactions, the above model is then a tight-binding model with q bands; in the limit $q \rightarrow \infty$, the lowest bands become flat and equally spaced in energy, corresponding to Landau levels. We will suppose that the average density of electrons is such that the lowest Landau level has a filling $\nu = 1/2m$.

We begin by performing an exact rewriting of the above model by decomposing c_i in terms of different bosonic and fermionic variables:

$$c_i = b_i f_i, \quad (13)$$

where b_i annihilates a boson carrying the electric charge, and f_i annihilates a neutral fermion. This introduces a $U(1)$ gauge symmetry associated with the local transformation

$$b_i \rightarrow e^{i\theta_i} b_i, \quad f_i \rightarrow e^{-i\theta_i} f_i, \quad (14)$$

which keeps the electron operator invariant. We see:

$$n_i = n_i^b n_i^f, \quad (15)$$

where $n_i^b = b_i^\dagger b_i$ and $n_i^f = f_i^\dagger f_i$. The states at each site are now labelled as $|n_i^b, n_i^f\rangle$. These states form an expanded Hilbert space; the physical states in the expanded Hilbert space are the gauge-invariant ones: $|0, 0\rangle$ and $|1, 1\rangle$, which correspond to the state with zero or one electron(s), respectively. We see that the physical states satisfy the constraint $n_i^b = n_i^f$, so

$$n_i = n_i^b = n_i^f. \quad (16)$$

Inserting this into the original Hamiltonian (11) gives:

$$H = \sum_{ij} [t_{ij} b_i^\dagger b_j e^{iA_{ij}} f_i^\dagger f_j - (\mu + A_i^0) \delta_{ij} n_j^b + V_{ij} n_i^b n_j^b]. \quad (17)$$

Now we can decouple the quartic terms in H using a self-consistent mean-field approximation:

$$b_i^\dagger b_j e^{iA_{ij}} f_i^\dagger f_j \approx \frac{1}{2} [\chi_{ij} b_i^\dagger b_j e^{iA_{ij}} + \eta_{ij} f_i^\dagger f_j + \chi_{ij} \eta_{ij}], \quad (18)$$

where

$$\chi_{ij} = \langle f_i^\dagger f_j \rangle, \quad \eta_{ij} = e^{iA_{ij}} \langle b_i^\dagger b_j \rangle. \quad (19)$$

In the mean-field ansatz, we will assume that the ground state is such that b forms a $\nu = 1/2m$ Laughlin FQH state while f forms a Fermi surface. This implies that η_{ij} and χ_{ij} are both real-valued and translationally invariant. It is simple to verify that such a mean-field state can be self-consistent, because if η_{ij} is real-valued, then f forms a Fermi surface with no average magnetic field, and its real-space averages will be real, which will imply that χ_{ij} is real. Similarly, if χ_{ij} is real, then b feels a magnetic field around a plaquette set by A_{ij} , and therefore η_{ij} can be real.

The fluctuations about the mean-field state can be included in the manner familiar from parton constructions of other states: the amplitude fluctuations of χ and η are gapped, so we will only include the phase fluctuations. This leads to the following Hamiltonian:

$$H = \sum_{ij} [t_{ij} (\chi_{ij} b_i^\dagger b_j e^{iA_{ij} + ia_{ij}} + \eta_{ij} e^{-ia_{ij}} f_i^\dagger f_j) - (\mu + A_i^0) n_i^b + a_i^0 (n_i^b - n_i^f) + V_{ij} n_i^b n_j^b]. \quad (20)$$

a_i^0 is introduced as a Lagrange multiplier to enforce the constraint that $n_i^b = n_i^f$. While we derived this effective theory from a mean-field ansatz and included the allowed long-wavelength fluctuations, it can also be derived somewhat differently, following the analysis of Ref. 32 for the $U(1)$ spin liquid, as a saddle-point of the path integral by introducing additional fields to exactly decouple the quartic terms. Such mean-field approximations can yield stable deconfined fixed points of the resulting gauge theory. Whether they correspond to a global minimum in the energy is a more detailed question of energetics that must be answered through numerical studies or by comparing the phenomenology of these theories with experimental observations.

In a $\nu = 1/2m$ Laughlin state, the insertion of $2m$ units of flux quanta at sufficiently long wavelengths will create a single unit of charge. Therefore, at energies well below the gap of the bosonic Laughlin state, the boson can be represented by the instanton operator:

$$b = \hat{M}^{2m}, \quad (21)$$

where $\hat{M}$ is the instanton operator which annihilates 2π flux of a . Therefore, at energies well below the gap of the bosonic state, the electron operator can be represented as

$$c = \hat{M}^{2m} f, \quad (22)$$

just as in the conventional CFL theory. In particular, this means that at low energies, the boson number density is

$$n^b = \frac{1}{2\pi} \frac{1}{2m} \epsilon^{ij} \partial_i a_j, \quad (23)$$

which means that the boson interaction term in the Hamiltonian (20) can be rewritten in terms of a .

We can describe the $\nu = 1/2m$ bosonic Laughlin state by using a $U(1)_{2m}$ CS theory associated with a second emergent gauge field⁴⁵. Then, the effective theory becomes, to lowest order in a continuum approximation,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{int}, \\ \mathcal{L}_b &= \frac{2m}{4\pi} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} (a_\mu + A_\mu) \partial_\nu \tilde{a}_\lambda, \\ \mathcal{L}_f &= f^\dagger (i\partial_t + a_0) f + \frac{1}{2m^*} f^\dagger (\partial + ia)^2 f, \\ \mathcal{L}_{int} &= \int \frac{d^2 r'}{(4\pi m)^2} V(r - r') (\epsilon^{ij} \partial_i a_j(r)) (\epsilon^{ab} \partial_a a_b(r')). \end{aligned} \quad (24)$$

The boson current j_b^μ is given in terms of $\tilde{a}$:

$$j_b^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda. \quad (25)$$

Note this is consistent with (23), because integrating out $\tilde{a}_0$ in (24) in the absence of A yields the constraint $\epsilon_{ij} \partial_i a_j = 2m \epsilon^{ij} \partial_i \tilde{a}_j$.

Relabeling $a \rightarrow -(a + A)$ and subsequently integrating out $\tilde{a}$ yields the following Lagrangian density:⁷⁹

$$\begin{aligned} \mathcal{L} &= f^\dagger (i\partial_t - a_0 - A_0) f + \frac{1}{2m} f^\dagger (\partial - iA - ia)^2 f \\ &+ \frac{1}{2} \frac{1}{4\pi m} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \dots \\ &+ \int \frac{d^2 r'}{(4\pi m)^2} V(r - r') (\epsilon^{ij} \partial_i a_j(r)) (\epsilon^{ab} \partial_a a_b(r')), \end{aligned} \quad (26)$$

which is precisely the conventional CFL Lagrangian (2). It immediately follows that this construction yields a state with all of the same thermodynamic and transport properties of the conventional CFL state. A point of subtlety is that this construction yields a compressible state, despite the fact that b forms an incompressible Laughlin FQH state. Formally, this is possible because the polarizability of the electrons, Π_e , is given by the Ioffe-Larkin sum rule⁵²

$$\Pi_e^{-1} = \Pi_b^{-1} + \Pi_f^{-1}, \quad (27)$$

where Π_b and Π_f are the polarizability tensors of the b and f particles, respectively. If the polarization tensors were diagonal, the inverse compressibilities and inverse conductivities of the slave particles would add to give the inverse compressibility or inverse resistivity of the electron system. However, in this case, the polarization

tensors have off-diagonal components, $\Pi_{0i} \neq 0$; thus, the compressibility of the electron system can be finite, despite the incompressibility of the boson sector. Less formally, the coupling to the gapless gauge flux of a allows the Laughlin state of the bosons to respond continuously to the introduction of extra particles.

The single-electron Green's function is

$$\begin{aligned} G_e(r - r', t - t') &= \langle T c(r, t) c^\dagger(r', t') \rangle \\ &= \langle T b(r, t) b^\dagger(r', t') f(r, t) f^\dagger(r', t') \rangle. \end{aligned} \quad (28)$$

In the slave-particle mean-field approximation, where we ignore gauge fluctuations,

$$G_e^{mf}(r, t) = G_b(r, t) G_f(r, t). \quad (29)$$

Since b forms a gapped FQH state, its bulk correlations will clearly decay exponentially in space and time, and therefore so will $G_e^{mf}(r, t)$. The result of Ref. 39 is recovered by considering gauge fluctuations. At low energies, we can represent b by the instanton operator, so that the electron operator is given by (22). Now the calculation of the electron Green's function reduces to that of Ref. 39.

This mean-field theory of the CFL state can actually be derived at any filling fraction ν by allowing the bosons to form more generic incompressible Abelian or non-Abelian FQH states and is therefore not limited to even-denominator fillings. Even staying at $\nu = 1/2$, it is possible to consider distinct incompressible states at $\nu = 1/2$ for the bosons instead of the $1/2$ Laughlin state. A possible example is the $\nu = 1/2$ bosonic orbifold state studied in Ref. 49. These states would have different topological orders in the boson sector, but would yield the same low energy dynamics described by (2) if the bosons are integrated out. The different possible topological orders for the bosonic sector suggest that the CFL should be viewed as a *topological non-Fermi liquid*. An open conceptual problem is to develop tools to characterize the topological order in such topological non-Fermi liquid metals.

A. CFL in the absence of an external magnetic field: partially-filled Chern bands and the 'factorization' of bandstructure

Recently, there has been wide interest in studying FQH states in lattice models without an external magnetic field, but with partially filled flat bands with non-trivial topology. Since a flat band with Chern number 1 is topologically equivalent to a Landau level, it is expected that when such a band is partially filled, it is possible to recover conventional FQH states. So far, the focus has mostly been on the incompressible FQH states, however one may expect that the CFL can also appear in such systems. This raises the question of how to construct the groundstate wavefunction and low-energy effective description for such a state. It is simple to see that the conventional flux attachment and flux-smearing mean-field

approximation fails for such a situation.⁸⁰ For example, consider a square lattice with a half-filled flat band with Chern number 1. Since the band is at half-filling, this implies that there is one electron for every two sites. In the flux attachment picture, we must attach a multiple of 4π flux to obtain composite fermions. However, attaching 4π flux to each electron is equivalent to adding 2π flux per plaquette. In the flux-smearing mean-field approximation, 2π flux per plaquette is equivalent to zero flux per plaquette, so the statistical transmutation appears to be ineffective.

In contrast, the projective construction of the previous section can straightforwardly be generalized to the case of a partially filled Chern band without an external magnetic field. In Refs. 26,27, a prescription was given for a mean field theory with the parton decomposition $c = f_1 f_2 \dots f_n$; the idea is that the hopping elements for the f s should be n th roots of those of c . This was justified by a strong coupling expansion of the parton lattice gauge theory. Here we wish to extend this to the parton decomposition $c = f b$; the hopping elements for f and b must then be *factors* of those of c , but need not be equal.

Consider a tight-binding model defined by the following hamiltonian

$$H = - \sum_{rr' \in \mathcal{E}} t_{rr'}^c c_r^\dagger c_{r'}' + \text{h.c.} + \text{interactions}. \quad (30)$$

Here $c^\dagger, c$ denote creation and annihilation operators for electrons; $\mathcal{E}$ denotes links in the lattice. Our mean field ansatz for the "factorization" of this lattice is

$$H_{\text{mf}} = - \sum_{rr' \in \mathcal{E}} \left(t_{rr'}^f f_r^\dagger f_{r'}' + t_{rr'}^b b_r^\dagger b_{r'}' + \text{h.c.} \right) \quad (31)$$

where we demand that for each link

$$t_{rr'}^c = t_{rr'}^f t_{rr'}^b h_{rr'}^{-1}. \quad (32)$$

The scale $h_{rr'}$ is a parameter of the parton lattice gauge theory representing the energy cost for allowing gauge flux along the link rr' ; the argument from strong coupling expansion proceeds as in Ref. 26, to which we refer the reader for the details. The phases in t^c which make the resulting bands topological must be distributed between t^f and t^b – this choice is a lattice analog of the choice in the continuum parton construction of where to assign the electric charge amongst the partons. The analog of letting the bosons carry the charge is to put all the phases in t^b .

For definiteness, consider an electron model at quarter-filling on the checkerboard lattice. Some relatively-flat Chern bands are realized by the following tight-binding model with next-nearest neighbor interactions⁵³. The Hamiltonian is $H = \sum_{k \in BZ} (h_{2N} + h_{3N})$ with

$$\begin{aligned} h_{2N} &= -t e^{i\varphi} \left[\beta_k^\dagger \alpha_k \left(e^{\frac{i}{2}(k_x + k_y)} + e^{\frac{i}{2}(-k_x + k_y)} \right) \right. \\ &\quad \left. \alpha_k^\dagger \beta_k \left(e^{\frac{i}{2}(k_x - k_y)} + e^{\frac{i}{2}(-k_x - k_y)} \right) \right] + \text{h.c.} \\ h_{3N} &= -\alpha_k^\dagger \alpha_k (t_1' e^{ik_x} + t_2' e^{ik_y}) - \beta_k^\dagger \beta_k (t_2' e^{ik_x} + t_1' e^{ik_y}) \end{aligned} \quad (33)$$

where α and β are fourier modes of the electron creation operators on the two sublattices:

$$\alpha_k = \frac{1}{\sqrt{N_a}} \sum_{x_a} e^{ikx_a} c_{x_a}, \quad \beta_k = \frac{1}{\sqrt{N_b}} \sum_{x_b} e^{ikx_b} c_{x_b}, \quad (34)$$

The bands may be further flattened by the addition of third-neighbor interactions⁵⁴. At quarter-filling, the electrons fill the lowest band halfway; this band has Chern number unity.

To construct a lattice analog of the HLR state, we take the ansatz:

$$t^b = te^{i\phi}, \quad (t'_2)^b = -(t'_1)^b, \quad t^f = t, \quad (t'_2)^f = +(t'_1)^f. \quad (35)$$

This puts the fermion in a state which half-fills a topologically-trivial band, and puts the boson in a state which half-fills the lowest topologically non-trivial band with Chern number one. Note that unlike the case of Ref. 26, no enlargement of the unit cell was required so far; if we wanted to explicitly describe the lattice analog of the $\nu = 1/2$ Laughlin wavefunction of this boson, we would have to use a second layer of parton construction which in which the unit cell is doubled.

IV. STATES PROXIMATE TO CFL: GAPLESS MOTT INSULATOR AND LANDAU FERMI LIQUID

The main utility of the projective construction of the CFL state in the previous section is that now, within the same theory, we can find saddle points associated with other many-body states.

For example, consider the $\nu = 1/2$ Landau level problem and suppose that we add a periodic potential, V_p , with a multiple of 2π flux per plaquette. V_p couples to the boson density (see (20)), and therefore will increase the bandwidth of the Landau level for b . When V_p is on the order of the boson gap, the bosons can undergo a transition into either a Mott insulating state for b or a superfluid.¹⁴ Alternatively, in the case of the partially filled Chern band without an external magnetic field, we can consider increasing the pressure to increase the hopping matrix elements. This can also cause transitions out of the bosonic Laughlin state and into the MI or the superfluid.

It is also possible to consider fractionalized Mott insulators for b , such as Z_2 topologically ordered states. In this paper, we will ignore these more complicated scenarios, as they are expected to be less likely. Instead we will focus on the simplest possible states for the bosons and the correspondingly simplest ways of transitioning out of the CFL and into the LFL.

A. Gapless Mott Insulator

When the strength of the periodic potential is on the order of the interaction strength, we expect the bosons to

transition out of the $\nu = 1/2$ Laughlin state. A simple, generic possibility is that the bosons continuously transition into a trivial Mott insulator without any topological order. For this to occur, the bosons must be at integer filling per unit cell; this can happen by either explicit or spontaneous translation symmetry breaking.

From the constraint $n_b = n_f$, integer filling of the bosons implies integer filling of the fermions. This means that the fermions can generally form either a band insulator or Fermi surfaces with equal area of particles and holes. In the mean-field Hamiltonian, the fermion hopping is given by

$$t_{ij}^f \propto \langle b_i^\dagger b_j \rangle. \quad (36)$$

Infinitely deep in the Mott insulating phase, it is possible that $\langle n_i^b \rangle \rightarrow 1$ or 0, depending on the lattice site. Similarly, it is possible that $\langle b_i^\dagger b_j \rangle \rightarrow 0$ for $i \neq j$. In this atomic limit, the fermions cannot form a Fermi surface; they are localized to the sites where the bosons are located, and do not disperse.

Closer to the $\nu = 1/2$ Laughlin and superfluid state, $\langle n_i^b \rangle$ is approximately uniform, and the bosons have a larger correlation length, so t_{ij}^f can be appreciable. In such a situation, the mean-field state of the fermions can be such that they form a Fermi surface. Since the fermions are at integer filling, the area of hole and particle pockets must be equal. This state is closely related to the gapless $U(1)$ spin liquid Mott insulators with spinon Fermi surface and an emergent $U(1)$ gauge field.^{31,32} The difference is that the gapless Mott insulator considered here consists of spinless electrons. While there is an integer number of electrons per unit cell, each unit cell consists of multiple sites; therefore, there is a pseudospin index that plays the role of spin and which can allow the system to have gapless degrees of freedom, despite the charge being localized and gapped.

Starting from the CFL, where the fermions form a Fermi surface, the introduction of a periodic potential will fold the Brillouin zone; if the Fermi surface has no nesting, as in the conventional scenarios, there will continue to be stable particle/hole pockets for the f fermions. As the bosons undergo a continuous transition to the MI state, the f sector changes gradually, as there is generically no change of spatial symmetry at the boson FQH -MI transition. Since there are equal area particle and hole Fermi pockets, depending on the precise correlations of the bosons, it is possible in principle to alter the fermion band structure in such a way that the Fermi surface gradually shrinks to zero and eventually the f fermions form a band insulator. For what follows we will consider the situation in which the Fermi surface never completely disappears, which is a generic possibility.

Physically, we can understand the appearance of the GMI state by analogy with the CFL state. In the CFL state, the electron fractionalizes into a boson b , which carries the electric charge, and the neutral f fermion. The boson b then forms an incompressible Laughlin FQH

state in order to minimize the strong interaction energy and the phase frustration induced by the magnetic field, while f is free to form a Fermi surface. It is therefore reasonable to imagine that instead of b forming a Laughlin FQH state, it will form a Mott insulator in order to minimize the strong repulsive interaction energies. The many-body wave function that describes such a state is the usual projected wave function⁴⁵:

$$\begin{aligned}\Psi_e(\{r_i\}) &= \langle 0 | \prod_i c(r_i) | \Phi_{mf} \rangle \\ &= \Psi_b(\{r_i\}) \Psi_f(\{r_i\}),\end{aligned}\quad (37)$$

where $|\Phi_{mf}\rangle$ is the mean-field ground state of b and f , where f is forming a Fermi surface and b is forming a Mott insulator. This is equivalent to taking the boson MI wave function and fermion FS wave function and projecting the bosons and fermions to the same location.

The effective theory for this state takes the form

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_a, \quad (38)$$

where $\mathcal{L}_b$ describes the gapped bosonic excitations, $\mathcal{L}_f$ describes the fermions in a Fermi sea and coupled to a , and $\mathcal{L}_a = \frac{1}{g^2}(\nabla \times a)^2 + \dots$ is the action for the $U(1)$ gauge field a . When the bosons are in the gapped, trivial MI state, they can be integrated out to give, to lowest order,

$$\mathcal{L} = \mathcal{L}_f + \frac{1}{g^2}(\nabla \times a)^2. \quad (39)$$

The gapless Mott insulator is electrically insulating, thermally conducting, and incompressible. As a result, its existence cannot be established through DC electrical transport measurements. Instead, it can be distinguished from the trivial Mott insulator by probing thermal behavior, such as specific heat or thermal conductivity. The specific heat and thermal conductivity of such a state scale as $C_v \sim T^{2/3}$, and $K/T \sim T^{2/3}$, respectively.^{55–59} The possibility of thermal conduction in this spinless Mott insulator comes from the fact that the thermal conductivity of the electrons is equal to the sum of the thermal conductivity of the b and f systems;⁵⁵ this is in contrast to the electrical transport, where typically it is the resistivities that add. The incompressibility of this state follows from the Ioffe-Larkin sum rule and the fact that the bosons have zero Hall conductance in this state.

The Fermi surface can be more directly probed through Friedel oscillations, as proposed recently in Ref. 60 for the $U(1)$ spin liquid Mott insulator; in both cases, the density-density correlation function has algebraic correlations displaying signatures of a Fermi surface.

B. Landau Fermi liquid

When the interaction energy is small compared to the bandwidth, then it will be preferable for b to condense

FIG. 2: Proposed phase diagram and renormalization-group flows including the Mott insulator, superfluid, and $\nu = 1/2$ Laughlin FQH state, for fixed average particle number.¹⁴ We have defined $m_{\pm} \equiv m_1 \pm m_2$ (see eq. 40 with $N_f = 2$); m_- can be viewed as a symmetry-breaking field, so the direct transition between the FQH state and the SF can occur if the symmetry is preserved. The red points on the horizontal and vertical axes indicate the three stable phases, while the blue points at the origin and the diagonals indicate the unstable critical fixed points.

into the bottom of its band and form a superfluid. In such a case, the emergent $U(1)$ gauge symmetry associated with a is spontaneously broken, and the resulting state of the electrons is described by Landau Fermi liquid theory. In the Landau Fermi liquid, the quasiparticle residue $Z \sim |\langle b \rangle|^2$.

Since time-reversal symmetry is broken, with interactions the boson superfluid state will in general consist of a normal component with non-vanishing orbital currents. The electron state will therefore be a Landau Fermi liquid with orbital loop currents and a non-zero Hall conductance, as time-reversal symmetry is explicitly broken.

If the electrons are at integer filling per unit cell of the periodic potential, then the resulting state is either a band insulator or a Landau Fermi liquid with equal area for electron and hole pockets. Within our construction, if we start from a CFL with non-nested Fermi surface and add a periodic potential, the metallic case will be the generic situation.

If the electrons are at fractional filling, then the resulting state is a Landau Fermi liquid with no constraints on electron and hole pockets, except for Luttinger's theorem.

V. CONTINUOUS TRANSITIONS

From this framework, we can now understand transitions out of the CFL state and ultimately into the Fermi liquid by considering transitions in the boson sector between the $\nu = 1/2$ Laughlin FQH state, the Mott insulator, and the superfluid. In Ref. 14, we found that in the absence of any special symmetries and for fixed particle number, there are continuous transitions between

the bosonic Laughlin FQH state and the Mott insulator, and between the boson Mott insulator and the superfluid. However, certain spatial symmetries, such as inversion, can stabilize a direct continuous transition between the FQH state and the superfluid. These critical points are described by massless Dirac fermions coupled to a $U(1)$ CS gauge field:

$$\mathcal{L}_{N_f,k} = \frac{N_f k}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \sum_{i=1}^{N_f} [\bar{\psi}_i \gamma^\mu D_\mu \psi_i + m_i \bar{\psi}_i \psi_i]. \quad (40)$$

It was found that the MI-SF transition is described as $m_1 \rightarrow 0^+$ in $\mathcal{L}_{1,1/2}$, the FQH-MI transition is described by $m_1 \rightarrow 0^+$ in $\mathcal{L}_{1,3/2}$, and the FQH-SF transition is described by $m_1 = m_2 \equiv m \rightarrow 0^+$ in $\mathcal{L}_{2,1/2}$ (see Fig. 2)^{8,14}. Note that for the above theory to have a well-defined lattice regularization, $N_f k$ must be an integer when N_f is even, and half-integer when N_f is odd. The critical theory at $m = 0$ may be modified in the presence of long-range interactions, depending on the value of k ¹⁰. It was found that in the large N_f limit, for $k > k_{c1}$, Coulomb interactions are marginally irrelevant, while for $k_{c2} < k < k_{c1}$, Coulomb interactions are relevant and cause a flow to a different stable fixed point with dynamic critical exponent $z = 1$ and correlation length exponent $\nu > 1$. For $k < k_{c2}$, Coulomb interactions are relevant and flow to strong coupling. It was found that $k_{c1} \approx 0.35$ and $k_{c2} \approx 0.28$. Thus for the cases of interest here, it is possible that Coulomb interactions will either be marginally irrelevant or flow to the controlled fixed point with $z = 1$ and $\nu > 1$.

We note that since time-reversal symmetry is strongly broken in the situation under consideration, a possibility is that at the FQH - SF critical point, the initial transition out of the FQH state is into a vortex state of the superfluid. If the vortices form a vortex lattice, then both translation and $U(1)$ charge symmetry are broken at the transitions. We expect that such a scenario would be multicritical and would not be described by (40); any translation symmetry breaking should generically occur away from the FQH to SF critical point.

Given the above critical points between the different bosonic states, it is possible that the electron system will also undergo continuous transitions between the CFL, gapless MI, and FL states as the bosonic sector of the theory undergoes transitions between the $\nu = 1/2$ Laughlin state, the Mott insulator, and the superfluid.

In order to analyze whether the resulting transitions of the electron system are continuous, we must analyze the coupling of the bosons to the fermions and the gauge field. The effective theory takes the form

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_a + \mathcal{L}_{bf}, \quad (41)$$

where $\mathcal{L}_b$ is the action for the boson sector, $\mathcal{L}_f$ is the action for the f fermions, which fill a Fermi surface, and

$$\mathcal{L}_a = \frac{1}{g^2} (\nabla \times a)^2, \quad (42)$$

and $\mathcal{L}_{bf}$ contains direct boson-fermion couplings; g is a phenomenological parameter in the effective theory. First, we will consider the transitions at the mean-field level, where we ignore fluctuations of the emergent $U(1)$ gauge field a . In the absence of $\mathcal{L}_{bf}$, then, the boson critical point will be described by $\mathcal{L}_{N_f,k}$, for a suitable choice of N_f and k . Let us consider the possible effects of $\mathcal{L}_{bf}$. An operator O from the boson sector can couple to the particle-hole continuum of the f Fermi surface at low momenta. We can use the arguments of Ref. 61 to see whether such a coupling is relevant at the boson critical point. Integrating out the f fermions gives rise to a perturbation

$$v \int \frac{|\omega|}{q} |O(q, \omega)|^2, \quad (43)$$

for $\omega \ll q$. The most relevant operator is expected to be $O = b^\dagger b$, which has scaling dimension $3 - 1/\nu$ at the boson critical point. For this operator, v is therefore irrelevant at the boson critical point if $\nu > 2/3$, and relevant if $\nu < 2/3$.

From the large N_f expansion of $\mathcal{L}_{N_f,k}$,^{8,10}

$$\nu^{-1} = 1 + \frac{512\phi(1-2\phi)}{3\pi^2(1+\phi)^3} \frac{1}{N_f} + O(1/N_f^2), \quad (44)$$

where $\phi = (\theta/16)^2$ and $\theta = 2\pi/k$. By comparing the value of $\nu_{1,1/2}$ with the known 3D XY value, we conclude that the large N_f expansion is unreliable for $N_f = 1$. For $N_f = 2$, we expect the large N_f expansion to be more accurate, and we find that to $O(1/N_f^2)$, $\nu_{2,1/2} = 1.4182 > 2/3$. Thus in the absence of gauge fluctuations and for short-ranged interactions, we see that the direct symmetry-protected CFL to FL transition is continuous according to the leading order $1/N_f$ approximation. Since we don't have accurate estimates of $\nu_{1,3/2}$, we cannot conclude that the CFL to GMI transition is also continuous. The GMI to FL transition is continuous: there, the boson transition is in the 3D XY universality class, for which $\nu_{3DXY} > 2/3$. Since both the GMI-FL and the CFL - FL are continuous, in what follows, we will consider the possibility that the CFL to GMI transition is also continuous and study the phenomenology of such a transition, along with that of the CFL - FL transition.

In the presence of the long-ranged Coulomb interactions, there are several possibilities. If the bosonic sector flows to the new fixed points found in Ref. 10, then $\nu > 1$, and so v will be an irrelevant perturbation in all cases. Alternatively, if the long-ranged Coulomb interactions are marginally irrelevant, then scaling functions will receive logarithmic corrections.

A. Effect of gauge fluctuations

In Ref. 30, the effect of gauge fluctuations has been studied in the case of the gapless MI to FL transition, where the bosons are undergoing a 3D XY transition and

FIG. 3: The red points on the horizontal and vertical axes indicate the three stable phases, while the blue points at the origin and the diagonals indicate the unstable critical fixed points.

time-reversal symmetry is preserved. The phenomenology here is similar; the main non-trivial differences arise from the non-zero Hall conductivity of the boson sector at the critical points. We note that the analysis of the gauge fluctuations relies on results obtained from RPA calculations, whose validity in large N expansions is discussed in Refs. 57–59,62.

First, consider integrating out all matter fields. The resulting action for the gauge field, to quadratic order in a , is

$$S_{eff}[a] = \frac{1}{2} \int a_\mu (\Pi_{\mu\nu}^f + \Pi_{\mu\nu}^b) a_\nu. \quad (45)$$

Since the boson critical point has dynamical scaling exponent $z = 1$ and rotation invariance, the general form of the boson polarization tensor $\Pi_{\mu\nu}^b$ at $T = 0$ is

$$\Pi_{\mu\nu}^b = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \Pi_+^b(k) + \epsilon^{\mu\nu\lambda} k_\lambda \Pi_-^b(k), \quad (46)$$

where Π_+^b and Π_-^b are functions of $k \equiv \sqrt{q^2 + \omega^2}$, and k_μ is a three-component vector $(\omega, \vec{q})$, where $\vec{q}$ is the wave vector. At the boson critical point, $\Pi_-^b(k) \sim \mathcal{O}(k^0)$ and $\Pi_+^b(k) \sim k$.⁸

In Coulomb gauge, $\partial \cdot a = 0$, the gauge field consists of $(a_0, a_\perp)$, where $a_\perp$ is the component transverse to the wave vector. The polarization tensors can then be written as 2×2 matrices for the time and transverse components:⁸¹

$$\Pi^b = \begin{pmatrix} \frac{q^2}{k} f_1(\xi k) & -qg(\xi k) \\ qg(\xi k) & k f_2(\xi k) \end{pmatrix}, \quad (47)$$

where ξ is the correlation length of the boson critical point. On the FQH and Mott insulating side of the critical points,

$$\begin{aligned} f_i(x \rightarrow \infty) &\rightarrow f_0, \\ f_i(x \rightarrow 0) &\rightarrow x, \\ g(x \rightarrow \infty) &\rightarrow \sigma_{xy}^c, \end{aligned} \quad (48)$$

where f_0 is a constant, and σ_{xy}^c is a constant setting the Hall conductivity at the critical point.

In the FQH state, $g(x \rightarrow 0) \rightarrow \frac{1}{2}$; in the Mott insulating state, $g(x \rightarrow 0) \rightarrow 0$. On the superfluid side of the critical points,

$$\begin{aligned} f_i(x \rightarrow \infty) &\rightarrow f_0, \\ f_i(x \rightarrow 0) &\rightarrow 1/x, \\ g(x \rightarrow \infty) &\rightarrow \sigma_{xy}^c, \\ g(x \rightarrow 0) &\rightarrow \sigma_{xy}^s, \end{aligned} \quad (49)$$

where we allow for the possibility of a non-zero Hall conductivity σ_{xy}^s in the superfluid phase. These limiting behaviors in the $x \rightarrow 0$ limit are standard responses for FQH, Mott insulating, and superfluid states, while the limit behaviors in the $x \rightarrow \infty$ limit are the properties of the critical points.^{8,14}

The fermion polarization Π_f is,³ for $q \ll k_F$ and $\omega \ll v_F q$,

$$\Pi_f = \begin{pmatrix} \kappa_f & \Pi_{f;xy}(q, \omega) \\ \Pi_{f;yx}(q, \omega) & \frac{k_0 |\omega|}{q} + \chi_d q^2 \end{pmatrix}, \quad (50)$$

where κ_f , χ_d , and k_0 are constants. We note that in general, since time-reversal symmetry is broken, the f fermions may have a Hall conductance σ_{xy}^f , so Π_f can also have off-diagonal components: $\Pi_{f;xy}(q, \omega = 0) = \Pi_{f;yx}^*(-q, \omega = 0) = -q\sigma_{xy}^f$.

The arguments of Ref. 30 imply that the gauge fluctuations also do not modify the boson critical point. This can be understood by observing that the gauge fluctuations, using the RPA propagator from (50), only lead to analytic corrections to the boson self-energy at low ω , q , and therefore do not modify the critical singularities coming from the boson sector. Alternatively, the ω/q term acts like a Higgs mass for the transverse gauge fluctuations in the boson sector, since ω and q scale the same way at the boson critical point.

Using (27), (47), and (50), we can obtain the compressibility κ_e at zero temperature close to the transition. The inverse electron polarizability satisfies

$$\Pi_e^{-1}(q, \omega = 0, T = 0) = \begin{pmatrix} \frac{\chi_d q^2}{|\Pi_f|} + \frac{q f_2}{|\Pi_b|} & \frac{q \sigma_{xy}^f}{|\Pi_f|} + \frac{q g}{|\Pi_b|} \\ -\frac{q \sigma_{xy}^f}{|\Pi_f|} - \frac{q g}{|\Pi_b|} & \frac{\kappa_f}{|\Pi_f|} + \frac{q f_1}{|\Pi_b|} \end{pmatrix}, \quad (51)$$

where $|\Pi_f| \propto q^2$ and $|\Pi_b| = q^2(f_1 f_2 + g^2)$ are the determinants of Π_f and Π_b . We are interested in the case where q and $q\xi$ are small, while ξ diverges. On the CFL side, in this limit $|\Pi_b| \propto q^2$, $f_i \sim q\xi$, and $g \sim \text{const.}$. On the LFL side, $f_i \sim 1/q\xi$, $g \sim \text{const.}$, and $|\Pi_b| \sim 1/\xi^2$. Therefore for small q and $q\xi$, on the CFL and LFL side, the dominant terms are

$$\Pi_e^{-1}(q, \omega = 0, T = 0) \approx \begin{pmatrix} \frac{q f_2}{|\Pi_b|} & \frac{\alpha}{\kappa_f} \\ -\frac{\alpha}{\kappa_f} & \frac{q f_1}{|\Pi_f|} \end{pmatrix}, \quad (52)$$

for some constant α , and so

$$\Pi_{e;00}(q, \omega = 0, T = 0) \sim \left(\frac{qf_2}{|\Pi_b|} + \frac{\alpha^2 |\Pi_f|}{q^2 \kappa_f} \right)^{-1}. \quad (53)$$

As $\xi \rightarrow \infty$, the first term dominates and we get $\kappa_e = \lim_{q \rightarrow 0} |\Pi_b/qf_2| \sim 1/\xi$.

Therefore, as the critical point is approached from either the CFL side or the LFL side, we find that the compressibility κ_e at zero temperature close to the transition is dominated by the correlation length of the boson sector:

$$\kappa_e \sim 1/\xi. \quad (54)$$

On the LFL side, $1/\xi \sim \rho_s$, while on the CFL side, $1/\xi \sim \Delta_b$, where Δ_b is the gap of the boson sector and ρ_s is the superfluid density.

B. Transport and thermodynamics at the critical points

From the Ioffe-Larkin composition rule (27), the inverse polarizability of the electrons is given by the sum of the inverse polarizabilities of f and b . The bosonic sector has a Hall conductance, $\Pi_{b,0i} \neq 0$ and therefore the inverse compressibilities and resistivities of b and f do not separately add to yield the electron inverse compressibility and resistivity. This implies that while the boson resistivity exhibits a universal jump,⁶³ there is also a jump in the resistivity of the electron system; in contrast to the time-reversal invariant case, the resistivity jump appears to be non-universal because of the off-diagonal terms in the polarization tensors.

At finite temperature and zero frequency, by scaling we have

$$\Pi_{b;\mu\nu}(q, \omega = 0, T) = T f_{\mu\nu}(q/T), \quad (55)$$

where $f_{\mu\nu}$ is a scaling function satisfying

$$0 < \lim_{q \rightarrow 0} |f_{\mu\nu}(q/T)| < \infty. \quad (56)$$

(56) can be understood by observing that the polarization function $\Pi_{\mu\nu}(q, \omega, T = 0)$ vanishes as $q, \omega \rightarrow 0$; at finite temperature, the current-current correlation functions should be more short-ranged in real space, and therefore should continue to vanish as $q, \omega \rightarrow 0$. Furthermore, generically $|f_{\mu\nu}(0)| > 0$ as there is no symmetry forcing it to be zero.

The fermion polarization at $\omega = 0$ and small q is temperature independent and given by (50).

Using (27), (56), (55) (50), the inverse electron polarizability is:

$$\Pi_e^{-1}(q, \omega = 0, T) = \frac{1}{T} f^{-1} + \frac{1}{|\Pi_f|} \begin{pmatrix} \chi_d q^2 & q\sigma_{xy}^f \\ -q\sigma_{xy}^f & \kappa_f \end{pmatrix}, \quad (57)$$

where the fermion polarization determinant $|\Pi_f| \propto q^2$. In the limit $q/T \rightarrow 0$ and $T \rightarrow 0$, the dominant terms are:

$$\Pi_e^{-1} \sim \begin{pmatrix} \frac{1}{T} (f^{-1})_{00} & \frac{q\sigma_{xy}^f}{|\Pi_f|} \\ \frac{-q\sigma_{xy}^f}{|\Pi_f|} & \frac{\kappa_f}{|\Pi_f|} \end{pmatrix}. \quad (58)$$

Therefore:

$$\Pi_{e;00} \sim \left(\frac{1}{T} (f^{-1})_{00} + \frac{q^2 (\sigma_{xy}^f)^2}{\kappa_f |\Pi_f|} \right)^{-1}. \quad (59)$$

The first term dominates at low temperatures, from which we find that the compressibility is $\kappa_e(T) \equiv \lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} \Pi_{e;00}(q, \omega, T)$,

$$\kappa_e \sim T + \mathcal{O}(T^2). \quad (60)$$

A remarkable prediction of the above scaling of the compressibility is that the critical point between the composite Fermi liquid and Landau Fermi liquid is incompressible at zero temperature, even though both the CFL and FL are compressible states.

The result that the critical point is incompressible despite the compressibility of the neighboring phases is quite surprising. Using (27), this can be seen to be a direct consequence of the divergent diamagnetic susceptibility of the boson sector at the critical point (ie the fact that $\Pi_{\perp\perp} \propto q$). Intuitively, we understand the incompressibility as follows. The mechanism for the compressibility of the composite Fermi liquid state follows from the Hall conductance of the boson sector, which implies that the gapless transverse gauge field fluctuations are tied to the boson number. However when the bosons have a divergent diamagnetic susceptibility, it is no longer possible to add a boson by adding a long-wavelength magnetic field, because doing so now costs an energy that grows with $1/q$. Therefore, the mechanism for the compressibility of the composite Fermi liquid state is destroyed as the critical point is approached, leading to an incompressible critical point. We would like to note that an incompressible critical point separating two compressible states is quite unusual, but there is a very simple example in 2+1 dimensions. When the chemical potential is at the Dirac point, the system is incompressible since the density of states vanishes linearly with energy, however as the chemical potential is tuned away from the Dirac point, a compressible Fermi liquid state is obtained.

At the critical point, the gauge propagator in RPA is equivalent to that considered in the Halperin-Lee-Read theory of CFL with long-range interactions. Since the specific heat of the boson sector is $C_v \sim T^2$, we therefore expect the specific heat to be dominated by the contribution of the fermion-gauge system:

$$C_v \sim T \ln\left(\frac{1}{T}\right). \quad (61)$$

The expectations for transport are very similar to those of Ref. 30,64.

C. Crossover out of criticality

An important signature of this class of transitions is that while the boson critical point has a quantum critical region associated with some crossover temperature scale $T^* \sim 1/\xi$, the gauge propagator is not significantly modified until a lower temperature scale, $T^{**} \sim \frac{1}{\xi^2 c k_0}$, where c is the characteristic velocity of excitations at the boson critical point (previously we had set $c = 1$), and k_0 is defined in (50).³⁰ Recall $\xi \sim 1/\Delta_b$, where Δ_b is the boson gap when the bosons are in the gapped FQH or MI states, and $\xi \sim 1/\rho_s$ when the bosons are in the superfluid state. This implies the existence of two finite temperature crossover regimes, each with distinctive properties.

In the crossover regime $T^{**} < T < T^*$, the gauge propagator still takes the form that it did in the quantum critical regime of the boson critical point. On the composite Fermi liquid side, this means that the theory formally is similar to the composite Fermi liquid theory with long-range Coulomb interactions and we expect the physics of this theory to dominate this crossover regime. In such a situation, the composite fermions form a “marginal” Fermi liquid, in the sense that the quasiparticle residue vanishes logarithmically at low energies (see eq. 64) and the quasiparticles are marginally unstable. If the problem does have long-range Coulomb interactions to begin with, there will be no significant modification of the physics on the CFL side as one crosses T^{**} . On the other hand, if the Coulomb interaction is screened by a gate, then T^{**} will mark a crossover between shorter-ranged and effectively long-ranged interactions. The fact that this crossover scale T^{**} can effectively be made to disappear and reappear with a screening gate is a non-trivial prediction of this theory.

The compressibility below T^* is dominated by the bosons and will be a temperature-independent constant $\kappa_e \sim 1/\xi$.

The crossover behaviors on the LFL and GMI sides of these transitions are spinless analogs of the cases studied in Ref. 30; we will briefly review a few of the key features here; the results are summarized in Figs. 4 and 5 respectively.

On the GMI side, in the second quantum critical regime, $T^{**} < T < T^*$, the specific heat is still behaving as

$$C_v \sim T \ln T. \quad (62)$$

The system can be viewed as a marginal Fermi liquid of the neutral f excitations, although the single-particle properties of f are not gauge invariant and cannot be directly measured. The thermal conductivity in this regime behaves as $K/T \sim 1/T$.

At the lowest temperatures away from the critical point, $T < T^{**}$, for short-range interactions the system has a specific heat

$$C_v \sim T^{2/3}, \quad (63)$$

FIG. 4: Schematic phase diagram showing finite temperature crossover regimes between the CFL and LFL. For $T > T^*$, the bosons are in their quantum critical regime. For $T^{**} < T < T^*$, on the CFL side the system is a marginal Fermi liquid of composite fermions; on the LFL side, the system is a marginal Fermi liquid of electrons.

and the electron single-particle Green’s function decays exponentially in frequency, indicating an exponential suppression of the electronic density of states. In this regime, the thermal conductivity behaves as $K/T \sim 1/T^{2/3}$.

On the Landau FL side, at $T = 0$ and $\frac{1}{\xi^2 c k_0} < \omega < 1/\xi$ the electron Green’s function has the marginal Fermi liquid form,⁶⁵ with a self-energy

$$\Sigma(\omega) \sim \omega \ln i\omega. \quad (64)$$

In the crossover temperature regime $T^{**} < T < T^*$, the specific heat behaves as

$$C_v \sim T \ln T, \quad (65)$$

while the thermal conductivity is

$$K/T \sim T. \quad (66)$$

The compressibility below T^* is dominated by the bosons and is a temperature-independent constant, $\kappa_e \sim 1/\xi$. As the transition is approached from the Landau FL side, the quasiparticle effective masses and Landau parameters are expected to diverge in the same manner as described in Ref. 30.

VI. ONSET OF FERMION PAIRING

Another possible neighbor of the composite Fermi liquid state of the half-filled Landau level is a paired superconducting state of the composite fermions. For example, when the composite fermions are paired into a topologically non-trivial $p_x + ip_y$ superconducting state, the result is a description of the non-Abelian Moore-Read Pfaffian state, which is a candidate state for the plateau

FIG. 5: Schematic phase diagram showing finite temperature crossover regimes between the CFL and GMI. For $T > T^*$, the bosons are in their quantum critical regime. For $T^{**} < T < T^*$, on the CFL side the system is a marginal Fermi liquid of composite fermions; on the GMI side, the system is a marginal Fermi liquid of the emergent f -fermions.

at $\nu = 5/2$ in GaAs quantum wells^{15,66}. The properties of the transition from CFL to the Moore-Read state are the subject of ongoing investigations by others.

In the context of this paper, it is possible in principle that the pairing transition of the composite fermions occurs before the transition out of the composite Fermi liquid state and into the gapless Mott insulator or the Landau Fermi liquid, depending on the interactions between the composite fermions. In this case, the Landau Fermi liquid is replaced by a paired electronic superconductor, while the gapless Mott insulator is replaced by a different topologically ordered state, consisting of the fermion condensate coupled to an emergent Z_2 gauge field. For example, the Moore-Read Pfaffian has a topological ground state degeneracy of six on the torus, and the gapless chiral edge theory consists of the Ising $\times$ U(1) chiral conformal field theory (CFT) with central charge $c = 3/2$. When the fermions of the gapless Mott insulator are paired into a topologically non-trivial (weak-paired) $p_x + ip_y$ state, we instead obtain a topological phase with a torus ground state degeneracy of three, and a $c = 1/2$ Majorana fermion edge state; such a state has the “Ising” topological order,⁶⁷ and is well-known as the non-Abelian state in the “B phase” of Kitaev’s honeycomb model.²⁸

While the superconducting state cannot exist in the presence of a strong magnetic field, these transitions may be induced by starting with the half-filled Landau level and turning on a periodic potential with 2π flux per plaquette. In the limit where the superconducting state is obtained, the effect of the magnetic field disappears because the electrons are confined to paths along the periodic potential and therefore do not feel a net magnetic field, as the phase through all closed paths will be a multiple of 2π . In these paired states, the U(1) emergent gauge symmetry is broken to Z_2 . Since the gauge fluctuations are gapped, at the critical point they can be integrated out to obtain short-range interactions among the

FIG. 6: Schematic phase diagram of the electron system when the fermions are paired into a topologically non-trivial $p_x + ip_y$ state. The transitions are driven by the bosonic sector undergoing $\nu = 1/2$ Laughlin FQH - MI - SF transitions. Since the fermions are gapped, the critical theories are the same as for the boson system, given by eq. (40).

bosons. At the boson critical points, these short-ranged interactions are irrelevant. Therefore, the critical theories separating these gapped states are the same critical theories (40) that separate the $\nu = 1/2$ Laughlin state, the Mott insulator, and the superfluid of bosons. Furthermore, since the gauge fluctuations are gapped, one obtains a single crossover temperature scale associated with the quantum critical points. Interestingly, since the fermions remain in a gapped state throughout, the electron operator is not a scaling operator at these quantum phase transitions and has exponentially decaying correlations at the quantum critical points.

The existence of a continuous bandwidth-tuned transition between the $\nu = 1/2$ Moore-Read Pfaffian state and the Kitaev B (Ising) phase, and the symmetry-protected quantum critical point between the $\nu = 1/2$ Moore-Read Pfaffian and the $p_x + ip_y$ electronic superconductor are highly non-trivial predictions of the theories presented here.

VII. DISCUSSION

In this paper, we developed a theory of transitions out of the composite Fermi liquid and ultimately into a Landau Fermi liquid. These transitions can be induced by tuning the bandwidth of the partially filled band. In the Landau level problem, this can be done by using an external periodic potential; if the composite Fermi liquid is instead realized in a partially filled Chern band, the bandwidth can be tuned with pressure.

We found that generically, the transition to the Landau Fermi liquid occurs through an intermediate gapless Mott insulating phase, with a Fermi surface of neutral fermions. In the presence of certain spatial symmetries, such as inversion, we found a direct continuous transition between the composite Fermi liquid and the Landau Fermi liquid, providing a highly non-trivial example of

a quantum critical point between a fractionalized non-Fermi liquid and a conventional Fermi liquid.

In order to establish the above results, we had to depart from the conventional understanding of the composite Fermi liquid, where the electron is viewed as a fermion attached to flux quanta. Instead, the electron fractionalizes into a boson b and a fermion f . When b forms a $\nu = 1/2$ incompressible Laughlin state, at low energies b is effectively created by inserting 2 flux quanta, so the electron can be viewed as 2 flux quanta attached to the fermions, which leads to the composite Fermi liquid description. Another natural possibility is that depending on the nature of the interactions and the bandwidth, b can form a Mott insulator to minimize the interaction energy. In this case, the state of the electrons is a Mott insulator, which is gapless when f has a Fermi surface. Finally, when the bandwidth is large enough, the bosons will condense into a superfluid, leading to a description of the Landau Fermi liquid.

The direct CFL – FL transition found here is not stable to disorder, as it relies on the presence of spatial symmetries that can protect a pair of Dirac cones. Nevertheless, in the presence of weak disorder and small but finite $T \neq 0$, the properties of the direct CFL – FL quantum critical point can determine the physics. Despite the fact that weak disorder will ultimately render the direct transition unstable at the lowest temperatures, the relevance of this depends on the relative strength of disorder, W , and temperature T . For $W \ll T$, this ultimate instability is of no measurable consequence.

The situation is somewhat less clear for the CFL - GMI transition, because there is currently no reliable estimate for its correlation length exponent ν . While the boson FQH - MI transition does not require any spatial symmetries, it is expected to be unstable to disorder if $\nu < 1$; in this case, the theory will flow to a different critical point, with $\nu \geq 1$, that is stable to disorder. Whether the putative clean critical point or the disordered one is the relevant one for describing the physics depends again on W/T .

Interestingly, the currently understood direct transition¹⁴ between the Laughlin state and the superfluid exists only for the $\nu = 1/2$ Laughlin state.⁸² This implies that the direct continuous transition be-

tween the CFL and the Fermi liquid can currently only be understood at half-filling. For other more general CFL states at other filling fractions, the route to the Landau Fermi liquid appears to always be separated by an intervening gapless Mott insulator, although there can be a multi-critical point directly separating the CFL and LFL. This difference in the properties between $\nu = 1/2$ and $\nu = 1/2m$ for $m > 1$ appears to form a counterexample to the law of corresponding states⁷ that was suggested for previously understood FQH transitions.

The experimentally observable phenomenology of the transitions includes the existence of two crossover temperature scales and resistivity jumps at the transition, and a vanishing compressibility at the critical points, similar to the $U(1)$ spin liquid Mott transition³⁰. We find that the composite Fermi liquid provides another experimentally promising venue where the physics of such slave-particle gauge theory transitions in the presence of a Fermi surface, and novel exotic fractionalized phases, can be studied.

The theory developed here assumes that the system does not hit a first-order transition out of the composite Fermi liquid. The validity of this assumption depends on microscopic details of the interactions. If there is a first-order transition, then both disorder and/or long-range interactions can change the first-order transition to a continuous one.⁶⁸ Such continuous transitions would exhibit completely different physics from that studied in this paper and would require a completely different theory.

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- ⁷⁷ By finite compressibility, it is meant that the zero frequency density-density correlation function $\chi_{\rho\rho}(q)$ is finite as the wave vector $q \rightarrow 0$ for short-range interactions. For long-range interactions $V(q)$, finite compressibility means that $\chi_{\rho\rho}(q) \sim 1/V(q)$.
- ⁷⁸ An orthogonal set of shortcomings of a more microscopic nature is addressed in eg Ref. 70–75.
- ⁷⁹ Note that integrating out $\tilde{a}$ this way is inconsistent on a genus $g \geq 1$ manifold and leads to a gauge anomaly associated with large gauge transformations, as there are multiple topological sectors that must be properly considered.^{42–44}
- ⁸⁰ We would like to thank Sri Raghu for useful conversations regarding this point
- ⁸¹ To see this, it is convenient to set $\vec{q} = q\hat{x}$, so that $\Pi_{0\perp} = \Pi_{0y}$, and $\Pi_{\perp\perp} = \Pi_{yy}$, where $\perp$ indicates the direction transverse to the momentum.
- ⁸² For other boson FQH states, the direct transition to the superfluid appears to be highly multi-critical.