Skyrmionic state and stable half-quantum vortices in chiral p-wave superconductors

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Skyrmionic State and Stable Half-Quanta Vortices in Chiral p-wave Superconductors

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Observability of half-quanta vortices and Skyrmions in p-wave superconductors is an outstanding open question. Under the most common conditions, fractional flux vortices vortices are not thermodynamically stable in bulk samples. Here we show that in chiral p-wave superconductors, there is a regime where, in contrast lattices of integer flux vortices are not thermodynamically stable. Instead Skyrmions made of spatially separated half-quanta vortices are the topological defects produced by an applied external field.

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Higher broken symmetries in p-wave superconductors inspired long-standing interest to realize topological defects more complicated than vortices. Much of the early discussions of various complex topological defects were in context of superfluid ^{3}He.[1] Recently the attention to these questions raised dramatically in connection with superconductors which are argued to have p-wave pairing, such as Sr_2RuO_4 . The highly interesting possibility there, is connected with half-quantum vortices.[2–8] Their statistics is non-Abelian and they could potentially be used for quantum computations.[9] Other kind of topological defects discussed in connection with spin-triplet superconductors are Skyrmions[10] and Hopfions.[11] In superconducting materials, creation of these topological excitations is highly nontrivial. Superconducting components are coupled by a gauge field and there are also symmetry-reducing inter-component interactions. As a consequence fractional vortices have logarithmically or linearly divergent energies (see *e.g.* Ref. 8), while integer flux vortices have finite energy per unit length. Consequently, under usual conditions, half-quanta vortices are thermodynamically unstable in bulk systems. It was argued that complex setups, such as mesoscopic samples, are needed for their creation.[2,8,12] Recently it was claimed that a half-quantum vortex was observed in mesoscopic sample of Sr_2RuO_4.[2] Other proposed routes to observe fractional vortices, invoke (i) thermal deconfinement,[3,6,13] (ii) potential materials with strongly reduced spin stiffness,[4] (iii) regimes very close to upper critical magnetic field, where gauge-field mediated half-quanta vortex confinement is weak.[5] In some more general systems it was shown that fractional vortices could be thermodynamically stable near boundaries.[14] Today the conditions under which half-quanta vortices and Skyrmions[10] could be experimentally created in bulk superconductors still remains an outstanding open question.

In this work we investigate the magnetic response of the Ginzburg-Landau model widely applied to Sr_2RuO_4.[15,16] Our considerations apply to two-dimensional systems or three-dimensional problems with translation invariance along the z-direction. Then the free energy density reads

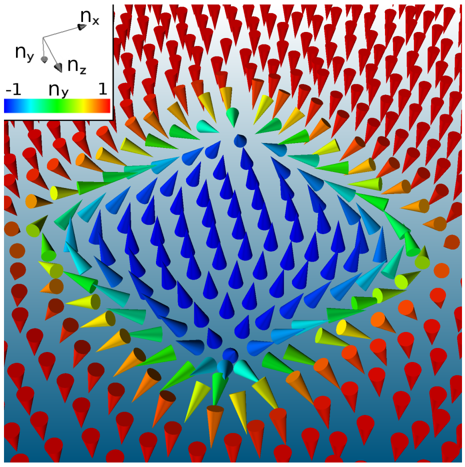

Figure 1. (Color on-line) – Numerically calculated texture of the pseudo-spin vector for a Skyrmion carrying with a topological charge $\mathcal{Q}=2$. As can be seen in the picture the skyrmionic topological charge density is confined in a closed domain-wall.

$$\mathcal{F}(\psi_a,\boldsymbol{A}) = |\nabla\times\boldsymbol{A}|^2 \tag{1a}$$
$$+|D_x\psi_1|^2+|D_y\psi_2|^2+\gamma|D_y\psi_1|^2+\gamma|D_x\psi_2|^2$$
$$+2\gamma\mathrm{Re}\left[(D_x\psi_1)^*D_y\psi_2+(D_y\psi_1)^*D_x\psi_2\right] \tag{1b}$$
$$+(2\gamma-1)|\psi_1|^2|\psi_2|^2+\sum_{a=1,2}-|\psi_a|^2+\frac{1}{2}|\psi_a|^4 \tag{1c}$$
$$+\gamma|\psi_1|^2|\psi_2|^2\cos(2(\varphi_2-\varphi_1))\,. \tag{1d}$$

The different components of the order parameter are denoted $\psi_{1,2}=|\psi_{1,2}|\mathrm{e}^{i\varphi_{1,2}}$; $\boldsymbol{D}=\nabla+ie\boldsymbol{A}$. The p-wave state is described here by a doublet of complex fields subjected to the the following symmetry breaking coupling : $\mathrm{Re}\left(\psi_1^{*\,2}\psi_2^2\right)=|\psi_1|^2|\psi_2|^2\cos(2(\varphi_2-\varphi_1))$. The ground state breaks the $U(1)\times\mathbb{Z}_2$ symmetry, since the ground state phase difference is either $\pi/2$ or $3\pi/2$. Gradient terms (1b) make this model clearly anisotropic in the xy-plane. The coefficient γ, controlling the anisotropy, should be $\gamma>1/3$ when specially considering Sr_2RuO_4 , according to.[15] The coupling constant e is a convenient quantity to parametrize the penetration depth of the magnetic field. The discrete $\mathbb{Z}_2$ symmetry dictates that

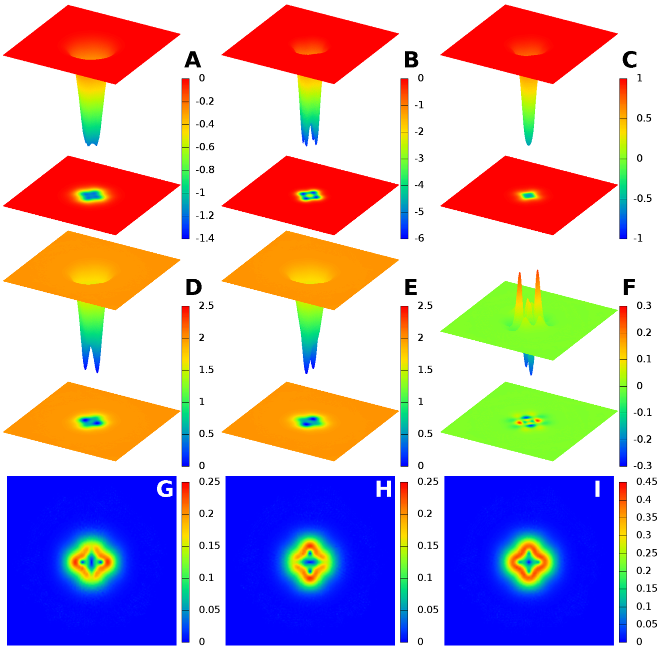

Figure 2. (Color on-line) – A thermodynamically stable Skyrmion carrying two flux quanta, with $e = 0.8$ and $\gamma = 0.5$. Displayed quantities are, magnetic flux (**A**), the (inverted) energy density (**B**) and the sine of the phase difference $\sin(\varphi_2 - \varphi_1)$ (**C**). On the second line, the densities of superconducting order parameter components $|\psi_1|^2$ (**D**), $|\psi_2|^2$ (**E**), and the 'doubled phase difference' $\mathrm{Im}(\psi_1^{*\,2}\psi_2^2)$ (**F**). Panels (**G**) and (*resp.* **H**) on the third line are the supercurrents associated with each component ψ_1 (*resp.* ψ_2) of the order parameter.[17] The last panel (**I**) shows the total supercurrent.

the system allows domain-wall solutions interpolating between two regions with different phase-locking. Such domain-walls are energetically expensive and thus not intrinsically stable. It was suggested that they could be observable if pinned by crystalline defects.[18] Also domain-walls formed as dynamic excitations inside vortex lattices were studies extensively in.[19] They could be experimentally observable in these setups since they pin half-quanta vortices.[18,19]

Returning to the discussion of vortices one can observe that the system (1) has $\mathrm{U}(1) \times \mathbb{Z}_2$ broken symmetry. Thus a single fractional vortex has linearly diverging energy and thus is not thermodynamically stable.[8] Since both components have similar ground state density, the fractional vortex excitation are half-quantum vortices, *i.e.* they carry a half of magnetic flux quantum. Also from this broken symmetry, the existence of skyrmionic excitations would not follow. The previous works required higher broken symmetry for the existence of Skyrmions.[10] However we show below that there is a considerable window of parameters where the system (1) possesses what we term as a "skyrmionic phase". In that phase, mostly because of favorable competition of field gradients, potential and magnetic energies, the system does have *thermodynamically stable* Skyrmions while ordinary integer flux vortex lattices are *not* thermodynamically stable. These Skyrmions are bound states of spatially separated half-quanta vortices, connected by domain-walls. Half-quanta vortices are linearly confined into integer vortices in a bulk sample because of the terms $|\psi_1|^2|\psi_2|^2\cos(2(\varphi_2 - \varphi_1))$. However on a (closed) domain-wall, a composite vortex should split along this wall, since the above-mentioned term has there, unfavorable values of the phase difference. Indeed, such deconfining allows to reduce energetically unfavorable values of the phase differences. Because of this vortex splitting and resulting repulsive interactions, vortices trapped on domain wall can prevent the collapse of a closed domain-wall. The main result of this paper is that we show that these objects are characterized by an integer-valued skyrmionic topological charge and that they can be energetically cheaper than vortices. Such a Skyrmion is displayed in Fig. 1, as a texture of a pseudo-spin vector field defined later on.

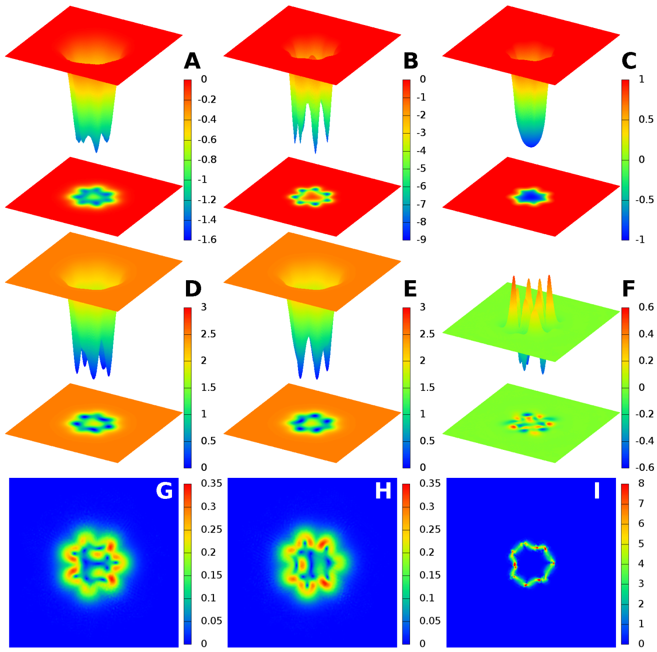

Figure 3. (Color on-line) – A Skyrmion carrying five flux quanta, with $e = 0.8$ and $\gamma = 0.4$. Displayed quantities are the same as in Fig. 2, except panel (**I**) showing the gradient of the phase difference $\nabla(\varphi_{12})$, which is non zero at the domain-wall. The Skyrmion consists of ten spatially separated half-quanta vortices. It assumes a complicated non-symmetric structure due to a competition of a preferred geometry of a Skyrmion with the anisotropies (1b).

We investigated structures carrying N flux quanta (*i.e.* with each phase winding $\oint \nabla\varphi_a = 2\pi N$) as functions of the gauge coupling e and the anisotropy parameter γ. Ground states, carrying a given number of magnetic flux quanta, are computed numerically by minimizing the energy within a finite element framework provided by the Freefem++ library.[20] See technical details in supplementary material.[17]

When the penetration length is sufficiently large (*i.e.* at small values of the coupling constant e), the system indeed forms ordinary Abrikosov vortices in external field. On the other hand for sufficiently large e the system behaves as a type-I superconductor. However there is a regime in a wide range of intermediate coupling constants e, where integer flux vortices are more expensive than bound states of spatially separated half-quanta vortices connected by closed domain-wall. Such configurations carrying different number of flux quanta are given in Figures 2, 3 and 4. The clearly visible preferred directions for supercurrents originate in the anisotropies (1b). The cores in different components do not coincide in space. This means fractionalization of vortices in this state. Each of the split cores carries a half of a flux quantum (for detailed calculations of fractional vortices flux quantization, see *e.g.* Ref. 8).

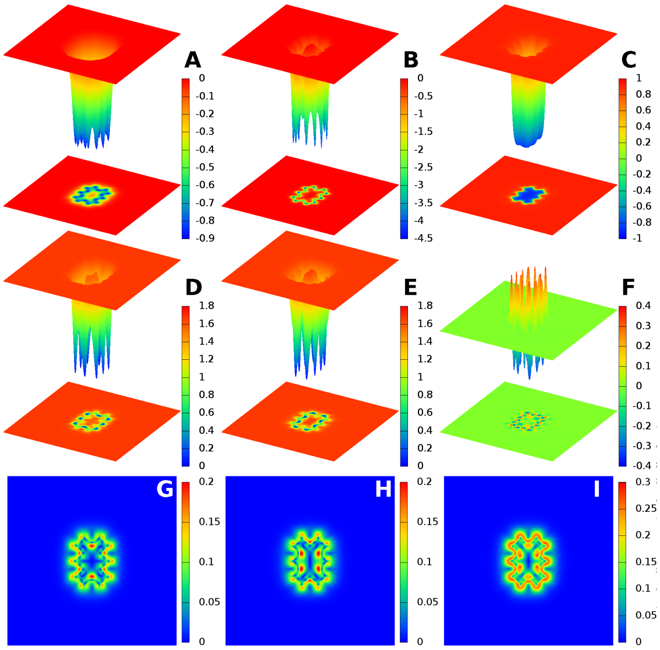

Figure 4. (Color on-line) – A Skyrmion with $N = 8$, $e = 0.6$ and in the case of higher anisotropy $\gamma = 0.6$. Displayed quantities are the same as in Fig. 2.

The configurations found here are actually Skyrmions, although it may not be obvious from the Figures 2, 3 and 4. To prove that the solutions are Skyrmions the two-component model (1) is mapped to an anisotropic non-linear σ-model.[21] In that mapping the superconducting condensates are projected on the Pauli matrices $\boldsymbol{\sigma}$ allowing to define the pseudo-spin vector $\mathbf{n}$:

$$\mathbf{n} \equiv (n_x, n_y, n_z) = \frac{\Psi^\dagger \boldsymbol{\sigma} \Psi}{\Psi^\dagger \Psi} \qquad \text{where} \qquad \Psi^\dagger = (\psi_1^*, \psi_2^*)\,. \tag{2}$$

The target space being a sphere, together with the one-point compactification of the plane defines the map $\mathbf{n} : S^2 \to S^2$. Such maps are classified by the homotopy class $\pi_2(S^2) \in \mathbb{Z}$, so there exists an integer valued topological charge

$$\mathcal{Q}(\mathbf{n}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathbf{n} \cdot \partial_x \mathbf{n} \times \partial_y \mathbf{n} \; dx dy\,. \tag{3}$$

For a Skyrmion, $\mathcal{Q} = N$, while $\mathcal{Q} = 0$ for ordinary vortices. The terms in (1c) and (1d), break the O(3) symmetry of the pseudo-spin $\mathbf{n}$ down to $\mathbb{Z}_2$. In a non-linear σ-model, such anisotropy would undermine stability of the Skyrmions. However this collapse does not occur in the Ginzburg-Landau model, because of the demonstrated below behaviour of gradient energy.

The numerically computed topological charge (3) is found to be integer (with a negligible relative error of the order 10^{-5}, due to the discretization) for the closed domain-wall/vortex systems which are therefore Skyrmions. The solutions shown in Figures 2, 3 and 4 have skyrmionic topological charge $\mathcal{Q} = 2$, $\mathcal{Q} = 5$, $\mathcal{Q} = 8$ correspondingly. The terminology Skyrmion is more intuitively obvious when the solutions are represented in terms of the pseudo-spin vector field $\mathbf{n}$, as in Fig. 1. However unlike Skyrmions in non-linear σ-model, here the skyrmionic topological charge density is mostly concentrated on the half-quanta vortices and on the domain-wall.

The main result of this work is that Skyrmions of the above type (and thus half-quanta vortices) can be *less energetic* than integer-flux ordinary vortices and *thermodynamically stable*, in the chiral p-wave superconductors. The critical external magnetic field H_{c1} for formation of a flux-carrying topological defect is determined by the condition where Gibbs free energy $G = E_d - 2\int \mathbf{B} \cdot \mathbf{H}_e \; dxdy$ becomes negative. Here E_d and $\mathbf{B}$ are the energy and magnetic field of the defect. $\mathbf{H}_e$ denotes the applied field. Thus $H_{c1} = E_d/2\Phi$ where Φ is the magnetic flux produced by the defect. The defects are thermodynamically stable if the critical external magnetic field's energy density H_{c1}^2 is smaller than the condensation energy. We investigated the energy dependence of the Skyrmions on the number of enclosed flux quanta N. The energy of an integer flux vortex is used as a reference point. As shown in Fig. 5 panels (**a**) and (**b**), for low N, the energy depends non-monotonically on N. This is because the preferred symmetry of small N configurations in some cases is in strong conflict with the anisotropies of the model. In the large-N limit the energy per flux quantum gradually tends to some value. The main point here is that the energy per flux quantum for Skyrmions is in certain cases smaller than that of vortices. This signals instability of vortex lattices with respect to Skyrmion formation.

Next, the thermodynamical stability of Skyrmions is investigated. Results for $N = 5$ quanta are reported as a characteristic example, in Fig. 5 (**c**). We find that there are three regimes on the resulting phase diagram. When penetration length is large (*i.e.* low e), the system shows usual type-II superconductivity. When penetration length is small, the system is a type-I superconductor. For intermediate values of the penetration length, depending on the underlying anisotropies γ, the external

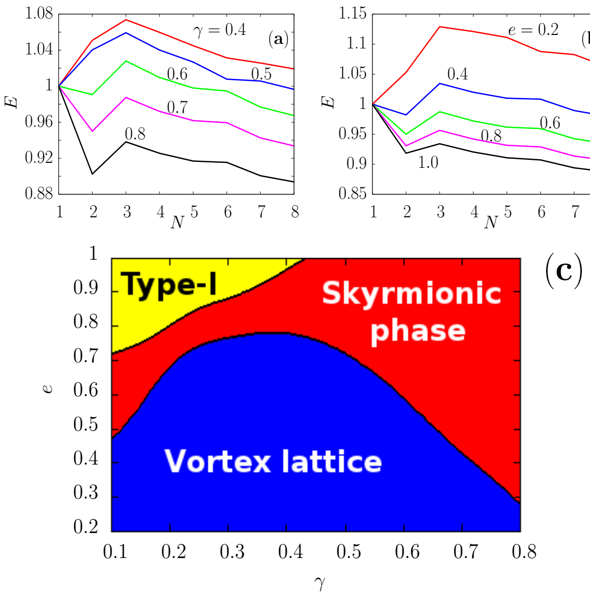

Figure 5. (Color on-line) – Upper panels show the dependence of the energy per flux quantum for Skyrmions of different topological charges $\mathcal{Q}$ (values are given in the units of the energy of one integer flux vortex). The $N = 1$ point at the origin corresponds to an ordinary vortex solution. Panel (**a**) shows calculations corresponding to different γ for fixed $e = 0.6$, while (**b**) displays how the energy per flux quantum changes with e and N for fixed anisotropy parameter $\gamma = 0.7$. The $\mathcal{Q} = 2$ Skyrmions are usually less energetically expensive than the $\mathcal{Q} = 3$. This is because the $\mathcal{Q} = 2$ Skyrmions can be better aligned with the underlying anisotropies, than the $\mathcal{Q} = 3$ Skyrmions.
The lower panel displays the *phase diagram*, calculated using energy characteristics of $\mathcal{Q} = 5$ Skyrmions. The different colors refer to different physical properties. The type-I region is shown by yellow shade. The lower part of the phase diagram shows regions where Skyrmions (red) or vortex lattices (blue) form in applied external field. The phase diagram retains similar structure in calculations with different topological charges. With the increasing of the skyrmionic charge $\mathcal{Q}$, the region where Skyrmions are energetically preferred over vortex lattices slightly grows. These results apply either for two-dimensional systems or three dimensional systems with translational invariance along z-axis. In the latter case the energy should be understood as the energy per unit length of a Skyrmion line (i.e. a Skyrmion texture in xy plane which is invariant under translation along z-axis). The discretization errors can be estimated by computing the total magnetic flux and comparing it to the exact value which follows from the quantization condition $2\pi N/e$. This gives the relative accuracy on the flux to be around 10^{-5}. From that, the accuracy on the energy is estimated to be at least three order of magnitude smaller than the energy difference between Skyrmions and vortices.

field produces Skyrmions rather than vortex lattices. To understand the instability of vortex lattices with respect to Skyrmion formation, different contributions to energy are investigated in Table I. In the skyrmionic state, vortex lattice decay into Skyrmions is driven by a win in gradient and potential energies although there is a loss in magnetic energy as well as the extra cost of producing a domain-wall.

	$E_{\rm total}$	$E_{\rm grad}$	$E_{\rm pot}$	$E_{\mathbb{Z}_2}$	$E_{\rm mag}$
Vortex	19.7759	10.7518	-12.0190	16.5195	4.5235
Skyrm.	18.9004	8.10522	-12.2301	17.6336	5.3916
Vortex	32.1684	19.3227	-19.0381	25.4445	6.4392
Skyrm.	37.6456	16.2529	-22.1474	32.8582	10.6818

Table I. Different contributions to the Skyrmion energy *per flux quantum*. $\mathcal{Q} = 5$ Skyrmions are considered in this example. The results are compared with the contributions to the energy of a single vortex (which determines the lower bound on vortex lattice energy near the first critical magnetic field H_{c1}). The gradient contribution $E_{\rm grad}$ is given by the integrated (1b), the magnetic energy $E_{\rm mag}$ by (1a). The potential energy $E_{\rm pot}$ is (1c) and $E_{\mathbb{Z}_2}$ is (1d). First block, for which $\gamma = 0.8$ and $e = 0.4$, corresponds to the state where Skyrmions are thermodynamically stable but vortex lattices are not. Second block is for $\gamma = 0.6$ and $e = 0.2$. It corresponds to a regime with standard Abrikosov vortex lattice. Here the Skyrmions are local minima of the free energy functional. They are more expensive than vortices but, if formed, they are protected against decay by a finite energy barrier. In the second example the win in the kinetic energy is too small to overcome extra energy cost associated with domain-wall formation and magnetic energy.

The Skyrmions we find are are structurally different from Skyrmions discussed in other kinds of superconductors[10] because of the different symmetry of the model. Other principal difference is the nature of the Skyrmionic state. Namely the works[10] proposed models where there are only skyrmionic solutions carrying two flux quanta. The latter forming stable lattices. In contrast, the model we consider supports Skyrmions with any integer value of topological charge. Importantly, here the energy per flux quantum is a sublinear function of the topological charge, which prohibits a ground state in the form of a lattice of simplest Skyrmions envisaged in.[10] Instead our model predicts more complicated high-topological-charge skyrmionic structures. Also in type-II regime our model predicts metastable states of coexisting vortices and Skyrmions.

In conclusion we have shown that the phase diagram of chiral p-wave superconductors has a thermodynamically stable skyrmionic phase between type-I and the usual type-II regimes. This is despite the fact that the model has $U(1) \times \mathbb{Z}_2$ broken symmetry where naive symmetry arguments would rule out skyrmionic excitations. In the skyrmionic phase, the long sought-after half-quanta vortices acquire thermodynamic stability. These objects can be detected with surface probes through their characteristic profile of magnetic field. The phase transition into a skyrmionic state should be first order, because the energy per flux quantum is decreasing with the skyrmionic topological charge.

We estimate that Sr_2RuO_4 which is frequently described by the model (1) has penetration length which is slightly too large to fall into the skyrmionic phase. However for these parameters the model predicts metastable skyrmionic excitations (which are slightly more energetic than vortices). Recently sporadic formation of objects with multiple flux quanta were reported in Fig. 2 of Ref. 22. Higher resolution scans of the magnetic field profile could confirm or rule out if the observed objects are Skyrmions.

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