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Nonlinear dynamics in a magnetic Josephson junction

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We theoretically consider a Josephson junction formed by a ferromagnetic spacer with a strong spin-orbit interaction or a magnetic spin valve, i.e., a bilayer with one static and one free layer. Electron spin transport facilitates a nonlinear dynamical coupling between the magnetic moment and charge current, which consists of normal and superfluid components. By phenomenologically adding reactive and dissipative interactions (guided by structural and Onsager symmetries), we construct magnetic torques and charge pumping, whose microscopic origins are also discussed. A stability analysis of our coupled nonlinear systems generates a rich phase diagram with fixed points, limit cycles, and quasiperiodic states. Our findings reduce to the known phase diagrams for current-biased nonmagnetic Josephson junctions, on the one hand, and spin-torque driven magnetic films, on the other, in the absence of coupling between the magnetic and superconducting order parameters.

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I. INTRODUCTION

Hybrid structures with ferromagnet (F)|normal-metal (N) interfaces have garnered much attention over the past few decades owing to their application in spintronic devices. Injecting a spin current into such a system exerts a torque on the magnet,¹ which can induce precession and even reversal,² allowing for manipulation of the magnetic order parameter in nanoscale structures without an external magnetic field.³ Because of the nonlinear nature of the ensuing magnetic dynamics, such devices offer observation of effects traditionally seen in nonlinear dynamical systems: phase locking, hysteresis, bifurcations, and chaos are readily observed.⁴

In consideration of a superconductor (S)|F|S heterostructure, one may expect the Josephson effect to be suppressed due to the rapid decay of a singlet pair inside the ferromagnet. Recent experiments,⁵ however, observed superconducting transport through a strong ferromagnet between two conventional (*s*-wave) superconductors. With the expectation that the triplet component of the superconducting condensate can penetrate long distances into a ferromagnet, the preservation of this signal suggests a spin singlet-to-triplet conversion at the interfaces.⁶ The unexpected persistence of a supercurrent through the magnet forecasts a new kind of spintronic device that manipulates the Josephson junction by the ferromagnet and, conversely, ferromagnetic layer by the superconducting condensate.^{7–9}

Previous analyses^{7–10} have considered *equilibrium* interactions between magnetic and superconducting order parameters, which naturally induce a reactive coupling. In contrast, in our description, we introduce *nonequilibrium* interactions consistent with the symmetries of the structure and obeying Onsager reciprocity.¹¹ This treatment allows the addition of both dissipative and reactive couplings between the magnet and superconductor that may in practice be crucial in the understanding of ferromagnetic Josephson junctions, analogous to the importance of Slonczewski^{1–3} and spin-pumping¹² terms in

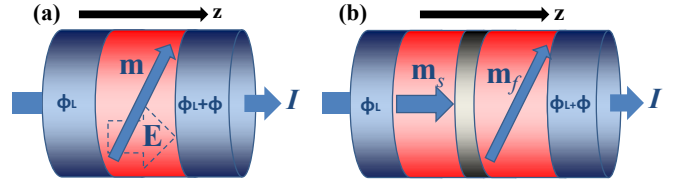


FIG. 1. Schematics of our magnetic Josephson junctions. The directions of junction layering, applied current I , internal Rashba field \mathbf{E} (a), and direction of the static ferromagnetic layer \mathbf{m}_s (b) all lie along the \mathbf{z} axis. ϕ is the phase difference between the superconducting leads.

the theory of spin-transfer torques. Such effects cannot be fully captured by quasiequilibrium free-energy considerations. We expect the dissipation to be governed by the quasiparticle excitations in the superconductors in concert with the microscopic processes in the ferromagnet (due to magnon interaction with electrons, phonons, or other magnons) that are responsible for their Gilbert damping (which, in turn, is known to persist down to very low temperatures¹³).

In order to provide specific examples, we consider (a) an S|F|S heterostructure with a Rashba spin-orbit interaction (SOI) in a thin ferromagnetic interlayer (neglecting the vector potential and associated phase shift caused by its magnetic moment⁹) and (b) S|F|N|F|S heterostructure wherein the SOI is replaced by a pinned ferromagnetic layer, \mathbf{m}_s . See Fig. 1. The corresponding spin-dependent Hamiltonians mix the singlet and triplet superconducting components,⁶ allowing the superfluid to penetrate into the magnet and exert spin torque and carry spin pumping (since a triplet Cooper pair is a spin-1 object) that are analogous to those associated with normal quasiparticles (spin-1/2 objects). In particular, as a simple model to demonstrate proof of concept, we take the device geometry to be rotationally symmetric along the axis associated with the Rashba interaction, as sketched in Fig. 1(a), or along the direction of the fixed

magnetic layer, as sketched in Fig. 1(b). By analyzing the stability and dynamics of our model, we outline a phase diagram of the coupled system as a function of applied magnetic field and current bias.

II. MODEL

The phenomenological equation of motion of an isolated ferromagnet sufficiently well below the Curie temperature is given by the Landau-Lifshitz-Gilbert (LLG) equation¹⁴

$$\dot{\mathbf{m}} = -\gamma \mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}}, \quad (1)$$

where γ is the gyromagnetic ratio and α is the dimensionless Gilbert damping. We use a normalized form of this equation, in which the (unit) magnetic direction vector $\mathbf{m} = \mathbf{M}/M_s$, $M_s = |\mathbf{M}|$ (saturation magnetization), is dimensionless. $\mathbf{H} = -\mathcal{V}^{-1} \partial F / \partial \mathbf{M}$ is the effective magnetic field and F , \mathbf{M} , and \mathcal{V} are the free energy, magnetization vector, and volume, respectively. In the spin-valve model, Fig. 1(b), \mathbf{m} will denote the free layer \mathbf{m}_f .

We consider the resistively-shunted junction (RSJ) model for the Josephson junction, wherein the device is composed of conventional superconductors with some Ohmic conductance σ in the junction.¹⁵ Additionally, we take the capacitance to be zero, which precludes RC -type delays in the coupled dynamics. The corresponding Josephson relations (for a static magnetization) are

$$\dot{Q} = I_c \sin \phi + \sigma V, \quad \dot{\phi} = \frac{2e}{\hbar} V, \quad (2)$$

where V is the voltage drop across the junction. ϕ is the phase difference between the superconducting reservoirs and Q is the charge transported by the junction. The supercurrent is proportional to the critical current, $I_c = (2e/\hbar)E_J$, where E_J parametrizes the Josephson energy $-E_J \cos \phi$. We note that Eq. (2) is dictated by gauge symmetry and, in anticipation of the arguments to follow, is a manifestation of Onsager reciprocity in the dynamics of Q and ϕ .

Under time reversal, $\dot{\mathbf{m}} \rightarrow \dot{\mathbf{m}}$ and $\alpha \mathbf{m} \times \dot{\mathbf{m}} \rightarrow -\alpha \mathbf{m} \times \dot{\mathbf{m}}$. The term proportional to α in the LLG equation thus reflects irreversible processes. We characterize such terms as dissipative. σ , likewise, parametrizes Ohmic dissipation of normal fluid. All other terms thus considered so far are reactive. Couplings between the free ferromagnet and superconductor at the level of the free energy, induced by the static magnetic layer or SOI, are restricted by the symmetries of our structure. Our device geometries, shown in Fig. 1, are assumed to be structurally invariant under arbitrary rotations about the \mathbf{z} axis as well as a parity transformation followed by π rotation about the \mathbf{x} (or \mathbf{y}) axis. Because both the exchange interaction between the magnetic layers of our spin-valve device and the Josephson energy are individually preserved under the symmetries of the combined system, the product

of these interactions must also be permitted.⁷ However, the interlayer F|N|F spin-valve exchange is usually very small (except for the thinnest N spacers),³ and will be disregarded in our study. One may, furthermore, show that any (time-reversal symmetric) bilinear cross term involving \mathbf{m} , Q , and ϕ does not respect the symmetries of our device geometry (keeping in mind that $\phi \rightarrow -\phi$ under time reversal and \mathbf{m} is a pseudovector under improper rotations). In particular, an interaction of the type¹⁰ $\cos(\phi + \Gamma m_z)$ is forbidden in our geometry. Thus neglecting interactions of \mathbf{m} , Q , and ϕ beyond quadratic order, the free energy remains uncoupled:

$$F[\mathbf{m}, Q, \phi] = F[\mathbf{m}] + F[Q] + F[\phi], \quad (3)$$

where $F[\mathbf{m}] = \mathcal{V} K M_z^2 / 2 - \mathcal{V} \mathbf{M} \cdot \mathbf{H}_a$, $F[Q] = -QV$, and $F[\phi] = -E_J \cos \phi$. The sign of the anisotropy constant, K , defines an easy plane or easy axis and is determined by the geometry of the device and crystalline anisotropies. \mathbf{H}_a is an applied external magnetic field.

The LLG equation of motion of the magnet is now complemented with interactions that are quasistationary (i.e., first order in frequency), up to quadratic order in the components of \mathbf{m} , preserving the magnitude of \mathbf{m} , and consistent with the structural symmetry of the device:

$$\begin{aligned} \dot{\mathbf{m}} = & -\gamma \mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}} \\ & + (\mu \dot{Q} + \lambda \dot{\phi}) \mathbf{m} \times \mathbf{z} + (\nu \dot{Q} + \kappa \dot{\phi}) \mathbf{m} \times \mathbf{z}. \end{aligned} \quad (4)$$

Hereafter, we are focusing on the spin-valve case, Fig. 1(b), where the phenomenological coupling coefficients μ , λ , ν , and κ may be taken to be angle-independent constants if we identify $\mathbf{z} \equiv \mathbf{m}_s$. [For the SOI device, Fig. 1(a), structural symmetries dictate these coefficients to be odd functions in m_z , similar to voltage-controlled torques at CoFeB/MgO interfaces,¹⁶ which are due to magnetic anisotropy induced by broken inversion symmetry.] Constants μ and ν characterize the strength of the coupling between the magnet and the total current \dot{Q} . Similarly, the strength of the coupling between the magnet and the dynamics of the superfluid condensate $\dot{\phi}$ is characterized by λ and κ . To the reader familiar with spin valves,³ Eq. (4) is reminiscent of the Landau-Lifshitz-Gilbert equation with the so-called Slonczewski and field-like torques, respectively, added on the second line of the right-hand side. In this case, sketched in Fig. 1(b), current is spin polarized by passing through the fixed magnetic layer. The resulting spin-polarized current impinging on a free ferromagnet induces torque due to conservation of angular momentum. In the case of a single magnetic layer with SOI, Fig. 1(a), a spin torque is generated via SOI inside this layer itself.^{17,18} Because the leads in our system are superconducting, we additionally generate a torque as a result of the dynamics of the superfluid condensate. Loosely speaking, the torque induced by both currents, normal current and supercurrent, through the junction produce two channels for driving magnetization dynamics (and thus two sets of terms, as compared to the usual normal-metal spin

torques). Appropriately, above the critical temperature of the superconductor, we recover the normal-metal limit, in which torque is generated by the Ohmic current (or, equivalently, voltage) alone.^{16–18}

The reaction of the current and superconducting phase dynamics to the magnet are not captured by the Josephson relations, Eq. (2), which would not be consistent with Eq. (4). One must extend Eq. (2) to include the pumping terms satisfying Onsager reciprocity, in order to obtain equations of motion for our coupled system that obey microscopic time-reversal symmetry.¹¹ Because the magnet flips under time reversal (upon invoking Onsager symmetry), one must additionally use the symmetries of the structure to relate the time-reversed state to the original. After straightforward manipulations, that are analogous to Ref. 18 for normal junctions, we construct the following equations in lieu of Eq. (2):

$$\begin{aligned}\dot{Q} &= \frac{2e}{\hbar} [E_J \sin \phi - \mathcal{S}(\lambda \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} + \kappa \dot{\mathbf{m}} \cdot \mathbf{z})] + \frac{\hbar \sigma}{2e} \dot{\phi}, \\ \dot{\phi} &= \frac{2e}{\hbar} [V - \mathcal{S}(\mu \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} + \nu \dot{\mathbf{m}} \cdot \mathbf{z})] - \rho \dot{Q},\end{aligned}\quad (5)$$

where $\mathcal{S} = \mathcal{V}M_s/\gamma$ is the total spin angular momentum of the ferromagnetic layer. These equations of motion now include both normal and superfluid pumping, which are Onsager reciprocal to the driving effects introduced in the generalized LLG equation, Eq. (4). Our theory includes two types of pumping as a result of the non-Ohmic relationship between current and voltage. The term with coefficient ρ causes current to drag phase across the device; ρ is a measure of the viscosity between the current and superfluid condensate. Although this term is not needed for consistency with Onsager reciprocity, we will see that it would generally have to be included in order to satisfy the second law of thermodynamics. We could also immediately notice that the coefficients ρ , ν , and μ should vanish in the limit of large superconducting reservoirs, recovering the ordinary ac Josephson effect (as expected based on the gauge invariance). Keeping these terms, on the other hand, would capture finite-size (mesoscopic) properties of the superconducting layers, which are of secondary interest to our ends.

We may write the equations of motion in a dimensionless form by measuring time, magnetic field, charge, voltage, and conductance in units of \mathcal{S}/E_J , $E_J/\gamma\mathcal{S}$, $2e\mathcal{S}/\hbar$, $E_J\hbar/2e\mathcal{S}$, and $\mathcal{S}(2e/\hbar)^2$, respectively:

$$\begin{aligned}\dot{\mathbf{m}} &= -\mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}} + \dot{\phi}(\lambda \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \kappa \mathbf{m} \times \mathbf{z}) \\ &\quad + \dot{Q}(\mu \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \nu \mathbf{m} \times \mathbf{z}), \\ \dot{Q} &= \sin \phi - \lambda \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} - \kappa \dot{\mathbf{m}} \cdot \mathbf{z} + \sigma \dot{\phi}, \\ \dot{\phi} &= V - \mu \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} - \nu \dot{\mathbf{m}} \cdot \mathbf{z} - \rho \dot{Q}.\end{aligned}\quad (6)$$

Additionally, allow us to absorb a factor of $\mathcal{V}M_s^2/E_J$ into the anisotropy constant, such that the free energy for the magnet reads $F[\mathbf{m}, Q, \phi] = E_J(Km_z^2/2 - \mathbf{m} \cdot \mathbf{H}_a - QV - \cos \phi)$. Under time reversal, the terms with coefficients ν and λ reverse sign in the LLG equation. Because $\dot{\mathbf{m}}$

does not change sign, these are dissipative. Likewise, the terms with coefficients μ and κ do not reverse sign and are thus nondissipative. σ is a dissipative coefficient, therefore ρ is as well.

Let us try to understand the microscopic origin of the dissipative terms in our theory. Consider momentarily only the RSJ subsystem: when enough energy is supplied (either thermally or by a bias), quasiparticles are able to overcome the superconducting gap and transport through the junction. Normal scattering of quasiparticles across the junction causes Ohmic resistance. Likewise, consider an isolated precessing ferromagnet. This is microscopically described by a coherent magnon state that can decay into phonons and incoherent magnons, processes which macroscopically give Gilbert damping. In the case of a metallic ferromagnet, the additional decay channel into the electron-hole continuum enhances further its Gilbert damping. Upon coupling these subsystems, energy is shared by the entire structure. Likewise, dissipation by microscopic mechanisms underlying Ohmic conductance and Gilbert damping can give rise to a dissipative (viscous) energy transfer between ferromagnetic and superconducting layers, as parametrized by new dissipative coefficients ν and λ . Phenomenologically, therefore, we may expect σ , α , and ρ to bound ν and λ , which is indeed verified below.

In the RSJ model, Eq. (2), if ϕ is static, we are in a superconducting (S) state because only dissipationless current is passing through the junction. Likewise if ϕ is not constant, the circuit must have a finite voltage drop. This is called a resistive (R) state. Notice that in our generalized model, Eqs. (6), a choice of dynamics that leave ϕ static can still generate dissipative current due to magnetic pumping. We will, nonetheless, keep referring to the static and dynamic states of ϕ as the superconducting (S) and resistive (R) states, respectively, even though this terminology is, in general, abusive, in the presence of the new spin-torque/pumping terms in Eqs. (6).

We distinguish between two regimes governed by the superconducting coherence length ξ . When ξ is smaller than the thickness of superconducting layers, the bulk properties of the superconductors will be largely detached from physics at the interfaces. Thus for superconducting *reservoirs*, a change in phase difference cannot be induced by transport through the junction. We expect the corresponding coefficients μ , ν , and ρ to scale inversely with the volume of the smaller of the superconducting layers then; these are representative of mesoscopic effects, as has already been inferred above. Because charge is a conserved hydrodynamic quantity, on the other hand, the electric current should be maintained in the bulk far from the junction. In particular, λ , κ , and σ should not depend on the size of superconducting reservoirs; these coefficients parametrize the properties of the Josephson junction itself and are thus of central interest to us.

In what follows we consider a dc current biased junction, $\dot{Q} = I$, where the applied magnetic field is along the axis of symmetry, $\mathbf{H}_a = H_a \hat{\mathbf{z}}$, and K is positive (which is

generically the case for films with magnetostatic energy dominating over crystalline anisotropy).

III. DECOUPLED JUNCTION

In the special case where $\lambda = \kappa = 0$ in Eqs. (6), the current-biased magnetic and superconducting dynamics decouple. We take this opportunity to recall the properties of magnetic spin valves and the RSJ model of superconductors, to which the decoupled equations map. Ignoring λ and κ ,

$$\begin{aligned}\dot{\mathbf{m}} &= -\mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}} + I(\mu \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \nu \mathbf{m} \times \mathbf{z}), \\ \sigma \dot{\phi} &= I - \sin \phi.\end{aligned}\quad (7)$$

The equation of motion for the magnet is thus the LLG equation for a spin valve, including Slonczewski (μ) and field-like (ν) torques, in the case that a fixed magnetic layer points along the z axis. The superconductor is described by the RSJ model with zero capacitance.

There are three possible stable states of the current-biased magnet in the presence of a static field in the z direction: pinned parallel to the z axis, antiparallel to the z axis, or precessing around the z axis, labeled p , a , and o , respectively. A pinned state is stable when $|(\mu/\alpha - \nu)I/K + h_a| \geq 1$, where $h_a \equiv H_a/K$. If $|(\mu/\alpha - \nu)I/K + h_a| < 1$, the magnet precesses at frequency $\omega_M = \mu I/\alpha$. The corresponding stability diagram with Hopf bifurcation lines is shown in Fig. 2. In the dimensionless form of the RSJ description, when $-1 \leq I \leq 1$, the junction is in the S state and the phase is fixed at $\phi = \sin^{-1} I$. When the current is raised beyond the critical current, $I > 1$, the Josephson junction is in the R state and ϕ oscillates with frequency $\omega_J = \sqrt{I^2 - 1}/\sigma$.¹⁵ For the RSJ model, a π junction is trivially impossible: $|\phi|$ cannot access values between $\pi/2$ and π . The inset of Fig. 2 displays the well-known phase diagram of the RSJ junction.

IV. COUPLED JUNCTION

Neglecting mesoscopic effects, we set ρ , ν , μ to zero:

$$\begin{aligned}\dot{\mathbf{m}} &= -\mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}} + \dot{\phi}(\lambda \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \kappa \mathbf{m} \times \mathbf{z}), \\ \sigma \dot{\phi} &= I - \sin \phi + \lambda \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} + \kappa \dot{\mathbf{m}} \cdot \mathbf{z}.\end{aligned}\quad (8)$$

Thermodynamic self consistency of our theory requires for the dissipation power $P = (E_J^2/S)(\alpha \dot{\mathbf{m}}^2 - 2\lambda \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z} \dot{\phi} + \sigma \dot{\phi}^2) \geq 0$. This bounds our phenomenological constant λ as $\lambda^2 \leq \alpha\sigma$ (while, clearly, $\alpha \geq 0$ and $\sigma \geq 0$). To proceed with the analysis, notice that, according to Eq. (8), the dynamics of m_z and ϕ decouple from the transverse dynamics ($m = m_x + im_y$), which can, in turn,

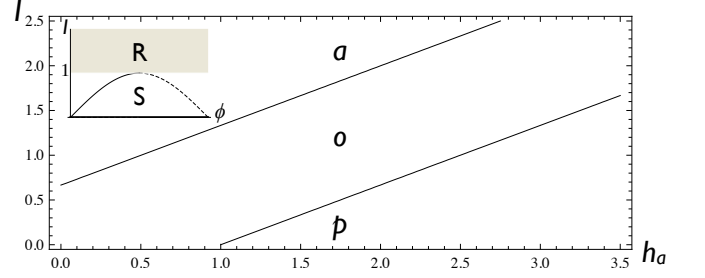


FIG. 2. Stability diagram as a function of the current and applied magnetic field of the decoupled magnet. $\mu = -1.5$, $\nu, \lambda, \kappa = 0$, and $K = 1$. p and a label the parallel and antiparallel states of the magnet, respectively. Inset: decoupled Josephson junction. The S state (unshaded) and R state (shaded) are separated by the line $I = 1$. Solid line is the value of ϕ for a 0 junction and dashed for the unstable π junction.

be expressed in terms of (m_z, ϕ) :

$$\begin{aligned}\dot{m}_z &= (1 - m_z^2) [\bar{\alpha}(H_a - Km_z) + \bar{\lambda}\dot{\phi}], \\ \dot{\phi} &= \frac{I - \sin \phi - \bar{\lambda}(H_a - Km_z)(1 - m_z^2)}{\sigma - (\lambda\bar{\kappa} + \kappa\bar{\lambda})(1 - m_z^2)}, \\ m &= \sqrt{1 - m_z^2} \exp \left[-\frac{i}{\alpha} \left(\lambda\phi + \frac{1}{2} \ln \frac{1 - m_z}{1 + m_z} \right) + i\varphi \right],\end{aligned}\quad (9)$$

where $\bar{\lambda} \equiv (\lambda - \alpha\kappa)/(1 + \alpha^2)$, $\bar{\kappa} \equiv (\kappa + \alpha\lambda)/(1 + \alpha^2)$, $\bar{\alpha} \equiv \alpha/(1 + \alpha^2)$ and φ determined by initial conditions. Consequently, the fixed points of the equations of motion for m_z and ϕ immediately determine the state of the full system.

When the current is below the critical current, $I \leq 1$, one can show that there are three stable fixed points: p_0 , a_0 , and o_0 which correspond to a Josephson 0-junction (defining a junction with $|\phi| < \pi/2$ to be in the “0 phase” and $\pi/2 < |\phi| < \pi$ in the “ π phase”) and magnetic direction parallel, antiparallel, and away from the z axis, respectively. In all these states ϕ is fixed by the applied current such that $\sin \phi = I$. As indicated by our stability diagram, Fig. 3, the state of our device is determined by the applied magnetic field. When $|h_a| \leq 1$, $m_z = h_a$ and m is fixed by initial conditions. By applying a sufficiently large external magnetic field, $|h_a| \geq 1$, o_0 is annihilated under a saddle-node bifurcation,¹⁹ and the sole stable state is p_0 or a_0 for positive or negative applied field, respectively, pinning the magnet along the z axis. A full linear stability analysis is discussed in the appendix, where we note, specifically, that the dissipation power bound precludes the existence of a stable π -junction.

If the current exceeds its critical value, $I > 1$, the superconducting phase and z component of the magnet become dynamic. This disappearance of all the fixed points is an infinite-period bifurcation.¹⁹ Because no fixed points exist, the Poincaré-Bendixson theorem implies any closed orbit on the cylinder, parameterized

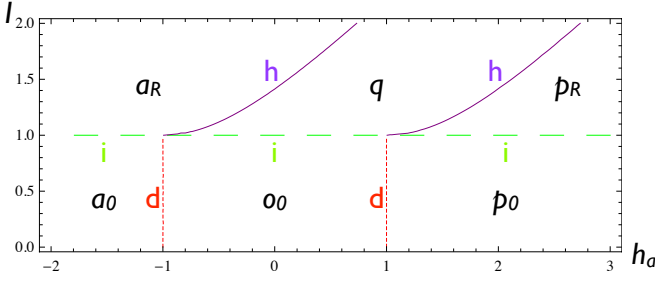


FIG. 3. Stability diagram as a function of the current, I , and applied magnetic field h_a . $\lambda = -0.1$, $\mu, \nu, \kappa = 0$, $K = 1$, $\alpha = 1$, and $\sigma = 0.1$. h labels the Hopf bifurcation (solid lines), i labels the infinite-period bifurcation (long-dashed lines), and d labels the saddle-node bifurcation (short-dashed lines).

by m_z and ϕ , is periodic and must go around the circumference of this cylinder. Supposing the frequency of this periodic motion is Ω_J , m_z may be written as a constant plus terms periodic in Ω_J . Likewise, we may express $\phi = n\Omega_J t$ (with nonzero $n \in \mathbb{Z}$) plus terms periodic in Ω_J . Therefore, the characteristic frequency of the system is given by the time average of $\dot{\phi}$. According to the equation for transverse component of the magnet, Eqs. (9), we find it undergoes rotations at frequency $n(\lambda/\alpha)\Omega_J$ that are superimposed with Ω_J oscillations. Therefore the magnet in general undergoes quasiperiodic motion, a state we label q .

To determine the full expression for Ω_J when $I > 1$ would require solving the system of differential equations Eqs. (9). For simplicity, consider the limit of small λ and κ , so that we can neglect quadratic terms in λ and κ . In this case, the characteristic frequency of the Josephson junction is given by the usual RSJ frequency $\omega_J = \sqrt{I^2 - 1}/\sigma$.¹⁵ In region q of our stability diagram, Fig. 3, m_z oscillates with frequency ω_J around the average value $\text{sign}(I)(\lambda/\alpha - \kappa)\omega_J + h_a/K$. Near the point $|\text{sign}(I)(\lambda/\alpha - \kappa)\omega_J + h_a/K| = 1$, a Hopf bifurcation¹⁹ (labeled h) is induced wherein the quasiperiodic orbit disappears and the magnet is parallel or antiparallel to the z axis, labeled p_R and a_R respectively, and the phase is dynamic. We anticipate the higher-order coupling in λ and κ to modify the frequency dependence on current. Furthermore, we expect that, near the line defining the Hopf bifurcation, there exists a phase of bimodal stability wherein the magnet can orient along the z axis or precess quasiperiodically, subject to the initial conditions. This is a natural consequence of the reciprocity of current-driven magnetic dynamics and pumping and persists even in the absence of any superconductivity (i.e., $E_J = 0$). Details of these rich coupled nonlinear dynamics are, however, beyond the scope of the present paper.

V. MESOSCOPIC JUNCTION

Finally, we analyze the properties of the general junction wherein we do not restrict any phenomenological

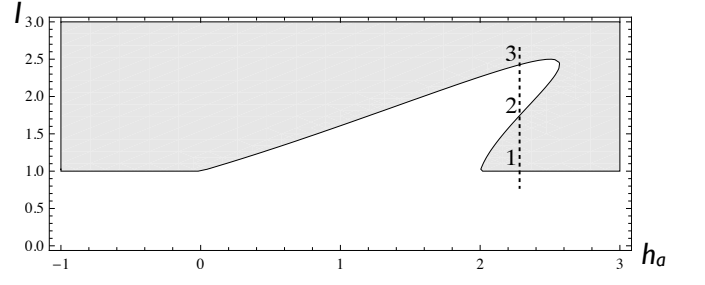


FIG. 4. Separation of the S state (white) and R state (grey) of the superconductor by a nonlinear function defined by $I' = 1$. The parameters of this system are $\mu = -1$, $\lambda = 0.6$, $\nu, \kappa = 0$, $K = \alpha = 1$, and $\sigma = 2$. The 1, 2, 3 labels along the dashed line show the three places where the Josephson junction switches between superconducting and resistive states.

parameters in Eqs. (6) to be zero, thus including mesoscopic effects. As previously, the transverse magnetization $m = m_x + im_y$ decouples from the (m_z, ϕ) dynamics:

$$\begin{aligned} \dot{m}_z &= (1 - m_z^2) \left[\bar{\alpha}(H_a - Km_z) + \bar{\lambda}\dot{\phi} + \bar{\mu}I \right], \\ \dot{\phi} &= \frac{I - \sin\phi - [\bar{\lambda}(H_a - Km_z) - I(\bar{\mu}\kappa + \bar{\nu}\lambda)](1 - m_z^2)}{\sigma - (\lambda\bar{\kappa} + \kappa\bar{\lambda})(1 - m_z^2)}, \end{aligned} \quad (10)$$

where $\bar{\mu} \equiv (\mu - \alpha\nu)/(1 + \alpha^2)$, $\bar{\nu} \equiv (\nu + \alpha\mu)/(1 + \alpha^2)$ and $\bar{\lambda} \equiv (\lambda - \alpha\kappa)/(1 + \alpha^2)$, $\bar{\kappa} \equiv (\kappa + \alpha\lambda)/(1 + \alpha^2)$, $\bar{\alpha} \equiv \alpha/(1 + \alpha^2)$, as before. One can show that the general solution for transverse components is (up to an arbitrary phase shift φ)

$$m = \sqrt{1 - m_z^2} \exp \left[-\frac{i}{\alpha} \left(\mu It + \lambda\phi + \frac{1}{2} \ln \frac{1 - m_z}{1 + m_z} \right) \right]. \quad (11)$$

The fixed points in the (m_z, ϕ) plane, as shown in the appendix, are

$$(\bar{m}_z, \bar{\phi}) = \begin{cases} (\pm 1, \sin^{-1} I), (\pm 1, \pi - \sin^{-1} I) \\ ((\mu/\alpha - \nu)I/K + h_a, \sin^{-1} I') \\ ((\mu/\alpha - \nu)I/K + h_a, \pi - \sin^{-1} I') \end{cases}, \quad (12)$$

where we have introduced

$$I' \equiv I [1 + (\mu/\alpha)\lambda(1 - \bar{m}_z^2)] \quad (13)$$

with $\bar{m}_z = (\mu/\alpha - \nu)I/K + h_a$ that itself depends on the current bias I . At the first four fixed points, the magnet is pinned parallel or antiparallel to the z axis and can be either a 0 or π junction. Hence we label these fixed points p_0 , a_0 , p_π , and a_π . The final two fixed points [which are possible when $|(\mu/\alpha - \nu)I/K + h_a| < 1$] are labeled by o_0 and o_π . These o_0 and o_π points are stationary in the (m_z, ϕ) plane but the transverse components of the magnet follow a circular orbit of radius $\sqrt{1 - \bar{m}_z^2}$ at frequency $\omega_M = \mu I/\alpha$.

The salient differences between these fixed points and those found studying the fixed points of Eqs. (9) are in

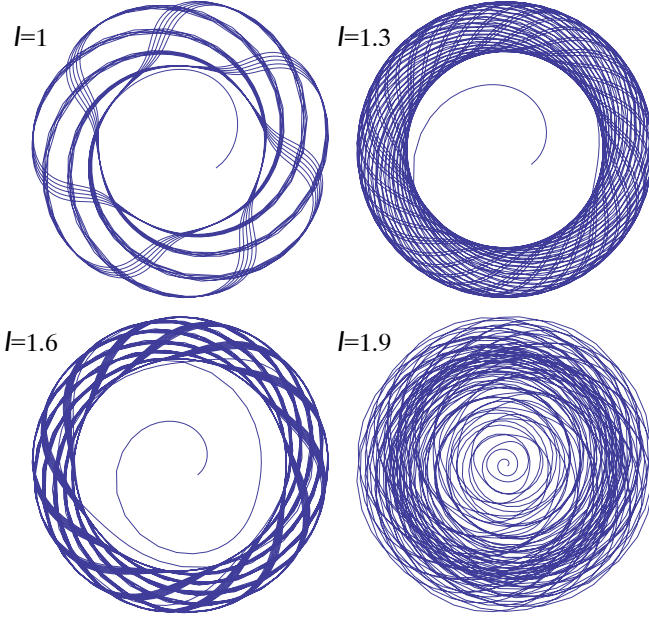


FIG. 5. Stereographic projection of the magnetization at different currents. Here, $\mu = 0.1$, $\lambda = 0.5$, $\nu, \kappa = 0$, $K = 1$, $\alpha = 1$, and $\sigma = 1$.

the properties of o_0 and o_π . First, the transverse component of the ferromagnet is dynamic when $\mu \neq 0$. Second, the static value of $\sin \phi$ is a nonlinear function of the current. This results in a change in shape of the boundary separating the S and R states of the superconductor: see, for example, Fig. 4, where the phase diagram develops a “foldover region.” Consider the current increase at fixed magnetic field along the dashed line in Fig. 4. The system undergoes changes from (1) S to R, (2) R to S, and (3) S to R again. Unlike in a conventional Josephson junction, our model has multiple values of the current for which the junction changes between superconducting and resistive states. Thus the junction has three ‘critical currents.’ Likewise at a particular fixed value of current, we can induce a change from R to S then S to R by increasing or decreasing the applied magnetic field. This has no analogy in the RSJ model. As a function of the applied current, a rich variety of the coupled dynamics generally emerges, as seen in Fig. 5, where we have plotted the stereographic projection of the magnetic direction. A detailed analysis of this motion is deferred to a future study.

VI. SUMMARY AND OUTLOOK

We have introduced a model of S|F|S and S|F|N|F|S heterostructures coupling the dynamics of the magnets to that of the superconductor via a Rashba SOI in single-layer junctions and via magnetic misalignment in spin-valve junctions. We expect such structures to be highly adaptable to uses in spintronics due to the versatility

with which one can in principle influence both the magnet and superconductor.

The S|F|S structures deposited on topological insulators (engendered by strong SOI) were studied in Refs. 20. With an appropriate modification of the Josephson effect in the presence of Majorana modes and, if necessary, revision of the structural symmetries, such systems can also be amenable to our phenomenology.

Chaos in ferrites and magnetic thin films is often attributed to spatially nonuniform magnetizations.⁴ Perhaps a simpler route towards chaos in our model is by applying a magnetic field perpendicular to the axis of cylindrical symmetry. As a result, the dynamic equations become three dimensional and thus no longer restricted by the Poincaré-Bendixson theorem to periodic orbits or fixed points.

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Appendix A: Linear stability analysis

Here, we explicitly calculate the fixed points and their associated stabilities for the equations of motion of m_z and ϕ . In particular, we point out that when $K > 0$, the existence of a π -junction is precluded. By dc biasing a general junction defined by Eqs. (6), we obtain the equations of motion for \mathbf{m} and ϕ :

$$\begin{aligned} \dot{\mathbf{m}} = & -\mathbf{m} \times \mathbf{H} + \alpha \mathbf{m} \times \dot{\mathbf{m}} + \dot{\phi}(\lambda \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \kappa \mathbf{m} \times \mathbf{z}) \\ & + I(\mu \mathbf{m} \times \mathbf{z} \times \mathbf{m} + \nu \mathbf{m} \times \mathbf{z}), \\ \sigma \dot{\phi} = & I - \sin \phi + \kappa \dot{m}_z + \lambda \dot{\mathbf{m}} \cdot \mathbf{m} \times \mathbf{z}, \end{aligned} \quad (\text{A1})$$

where $I = \dot{Q}$ is fixed. If the applied magnetic field is strictly along the axis of rotational symmetry of our junction, $\mathbf{H} = H_a \hat{\mathbf{z}}$, the transverse equations of motion decouple from (m_z, ϕ) and we obtain Eqs. (10):

$$\begin{aligned} \dot{m}_z = & (1 - m_z^2) \left[\bar{\alpha}(H_a - K m_z) + \bar{\lambda} \dot{\phi} + \bar{\mu} I \right], \\ \Sigma \dot{\phi} = & I - \sin \phi - \left[\bar{\lambda}(H_a - K m_z) - I(\bar{\mu} \kappa + \bar{\nu} \lambda) \right] (1 - m_z^2). \end{aligned}$$

Note that $\Sigma \equiv \sigma - (\lambda \bar{\kappa} + \kappa \bar{\lambda})(1 - m_z^2)$ is guaranteed to be nonnegative by the thermodynamic bound $\lambda^2 \leq \alpha \sigma$. We see this by setting κ to be λ/α , which maximizes the quantity $\lambda \bar{\kappa} + \kappa \bar{\lambda}$ at $\lambda^2/\alpha \leq \sigma$. Furthermore, according to Sylvester’s criterion, the thermodynamic bound requires the additional condition $\alpha \rho \sigma - \nu^2 \sigma - \lambda^2 \rho \geq 0$ in the general mesoscopic regime.

The corresponding fixed points of Eqs. (10) are given by Eq. (12). Linearizing around a fixed point $(\bar{m}_z, \bar{\phi})$, we

obtain

$$\begin{aligned}\dot{m}_z &= (\bar{m}_z^2 - 1)(\bar{\alpha}K\bar{m}_z - \bar{\lambda}\dot{\phi}) \\ &\quad + 2m_z\bar{m}_z(\bar{\alpha}K\bar{m}_z - \bar{\alpha}H_a - \bar{\mu}I), \\ \bar{\Sigma}\dot{\phi} &= -\phi \cos \bar{\phi} + \bar{\lambda}K\bar{m}_z(1 - \bar{m}_z^2) \\ &\quad - 2m_z\bar{m}_z [\bar{\lambda}(K\bar{m}_z - H_a) + \lambda\bar{\nu}I + \kappa\bar{\mu}I], \quad (\text{A2})\end{aligned}$$

where we have introduced $\bar{\Sigma}$ as Σ evaluated at the fixed point \bar{m}_z . Recall that the stability properties of a system of linear differential equations, $\dot{\mathbf{x}} = \hat{\mathbf{A}}\mathbf{x}$, around a fixed point, $\bar{\mathbf{x}}$, of a pair variables, $\mathbf{x} = \{x_1, x_2\}$, are classified according to the following criteria: (1) $\det \hat{\mathbf{A}} < 0$, $\bar{\mathbf{x}}$ is a saddle point, (2) $\det \hat{\mathbf{A}} > 0$ and $\text{tr} \hat{\mathbf{A}} < 0$, $\bar{\mathbf{x}}$ is stable, and (3) $\det \hat{\mathbf{A}} > 0$ and $\text{tr} \hat{\mathbf{A}} > 0$, $\bar{\mathbf{x}}$ is unstable.⁴ The matrix

defining the linearization around the fixed points with $\bar{m}_z = \pm 1$ is

$$\begin{bmatrix} 2[\bar{\alpha}K \mp (\bar{\alpha}H_a + \bar{\mu}I)] & 0 \\ -2/\sigma [\bar{\lambda}K \pm (\lambda\bar{\nu}I + \kappa\bar{\mu}I - \bar{\lambda}h_a)] & -\sigma^{-1} \cos \bar{\phi} \end{bmatrix}. \quad (\text{A3})$$

Because only the diagonal terms of the matrix in Eq. (A3), which are independent of λ and κ , contribute to the trace and determinant, the stability analysis proceeds as in the decoupled regime. Specifically, if the current and magnetic field are not large enough to overcome the anisotropy, $\bar{\alpha}K > |\bar{\alpha}H_a + \bar{\mu}I|$, then these points are unstable π -junctions or saddle point 0-junctions. If the external sources are large enough to overcome the anisotropy, $\bar{\alpha}K < |\bar{\alpha}H_a + \bar{\mu}I|$, the π -junction is a saddle point and the 0-junction is stable node. When $\bar{m}_z = (\mu/\alpha - \nu)I/K + h_a$, the linearization matrix

$$\hat{\mathbf{A}} = \begin{bmatrix} (\bar{m}_z^2 - 1)\bar{\alpha}K - \bar{\lambda}\bar{\Sigma}^{-1}(\bar{m}_z^2 - 1) [\bar{\lambda}K(1 - \bar{m}_z^2) - 2\mu\lambda I\bar{m}_z/\alpha] & \bar{\lambda}\bar{\Sigma}^{-1}(\bar{m}_z^2 - 1) \cos \bar{\phi} \\ \bar{\Sigma}^{-1} [\bar{\lambda}K(1 - \bar{m}_z^2) - 2\lambda\mu I\bar{m}_z/\alpha] & -\bar{\Sigma}^{-1} \cos \bar{\phi} \end{bmatrix}. \quad (\text{A4})$$

One may show that the determinant of this matrix is equal to that of an uncoupled junction and, because we have chosen $K > 0$, π -junctions are always saddle nodes and therefore cannot be realized in this device geometry.

We point out, however, that for $K < 0$ a π -junction can be stable or unstable subject to the trace of the above matrix. Because $\bar{\alpha} > (1 - \bar{m}_z^2)\bar{\lambda}^2/\bar{\Sigma}$ as a result of the thermodynamic bound on λ , a stable π -junction is precluded in the absence of mesoscopic effects. However, in general, there is an additional term that contributes to stability in Eq. (A4): $2\bar{m}_z(\bar{m}_z^2 - 1)\mu\lambda\bar{\lambda}I/\alpha\bar{\Sigma}$. We find that for sufficiently large κ and conductance, and thereby sufficiently large λ , an applied magnetic field and current can stabilize a π -junction in the mesoscopic regime.

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