

This is the accepted manuscript made available via CHORUS. The article has been published as:

Scattering of hole excitations in a one-dimensional spinless quantum liquid

K. A. Matveev and A. V. Andreev

Phys. Rev. B **86**, 045136 — Published 30 July 2012

DOI: [10.1103/PhysRevB.86.045136](https://doi.org/10.1103/PhysRevB.86.045136)

Scattering of hole excitations in a one-dimensional spinless quantum liquid

K. A. Matveev¹ and A. V. Andreev²

¹*Materials Science Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*

²*Department of Physics, University of Washington, Seattle, Washington 98195, USA*

(Dated: June 28, 2012)

Luttinger liquid theory accounts for the low energy boson excitations of one-dimensional quantum liquids, but disregards the high energy excitations. The most important high energy excitations are holes which have infinite lifetime at zero temperature. At finite temperatures they can be scattered by thermally excited bosons. We describe the interaction of the hole with the bosons by treating it as a mobile impurity in a Luttinger liquid. This approach enables us to evaluate the scattering probability at arbitrary interaction strength. In general, the result is expressed in terms of the hole spectrum, its dependence on the density and momentum of the fluid, and the parameters of the Luttinger liquid Hamiltonian. In the special case of Galilean invariant systems the scattering probability is expressed in terms of only the hole spectrum and its dependence on the fluid density. We apply our results to the problem of equilibration of one-dimensional quantum liquids.

PACS numbers: 71.10.Pm

I. INTRODUCTION

The Luttinger liquid theory describes low energy properties of one-dimensional systems of interacting fermions^{1,2} or bosons^{3,4} in terms of excitations with Bose statistics. The latter have an acoustic spectrum and correspond to long wavelength density fluctuations. Despite the simplicity of the Hamiltonian, the Luttinger liquid theory successfully describes the nontrivial power law behavior⁵ of various correlation functions of one-dimensional systems at small frequencies and wavenumbers. Many of them, such as the power law energy dependence of the tunneling density of states, have been experimentally confirmed.⁶

The power law behavior of correlators in one-dimensional systems is not limited to small frequencies and wavenumbers. For example, even at large wave numbers $q \sim n_0$, where n_0 is the fluid density, the zero temperature dynamic structure factor (the Fourier transform of the density-density correlator) exhibits a power law singularity $S(q, \omega) \propto [\hbar\omega - \varepsilon(q)]^\alpha \theta(\hbar\omega - \varepsilon(q))$. Its location $\varepsilon(q)$ defines the spectral edge below which the system cannot absorb excitations. It corresponds to the lowest energy state of the system with momentum $\hbar q$. The position of the spectral edge $\varepsilon(q)$ is periodic in q with the period $2\pi n_0$ (in the spinless case considered here), as illustrated in Fig. 1.

The nature of states corresponding to the spectral edge and periodicity of the latter may be illustrated by the simple example of noninteracting fermions in one dimension. Because the energy cost associated with transferring a particle between the two Fermi points vanishes in the thermodynamic limit, the minimal energy states with momenta differing by an integer multiple of $2\hbar k_F = 2\pi\hbar n_0$ are degenerate, resulting in the periodicity of the spectral edge. It is thus sufficient to consider states with wave vectors in the fundamental domain $0 < q < 2\pi n_0$. The states at the spectral edge correspond to hole excitations which are obtained by moving

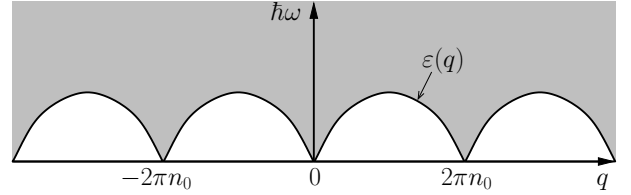


FIG. 1: At a given momentum $\hbar q$ the energy of a one-dimensional system is limited from below by the spectral edge $\varepsilon(q)$. Thus the imaginary part of the structure factor $S(q, \omega)$, describing the dissipation, vanishes outside the shaded region.

a fermion with the wave vector $k_F - q$ to the right Fermi point $k = k_F$.

In the presence of interactions between the fermions the energy cost associated with the transfer of a particle between the opposite Fermi points still vanishes in the thermodynamic limit, and the spectral edge $\varepsilon(q)$ remains periodic in q with the period $2\pi n_0$. A state at the spectral edge with a wave vector in the range $0 < q < 2\pi n_0$ may again be viewed as the $q = 0$ ground state with an additional hole. However, the hole is now dressed by the interactions, and its energy $\varepsilon(q)$ is renormalized. The above picture of the spectral edge also applies to bosonic fluids, where holes are known as Lieb's type-II excitations.⁷

At $q \rightarrow 0$ and $q \rightarrow 2\pi n_0$ the energy $\varepsilon(q)$ is small and the hole can be decomposed into the bosonic excitations of the Luttinger liquid. In contrast, the high energy holes with $q \sim n_0$ are not described by the Luttinger liquid theory and should be viewed as distinct from the Luttinger liquid bosons. As noted above, the states with a single hole are lowest energy states of the system with a given momentum. Therefore they cannot decay into bosons and their lifetime is infinite.

High energy hole excitations may be produced in the system by external probes (e.g., optically or by tunneling) or by thermal fluctuations. Therefore the problem of

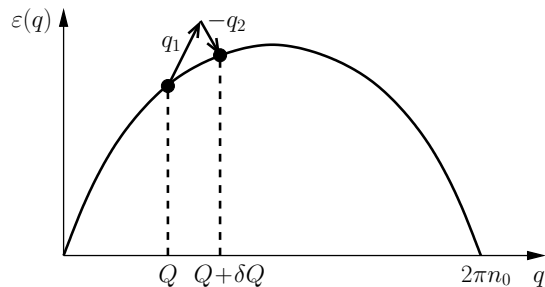


FIG. 2: Scattering of a hole at wavenumber Q to a state $Q + \delta Q$ involves absorption of one bosonic excitations and emission of another.

their dynamics is of considerable interest. Although the lifetime of the hole states is infinite at zero temperature, the situation changes at $T > 0$. In this case the hole can scatter off thermally excited bosons. Since the resulting energy change $\delta\epsilon \lesssim T$, a hole with energy $\epsilon(q) \gg T$ remains distinct from the Luttinger liquid bosons and may be treated as a mobile impurity in a Luttinger liquid.^{8–16}

In this paper we evaluate the probability of scattering of a hole by the bosonic excitations. The scattering event will be assumed to change the wavenumber of the hole Q by a small amount δQ . In order to conserve both the energy and momentum of the system, the hole has to absorb one boson and emit another, see Fig. 2. Such processes were first considered by Castro Neto and Fisher⁹ in the context of the dynamics of mobile impurities in the Luttinger liquid. The scattering probability near the top of the spectrum, $Q = \pi n_0$, determines the rate of equilibration of one-dimensional quantum liquids studied in Ref. 17. In the limit of strong repulsion the scattering processes of Fig. 2 were studied in the context of equilibration of the one-dimensional Wigner crystal,^{18,19} while for weakly interacting bosons they are responsible for the decay of the so-called dark soliton.¹⁵ The mobile impurity approach enables us to treat this problem at any interaction strength. We express the scattering probability in terms of the hole spectrum $\epsilon(q)$ and its dependence on the density and velocity of the fluid. For Galilean invariant systems the result can be expressed in terms of only the spectrum and its dependence of the fluid density.

The standard bosonization of Luttinger liquids,^{1,2} used in the previous treatments of mobile impurities,^{8–14,20–25} corresponds to the Eulerian description of the liquid. In this approach the dynamical degrees of freedom describing an element of the fluid are given as functions of time t and the instantaneous position x of the element in the laboratory frame. Alternatively, the liquid may be described using Lagrangian variables. In the latter approach the dynamical degrees of freedom are labeled by the position y of the fluid element in a reference state of uniform density n_0 . For a liquid moving with a uniform velocity the transformation from Eulerian to Lagrangian variables is equivalent to a Galilean transformation from a laboratory frame to a reference frame moving with the

fluid. Therefore one expects that Galilean invariant systems are more naturally described in Lagrangian variables. This expectation is borne out: the use of Lagrangian variables considerably simplifies evaluation of the scattering amplitude.

On the other hand, Galilean invariance is not a universal property of Luttinger liquids. For instance, electrons in solids move in the periodic potential of the lattice which gives rise to the band structure of the spectrum. In this case the quadratic spectrum $p^2/2m$ required for Galilean invariance appears only near the band edges. A similar situation arises when the concept of Luttinger liquid is applied to spin chains and one-dimensional systems of cold atoms in optical lattices. In the absence of Galilean invariance Lagrangian variables offer no obvious advantages. To address this regime we develop the theory of hole scattering using the standard Luttinger liquid theory^{1,2} based on the Eulerian approach to fluid dynamics.

The paper is organized as follows. In Sec. II we consider the Galilean invariant case, and develop theoretical description of quantum liquids in Lagrangian variables. We obtain the scattering probability of the hole in terms of its spectrum. In Sec. III we consider the general case, where Galilean invariance is not assumed, and use the conventional approach based on the Eulerian variables to obtain the scattering probability of the excitation. In Sec. IV we verify that the two approaches give the same result in the case of a Galilean invariant system and adapt our calculations to the problem of a massive mobile impurity in a Luttinger liquid. We also discuss the implications of our results for the problem of equilibration of the Luttinger liquid.

II. SCATTERING OF HOLES IN GALILEAN INVARIANT SYSTEMS

In this section we consider systems which possess Galilean invariance. Evaluation of the hole scattering probability is simplified by developing the theory in Lagrangian variables,²⁶ which are frequently used to describe one-dimensional flows in classical hydrodynamics.

A. Hamiltonian of a quantum liquid in Lagrangian variables

We first consider a liquid in which only low energy excitations are present. These excitations are essentially sound waves with small wavenumbers $q \ll n_0$. In this regime the discreteness of particles is not important and the liquid may be described as a continuum in the spirit of hydrodynamics.

The Hamiltonian describing the long wavelength excitations may be obtained by coarse-graining the fluid into small elements. The Lagrangian coordinate y of the fluid element corresponds to its position in a reference state

of uniform density n_0 . In the presence of sound waves the particle density $n(y)$ inside each element may deviate from n_0 . The length of each element in the reference state Δy is assumed to be sufficiently small, $q\Delta y \ll 1$, so that the variations of the fluid density inside it are negligible. On the other hand, the number of particles in the element is assumed to be large, $n_0\Delta y \gg 1$.

Taking advantage of the Galilean invariance we write the energy of the fluid element as a sum of the kinetic energy of its center of mass motion and the internal energy,

$$\Delta E = \frac{(\Delta P)^2}{2mn_0\Delta y} + U(n)n_0\Delta y. \quad (1)$$

Here ΔP is the momentum of the element and m is the mass of the particles. In the absence of high energy excitations the internal energy is given by the energy per particle in the ground state $U(n)$ multiplied by the number of particles $n_0\Delta y$.

The Hamiltonian of the liquid is written in terms of two dynamical variables, the displacement of the fluid elements from their reference positions $u(y)$ and the conjugate momentum density $p(y) = \Delta P/\Delta y$, satisfying the commutation relation $[u(y), p(y')] = i\hbar\delta(y - y')$. Summing the energies (1) of the fluid elements we obtain

$$H_L = \int \left[\frac{p^2}{2mn_0} + n_0U(n) \right] dy. \quad (2)$$

The Lagrangian variable $u(y)$ enters the Hamiltonian via the particle density

$$n(y) = \frac{n_0}{1 + u'(y)}, \quad (3)$$

where prime denotes the derivative. Equation (3) follows immediately from the relation

$$x = y + u(y) \quad (4)$$

between the physical coordinate x of the fluid element and the Lagrangian variable y .

For the subsequent discussion it is sufficient to expand the Hamiltonian of the liquid to third order in the deformation u' ,

$$H_L = \int \left(\frac{p^2}{2mn_0} + \frac{mn_0v^2}{2} u'^2 - \alpha u'^3 \right) dy. \quad (5)$$

The speed of sound in the liquid v and the coefficient α in the anharmonic term are given by

$$v(n_0) = \left\{ \frac{1}{m} [2n_0U'(n_0) + n_0^2U''(n_0)] \right\}^{1/2}, \quad (6)$$

$$\alpha(n_0) = n_0^2U'(n_0) + n_0^3U''(n_0) + \frac{1}{6}n_0^4U'''(n_0), \quad (7)$$

where derivatives are again denoted by prime.

The quadratic part of Eq. (5) is the Luttinger liquid Hamiltonian in Lagrangian variables. It can be brought

to the diagonal form $\sum \hbar v|q|b_q^\dagger b_q$ by introducing the boson operators b_q via the standard procedure^{1,2}

$$u(y) = \sum_q \sqrt{\frac{\hbar}{2mn_0Lv|q|}} (b_q e^{iqy} + b_q^\dagger e^{-iqy}), \quad (8)$$

$$p(y) = -i \sum_q \sqrt{\frac{\hbar mn_0v|q|}{2L}} (b_q e^{iqy} - b_q^\dagger e^{-iqy}). \quad (9)$$

where L is the system size.

Because of the continuum approximation made in the description of the liquid the sums in the above equations include only small wavevectors $|q| \ll q_0$, where $q_0 = 1/\Delta y$. Therefore if a high energy hole is present in the liquid, it needs to be treated as an additional entity.

B. Description of the hole in Lagrangian variables

Let us now consider the fluid with a single high energy hole excitation. The Lagrangian coordinate Y of the hole refers to the fluid element containing it. As discussed in the introduction the hole excitation is obtained by moving a particle from state $k_F - q$ to the Fermi point k_F . Thus the presence of the hole has no effect on the mass of the fluid element, and the expression for its kinetic energy given by the first term in Eq. (1) remains unchanged. On the other hand, the internal energy increases by the excitation energy $\varepsilon(q, n)$. (Here we take into account the dependence of the latter on the fluid density n .)

It is important to note that in the presence of sound waves the physical size of the fluid element Δx differs from its size in the reference state, $\Delta x = (n_0/n)\Delta y$. As a result the physical wavenumber q differs from the wavenumber Q corresponding to the Lagrangian coordinate of the excitation, $q = (n/n_0)Q$. Therefore it is convenient to introduce the excitation spectrum $\epsilon(Q, n)$ with respect to the Lagrangian wavenumber Q , which is related to the physical spectrum $\varepsilon(q, n)$ by

$$\epsilon(Q, n) = \varepsilon(nQ/n_0, n). \quad (10)$$

We now conclude that the presence of a hole at point Y is accounted for by the contribution to the Hamiltonian in the form

$$H_h = \epsilon(-i\partial_Y, n(Y)). \quad (11)$$

The density $n(Y)$ at the location of the hole is affected by the boson excitations, see Eqs. (3) and (8). This results in the interaction of the hole with the bosons.

C. Scattering probability

We now consider scattering processes shown in Fig. 2 and evaluate their rate $W_{Q, Q+\delta Q}$. To this end we expand

the excitation energy to second order in u'

$$\epsilon^{(2)}(Q, n(Y)) = \epsilon_Q - n_0 \partial_n \epsilon_Q u'(Y) + \left[n_0 \partial_n \epsilon_Q + \frac{1}{2} n_0^2 \partial_n^2 \epsilon_Q \right] u'^2(Y). \quad (12)$$

In this equation and the subsequent results the hole energy $\epsilon_Q = \epsilon(Q, n)$ and its partial derivatives are evaluated at $n = n_0$. The corresponding Hamiltonian $H_h^{(2)}$ is obtained by substituting $Q = -i\partial_Y$ and symmetrizing the operators. Because of the relation (8) $H_h^{(2)}$ contains both linear and quadratic coupling between the bosons and the hole.

The scattering process illustrated in Fig. 2 involves two bosons. Its amplitude may be obtained using perturbation theory in the coupling between the hole and the bosons. One contribution to the scattering amplitude arises from the quadratic coupling in the second line of Eq. (12) taken in the first order of perturbation theory. Keeping in mind Eq. (8) one easily concludes that such amplitude is proportional to the small momentum transfer δQ .

Another contribution arises from the linear coupling term which scales as $\sqrt{\delta Q}$. The process shown in Fig. 2 is realized in the second order perturbation theory in the linear coupling. Since the energy denominator is proportional to δQ one may naively expect the respective amplitude to scale as $(\sqrt{\delta Q})^2 / \delta Q = (\delta Q)^0$. However, there are two processes in which the two participating bosons are created and destroyed in opposite order. Since the corresponding energy denominators have opposite signs, the amplitudes of such processes cancel each other in leading order in δQ . The remaining subleading contribution again scales as δQ .

Finally there is a contribution in the second order perturbation theory, which arises from the combination of the linear coupling of the bosons to the hole and the $\alpha u'^3$ term in Eq. (5). The latter scales as $(\delta Q)^{3/2}$, resulting in the scattering amplitude linear in δQ .

One can avoid the somewhat tedious calculation outlined above by performing a unitary transformation $U^\dagger (H_L + H_h^{(2)}) U$ of the Hamiltonian, which eliminates the linear coupling between the hole and the bosons to leading order in δQ . The operator U should be chosen in the form

$$U = \exp \left(\frac{i}{\hbar} f_u u(Y) + \frac{i}{\hbar} f_p \int_{-\infty}^Y p(y) dy \right). \quad (13)$$

Here the coefficients f_u and f_p are given by

$$f_u = -\frac{n_0 v_Q \partial_n \epsilon_Q}{v^2 - v_Q^2}, \quad f_p = \frac{\partial_n \epsilon_Q}{m(v^2 - v_Q^2)}, \quad (14)$$

where v_Q is shorthand notation for the velocity of the hole,

$$v_Q = v(Q, n) = \frac{1}{\hbar} \partial_Q \epsilon(Q, n). \quad (15)$$

Because in the above unitary transformation Q denotes the initial momentum of the hole rather than the operator $-i\partial_Y$ the linear coupling between the hole and the bosons is removed only to leading order in δQ . However, in its absence the subleading contribution, which scales as $(\delta Q)^{3/2}$, gives a negligible correction to the scattering amplitude, $[(\delta Q)^{3/2}]^2 / \delta Q = (\delta Q)^2$. The dominant contribution to the scattering amplitude arises from the terms in the transformed Hamiltonian that couple the hole to second powers of boson fields, $u'^2(Y)$ and $p^2(Y)$. This coupling has the form

$$\left(-3\alpha f_p + n_0 \partial_n \epsilon_Q + \frac{1}{2} n_0^2 \partial_n^2 \epsilon_Q - v_Q f_u - n_0 \partial_n v_Q f_u - \frac{f_u^2}{2m_Q^*} \right) u'^2(Y) - \frac{f_p^2}{2m_Q^*} p^2(Y), \quad (16)$$

where the momentum dependent effective mass m_Q^* is defined in terms of the curvature of the spectrum

$$\frac{1}{m_Q^*} = -\frac{1}{\hbar^2} \partial_Q^2 \epsilon_Q. \quad (17)$$

In addition we introduced the notation

$$v_Q = v(Q, n) = v(nQ/n_0, n) \quad (18)$$

for the velocity of the hole as a function of the Lagrangian wavenumber, cf. Eq. (10). Although $v_Q = v_Q$ at $n = n_0$, their dependences on density are different. In particular,

$$\partial_n v_Q = \partial_n v_Q - \frac{\hbar Q}{m_Q^* n_0} \quad (19)$$

at $n = n_0$.

The matrix element for the scattering process in which a boson with momentum q_1 is annihilated and a boson with momentum q_2 is created is obtained by expressing the deformation u' and the momentum density p in the above expression in terms of the boson creation and annihilation operators using Eqs. (8) and (9),

$$t_{q_1 q_2} = \frac{\hbar \sqrt{|q_1 q_2|}}{m n_0 L v} \left[3\alpha f_p - n_0 \partial_n \epsilon_Q - \frac{1}{2} n_0^2 \partial_n^2 \epsilon_Q + v_Q f_u + n_0 \partial_n v_Q f_u + \frac{f_u^2 - (m n_0 v f_p)^2}{2m_Q^*} \right]. \quad (20)$$

Here we assumed that $q_1 q_2 < 0$, see Fig. 2.

The coefficient α may be expressed in terms of the density dependence of sound velocity v with the aid of Eqs. (6) and (7),

$$3\alpha(n) = m n \left(v^2 + \frac{1}{2} n \partial_n v^2 \right). \quad (21)$$

Using this relation and Eq. (14) we can express the scattering amplitude in terms of the spectrum of the hole,

$$t_{q_1 q_2} = \frac{\hbar n_0 \sqrt{|q_1 q_2|}}{2m L v (v^2 - v_Q^2)} \Upsilon_Q, \quad (22)$$

where we introduced the notation

$$\Upsilon_Q = (\partial_n \epsilon_Q) \partial_n (v^2 - v_Q^2) - (v^2 - v_Q^2) \partial_n^2 \epsilon_Q - \frac{1}{m_Q^*} (\partial_n \epsilon_Q)^2. \quad (23)$$

The scattering probability per unit time is given by the Fermi golden rule,

$$W_{Q,Q+\delta Q} = \frac{2\pi}{\hbar} \sum_{q_1, q_2} |t_{q_1, q_2}|^2 N_{q_1} (N_{q_2} + 1) \delta(q_1 - q_2 - \delta Q) \times \delta(\epsilon_Q - \epsilon_{Q+\delta Q} + \hbar v |q_1| - \hbar v |q_2|), \quad (24)$$

where N_q is the occupation number of the boson state q . We have normalized the probability $W_{Q,Q+\delta Q}$ in such a way that the total scattering rate is given by the integral

$$W_Q = \int W_{Q,Q+\delta Q} d\delta Q. \quad (25)$$

Using Eq. (22) we immediately obtain

$$W_{Q,Q+\delta Q} = \frac{n_0^2 N_{q_1} (N_{q_2} + 1)}{64\pi m^2 v^5 (v^2 - v_Q^2)} \Upsilon_Q^2 (\delta Q)^2. \quad (26)$$

Here q_1 and q_2 are given by the relations

$$q_1 = \frac{1}{2}\delta Q + \frac{v_Q}{2v}|\delta Q|, \quad q_2 = -\frac{1}{2}\delta Q + \frac{v_Q}{2v}|\delta Q|, \quad (27)$$

arising from conservation of energy and momentum.

Equations (23) and (26) express the scattering rate of a high energy hole excitation in a Galilean invariant quantum fluid in terms of the hole spectrum (10). They are the main result of this section.

III. SCATTERING OF HOLES IN THE ABSENCE OF GALILEAN INVARIANCE

We now turn to the more general situation in which the quantum liquid is not assumed to possess Galilean invariance. As discussed above, in this case Lagrangian variables offer no obvious advantages. Therefore we apply the standard (Eulerian) theory^{1,2} of the Luttinger liquid.

A. Eulerian description of one-dimensional quantum liquids

The standard theory of a Luttinger liquid describes the system by two bosonic fields $\phi(x)$ and $\theta(x)$ satisfying the canonical commutation relations

$$[\phi(x), \nabla\theta(x')] = i\pi\delta(x - x'). \quad (28)$$

In contrast to the coordinate y in the Lagrangian approach of Sec. II, the coordinate x denotes the real space position, see Eq. (4).

The field ϕ is defined in terms of the particle density

$$n(x) = n_0 + \frac{1}{\pi} \nabla\phi(x). \quad (29)$$

The field θ accounts for the motion of the liquid. The latter may be characterized by the momentum per particle

$$\kappa(x) = \frac{1}{2} \left(\frac{1}{n(x)} p(x) + p(x) \frac{1}{n(x)} \right). \quad (30)$$

Here the momentum density

$$p(x) = \frac{1}{2} \sum_l [p_l \delta(x - x_l) + \delta(x - x_l) p_l] \quad (31)$$

is defined in terms of the coordinates x_l and momenta p_l of the physical particles. Using the commutators $[x_l, p_{l'}] = i\hbar \delta_{ll'}$ and definition of the particle density

$$n(x) = \sum_l \delta(x - x_l), \quad (32)$$

one easily obtains the commutation relation

$$[n(x), \kappa(x')] = -i\hbar \nabla\delta(x - x'). \quad (33)$$

Comparing this relation with Eqs. (28) and (29) we identify

$$\kappa(x) = -\hbar \nabla\theta(x). \quad (34)$$

Thus the gradient of the boson field θ determines momentum per particle in the fluid.

It is worth mentioning that in the Galilean invariant case $\kappa(x) = mV(x)$ and $\nabla\theta(x)$ gives the expression

$$V(x) = -\frac{\hbar}{m} \nabla\theta(x) \quad (35)$$

for the fluid velocity. The relation between the fields $\phi(x)$, $\theta(x)$ and the Lagrangian variables $u(y)$, $p(y)$ used in Sec. II is given by Eqs. (3), (29), (30), and (34), as well as Eq. (4) which expresses the physical coordinate x in terms of the Lagrangian coordinate y .

B. Hamiltonian of the liquid in the presence of the hole

The standard form of the Hamiltonian of a Luttinger liquid is

$$H_0 = \frac{\hbar v}{2\pi} \int dx [K(\nabla\theta)^2 + K^{-1}(\nabla\phi)^2] \quad (36)$$

(see, e.g., Sec. 3.1 of Ref. 2). Here the velocity v and the dimensionless Luttinger liquid parameter K depend on the density of the particles and the interactions between

them. This Hamiltonian describes noninteracting bosons with an acoustic spectrum. It can be brought to the form

$$H_0 = \sum_q \hbar v |q| \left(b_q^\dagger b_q + \frac{1}{2} \right) \quad (37)$$

with the help of the relations

$$\nabla \phi(x) = -i \sum_q \sqrt{\frac{\pi K |q|}{2L}} \operatorname{sgn}(q) (b_q e^{iqx} - b_q^\dagger e^{-iqx}), \quad (38)$$

$$\nabla \theta(x) = i \sum_q \sqrt{\frac{\pi |q|}{2KL}} (b_q e^{iqx} - b_q^\dagger e^{-iqx}), \quad (39)$$

which express the fields ϕ and θ in terms of the boson creation and annihilation operators b_q^\dagger and b_q .

The Hamiltonian H_0 describes the low energy properties of the system and may be viewed as the fixed point Hamiltonian in the renormalization group sense. The leading irrelevant perturbation is given by cubic in the boson fields corrections,^{1,27}

$$H_\alpha = \int dx [\alpha_\theta (\nabla \phi)(\nabla \theta)^2 + \alpha_\phi (\nabla \phi)^3]. \quad (40)$$

Here we assumed that the system is invariant under inversion, $x \rightarrow -x$. In this case the Hamiltonian cannot contain odd powers of $\nabla \theta$. The parameters α_θ and α_ϕ can be expressed in terms of density dependent parameters of the fixed point Hamiltonian (36). Indeed, according to Eq. (29) a small change of density δn shifts $\nabla \phi \rightarrow \nabla \phi + \pi \delta n$. As a result the cubic perturbation (40) generates a small correction to the quadratic Hamiltonian (36). The corresponding change in its parameters is $\delta(\hbar v K / 2\pi) = \pi \alpha_\theta \delta n$, and $\delta(\hbar v / 2\pi K) = 3\alpha_\phi \delta n$. We thus find the expressions

$$\alpha_\theta = \frac{\hbar}{2\pi^2} \partial_n (vK), \quad \alpha_\phi = \frac{\hbar}{6\pi^2} \partial_n \left(\frac{v}{K} \right) \quad (41)$$

for the parameters of the Hamiltonian (40).

Let us now consider a hole excitation at point X . In general its energy ε depends not only on the wave number $Q = -i\partial_X$ but also on the density of the liquid $n(X)$ and its motion quantified by $\kappa(X)$. If the spectrum of the hole in a uniform liquid $\varepsilon(Q, n, \kappa)$ is known, the correction to the Hamiltonian of the system due to the presence of the hole may be written as

$$H_h = \varepsilon(-i\partial_X, n(X), \kappa(X)). \quad (42)$$

The right hand side here is assumed to be symmetrized with respect to the operators $-i\partial_X$, $n(X)$, and $\kappa(X)$. In view of the relations (29) and (34) this Hamiltonian describes the hole interacting with the boson fields ϕ and θ .

C. Scattering probability

The Hamiltonian of the liquid in the presence of the hole given by Eqs. (36), (40) and (42) is similar to that

obtained in Lagrangian variables for the Galilean invariant case, Eqs. (5) and (11). The main difference is that the spectrum of the hole in the Eulerian description depends not only on the fluid density n but also on its momentum κ . The rate $W_{Q, Q+\delta Q}$ of scattering of the hole by bosons, Fig. 2, can be found by repeating the steps of Sec. II C. First we perform a unitary transformation $U^\dagger H U$ of the Hamiltonian with the operator

$$U = \exp [i f_\phi \phi(X) + i f_\theta \theta(X)]. \quad (43)$$

Upon such a transformation the Hamiltonian takes the form $H_0 + H_\alpha + \tilde{H}_h$ where

$$\begin{aligned} \tilde{H}_h = & \varepsilon(-i\partial_X + f_\phi \nabla \phi + f_\theta \nabla \theta, n_0 + \nabla \phi / \pi, -\hbar \nabla \theta) \\ & + \hbar v K f_\phi \nabla \theta + \frac{\hbar v}{K} f_\theta \nabla \phi \\ & + \pi f_\theta [\alpha_\theta (\nabla \theta)^2 + 3\alpha_\phi (\nabla \phi)^2] \\ & + 2\pi f_\phi \alpha_\theta \nabla \phi \nabla \theta. \end{aligned} \quad (44)$$

Here the gradients of the boson fields are evaluated at point X .

The parameters f_ϕ and f_θ should be chosen such that the linear terms in the expansion of Eq. (44) in powers of $\nabla \phi$ and $\nabla \theta$ vanish to leading order in $-i\partial_X - Q$, where Q is the initial wavenumber of the hole, Fig. 2. This yields

$$f_\phi = \frac{1}{\pi \hbar K} \frac{v_Q K \partial_n \varepsilon_Q + \pi \hbar v \partial_\kappa \varepsilon_Q}{v^2 - v_Q^2}, \quad (45)$$

$$f_\theta = -\frac{1}{\pi \hbar} \frac{v K \partial_n \varepsilon_Q + \pi \hbar v_Q \partial_\kappa \varepsilon_Q}{v^2 - v_Q^2}, \quad (46)$$

where we used the shorthand notations $\varepsilon_Q = \varepsilon(Q, n, \kappa)$ and $v_Q = \partial_Q \varepsilon_Q / \hbar$. In Eqs. (45), (46), and the subsequent results ε_Q and its partial derivatives are evaluated at $n = n_0$ and $\kappa = 0$.

With the above choice of f_ϕ and f_θ the Hamiltonian (44) takes the form

$$\begin{aligned} \tilde{H}_h = & (\nabla \theta)^2 \left(\pi \alpha_\theta f_\theta - \frac{\hbar^2 f_\theta^2}{2m_Q^*} + \frac{\hbar^2}{2} \partial_\kappa^2 \varepsilon_Q - \hbar^2 f_\theta \partial_\kappa v_Q \right) \\ & + (\nabla \phi)^2 \left(3\pi \alpha_\phi f_\theta - \frac{\hbar^2 f_\phi^2}{2m_Q^*} + \frac{\partial_n^2 \varepsilon_Q}{2\pi^2} + \frac{\hbar f_\phi}{\pi} \partial_n v_Q \right) \\ & + \varepsilon_Q + \dots \end{aligned} \quad (47)$$

Here we omitted linear in $\nabla \phi$ and $\nabla \theta$ terms with coefficients small in $-i\partial_X - Q$, the quadratic term proportional to $(\nabla \phi)(\nabla \theta)$, and higher powers of $\nabla \phi$ and $\nabla \theta$. Such perturbations do not affect the amplitude of the scattering process depicted in Fig. 2 to leading order in δQ . In Eq. (47) the effective mass of the hole m_Q^* is defined by

$$\frac{1}{m_Q^*} = -\frac{1}{\hbar^2} \partial_Q^2 \varepsilon(Q, n_0, 0). \quad (48)$$

We now substitute relations (38) and (39) into Eq. (47) and extract the matrix element $t_{q_1 q_2}$ corresponding to the

scattering process in Fig. 2. Using Eqs. (41), (45), and (46) we express the matrix element $t_{q_1 q_2}$ in the form

$$t_{q_1 q_2} = \frac{K}{2\pi L} \frac{\sqrt{|q_1 q_2|}}{v^2 - v_Q^2} \Upsilon_Q, \quad (49)$$

where

$$\begin{aligned} \Upsilon_Q = & -\frac{1}{m_Q^*} \left[(\partial_n \varepsilon_Q)^2 - \left(\frac{\pi \hbar}{K} \right)^2 (\partial_\kappa \varepsilon_Q)^2 \right] \\ & - (v^2 - v_Q^2) \left[\partial_n^2 \varepsilon_Q - \left(\frac{\pi \hbar}{K} \right)^2 \partial_\kappa^2 \varepsilon_Q \right] \\ & + 2 \left(\frac{\pi \hbar}{K} \partial_\kappa v_Q - \frac{v}{K} \partial_n K \right) \left(v \partial_n \varepsilon_Q + v_Q \frac{\pi \hbar}{K} \partial_\kappa \varepsilon_Q \right) \\ & - 2(\partial_n v_Q) \left[v_Q \partial_n \varepsilon_Q + v \frac{\pi \hbar}{K} \partial_\kappa \varepsilon_Q \right]. \end{aligned} \quad (50)$$

Substituting Eq. (49) into the Fermi golden rule expression (24) we obtain the scattering rate

$$W_{Q, Q+\delta Q} = \frac{K^2 N_{q_1} (N_{q_2} + 1)}{64\pi^3 \hbar^2 v^3 (v^2 - v_Q^2)} \Upsilon_Q^2 (\delta Q)^2, \quad (51)$$

where q_1 and q_2 are again given by Eq. (27).

Equations (51) and (50) express the scattering rate of the hole in terms of its spectrum $\varepsilon(Q, n, \kappa)$ and the parameters of the Luttinger liquid. They are the main result of this section.

IV. DISCUSSION OF THE RESULTS

In sections II and III we obtained expressions for the rate $W_{Q, Q+\delta Q}$, of scattering processes illustrated in Fig. 2. In Sec. II we considered Galilean invariant systems and obtained the hole scattering rate in the form of Eqs. (26) and (23). Our consideration in Sec. III did not assume Galilean invariance and yielded the scattering rate in a somewhat more complicated form, Eqs. (51) and (50). Below we compare these results and discuss their applications to the problem of equilibration of one-dimensional quantum liquids and to the problem of dynamics of a mobile impurity in a Luttinger liquid.

A. Galilean invariant systems

To obtain the expressions (26) and (23) for the scattering rate we described the liquid in Lagrangian coordinates. The state of the hole was parameterized by the Lagrangian wave number conjugate to the Lagrangian coordinate of the hole Y , rather than the conventional wave number conjugate to the spatial coordinate X . Accordingly, the excitation energy $\epsilon_Q = \epsilon(Q, n)$ and velocity $v_Q = v(Q, n)$ were expressed in terms of the Lagrangian wavenumber using Eqs. (10) and (18).

To illustrate this point let us consider the simple special case of noninteracting fermions. Recalling that the hole excitation is created by moving a fermion from state $k_F - Q$ to state k_F we find its energy and velocity in the form

$$\varepsilon_Q = \frac{\hbar^2}{2m} Q(2\pi n - Q), \quad v_Q = \frac{\hbar}{m} (\pi n - Q). \quad (52)$$

To express the energy and velocity in terms of the Lagrangian wave number we substitute $Q \rightarrow nQ/n_0$, see Eqs. (10) and (18), and find

$$\epsilon_Q = \frac{\hbar^2 n^2}{2mn_0^2} Q(2\pi n_0 - Q), \quad v_Q = \frac{\hbar n}{mn_0} (\pi n_0 - Q). \quad (53)$$

Substituting (53) into (23) we obtain $\Upsilon_Q = 0$. This is the expected result, as no scattering of excitations may occur in a system of noninteracting fermions. An erroneous substitution of the ε_Q and v_Q given by Eq. (52) for ϵ_Q and v_Q into Eq. (23) would result in $\Upsilon_Q \neq 0$.

Although less common than the Eulerian variables in hydrodynamics, Lagrangian variables arise naturally in the theory of elasticity. In particular, in the case of a one-dimensional Wigner crystal (i.e. quantum anharmonic chain) the Lagrangian coordinate is essentially the number of a site of the Wigner lattice. Thus the phonon spectrum ω_Q of the Wigner crystal is naturally expressed in terms of the Lagrangian wavenumber. The weakness of interactions between the phonons in the Wigner crystal enables one to develop a microscopic theory of scattering of a high energy phonon by acoustic modes.^{18,19} On the other hand, the Wigner crystal is simply the limiting case of a quantum fluid in the regime of extremely strong repulsion between the particles. Thus the scattering of a high energy phonon can also be studied phenomenologically using the approach of Sec. II. Indeed, substituting the phonon energy $\epsilon(Q, n) = \hbar\omega_Q$ into Eq. (23) we recover the result of Ref. 19.

Experimentally, it is easier to measure the excitation spectrum $\varepsilon(Q, n)$ as a function of the conventional wavenumber. To obtain the scattering rate in terms of $\varepsilon(Q, n)$ using Eq. (26) one should substitute the relations (10) and (18) into Eq. (23). This yields,

$$\begin{aligned} \Upsilon_Q = & (\partial_n \varepsilon_Q) \partial_n (v^2 - v_Q^2) - (v^2 - v_Q^2) \partial_n^2 \varepsilon_Q \\ & - \frac{1}{m_Q^*} (\partial_n \varepsilon_Q)^2 + \frac{\hbar^2}{m_Q^*} \frac{v^2 Q^2}{n_0^2} \\ & + \frac{2\hbar v_Q}{n_0} (v_Q \partial_n v - v \partial_n v_Q). \end{aligned} \quad (54)$$

As before, the right hand side here is evaluated at $n = n_0$. Substitution of the expressions (52) for the energy and the velocity of the hole in a noninteracting Fermi gas into Eq. (54) gives the correct result $\Upsilon_Q = 0$.

Our discussion of hole scattering in Sec. III did not assume Galilean invariance of the system. The resulting scattering rate $W_{Q, Q+\delta Q}$ is given by Eqs. (51) and (50). For systems that possess Galilean invariance these results

should agree with those of Sec. II, Eqs. (26) and (23). The Luttinger liquid parameter K in Galilean invariant systems can be expressed in terms of the particle density n and the velocity v of excitations as

$$K = \frac{\pi \hbar n}{mv}. \quad (55)$$

Substituting this expression into Eq. (51) and setting $n = n_0$ we recover Eq. (26). To demonstrate that the resulting scattering rates are equal, we also need to derive the expression (23) for Υ_Q from Eq. (50).

In Sec. III the hole was described by the dependence of its energy on the wavenumber Q , particle density n and the momentum per particle κ . In Galilean invariant systems $\kappa = mV$, where V is the fluid velocity. The energy of the excitation $\varepsilon(Q, n, \kappa)$ in the moving fluid differs from its energy $\varepsilon(Q, n)$ in the stationary fluid by $\hbar QV$ (see e.g. Ref. 28), resulting in

$$\varepsilon(Q, n, \kappa) = \varepsilon(Q, n) + \hbar Q \frac{\kappa}{m}. \quad (56)$$

Substituting Eqs. (55) and (56) into Eq. (50) we obtain the result (54), which is equivalent to Eq. (23).

B. Hole vs. particle-hole excitation

Throughout this paper we considered hole excitations created by moving a particle from a state $k_F - Q$ to the Fermi point k_F . On the other hand, our discussion of hole scattering in Sec. III did not rely on this physical picture and thus should be applicable to a hole excitation obtained by removing a particle from the system. The wavenumber \tilde{Q} of such an excitation is related to Q by

$$\tilde{Q}(n, \kappa) = Q - k_F^R(n, \kappa), \quad (57)$$

where k_F^R is the wavenumber of the particle at the right Fermi point. Similarly, its energy $\tilde{\varepsilon}$ is related to ε by

$$\varepsilon(Q, n, \kappa) = \mu^R(n, \kappa) + \tilde{\varepsilon}(Q - k_F^R(n, \kappa), n, \kappa), \quad (58)$$

where μ^R is the energy of the particle at the right Fermi point. Since the two physical pictures of the hole excitation are equivalent, one should expect to find the same scattering rate for the particle-hole excitation with energy $\varepsilon(Q, n, \kappa)$ as for the hole (missing particle) with energy $\tilde{\varepsilon}(Q - \pi n_0, n, \kappa)$.

To verify such a feature of Eq. (50) we need to obtain the expressions for $k_F^R(n, \kappa)$ and the derivatives of $\mu^R(n, \kappa)$. To this end we express n and κ in terms of the numbers N^R and N^L of the right- and left-moving particles in a uniform system,

$$n = \frac{N^R + N^L}{L}, \quad \kappa = \pi \hbar \frac{N^R - N^L}{L}. \quad (59)$$

Then from $k_F = (2\pi/L)N^R$ we find

$$k_F^R(n, \kappa) = \pi n + \frac{\kappa}{\hbar}. \quad (60)$$

To find the derivatives of $\mu^R(n, \kappa)$ we substitute $\nabla\phi = \pi(n - n_0)$ and $\nabla\theta = -\kappa/\hbar$ into the Hamiltonian of the liquid given by Eqs. (36) and (40). Differentiating the resulting expression for the energy of uniform liquid with respect to N^R we find the expression

$$\begin{aligned} \mu^R(n, \kappa) &= \mu^R(n_0, 0) + \frac{\pi \hbar v}{K}(n - n_0) + vK\kappa \\ &+ \frac{\pi \alpha_\theta}{\hbar^2}[\kappa^2 + 2\pi \hbar(n - n_0)\kappa] \\ &+ 3\pi^3 \alpha_\phi (n - n_0)^2 + \dots \end{aligned} \quad (61)$$

valid to second order in $n - n_0$ and κ . From the first line one then immediately obtains the first derivatives of μ^R in the form

$$\partial_n \mu^R = \pi \hbar \frac{v}{K}, \quad \partial_\kappa \mu^R = vK. \quad (62)$$

The most convenient expression for the second derivative $\partial_n^2 \mu^R$ is obtained simply by differentiating the first of the expressions (62). To find $\partial_\kappa^2 \mu^R$ we notice that according to the second line of Eq. (61) we have $\partial_\kappa^2 \mu^R = (\pi \hbar)^{-1} \partial_n (\partial_\kappa \mu^R)$. Then from Eq. (62) we obtain

$$\partial_n^2 \mu^R = \pi \hbar \partial_n \frac{v}{K}, \quad \partial_\kappa^2 \mu^R = \frac{\partial_n (vK)}{\pi \hbar}. \quad (63)$$

Using the expressions (60), (62), and (63) one can show that Υ_Q obtained by the substitution of $\varepsilon(Q, n, \kappa)$ in the form (58) into Eq. (50) coincides with that obtained by simply replacing $\varepsilon(Q, n, \kappa) \rightarrow \tilde{\varepsilon}(Q - \pi n_0, n, \kappa)$ in Eq. (50).

C. Scattering of a mobile impurity in a Luttinger liquid

Apart from a hole excitation created by either moving a particle to the Fermi point or removing it from the system, the results of Sec. III apply to any mobile impurity in a Luttinger liquid. Our result for the scattering rate given by Eqs. (51) and (50) is applicable regardless of Galilean invariance of the system, but requires the knowledge of the energy of the mobile impurity $\varepsilon_p(Q, n, \kappa)$ as a function of the density and momentum of the liquid. Similar to the case of a hole excitation, Eq. (56), this relation simplifies in the presence of Galilean invariance. One should note, however, that unlike an intrinsic particle-hole excitation, a foreign particle has a non-zero mass M , which affects the dynamics of the liquid. The dependence of the energy $\varepsilon_p(V)$ of the impurity with momentum p on the velocity V of the fluid can be obtained by comparing the expressions for the energy of the system in the stationary frame and that moving with the fluid,

$$\varepsilon_p(V) = \varepsilon_{p-MV}(0) + pV - \frac{1}{2}MV^2, \quad (64)$$

see Ref. 29.

Denoting the momentum of the impurity $p = \hbar Q$, the velocity of the fluid $V = \kappa/m$, and expanding to second order in κ , we obtain

$$\varepsilon(Q, n, \kappa) \simeq \varepsilon(Q, n) + (\hbar Q - Mv_Q) \frac{\kappa}{m} - \frac{M}{2} \left(1 + \frac{M}{m_Q^*} \right) \frac{\kappa^2}{m^2}. \quad (65)$$

Substituting this expression into Eq. (50) we find

$$\begin{aligned} \Upsilon_Q = & (\partial_n \varepsilon_Q) \partial_n (v^2 - v_Q^2) - (v^2 - v_Q^2) \left(\partial_n^2 \varepsilon_Q + \frac{Mv^2}{n_0^2} \right) \\ & - \frac{1}{m_Q^*} \left(\partial_n \varepsilon_Q - \frac{Mv^2}{n_0} \right)^2 + \frac{\hbar^2 Q^2}{m_Q^*} \frac{v^2}{n_0^2} \\ & + \frac{2v}{n_0} (\hbar Q - Mv_Q) (v_Q \partial_n v - v \partial_n v_Q). \end{aligned} \quad (66)$$

This expression can be viewed as a generalization of our earlier result (54) to the case of impurity with non-zero mass M . The full expression for the rate of scattering of such an impurity is given by the combination of Eqs. (66) and (26). The same result can, of course, be obtained by generalizing the approach based on Lagrangian variables, Sec. II, to the case of a massive particle. The respective calculation is outlined in the Appendix.

As a simple check we consider an impurity completely decoupled from the liquid. In this case the scattering probability must vanish, and we expect to obtain $\Upsilon_Q = 0$. This is easily verified by substituting $\partial_n \varepsilon_Q = 0$, $v_Q = \hbar Q/M$ and $m_Q^* = M$ into Eq. (66).

The problem of dynamics of a mobile impurity in a Galilean invariant fluid was recently addressed by Schecter, Gangardt, and Kamenev.¹⁶ Comparing our expression (66) with the Eq. (75) of Ref. 16 we find an agreement, provided that their matrix element Γ_{+-} is related to Υ_Q as

$$\Gamma_{+-} = \frac{n_0}{mv} \frac{\Upsilon_Q}{(v^2 - v_Q^2)^2}. \quad (67)$$

The authors of Ref. 16 characterized the impurity by the number of particles N it expels from the fluid and the superfluid phase Φ . The relation (67) was obtained by expressing N and Φ in terms of the energy and velocity of the impurity with the help of Eq. (21) of Ref. 16.

D. Dissipative dynamics of holes and equilibration of one-dimensional quantum liquids

In Secs. II and III we have evaluated the rate $W_{Q,Q+\delta Q}$ of scattering of a hole excitation by the low energy bosons. Our results (26) and (51) enable one to study the dynamics of the hole excitation, provided the occupation numbers N_q of bosonic states are known. In the most interesting case when the Luttinger liquid is in equilibrium

at temperature T , the rate $W_{Q,Q+\delta Q}$ falls off exponentially at $|\delta Q| \gg T/(v - |v_Q|)$. If a single hole with energy $\varepsilon \gg Tv/(v - |v_Q|)$ is excited in the Luttinger liquid, the collisions with bosons give rise to a gradual change of its momentum at the rate

$$F = \left\langle \frac{d}{dt} \hbar Q \right\rangle = \int \hbar \delta Q W_{Q,Q+\delta Q} d\delta Q. \quad (68)$$

Using our most general result (51) for the scattering rate we obtain the force acting on the hole in the form

$$F = - \frac{2\pi K^2 \Upsilon_Q^2 T^4}{15\hbar^5} \frac{(v^2 + v_Q^2)v_Q}{(v^2 - v_Q^2)^5 v^2}. \quad (69)$$

The negative sign in this expression indicates that as a result of scattering by the bosons the wavenumber of the hole approaches one of the Fermi points, $Q = 0$ or $Q = 2\pi n_0$, i.e., the hole is eventually absorbed into the boson gas.

As we stated in Sec. IV C our results can also be applied to a mobile impurity in a Luttinger liquid. An expression for the force acting on such an impurity in the case of Galilean invariant system was found in Ref. 16. Using Eqs. (55) and (67) we find that our result (69) recovers Eq. (73) of Ref. 16.

The collisions of the hole with the bosons result in a stochastic motion which should be described in terms of the hole distribution function f_Q . Assuming again that the bosons are in thermal equilibrium and Q is sufficiently far from the Fermi points, the evolution of the distribution function is controlled by the collision integral

$$\frac{df_Q}{dt} = - \int dQ' [f_Q W_{Q,Q'} - f_{Q'} W_{Q',Q}]. \quad (70)$$

One can now use our results for $W_{Q,Q+\delta Q}$ to study the evolution of the distribution function towards the equilibrium $f_Q^{(0)} = e^{-\varepsilon(Q)/T}$.

The collision integral (70) takes a particularly simple form for Q in the vicinity of $Q_0 = \pi n_0$, where the energy $\varepsilon(Q)$ takes the maximum value, Fig. 2. In this case the typical change of energy in a collision $\delta\varepsilon \sim v_Q \delta Q \sim (v_Q/v)T \ll T$. Thus the distribution function changes very little after each collision. This enables one to bring the collision integral (70) to the Fokker-Planck form

$$\frac{d}{dt} f = -\partial_Q \left(A(Q) f - \frac{1}{2} \partial_Q [B(Q) f] \right), \quad (71)$$

where

$$A(Q) = \int \delta Q W_{Q,Q+\delta Q} d\delta Q, \quad (72)$$

$$B(Q) = \int (\delta Q)^2 W_{Q,Q+\delta Q} d\delta Q, \quad (73)$$

see, e.g., Ref. 30.

The Fokker-Planck equation for the evolution of the hole distribution function has been applied earlier to the

problem of equilibration of a one-dimensional system of interacting electrons.^{17–19,31} At low temperatures the collisions lead to relatively fast thermalization of excitations near each Fermi point. However, the full equilibration of the system includes exchange of particles between the right- and left-moving branches, which equilibrates the respective chemical potentials. This process involves diffusion of a hole excitation in momentum space from one Fermi point to the other.¹⁷ In the case of arbitrary interaction strength, when the one-dimensional system is treated as a Luttinger liquid, the equilibration rate is expressed as¹⁷

$$\tau^{-1} = \frac{3\hbar n_0^2 B}{\sqrt{2\pi m^* T}} \left(\frac{\hbar v}{T} \right)^3 e^{-\Delta/T}. \quad (74)$$

Here Δ , m^* , and B are given, respectively, by the hole energy $\varepsilon(Q_0)$, effective mass $m_{Q_0}^*$, and $B(Q_0)$ at the maximum $Q_0 = \pi n_0$. Our results for $W_{Q,Q+\delta Q}$ allow us to use Eq. (73) to evaluate B for any Luttinger liquid, regardless of the presence of Galilean invariance. Using Eq. (51) we find

$$B = \frac{4\pi}{15} \frac{K^2 \Upsilon_{Q_0}^2}{\hbar^7 v^{10}} T^5. \quad (75)$$

In the case of a Galilean invariant system, Eq. (75) recovers the result of Ref. 17.

E. Integrable models

The excitation spectrum of a one-dimensional quantum liquid can be obtained exactly for integrable models. A simple example of such a model is that of spinless fermions of mass m interacting with potential decaying as inverse square of the distance between particles, $V(r) = g/r^2$. The hole spectrum of this Calogero-Sutherland model is given by³²

$$\varepsilon(Q, n) = \frac{\hbar^2 \lambda}{2m} Q(2\pi n - Q), \quad (76)$$

where $\lambda(\lambda - 1) = gm/\hbar^2$. The velocity of holes is easily obtained from Eq. (76),

$$v(Q, n) = \frac{\hbar \lambda}{m} (\pi n - Q), \quad (77)$$

and the velocity of bosonic excitations $v = v(0, n) = \pi \hbar \lambda n / m$. Upon substitution of these values into the expression (54) one finds $\Upsilon_Q = 0$, which points to the absence of scattering of holes by bosonic excitations.

The absence of scattering of excitations is widely believed to be a universal property of integrable models.^{12,15–19,33–35} Indeed, integrability implies the presence of a large number of integrals of motion, which precludes scattering of excitations. Given the expressions (26) and (51) for the scattering probability, we expect that $\Upsilon_Q = 0$ for any integrable model.

V. SUMMARY

To summarize, in this paper we developed a phenomenological theory of scattering of hole excitations in one-dimensional spinless quantum liquids. We expressed the scattering rate in terms of the spectrum of the hole and its dependence on the fluid density and velocity. We considered liquids which may or may not possess Galilean invariance.

Our approach is based on the concept of Luttinger liquid and thus applies at any strength of the interactions between the particles. We used two alternative descriptions of the liquid, based on either Lagrangian or Eulerian variables. The former has the advantage of simplicity when applied to Galilean invariant systems. The latter is somewhat more complicated but provides a more natural description of systems without Galilean invariance.

Our most general result for the scattering probability of a hole $W_{Q,Q+\delta Q}$ is given by Eqs. (51) and (50). It simplifies considerably for Galilean invariant systems. In this case $W_{Q,Q+\delta Q}$ is given by Eq. (26) where Υ_Q may be expressed in two equivalent ways. It takes the form (23) if the energy of the hole is given as a function of the Lagrangian wavenumber. This description arises naturally in the limit of strong repulsion between particles, when the system forms a Wigner crystal.^{18,19} Alternatively, if the energy of the hole is known as a function of the Eulerian wavenumber the expression (54) for Υ_Q is more convenient.

In our theory the hole excitation is treated as a mobile impurity. Consequently, our result for the scattering probability (51) and (50) applies not only to the hole, but any impurity, including a foreign particle introduced into the system. In the Galilean invariant case, dynamics of such a particle was studied recently by Schechter, Gangardt, and Kamenev.¹⁶ We have verified that their expression for the force acting on the particle agrees with our Eq. (69).

The scattering of intrinsic hole excitations controls equilibration of one-dimensional quantum liquids. Apart from fundamental importance, the equilibration determines the conductance of long uniform quantum wires.^{31,36} Previous calculations of the equilibration rate^{17–19} assumed Galilean invariance. Our discussion in Sec. IV D extends these results to the general case.

Acknowledgments

The authors are grateful to A. Kamenev for discussions. This work was supported by the U.S. Department of Energy under Contracts No. DE-AC02-06CH11357 and DE-FG02-07ER46452.

Appendix: Description of a mobile impurity in Lagrangian variables

In Sec. IV C we discussed scattering of a massive mobile impurity in a Luttinger liquid. Our approach was based on the conventional description of the system in terms of Eulerian variables. In this Appendix we derive the main result (66) using Lagrangian variables, see Sec. II. We start by generalizing the expression (11) for the Hamiltonian of the excitation to account for non-vanishing mass M of the mobile impurity.

Following the approach of Sec. II, we consider an element of the liquid of length Δy which contains the impurity. We write the energy of the element as a sum of the kinetic energy of center of mass of the system and the energy in the center of mass frame,

$$\Delta H = \frac{(\Delta P)^2}{2(mn_0\Delta y + M)} + U(n)n_0\Delta y + \epsilon(P_Y/\hbar, n), \quad (\text{A.1})$$

cf. Eqs. (1) and (11). Expanding Eq. (A.1) to first order in $M/(mn_0\Delta y) \ll 1$, we obtain

$$\begin{aligned} \Delta H = & \frac{(\Delta P)^2}{2mn_0\Delta y} + U(n)n_0\Delta y \\ & + \epsilon(P_Y/\hbar, n) - \frac{1}{2}M \left(\frac{\Delta P}{mn_0\Delta y} \right)^2. \end{aligned} \quad (\text{A.2})$$

Here P_Y and ΔP are the momenta conjugated to the Lagrangian coordinate Y of the impurity and the center of mass coordinate R of the element, respectively. Because of the non-vanishing mass of the impurity, the latter is no longer equivalent to the displacement u of the fluid,

$$R = \frac{mn_0\Delta y[y + u(y)] + M[Y + u(Y)]}{mn_0\Delta y + M}. \quad (\text{A.3})$$

Here y is the Lagrangian coordinate of the fluid element containing the impurity, $y - \Delta y/2 < Y < y + \Delta y/2$. A small change in the positions of the element du and the particle dY results in the shift of the center of mass by

$$dR = du + \frac{M[1 + u'(Y)]}{mn_0\Delta y + M} dY. \quad (\text{A.4})$$

Unlike the case of $M = 0$ considered in Sec. II, the center of mass position is affected by the motion of the impurity. Despite that, the momentum ΔP is still expressed in terms of the operator ∂_u ,

$$\Delta P = -i\hbar\partial_R|_{Y=\text{const}} = -i\hbar\partial_u = \Delta y p(y), \quad (\text{A.5})$$

where $p(y)$ is the momentum of the liquid per unit length introduced in Sec. II A. On the other hand, the momentum $P_Y = -i\hbar\partial_Y|_{R=\text{const}}$ of the relative motion of the impurity and the liquid differs from the momentum $\hbar Q = -i\hbar\partial_Y|_{u=\text{const}}$ of the impurity in a stationary liquid,

$$P_Y = \hbar Q - \frac{M[1 + u'(Y)]}{mn_0\Delta y + M} (-i\hbar\partial_u). \quad (\text{A.6})$$

We now neglect in the denominator the mass of the impurity M compared to the much larger mass $mn_0\Delta y$ of the fluid element and use Eq. (A.5) to exclude ∂_u . This yields

$$P_Y = \hbar Q - \frac{M}{m} \frac{p(Y)}{n(Y)}, \quad (\text{A.7})$$

see Eq. (3). Since $p(Y)$ is the momentum density of the liquid, at $u' = 0$ one can interpret the last term in the right-hand side as $-MV$, where V is the velocity of the liquid.

Our theory is constructed in terms of the displacement of the fluid u rather than the center of mass coordinate R . Thus we substitute Eqs. (A.5) and (A.7) into the expression (A.2) for the Hamiltonian of the fluid element. Then the first line of Eq. (A.2) gives the energy of the liquid without the impurity, cf. Eq. (2), whereas the second line gives the Hamiltonian of the impurity in the form

$$H_i = \epsilon \left(Q - \frac{M}{m} \frac{p(Y)}{\hbar n(Y)}, n(Y) \right) - \frac{1}{2}M \left[\frac{p(Y)}{mn_0} \right]^2. \quad (\text{A.8})$$

At $M = 0$ this Hamiltonian recovers that of the massless hole, Eq. (11).

To obtain the probability $W_{Q, Q+\delta Q}$ of scattering of the impurity from state Q to $Q + \delta Q$ per unit time, we repeat the steps outlined in Sec. II C using the Hamiltonian (A.8) instead of (11). The first step is to perform the unitary transformation $U^\dagger(H_L + H_i)U$ of the form (13). The latter transforms the relevant operators as follows

$$U^\dagger u'(y)U = u'(y) + f_p \delta(y - Y), \quad (\text{A.9})$$

$$U^\dagger p(y)U = p(y) + f_u \delta(y - Y), \quad (\text{A.10})$$

$$U^\dagger QU = Q + \frac{1}{\hbar} f_u u'(Y) + \frac{1}{\hbar} f_p p(Y). \quad (\text{A.11})$$

We then choose the coefficients f_u and f_p such that the linear in the bosonic fields u and p contribution

$$\begin{aligned} H^{(1)} = & (mn_0 v^2 f_p + v_Q f_u - n_0 \partial_n \epsilon_Q) u'(Y) \\ & + \left(\frac{f_u}{mn_0} + v_Q f_p - \frac{M v_Q}{mn_0} \right) p(Y) \end{aligned} \quad (\text{A.12})$$

to the transformed Hamiltonian $U^\dagger(H_L + H_i)U$ vanishes. This gives

$$f_u = -\frac{v_Q(n_0 \partial_n \epsilon_Q - M v^2)}{v^2 - v_Q^2}, \quad f_p = \frac{n_0 \partial_n \epsilon_Q - M v_Q^2}{mn_0(v^2 - v_Q^2)}. \quad (\text{A.13})$$

The next step is to expand the transformed Hamiltonian to second order in bosonic fields,

$$\begin{aligned} H^{(2)} = & \left[-3\alpha f_p + n_0 \partial_n \epsilon_Q + \frac{1}{2} n_0^2 \partial_n^2 \epsilon_Q \right. \\ & \left. - v_Q f_u - n_0 \partial_n v_Q f_u - \frac{f_u^2}{2m_Q^*} \right] u'^2(Y) \\ & - \left[\frac{(mn_0 f_p - M)^2}{2m_Q^*} + \frac{M}{2} \right] \frac{p^2(Y)}{m^2 n_0^2} + \dots \end{aligned} \quad (\text{A.14})$$

Here ellipses denote the omitted terms of the form $u'(Y)p(Y)$ which generate coupling of the impurity to two bosons on the same branch and are therefore not relevant for our scattering problem, see Fig. 2.

We now substitute the expressions (8) and (9) into (A.14) and extract the matrix element corresponding to annihilation of boson q_1 and creation of boson q_2 ,

$$t_{q_1 q_2} = \frac{\hbar \sqrt{|q_1 q_2|}}{mn_0 L v} \left(3\alpha f_p - n_0 \partial_n \epsilon_Q - \frac{1}{2} n_0^2 \partial_n^2 \epsilon_Q \right. \\ \left. + v_Q f_u + n_0 \partial_n v_Q f_u + \frac{f_u^2}{2m_Q^*} \right. \\ \left. - \frac{(mn_0 f_p - M)^2 v^2}{2m_Q^*} - \frac{M v^2}{2} \right). \quad (\text{A.15})$$

Substituting relations (21) and (A.13) we bring this ex-

pression to the form (22) with

$$\Upsilon_Q = (\partial_n \epsilon_Q) \partial_n (v^2 - v_Q^2) - (v^2 - v_Q^2) \left(\partial_n^2 \epsilon_Q + \frac{M v^2}{n_0^2} \right) \\ - \frac{1}{m_Q^*} \left(\partial_n \epsilon_Q - \frac{M v^2}{n_0} \right)^2 \\ + \frac{2M}{n_0} v v_Q (v \partial_n v_Q - v_Q \partial_n v). \quad (\text{A.16})$$

This expression generalizes our earlier result (23) to the case of a massive mobile impurity. One can now substitute Eqs. (10) and (15) to express Υ_Q in terms of the physical energy ε of the impurity in the liquid. This procedure transforms the expression (A.16) to the form (66).

-
- ¹ F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).
 - ² T. Giamarchi, *Quantum Physics in One Dimension* (Clarendon Press, Oxford, 2004).
 - ³ V. N. Popov, Theor. Math. Phys. **11**, 565 (1972).
 - ⁴ F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
 - ⁵ I. E. Dzyaloshinskii and A. I. Larkin, Sov. Phys. JETP **38**, 202 (1974).
 - ⁶ M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, and P. L. McEuen, Nature **397**, 598 (1999).
 - ⁷ E. H. Lieb, Phys. Rev. **130**, 1616 (1963).
 - ⁸ T. Ogawa, A. Furusaki, and N. Nagaosa, Phys. Rev. Lett. **68**, 3638 (1992).
 - ⁹ A. H. Castro Neto and M. P. A. Fisher, Phys. Rev. B **53**, 9713 (1996).
 - ¹⁰ M. Pustilnik, M. Khodas, A. Kamenev, and L. I. Glazman, Phys. Rev. Lett. **96**, 196405 (2006).
 - ¹¹ M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, Phys. Rev. B **76**, 155402 (2007).
 - ¹² R. G. Pereira, S. R. White, and I. Affleck, Phys. Rev. B **79**, 165113 (2009).
 - ¹³ A. Imambekov and L. I. Glazman, Science **323**, 228 (2009).
 - ¹⁴ A. Imambekov and L. I. Glazman, Phys. Rev. Lett. **102**, 126405 (2009).
 - ¹⁵ D. M. Gangardt and A. Kamenev, Phys. Rev. Lett. **104**, 190402 (2010).
 - ¹⁶ M. Schecter, D. M. Gangardt, and A. Kamenev, Ann. Phys. **327**, 639 (2012).
 - ¹⁷ K. A. Matveev and A. V. Andreev, Phys. Rev. B **85**, 041102(R) (2012).
 - ¹⁸ K. A. Matveev, A. V. Andreev, and M. Pustilnik, Phys. Rev. Lett. **105**, 046401 (2010).
 - ¹⁹ K. A. Matveev, A. V. Andreev, and M. Pustilnik, Physica B **407**, 1898 (2012).
 - ²⁰ L. Balents, Phys. Rev. B **61**, 4429 (2000).
 - ²¹ M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, Phys. Rev. Lett. **99**, 110405 (2007).
 - ²² M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, Phys. Rev. Lett. **99**, 240404 (2007).
 - ²³ M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, Phys. Rev. B **80**, 201102 (2009).
 - ²⁴ A. Kamenev and L. I. Glazman, Phys. Rev. A **80**, 011603 (2009).
 - ²⁵ V. V. Cheianov and M. Pustilnik, Phys. Rev. Lett. **100**, 126403 (2008).
 - ²⁶ L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, 1987).
 - ²⁷ In some lattice systems, such as that of one-dimensional fermions at half filling, perturbations in the form of cosines of bosonic fields appear. In this case the problem of scattering of hole excitations is more complicated. We leave this problem for future studies.
 - ²⁸ E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part II* (Butterworth-Heinemann, 1980).
 - ²⁹ G. Baym and C. Ebner, Phys. Rev. **164**, 235 (1967).
 - ³⁰ N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, 1992).
 - ³¹ T. Micklitz, J. Rech, and K. A. Matveev, Phys. Rev. B **81**, 115313 (2010).
 - ³² B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).
 - ³³ I. E. Mazets, T. Schumm, and J. Schmiedmayer, Phys. Rev. Lett. **100**, 210403 (2008).
 - ³⁴ S. Tan, M. Pustilnik, and L. I. Glazman, Phys. Rev. Lett. **105**, 090404 (2010).
 - ³⁵ A. Imambekov, T. Schmidt, and L. I. Glazman, arXiv:1110.1374.
 - ³⁶ K. A. Matveev and A. V. Andreev, Phys. Rev. Lett. **107**, 056402 (2011).