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Dissipationless Phonon Hall Viscosity

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We study the acoustic phonon response of crystals hosting a gapped time-reversal symmetry breaking electronic state. The phonon effective action can in general acquire a dissipationless "Hall" viscosity, which is determined by the adiabatic Berry curvature of the electron wave function. This Hall viscosity endows the system with a characteristic frequency, ω_v; for acoustic phonons of frequency ω, it shifts the phonon spectrum by an amount of order $(\omega/\omega_v)^2$ and it mixes the longitudinal and transverse acoustic phonons with a relative amplitude ratio of ω/ω_v and with a phase shift of $\pm\pi/2$, to lowest order in ω/ω_v. We study several examples, including the integer quantum Hall states, the quantum anomalous Hall state in $Hg_{1-y}Mn_yTe$ quantum wells, and a mean-field model for $p_x + ip_y$ superconductors. We discuss situations in which the acoustic phonon response is directly related to the gravitational response, for which striking predictions have been made. When the electron-phonon system is viewed as a whole, this provides an example where measurements of Goldstone modes may serve as a probe of adiabatic curvature of the wave function of the gapped sector of a system.

I. INTRODUCTION

One of the most important discoveries in condensed matter physics has been that there are distinct states of matter that are distinguished not by their patterns of symmetry breaking, but by their topological order.[1] Such topological states of matter (TSM) cannot be described by local order parameters, but can sometimes be characterized by quantized topological responses to external fields. For example, the quantum Hall states[2,3]—the first topological states discovered in nature—can be characterized by their quantized Hall conductance. The three-dimensional time-reversal invariant topological insulators discovered more recently[4–6] can be characterized by a topological magneto-electric effect[7,8]. More generically, topological insulators in arbitrary dimensions can be characterized by topological responses to electromagnetic fields[7]. However, many TSM cannot be characterized by electromagnetic response. For example in topological superconductors the charge conservation symmetry is effectively broken, and the electromagnetic field is screened. Thus more generic response properties need to be investigated in order to distinguish different topological states.[9–11]

In this paper, we propose a response property, the "phonon Hall viscosity," for gapped states that break time-reversal symmetry. For a quantum liquid, the viscosity tensor η_{ijkl} is defined by the linear response $T_{ij} = -p_{ij}\eta_{ijkl}v_{kl}$, with T_{ij} the stress tensor, p_{ij} the pressure tensor, and $v_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ the gradient of the velocity field v_i. Usually, a finite viscosity indicates dissipation in the system, similar to a finite resistivity. However the viscosity can have a dissipationless component, associated with the part of η_{ijkl} that is anti-symmetric under exchange of the first and second pair of indices.[12] This Hall viscosity can only exist in a system that breaks time-reversal symmetry and is analogous to the dissipationless Hall resistivity. The Hall viscosity has appeared in the hydrodynamic theory of the A-phase of He-3[13] and was studied for quantum Hall liquids by J. E. Avron *et. al.*[14,15]. It has since been studied for various $(2+1)$-d topological states including IQH states[14], the $(2+1)$-d Dirac model[16], fractional quantum Hall (FQH) states[17–20] and $p_x + ip_y$-paired topological superconductors[17]. The Hall viscosity provides a probe of gapped time-reversal symmetry breaking states in the charge neutral channel, which does not require charge conservation and thus may be a suitable response for TSC and more generic TSM that cannot be characterized by topological electromagnetic response. In particular, it was recently proposed that the Hall viscosity of a rotationally invariant system is related to the angular momentum carried by each quasi-particle of the system[17,21], which is in turn proportional to the "topological shift" of the topological field theory on the sphere[22]. However, the Hall viscosity is only defined for a liquid in continuum, since the stress tensor is a momentum current which is ill-defined if continuous translation symmetry is broken by the lattice. The discussions of Hall viscosity in the literature have been treating the electron system as a liquid without explicitly considering the lattice effects. This approximation is in general problematic since the Hall viscosity intrinsically depends on a length scale, and there are two natural length scales in a gapped system: a length scale associated with the energy gap, and a different length scale associated with the electron density. In general, the Hall viscosity will depend on both of these length scales and will therefore depend on non-universal short range physics. Even if the results in the continuum approximation are applicable in some cases, we are facing another problem of how to observe the Hall viscosity in general since it is difficult to measure the velocity and stress of the electron liquid. To distinguish the Hall viscosity defined in this traditional way with the phonon Hall viscosity that we study in this work, we refer to the Hall viscosity of the continuum electron liquid as the gravitational Hall viscosity since the viscosity tensor of the electron liquid can be considered as a response to an external deformation of the spatial metric g_{ij}[14].

To solve these problems of the gravitational Hall vis-

cosity, we alternatively define the phonon Hall viscosity, which is the adiabatic response of the electron state to the deformation of the crystal, *i.e.*, to acoustic phonons. Instead of the stress tensor which couples to the deformation of the spatial metric and is only well-defined in the continuum, the deformation of the crystal and the electron-phonon coupling are always well-defined. The linear response of the electron liquid to the crystal deformation leads to a correction to phonon dynamics. The strain field $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ plays the role of the spatial metric g_{ij}, with u_i the displacement field of the nuclei. The phonon Hall viscosity is defined by the linear response $\left\langle \frac{\partial H}{\partial u_{ij}} \right\rangle = \lambda_{ijkl} u_{kl} + \eta_{ijkl} \dot{u}_{kl}$ with H the electron Hamiltonian depending on the lattice strain u_{ij}, λ_{ijkl} the elastic moduli, and η_{ijkl} the phonon Hall viscosity. The phonon Hall viscosity is well-defined for all gapped states with a regular lattice background, and is physically observable through phonon dynamics.

In the rest of this paper, we will first give a general definition of the phonon Hall viscosity in Sec. II and show how it relates to the adiabatic Berry curvature of the many-body electron wave function. In Sec. III we study the phonon Hall viscosity in several example systems such as quantum Hall states, quantum anomalous Hall states and $(p+ip)$ superconductors. We make comparisons between the phonon Hall viscosity and the gravitational Hall viscosity in the continuum limit, and in certain examples, we analyze features of the Hall viscosity that are universal or depend only on the low-energy theory of the electronic system. Finally in Sec. IV we will discuss the physical consequences of the phonon Hall viscosity, give order of magnitude numerical estimates, and discuss experimental prospects for the observation of phonon Hall viscosity.

II. DEFINITION OF THE PHONON HALL VISCOSITY

A. Effective action of acoustic phonons

In this section we will define the phonon Hall viscosity based on a generic discussion of acoustic phonon dynamics. The dynamics of acoustic phonons can be described using the long-wavelength effective action in terms of the displacement fields $\boldsymbol{u}(\boldsymbol{r})$, which describe the displacement of an atom from its original location. For a gapless state, such as a metal or a magnet, the phonon effective action will not be a local theory in terms of the strain fields. For an insulator, dissipation can be ignored at frequencies below the energy gap and the long-wavelength elastic response is described by an effective action for the displacement fields $\boldsymbol{u}$, which can be obtained by integrating out the electrons: $e^{-S_{eff}[\boldsymbol{u}]} = \int \mathcal{D}c^\dagger \mathcal{D}c e^{-S[\boldsymbol{u},c,c^\dagger]}$. For a time-reversal invariant insulator, the phonon effective action is, to lowest order,

$$S_{eff} = \frac{1}{2}\int d^d x dt (\rho \partial_t u_j \partial_t u_j - \lambda_{ijkl}\partial_i u_j \partial_k u_l), \quad (1)$$

where $\lambda_{ijkl} = \lambda_{klij} = \lambda_{jikl}$ are the elastic moduli; its symmetry under interchange of the first or second pair of indices follows from invariance of the energy under rigid rotations. In the presence of time-reversal symmetry breaking, additional "Hall" viscosity terms are allowed:

$$\delta S_H = \frac{1}{2}\int d^d x dt \eta_{ijkl}\partial_i u_j \partial_k \dot{u}_l, \quad (2)$$

where $\eta_{ijkl} = -\eta_{klij}$. The total effective action is given by $S_{eff} + \delta S_H$. As we discuss further in the following sections, we expect that for a gapped system, such time-reversal symmetry breaking terms in the phonon effective action will be dominated by contributions from the many-body electron state. For an inversion symmetric system, this is the only additional term that can be added, to cubic order in momenta. Anharmonic corrections to the acoustic phonon dynamics go like $(\partial u)^4$, so they are $\mathcal{O}(k^4)$, implying that the Hall viscosity term may be distinctly measurable since it is lower order in momenta. Impurities in the crystal may be difficult to treat, but as we discuss in the Section IV B, their contributions are not sensitive to the sign of the time-reversal symmetry breaking of the electronic system and may be separated from Hall viscosity contributions.

Up to total derivatives, (2) only depends on $\eta_{ijkl} + \eta_{kjil}$:

$$\delta S_H = \frac{1}{2}\int d^d x dt [\eta^+_{ijkl}\partial_i u_j \partial_k \dot{u}_l + \eta^-_{ijkl}\partial_i (u_j \partial_k \dot{u}_l)], \quad (3)$$

where $\eta^\pm_{ijkl} = \frac{1}{2}(\eta_{ijkl} \pm \eta_{kjil})$. The boundary terms may have interesting consequences for the surface waves of a medium with such a Hall viscosity, but we will ignore them in this paper. In some cases it will be conceptually more clear to write the above action in terms of the strain tensor $u_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ and the rotation tensor $m_{ij} \equiv \frac{1}{2}(\partial_i u_j - \partial_j u_i)$:

$$\delta S_H = \frac{1}{2}\int d^d x dt [\eta^{SS}_{ijkl} u_{ij}\dot{u}_{kl} + \eta^{AA}_{ijkl} m_{ij}\dot{m}_{kl} + 2\eta^{SA}_{ijkl} u_{ij}\dot{m}_{kl}], \quad (4)$$

where $\eta^{SS}_{ijkl} = \eta^{SS}_{jikl} = -\eta^{SS}_{klij}$, $\eta^{AA}_{ijkl} = -\eta^{AA}_{jikl} = -\eta^{AA}_{klij}$, and $\eta^{SA}_{ijkl} = \eta^{SA}_{jikl} = -\eta^{SA}_{ijlk}$ can all be deduced from η_{ijkl}. For an isotropic three-dimensional system, η_{ijkl} must vanish. In two dimensions, η^{AA} always vanishes; for an isotropic 2D system, or one with $\pi/4$ rotation symmetry, η^{SS} and η^{SA} each reduce to a single number: $\eta^H \equiv \eta^{SS}_{xxxy} = \eta^{SS}_{xyyy}$ [14,15] and $\eta^M \equiv \eta^{SA}_{xxxy} = \eta^{SA}_{yyxy}$. In this case,

$$\delta S_H = 2\int d^2 x dt [\eta^H (u_{xx} - u_{yy})\dot{u}_{xy} + \eta^M (u_{xx} + u_{yy})\dot{m}_{xy}], \quad (5)$$

where $\eta^H = \frac{1}{2}(\eta_{xxxy} + \eta_{xxyx}) = \frac{1}{2}(\eta_{xyyy} + \eta_{yxyy})$ and $\eta^M = \frac{1}{2}(\eta_{xxxy} - \eta_{xxyx}) = \frac{1}{2}(\eta_{xyyy} - \eta_{yxyy})$. It also follows that, up to boundary terms, (5) is equivalent to:

$$\delta S_H = \int d^2x dt [\eta_{xxxy}(u_{xx} - u_{yy})\dot{u}_{xy} + \eta_{xxxy}(u_{xx} + u_{yy})\dot{m}_{xy}]. \quad (6)$$

To obtain this, we have used the fact that, up to boundary terms, η_{xxyx} does not contribute to the action.

B. Phonon Hall viscosity as a response property of the electron system

In the adiabatic approximation, it is assumed that the motion of the lattice is infinitely slow compared with the motion of the electrons, so that at any moment, the electrons are in their ground state with respect to that particular instantaneous configuration of the lattice. Within this approximation, the effect of lattice displacements is to alter the parameters in the effective Hamiltonian of the electron system. The dependence of these parameters on the atom positions can be calculated using standard *ab initio* methods. Thus, in the adiabatic approximation, the electrons will be described by an effective Hamiltonian where the lattice displacements appear as external parameters: $H[\{\boldsymbol{u}_i\}]$. $\boldsymbol{u}_n$ is the displacement of the nth atom from its original location. We may also view H as a function of the Fourier components $\boldsymbol{u}_{\boldsymbol{q}} = \frac{1}{\sqrt{N_{site}}} \sum_{\boldsymbol{n}} \boldsymbol{u}_{\boldsymbol{n}} e^{i\boldsymbol{q}\cdot\boldsymbol{n}}$. Then, the following linear response formula

$$\eta_{ab}(\boldsymbol{q}, \omega) = \frac{1}{\omega}\frac{1}{L^d}\int dt e^{i\omega t}\left\langle \left[\frac{\partial H}{\partial u_{\boldsymbol{q},a}}(t), \frac{\partial H}{\partial u_{-\boldsymbol{q},b}}(0)\right]\right\rangle \quad (7)$$

gives an additional term to the acoustic phonon effective action of the form

$$\delta S = \frac{1}{2}\int d^{d+1}x d^{d+1}x' \eta_{ab}(x - x')u_a(x)\dot{u}_b(x'), \quad (8)$$

where x here is a $(d+1)$-component vector including space and time, $\eta_{ab}(x)$ is the Fourier transform into real space-time of $\eta_{ab}(\boldsymbol{q}, \omega)$, and we have taken the continuum limit to get the displacement field $\boldsymbol{u}(x)$. The leading order term that is independent of uniform displacements $\boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{a}$ is given by (2). Starting from (8), we find

$$\eta_{ijkl} = \frac{1}{2}\lim_{\omega\to 0}\lim_{q\to 0}\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_k}\eta_{jl}(\boldsymbol{q}, \omega) \quad (9)$$

For spatially homogeneous deformations, the distortion tensor $w_{ij} \equiv \partial_i u_j$ is a constant. To calculate η_{ijkl}, it will be more convenient to take w_{ij} to be a constant and to treat it as a parameter in H. Then,

$$\eta_{ijkl} = \frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\frac{1}{L^d}\int dt e^{i\omega t}\left\langle \left[\frac{\partial H}{\partial w_{ij}}(t), \frac{\partial H}{\partial w_{kl}}(0)\right]\right\rangle + (i \leftrightarrow k) \quad (10)$$

For spatially inhomogenous deformations, we can continue to use the DC response (10) instead of the exact AC response, as long as the acoustic phonon frequency is much less than the electronic energy gap.

It is well-known that the adiabatic response of a Hamiltonian to changes in some parameter is directly related to Berry curvature[23] of the ground state wave function. For a Hamiltonian $H[\{\lambda_i\}]$ that depends on a set of parameters $\{\lambda_i\}$, we have

$$\left\langle \frac{\partial H}{\partial \lambda_i}\right\rangle = \frac{\partial E}{\partial \lambda_i} + \Omega_{ij}\dot{\lambda_j}, \quad (11)$$

where Ω_{ij} is the Berry curvature of the ground state wave function. Thus η_{ijkl} is given by the Berry curvature associated with adiabatically varying the distortion tensor $w_{ij} \equiv \partial_i u_j$ as external parameters:

$$i\eta_{ijkl} = \frac{1}{2}\left(\frac{\partial}{\partial w_{ij}}\langle\psi|\frac{\partial}{\partial w_{kl}}|\psi\rangle - \frac{\partial}{\partial w_{kl}}\langle\psi|\frac{\partial}{\partial w_{ij}}|\psi\rangle + (i \leftrightarrow k)\right), \quad (12)$$

where $|\psi\rangle$ is the ground state of the Hamiltonian H.

III. EXAMPLES

In this section, we will study some examples of systems with a phonon Hall viscosity, including electrons hopping among the s-orbitals of a square lattice in a background magnetic field, a simple model for the quantum anomalous Hall state in HgMnTe quantum wells, and a simple mean-field model of a spinless $p_x + ip_y$ superconductor. In certain limits we compare the phonon Hall viscosity of these systems with their conventional Hall viscosity studied in the literature.

A. Hofstadter Model

Consider a square lattice with nearest and next-nearest neighbor hopping:

$$H = -\frac{1}{2}\sum_{\langle ij\rangle} t_{ij}e^{iA_{ij}}c_i^\dagger c_j - \frac{1}{2}\sum_{\langle\langle ij\rangle\rangle}\tilde{t}_{ij}e^{iA_{ij}}c_i^\dagger c_j + h.c. \quad (13)$$

Consider hopping among s-wave orbitals, in which case t_{ij} and $\tilde{t}_{ij}$ depend only on the distance $|\mathbf{r}_j - \mathbf{r}_i|$ between atoms. To leading order in the crystal deformations, $t_{i,i+\hat{\mathbf{x}}} \simeq t + t'u_{xx}$, $t_{i,i+\hat{\mathbf{y}}} \simeq t + t'u_{yy}$, and $\tilde{t}_{i,i+\hat{\mathbf{x}}\pm\hat{\mathbf{y}}} = \tilde{t} + \tilde{t}'(\frac{1}{2}(u_{xx} + u_{yy}) \pm u_{xy})$. If $t(r)$ is the hopping matrix element between the s-wave orbitals that are a distance r apart, $t \equiv t(a)$, $\tilde{t} = t(\sqrt{2}a)$, $t' = a\frac{\partial t}{\partial r}|_a$, $\tilde{t}' \equiv \sqrt{2}a\frac{\partial t}{\partial r}|_{\sqrt{2}a}$, where a is the lattice spacing in the absence of lattice deformations. In the absence of a background electromagnetic field and for constant lattice displacements, the Hamiltonian can be written in momentum space as $H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$. We note that a significant

effect of the strain fields on the energy of the system is to change the on-site energy of atomic orbitals. However, this contribution does not affect the Hall viscosity below, so we ignore it.

In the continuum limit, the dispersion is, up to a constant,

$$\epsilon_k \simeq \frac{1}{2m^*} k_i k_j g_{ij} - (t' + \tilde{t}')(u_{xx} + u_{yy}), \tag{14}$$

where the effective mass is defined by $\frac{1}{2m^*} \equiv (t/2 + \tilde{t})$, and $g_{ij} = \delta_{ij} + \delta g_{ij}$,

$$\delta g = 2m^* \frac{\tilde{t}'}{2} \begin{pmatrix} (1 + \frac{t'}{\tilde{t}'})u_{xx} + u_{yy} & 2u_{xy} \\ 2u_{xy} & (1 + \frac{t'}{\tilde{t}'})u_{yy} + u_{xx} \end{pmatrix} \tag{15}$$

In the presence of a gauge field, we take $k \to -iD \equiv -i(\partial - iA)$, so the effective theory becomes

$$H = -\frac{1}{2m^*} g_{ij} D_i D_j - (t' + \tilde{t}')(u_{xx} + u_{yy}). \tag{16}$$

The phonon Hall viscosity is then related directly to the gravitational Hall viscosity of the electronic fluid[14]. We find

$$\eta^H \equiv \eta^{SS}_{xxxy} = (t/2 + \tilde{t})^{-2} \frac{\tilde{t}' t'}{4} \eta^H_{gr}. \tag{17}$$

For N_L filled Landau levels, $\eta^H_{gr} = N_L \hbar n/4$, where n is the density of electrons, is the Hall viscosity of the electron liquid when the crystal is ignored.[14,17] The prefactor $(t/2+\tilde{t})^{-2}\tilde{t}'t'$ can explicitly be verified to be of order one for typical s-wave orbitals and typical separations between atoms; it can also be calculated fairly precisely using *ab initio* methods. Note that since the Hamiltonian depends only on the strain field u_{ij}, the phonon effective action also only depends on u_{ij} and there is no dependence on the rotation tensor m_{ij}; i.e. $\eta^{SA} = \eta^{AA} = 0$.

Therefore, we see that the result of[14,24] for integer quantum Hall (IQH) states has a direct effect in the phonon response, which, as will be discussed in Sec. IV, is a directly measurable physical quantity. For higher density systems, we cannot take the continuum limit; the phonon Hall viscosity is still a well-defined quantity that is calculable through linear response theory, but the previously defined "gravitational Hall viscosity"of Ref. 14 and 24 is not well-defined.

B. Quantized anomalous Hall state and $Hg_{1-y}Mn_yTe$ quantum wells

Here we will calculate the phonon Hall viscosity for $Hg_{1-y}Mn_yTe$ quantum wells, which exhibit a quantized anomalous Hall (QAH) state for certain thicknesses of the quantum well and spin polarization of the Mn ions[25]. The QAH state is a band insulator that exhibits a quantized Hall conductance in the absence of a net magnetic field. The first lattice model for such a state was introduced by Haldane[26], and since then it has been proposed to be realized in $Hg_{1-y}Mn_yTe$ quantum wells.[27–29] As the quantum well thickness and the magnetization of the Mn ions is tuned, the system can be tuned between different topological states: a quantum spin Hall state, QAH states, and the topologically trivial state.

At the topological phase transitions, the phonon Hall viscosity exhibits non-analyticities that can be accounted for in the continuum Dirac approximation. In what follows, we will calculate the phonon Hall viscosity for physically realistic parameters, we will isolate the universal contributions that depend only on the low energy physics near the Dirac cones and we make contact with the calculations of Ref. 16 for the regularized gravitational Hall viscosity of the continuum Dirac model.

The model for $Hg_{1-y}Mn_yTe$ quantum wells is given by a four-band Bloch Hamiltonian:

$$H(\boldsymbol{k}) = \begin{pmatrix} h_+(\boldsymbol{k}) & 0 \\ 0 & h_-(\boldsymbol{k}) \end{pmatrix}, \tag{18}$$

where the two-band Bloch Hamiltonians can be expanded in terms of Pauli matrices $h_\pm(\boldsymbol{k}) = \epsilon_\pm(\boldsymbol{k})\mathbb{I} + \boldsymbol{d}_\pm(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$, and $h_-(\boldsymbol{k}) = h_+^*(-\boldsymbol{k})$. In the continuum limit and in the absence of lattice distortions, expanding near the Γ point $\boldsymbol{k} = (0,0)$, we have

$$\begin{aligned} d_{\pm,x} + i d_{\pm,y} &= A(\pm k_x + i k_y), \\ d_{\pm,z} &= M_\pm - B(k_x^2 + k_y^2), \\ \epsilon(k) &= C_\pm - D(k_x^2 + k_y^2), \end{aligned} \tag{19}$$

where $M_\pm = M \pm \delta M/2$. The parameters A, B, C, D, M, and the lattice spacing a are given in Ref. 30 and 31 for HgCdTe/HgTe quantum wells and the relevant ones are listed in Table I. δM depends on Mn doping and spin polarization, as discussed in Ref. 25.

The phonon Hall viscosity will be a sum of the contributions of each of the two blocks:

$$\eta_{ijkl} = \eta^+_{ijkl} + \eta^-_{ijkl}, \tag{20}$$

where

$$\begin{aligned} \eta^\pm_{ijkl} =& \frac{1}{2} \frac{\hbar}{8\pi^2} \frac{1}{a^2} \int d^2k \hat{\mathbf{d}}_\pm \cdot \left(\frac{\partial \hat{\mathbf{d}}_\pm}{\partial(\partial_i u_j)} \times \frac{\partial \hat{\mathbf{d}}_\pm}{\partial(\partial_k u_l)} \right) \\ &+ (i \leftrightarrow k) \end{aligned} \tag{21}$$

In order to calculate η_{ijkl}, we need to obtain the Hamiltonian as a function of lattice distortions. To do this, observe that the blocks $h_\pm(\boldsymbol{k})$ are composed of the spin-orbit coupled states $|s, \pm\frac{1}{2}\rangle$ and $|p_x \pm i p_y; \pm\frac{1}{2}\rangle$.[30] Concentrating on a single 2×2 block – for definiteness consider h_+ – the Hamiltonian is written as:

$$\begin{aligned} H_+ = \frac{1}{2} \sum_{n,i} & c_n^\dagger (\tilde{t}_i \mathbb{I} + t_i \sigma^z + \boldsymbol{e}_i \cdot \boldsymbol{\sigma}) c_{n+\hat{i}} \\ &+ m_+ \sum_n c_n^\dagger \sigma^z c_n + h.c., \end{aligned} \tag{22}$$

where $i = x, y$ and n labels the sites of a two-dimensional square lattice. $\boldsymbol{\sigma}$ is the vector of Pauli matrices and the hopping parameters are, to first order in lattice distortions,

$$\begin{aligned} \tilde{t}_i &= \tilde{t} + a\tilde{t}'\partial_i u_i, \\ t_i &= t + at'\partial_i u_i, \\ \boldsymbol{e}_x &= i(\lambda + a\lambda'\partial_x u_x)(\hat{x} + \partial_x u_y \hat{y}) \\ \boldsymbol{e}_y &= i(\lambda + a\lambda'\partial_y u_y)(\partial_y u_x \hat{x} + \hat{y}) \end{aligned} \tag{23}$$

The lattice parameters λ, t, $\tilde{t}$, and m used above are related to the continuum parameters A, B, C, D, and M through: $\lambda = A/a$, $M_\pm = m_\pm + 2t$, $B = a^2 t/2$, $C = 2\tilde{t}$, and $D = \tilde{t}/2$. The hopping parameters are functions of the distance between neighboring atoms; the prime indicates a derivative with respect to this distance. The upper 2×2 block has topological phase transitions as $\frac{a^2 M_+}{2B}$ is tuned. The Chern number C_1 of the lower of the two bands is:

$$C_1 = \begin{cases} 1 & \text{for} \quad 2 < \frac{a^2 M_+}{2B} < 4 \\ -1 & \text{for} \quad 0 < \frac{a^2 M_+}{2B} < 2 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

and similarly for the lower 2×2 block.

From the Kubo formula (21), we see that the only non-zero terms are $\eta^+_{xxxy} = -\eta^+_{yyyx}$. The full effective action is given by (6); making the physically reasonable approximation $a\lambda' \approx \lambda$ and $at' \approx t$, we find

$$\eta^+_{xxxy} = \frac{\hbar}{8\pi a^2} f\left(\frac{aA}{2B}, \frac{a^2 M_+}{2B}\right), \tag{25}$$

where f is a function of dimensionless parameters:

$$f(\alpha, \beta) = \int \frac{\alpha^2 \sin^2 k_x (\beta - 2 + \cos k_y) d^2 k}{[\alpha^2(\sin^2 k_x + \sin^2 k_y) + (\beta - 2 + (\cos k_x + \cos k_y))^2]^{3/2}}. \tag{26}$$

In Fig. 1, we plot the function $\frac{1}{8\pi^2} f(\alpha, \beta)$ as a function of β for various choices of α. Note f has non-analyticities at the quantum phase transitions $\beta = 0, 2$, and 4.

The full Hall viscosity is given by the sum of contributions from the two blocks (see 20), which corresponds to taking the difference of the curve in Fig. 1 for two different values of m:

$$\eta_{xxxy} = \frac{\hbar}{8\pi^2} \frac{1}{a^2} (f_+ - f_-), \tag{27}$$

where $f_\pm = f\left(\frac{Aa}{2B}, \frac{a^2 M_\pm}{2B}\right)$. We may view η_{xxxy} as a function of $M = (m_+ + m_-)/2 - 4B/a^2$ and $\delta M \equiv m_+ - m_-$. In Figs. 2 - 3, we plot η_{xxxy} as a function of either M or δM, using physically realistic parameters (Table I) and focusing on the region near one of the transitions into the QAH state.

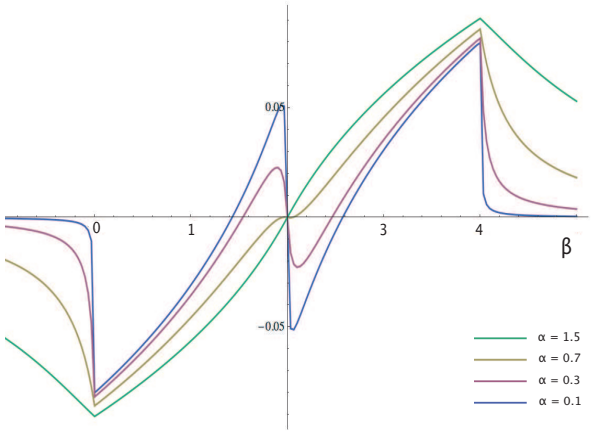

FIG. 1. The function $\frac{1}{8\pi^2} f(\alpha, \beta)$, plotted as a function of β for various choices of α.

A (eV · Å)	B (eV · Å^2)	a (Å)	M (eV)
3.645	-68.6	6.46	-0.01

TABLE I. Realistic parameters for HgTe quantum wells, taken from Ref. 31

In Fig. 2, we fix M and plot $\eta_{xxxy}(\delta M)$; experimentally this can be done by tuning an external magnetic field. We observe a discontinuity in the slope of η_{xxxy} as a function of δM at the transition into the QAH state. We denote this discontinuity $\Delta \frac{\partial \eta_{xxxy}}{\partial \delta M}$. In Fig. 3, we fix $a^2 \delta M / 2B \ll 1$ and plot $\eta_{xxxy}(M)$. This shows a discontinuity $\Delta\eta(M)$. While a full lattice calculation is required to compute the Hall viscosity, the properties of these non-analyticities can be accounted for in the continuum Dirac approximation to the above lattice model. In this approximation, we take $\sin k \approx k$ and $\cos k \approx 1 - k^2/2$ in (26); near the topological phase transition at $\beta = 0$, we have:

$$f(\alpha, \epsilon) \approx -\frac{2\pi\alpha^2 |\epsilon|(\epsilon - 1)}{(\alpha^2 - \epsilon)^2} + O(\Lambda), \tag{28}$$

where Λ is a high-energy cutoff. While the Hall viscosity will in general depend on Λ, the first term above is responsible for the non-analyticities of f and consequently of η_{xxxy}. Using (28), we can estimate the discontinuity

in $\partial\eta_{xxxy}/\partial\delta M$ at the transition, as shown in Fig. 2:

$$\left|\Delta\frac{\partial\eta_{xxxy}}{\partial\delta M}\right| = \frac{\hbar}{\pi}\frac{|B|}{a^2A^2}. \tag{29}$$

Similarly, the discontinuity in $\eta_{xxxy}(M)$ for fixed $a^2\delta M/2B \ll 1$ at the transition is found to be

$$|\Delta\eta_{xxxy}| = \left|\frac{\hbar}{\pi}\frac{B}{a^2A^2}\delta M\right|. \tag{30}$$

Finally, we note that the discontinuity in the derivative of $\eta_{xxxy}(M)$ can also be computed using (28): $\Delta\frac{\partial\eta_{xxxy}}{\partial M} \propto [\frac{\partial^2 f}{\partial\epsilon^2}|_{\epsilon=0^+} - \frac{\partial^2 f}{\partial\epsilon^2}|_{\epsilon=0^-}]$. We find:

$$\Delta\frac{\partial\eta_{xxxy}}{\partial M} \approx \frac{\hbar}{\pi}\delta M\left(\frac{2}{A^2} - \frac{a^2}{4B^2}\right). \tag{31}$$

The first term above depends only on parameters of the low-energy theory of the state and is independent of the lattice spacing and other high-energy details. This is essentially the contribution that was found for the gravitational Hall viscosity in the regularized Dirac model studied in Ref. 16. We see that the other contributions to the Hall viscosity that we find are dependent on the lattice spacing and other high energy details, which is why they are missed in the regularization of the continuum Dirac model of Ref. 16.

C. Interacting states and $p_x + ip_y$ superconductors

For interacting systems, while there is no universal relationship between phonon Hall viscosity and the gravitational Hall viscosity, a major exception occurs in systems with only on-site interactions: $H = \sum_{ij}(t_{ij}e^{iA_{ij}}c_i^\dagger c_j + h.c.) + U\sum_i n_{i\uparrow}n_{i\downarrow}$. If the hopping involves s-wave orbitals, then in the dilute limit, where the system can be described by a continuum interacting theory, the effect of a strain in the lattice is equivalent to a deformation of the gravitational metric. For such systems, the phonon Hall viscosity is then directly related to the gravitational Hall viscosity, through a proportionality factor of order unity, as in (17).

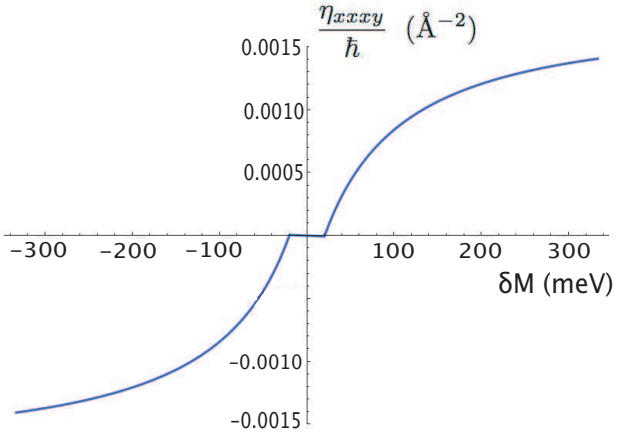

FIG. 2. Plot of $\eta_{xxxy}/\hbar$ as a function of δM for fixed M and for realistic parameters (Table I). There is a discontinuity in the slope at the transition to the quantum anomalous Hall state, which occurs at $\delta M = \pm 2M$.

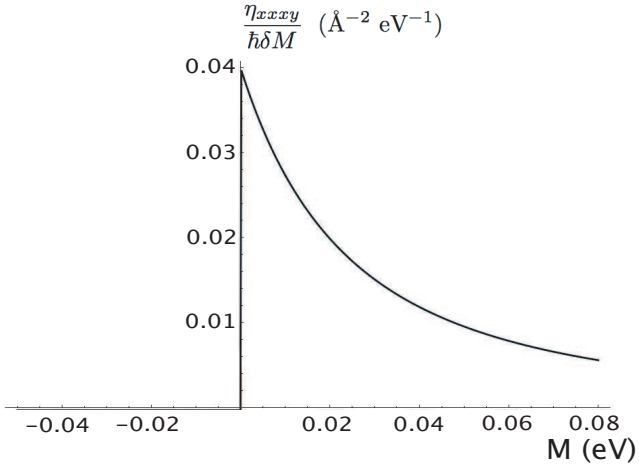

FIG. 3. Plot of $\eta_{xxxy}/\hbar\delta M$ as a function of M for fixed $\delta M = 10^{-7}eV$ and for realistic parameters (Table I). There is a discontinuity at the transition to the quantum anomalous Hall state, which occurs at $\delta M = \pm M$.

As an example of an interacting state with a phonon Hall viscosity, we consider a BCS mean-field description for a $p_x + ip_y$ superconductor, since such a model may be relevant for the chiral superconductor Sr_2RuO_4[32]. First consider the interaction between the nearest and the next nearest neighbors,

$$U = -V\sum_{\langle ij\rangle} c_i^\dagger c_i c_j^\dagger c_j - \tilde{V}\sum_{\langle\langle ij\rangle\rangle} c_i^\dagger c_i c_j^\dagger c_j. \tag{32}$$

Taking only the Cooper channel of the interaction yields:

$$U = -\sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c_{\mathbf{k}'}^\dagger c_{-\mathbf{k}'}^\dagger c_{-\mathbf{k}} c_{\mathbf{k}}, \tag{33}$$

where

$$\begin{aligned} V_{\mathbf{k}\mathbf{k}'} =& \frac{1}{N}[2(V+\tilde{V}) + (V'+\tilde{V}')(u_{xx}+u_{yy}) \\ & - (V/2+\tilde{V})(k_i - k_i')(k_j - k_j')\tilde{g}_{ij}]. \end{aligned} \tag{34}$$

$\tilde{g}_{ij} = \delta_{ij} + \frac{\tilde{V}'}{2(V/2+\tilde{V})}\delta\tilde{g}_{ij}$ has the same form as g_{ij} in eq. (15), with t and $\tilde{t}$ replaced by V and $\tilde{V}$:

$$\delta\tilde{g}_{ij} = \begin{pmatrix} (1+\frac{V'}{\tilde{V}'})u_{xx} + u_{yy} & 2u_{xy} \\ 2u_{xy} & (1+\frac{V'}{\tilde{V}'})u_{yy} + u_{xx} \end{pmatrix}. \tag{35}$$

$V' = a\frac{\partial V}{\partial r}|_{r=a}$ and $\tilde{V}' = \sqrt{2}a\frac{\partial\tilde{V}}{\partial r}|_{r=\sqrt{2}a}$, where $V(r)$ and $\tilde{V}(r)$ are the nearest and next-nearest neighbor interactions, which only depend on the distance r between the

nearest or next-nearest neighbor sites. As will be discussed below, the phonon Hall viscosity is proportional to the gravitational Hall viscosity only when the two metrics, g_{ij} and $\tilde{g}_{ij}$ are the same.

To study the simplest possible scenario, we assume the electrons hop among a single s-wave orbital of the atoms. The BCS mean-field Hamiltonian is then

$$H_{BCS} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{k}} [\Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} + H.c.], \quad (36)$$

where ϵ_k is given by Eq.(14). For $p_x + ip_y$ pairing, we take the order parameter to be,

$$\begin{aligned} \Delta_{\mathbf{k}} =& \Delta(\sin k_x + i \sin k_y) \\ &+ \tilde{\Delta}(\sin k_x \cos k_y + i \sin k_y \cos k_x). \end{aligned} \quad (37)$$

The order parameter must satisfy a self-consistency equation:

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}, \quad (38)$$

where $V_{\mathbf{k}\mathbf{k}'}$ is the Cooper channel of the interaction and $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2}$.

To calculate the phonon Hall viscosity for this system, we need to obtain the effect of the lattice deformation on the order parameter Δ_k. For simplicity, we consider the long-wavelength continuum limit, where

$$H_{BCS} = \frac{1}{2} \sum_k \Psi^\dagger_k H_{BdG}(k) \Psi_k, \quad (39)$$

where $\Psi^\dagger_k = (c^\dagger_k, c_{-k})$, and

$$\hat{H}_{BdG} = \begin{bmatrix} \epsilon_k - \mu & \Delta(|k|)(\hat{k}_i e_{xi} + i\hat{k}_j e_{yj}) \\ \Delta^*(|k|)(\hat{k}_i e_{xi} - i\hat{k}_j e_{yj}) & -\epsilon_k + \mu \end{bmatrix}. \quad (40)$$

$\hat{k}_i$ is a unit vector, and $\Delta(|k|)$ is chosen to fall to zero far away from the Fermi surface (the cutoff scheme is explain in Fig.4), μ is their chemical potential, and $e_{ij} = \delta_{ij} + \delta e_{ij}$ where δe_{ij} is a linear combination of the lattice distortions $w_{kl} \equiv \partial_k u_l$. It is convenient to define an "order parameter metric" $g^\Delta_{ij} = e_{ia} e_{ja}$, which we can fix in terms of g and $\tilde{g}$ using the BCS self-consistency equation. In the continuum limit, where the system is rotationally invariant, it is simple to show (see Appendix A) that

$$g^\Delta_{ij} = \gamma g_{ij} + (1 - \gamma) \tilde{g}_{ij}, \quad (41)$$

where γ is a constant that can be determined from the self-consistency equations (see Appendix A). To first order in w_{ij}, we have $g^\Delta_{ij} = \delta_{ij} + (\delta e_{ij} + \delta e_{ji})$, so the above equation does not fix $\delta e_{ij} - \delta e_{ji}$. We may fix $\delta e_{ij} - \delta e_{ji}$ by observing that the only affect of a rigid rotation of the crystal should be to rotate k_x and k_y into each other. Thus: $e_{ij} - e_{ji} = 2m_{ij}$. These considerations fix the dependence of the order parameter on the lattice deformations. Thus we can now use the Kubo formula and explicitly obtain the Hall viscosity:

$$\eta_{ijkl} = \frac{1}{2} \frac{\hbar}{8\pi^2} \int d^2k \hat{d} \cdot (\frac{\partial \hat{d}}{\partial w_{ij}} \times \frac{\partial \hat{d}}{\partial w_{kl}}) + (i \leftrightarrow k) \quad (42)$$

In Appendix B, we present some details of the calculation. In the case where $\tilde{g}_{ij} = g_{ij}$, the calculation simplifies considerably and we find a simple result

$$\eta^H = \frac{\tilde{t}' t'}{4(t/2 + \tilde{t})^2} \frac{1}{4} \hbar n = \frac{\tilde{t}' t'}{4(t/2 + \tilde{t})^2} \eta^H_{gr}, \quad (43)$$

where the gravitational Hall viscosity is $\eta^H_{gr} = \frac{1}{4}\hbar n$. The constant of proportionality η^H / η^H_{gr} is typically of order one. In Fig. 4, we show the Hall viscosity calculated from the results presented in Appendix B for $g_{ij} \neq \tilde{g}_{ij}$. While this does not modify the Hall viscosity in the weak pairing limit much, the behavior close to the weak to strong pairing transition is dependent on completely non-universal features, such as the frequency dependence of the pairing gap.

For a system with a circular Fermi surface, the $p_x + ip_y$ state has another interesting feature: the $U(1)_\phi \times U(1)_{L_z}$ symmetry associated with particle number and angular momentum conservation is spontaneously broken to a diagonal subgroup, $U(1)_{L_z - \phi}$. This implies that in the effective action, m_{xy} and ϕ, the angle through which the crystal is rotated about the z direction and the overall phase of the order parameter, respectively, should appear together as $m_{xy} + \phi$. This means that the effective action of the crystal involves the phase of the order parameter as well:

$$\begin{aligned} \delta S_H = 2 \int d^d x dt [& \eta^H (u_{xx} - u_{yy}) \dot{u}_{xy} \\ &+ \eta^M (u_{xx} + u_{yy})(\dot{m}_{xy} + \dot{\phi})]. \end{aligned} \quad (44)$$

Physically the $U(1)_{L_z - \phi}$ symmetry requires L_z to increase by $\hbar$ when two electrons are adiabatically added. Since m_{xy} and ϕ are conjugate variables to L_z and the number of Cooper pairs, the above effective action contains the Berry phase term for this adiabatic process:

$$\hbar \frac{\delta n}{2} = \frac{\partial \mathcal{L}_H}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}_H}{\partial \dot{m}_{xy}} = \delta L_z. \quad (45)$$

The physical consequences of the coupling between the uniform compression, $u_{xx} + u_{yy}$, and ϕ means that the change in particle density is given by

$$\delta n = 4\eta^M (u_{xx} + u_{yy}) = 4\eta^M \frac{\delta A}{A}, \quad (46)$$

where A is the area of the 2D system and n is the particle density, when μ is held constant. A generic superconductor of course has

$$\delta n / n = \alpha(\mu) \delta A / A, \quad (47)$$

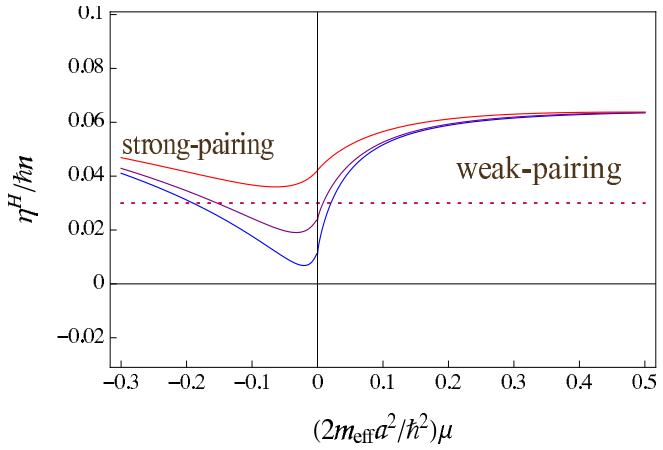

FIG. 4. The $p_x + ip_y$ superconductor phonon Hall viscosity. The filled curves are for the cases with $g_{ij} \neq \tilde{g}_{ij}$, while the dotted curves are for $g_{ij} = \tilde{g}_{ij}$. We have set $V'/V = 2t'/t$, $\tilde{t}/t = -\tilde{t}'/t = 0.75$, and $\tilde{V}/V = -\tilde{V}'/V = 0.40$. The pairing gap was set to have a Gaussian cutoff $\Delta = \hat{\Delta} k \exp[-k^2/2(\delta k)^2]$, with $\delta k = 1/a$ and $\hat{\Delta}(2m_{eff}a/\hbar^2) = 0.3, 0.4, 0.6$ for the blue, purple and red curves respectively. Note that while η^H converges to the same asymptotic values in the weak-pairing limit, in the strong-pairing phase and near the quantum phase transition, it is not proportional to the density n when $g_{ij} \neq \tilde{g}_{ij}$.

where $\alpha(\mu)$ is a constant depending on the chemical potential; note that we would have $\alpha = -1$ if the total particle number had been fixed. The additional symmetry in this problem, which relates phase rotations to rotations of the crystal, sets $\alpha(\mu) = -4\eta^M/n$. This suggests a way to experimentally measure η^M in systems that have an additional symmetry involving spatial rotations of the crystal.

IV. PHYSICAL CONSEQEUNCES OF PHONON HALL VISCOSITY

A. Acoustic phonon dynamics

Consider the effective long-wavelength elasticity theory of a crystal, given by (1) and (2). Note that this is an expansion in the displacement fields and its gradients. Since for the sound waves $\omega \propto |k| + \cdots$, the Hall viscosity terms are actually of order k^3, so for consistency one must also include a term of the form $\delta S_3 = \int d^dx dt \lambda_{ijklm} \partial_m \partial_i u_j \partial_k u_l$, but such a term vanishes in the presence of inversion symmetry. As we noted in Section II A, anharmonic effects are $\mathcal{O}(k^4)$, so Hall viscosity may be distinctly measurable because its effects appear at lower order in k. We briefly mention the effects of impurities later. The physical consequences of Hall viscosity terms can be analyzed most simply by considering 2D systems whose long-wavelength elastic theory is isotropic. This would be directly physically relevant for 2D systems with square lattice symmetry; the considerations directly apply for layered 3D crystals as well, where the 2D layers have a square lattice symmetry and where we consider phonons with wave-vector oriented parallel to the 2D layers. For such systems, the elastic theory simplifies and one obtains for the equation of motion:

$$\ddot{u}_i = c_t^2 \nabla^2 u_i + (c_l^2 - c_t^2)\partial_i \nabla \cdot \boldsymbol{u} + \eta \nabla^2 \epsilon^{ij} \dot{u}_j/\rho, \quad (48)$$

where the indices i, j run over the 2D spatial coordinates, c_t and c_l are the transverse and longitudinal sound velocities, respectively, and $\eta \equiv \eta_{xxxy}$ is the Hall viscosity. It is simple to show that such a wave equation does not admit purely transverse or purely longitudinal solutions. Let us denote $e_\pm$ as the eigenmodes of the system, and let the basis $\begin{pmatrix}1 & 0\end{pmatrix}^T$ and $\begin{pmatrix}0 & 1\end{pmatrix}^T$ correspond to the longitudinal and transverse acoustic phonon modes, respectively. In this basis, the eigenmodes of the system in the presence of the Hall viscosity are

$$e_+ \propto \begin{pmatrix}1\\ -ix\end{pmatrix} + \mathcal{O}(x^2), \qquad e_- \propto \begin{pmatrix}-ix\\ 1\end{pmatrix} + \mathcal{O}(x^2), \quad (49)$$

where $x \equiv \frac{\omega}{\omega_v} = \frac{\eta\omega}{\rho(c_t^2 - c_l^2)}$ is a dimensionless parameter. This defines the characteristic frequency $\omega_v = \frac{\rho(c_t^2 - c_l^2)}{\eta} \sim \frac{B}{\eta}$, where B is the bulk modulus of the crystal. (Note that for a crystal, $c_l > \alpha c_t$, for some constant α of order unity, which is why $\rho(c_t^2 - c_l^2) \sim \rho c_l^2 \sim B$). Observe that to linear order, there is a $\frac{\pi}{2}\mathrm{sgn}(\eta)$ phase shift between the longitudinal and transverse modes. The dispersion relation is

$$\omega^2 = \frac{k^2}{2}[c_l^2 + c_t^2 + \eta^2 k^2/\rho^2 \pm \sqrt{c_l^4 + (c_t^2 + k^2\eta^2/\rho^2)^2 + 2c_l^2(k^2\eta^2/\rho^2 - c_t^2)}]. \quad (50)$$

The shift in frequency for a given acoustic phonon mode for finite η is $\Delta\omega/\omega(\eta = 0) \sim x(\omega(\eta = 0))^2$. In principle then the shift in frequency can determine η. However, since this is not sensitive to the sign of the Hall viscosity, it may not be a useful method in practice for determining the Hall viscosity.

An analysis of surface (Rayleigh) waves of a 3D medium with non-zero $\eta_{xxxy} = \eta_{yyyx}$ displays similar behavior. For a medium with surface at $z = 0$, a surface wave travelling in the x-direction must have $u_y = 0$ in the absence of Hall viscosity, due to stress-free boundary conditions. In the presence of a Hall viscosity, the surface wave acquires a u_y component, which to linear order in $\eta\omega/\rho c^2$ differs by a phase shift of $\pi/2$ and has a relative amplitude of $\eta\omega/\rho c^2$.

B. Numerical estimates and discussion of possible experimental detection

As explained above, the physical consequences of a Hall viscosity in the phonon effective action is the mixing of

longitudinal and transverse sound modes, which at a frequency ω is determined by ω/ω_v, with $\omega_v = \rho c^2/\eta^H$ a characteristic frequency scale associated with the Hall viscosity. To lowest order in ω/ω_v, there is a phase shift of $\pi/2$. For the bulk modes, a more precise value of ω_v is $\omega_v = \rho(c_l^2 - c_t^2)/\eta^H$, and the amplitude ratio between the two modes in the elliptical polarization is ω/ω_v. It is not clear what the minimal experimentally measurable value of the Hall viscosity is. From elementary considerations, we can put a rough bound on what may realistically be measured. First, for the sound waves to not destroy the crystal, we expect that the strain is small: $|\partial u| \ll 1$, which implies that $|ku| \ll 1$. Since we have roughly $\omega \sim ck$, this implies $\omega u \ll c$. The amount of mixing is determined by ω/ω_v, so the amount of amplitude from the other mode that is mixed in is $(\omega/\omega_v)u$. For this to be realistically measurable, this should be much larger than the size of the quantum fluctuations of the wave function of an atom, which is on the order of 0.1 Å. That is, $|\omega u| \gg \omega_v \times 0.1\mathring{A}$. Thus we have the conditions $\omega_v \times 0.1\mathring{A} \ll \omega u \ll c$. For a typical sound velocity of 5×10^5 cm/s, this implies

$$\omega_v \ll 5 \times 10^{14} s^{-1}. \tag{51}$$

For smaller sound velocities, this bound will be smaller. This is not a fundamental bound, but a practical one. This is because in principle it is possible to measure oscillations of the center of the wave function of an atom at a resolution that is smaller than the characteristic size of its wave function.

The two-dimensional mass density of the crystal is $\rho \propto Am_p/a^2$, where A is the atomic number of atoms of the crystal. Typically, $A \sim 10$, and $a \sim 4 \times 10^{-8} cm$, so $\rho \sim 10^{-8} g/cm^2$. Furthermore, $c_l^2 - c_t^2 \sim (\alpha^2 - 1)10^{10} cm^2/s^2$, where $\alpha = c_l/c_t$ and typically $\alpha \sim 2$. $\eta^H \sim \eta^H_{gr} \sim \hbar n_e$, where n_e is the electron density. Thus, for a typical 2D electron density of $10^{15} cm^{-2}$, with $\alpha \sim 2$, we see $\omega_v \sim 10^{14} s^{-1}$. For a 1 GHz measurement, $\omega/\omega_v \sim 10^{-5}$; at 100 GHz, $\omega/\omega_v \sim 10^{-3}$ and, depending on material parameters, could be closer to 10^{-2}. Note that the acoustic phonon frequencies must be much less than the energy gap of the electronic state, which for a 10 K gap translates to approximately 0.2 THz, and also less than the phonon Debye frequency, which is close to 10 THz.

For quantum Hall states induced by an external magnetic field, the electron densities are usually low, $n_e \sim 10^{11} cm^{-2}$, yielding immeasurably small values for ω/ω_v. An exception may be graphene, where recent advancements in applying extremely large gate voltages may allow for much larger densities of electrons participating in quantum Hall states.[33] The necessary values of $n_e \sim 10^{15} cm^{-2}$ usually appear in states that spontaneously break time-reversal symmetry, such as quantum anomalous Hall states, ferromagnetic insulators, or chiral superconductors, where the effective magnetic moment per lattice site is much larger than could be produced by an external magnetic field.

The effects discussed here would most easily be measurable in bulk, layered 3D crystals, for phonons propagating along an in-plane high-symmetry direction. While there are a number of examples of 3D IQH states[34–39], the value of the Hall viscosity is probably too small to be measured, since the particle density is too small, though not typically as low as in 2D quantum wells. More promising systems are those that spontaneously break time-reversal symmetry, because those typically will have much higher angular momentum densities. One promising candidate may be the chiral superconductor Strontium Ruthenate, which exists as a 3D crystal and may have a large enough Hall viscosity because it spontaneously breaks time-reversal. Another promising set of materials to measure a phonon Hall viscosity are 3D ferromagnetic insulators, for example those discussed in Ref. 40–42 . Note that in cases where the spin gap is small, an external magnetic field can be used to ensure the electronic state is fully gapped.

In principle, one way to measure such an effect would be through pulsed echo ultrasound meaurements, which have been successful in detecting circular polarization between transverse sound waves. However, while bulk pulsed ultrasound seems to be limited to frequencies on the order of 1 GHz, it is not clear what the ultimate bounds are on an experimentally accessible amplitude ratio between transverse and longitudinal waves. A more promising experimental technique appears to be time-dependent x-ray diffraction[43]. Such techniques have been developed only recently over the last decade and have been used to directly image acoustic phonon modes[44,45].

One complication of measuring the phonon Hall viscosity is related to the effects of crystal disorder, which can also mix transverse and longitudinal waves. However, the effects of disorder are not sensitive to the sign of the time-reversal symmetry breaking of the electronic state; this dependence on the sign of the time-reversal symmetry breaking is unique to the Hall viscosity, and can be used to extract the phonon Hall viscosity even for imperfect crystals.

We would like to point out that related phenomena occur in various other time-reversal breaking systems. The phonons in a ferromagnet, for example, can exhibit acoustic Faraday rotation, where the two transverse modes acquire a circular or elliptic polarization[46,47]. However such systems cannot be described by a simple local effective action in terms of the strain fields because they are coupled to magnons, which are gapless; integrating out the magnons will result in non-local terms in the crystal effective action. The physical manifestations of such phenomena are also quite different; they occur as resonances when the frequency and wavelength of the phonons and magnons are matched. More directly related phenomena have been considered in the case of ionic crystals in an external magnetic field[48], and in Tkachenko modes of vortex lattices in rotating superfluids[49–51]. In these situations, one obtains a related equation of motion as in (48).

V. CONCLUSION

We have proposed the acoustic phonon Hall viscosity as a novel probe into the adiabatic Berry curvature of the many-body electron wave function for gapped, time-reversal symmetry breaking electronic states, and we have computed it and studied its behavior in a number of theoretical models. It is important to note that while the phonon Hall viscosity appears as a general higher order term in the acoustic phonon effective action, its contribution is dominated by the Berry curvature of the electronic wave function. Additional contributions besides the electronic one would require the atoms of the crystal to directly couple to a time-reversal breaking field; such an effect may be appreciable only in ionic crystals in a large magnetic field, where locally the charge of the ions is not entirely screened by the electrons. In more generic situations, there is no net charge locally as it is completely screened by the electrons, and therefore the only time-reversal symmetry breaking effects occur through the adiabatic change of the electron Hamiltonian as the phonons are excited.

In some simple cases, we have found that the phonon Hall viscosity is proportional, with a numerical factor of order 1, to the gravitational Hall viscosity of the continuum electronic theory. Our numerical estimates indicate that this is a measurable effect and there may be a number of materials, particularly the ferromagnetic insulators or chiral superconductors, which might be suitable candidates for experimentally detecting the phonon Hall viscosity by measuring time-reversal symmetry breaking corrections to acoustic phonon dynamics. Since the effects are expected to be small and their measurement would require high spatial resolution, it appears that time-dependent x-ray diffraction may be currently the most promising probe. As phonon Hall viscosity is developed into a more mature experimental probe, we hope that it can eventually be useful as a novel lens into the possible topological behavior of electron systems.

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Appendix A: $p+ip$ BCS self-consistency

The BCS mean-field Hamiltonian is:

$$H_{BdG} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c_{-\mathbf{k}} c_{\mathbf{k}}, \tag{A1}$$

where

$$\begin{aligned}\Delta_{\mathbf{k}} =& \Delta(\sin k_x + i \sin k_y) \\ &+ \tilde{\Delta}(\sin k_x \cos k_y + i \sin k_y \cos k_x)\end{aligned} \tag{A2}$$

and the kinetic energy is that of Eq.(14). $\Delta_{\mathbf{k}}$ satisfies a self-consistency equation:

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}, \tag{A3}$$

where $V_{\mathbf{k}\mathbf{k}'}$ is the Cooper channel of the interaction and $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2}$.

When the crystal is strained, the order parameter will take the following form in the continuum limit:

$$\Delta_k = \Delta(\hat{k}_i e_{xi} + i\hat{k}_i e_{yi}), \tag{A4}$$

where $e_{ab}[\{\partial_i u_j\}]$ are functions of the distortion tensor $w_{ij} \equiv \partial_i u_j$. In order to obtain the phonon response, we need to obtain this function to linear order in $\partial_i u_j$. First, observe that due to the $\mathrm{U}(1)_{L_z} \times \mathrm{U}(1)_\phi \to \mathrm{U}(1)_{L_z - \phi}$ symmetry breaking, ϕ spatial rotation is equivalent to the gauge transformation $\Delta \to \Delta \exp(i\phi)$. Therefore

$$e_{xy} - e_{yx} = w_{xy} - w_{yx}. \tag{A5}$$

Next, observe that e_{xx}, e_{yy}, and $e_{xy} + e_{yx}$ are symmetric in x and y, so they can only depend on the strain tensor u_{ij}. The self-consistency equation can be thought of as a constraint on the "order parameter metric" g^Δ:

$$f(g^\Delta, g, \tilde{g}) = 0. \tag{A6}$$

Considering the variations of this:

$$\delta g^\Delta_{ij} \frac{\partial f}{\partial g^\Delta_{ij}} + \delta g_{ij} \frac{\partial f}{\partial g_{ij}} + \delta \tilde{g}_{ij} \frac{\partial f}{\partial \tilde{g}_{ij}} = 0. \tag{A7}$$

For a rotationally invariant system, $\partial f / \partial g_{ij} \propto \delta_{ij}$, and similarly for $\partial f / \partial \tilde{g}_{ij}$. This implies that $\delta g^\Delta_{ii} = \gamma \delta g_{ii} + \tilde{\gamma} \delta \tilde{g}_{ii}$, where γ and $\tilde{\gamma}$ are constants. Now observe that when $g_{ij} = \tilde{g}_{ij}$, we are merely implementing a coordinate transformation, so we should have $g^\Delta_{ij} = g_{ij}$, which implies $\gamma + \tilde{\gamma} = 1$. Furthermore, for a rotationally invariant system, the deformations $\delta g_{xx} = -\delta g_{yy} = e$ and $\delta \tilde{g}_{xx} = -\delta \tilde{g}_{yy} = \tilde{e}$ are equivalent to the deformations $\delta g_{xy} = e$ and $\delta \tilde{g}_{xy} = \tilde{e}$, because for a rotationally invariant system, the two types of deformations simply differ by a rotation. Thus, we conclude:

$$g^\Delta_{ij} = \gamma g_{ij} + (1 - \gamma) \tilde{g}_{ij}. \tag{A8}$$

The constant γ can be found from the self-consistency equation.

To actually calculate γ from the self-consistency equation, we note that the assumption we made above tells us that the effect of change in the kinetic metric g_{ij} should be proportional to the effect of coordinate transformation $\delta_{ij} \to g_{ij}$. This means that, if we consider the case $\delta g_{xx} = -\delta g_{yy} = e_1$ and $\delta \tilde{g} = 0$, the change in the order parameter should come out as $\delta \Delta_{\mathbf{k}} = \gamma e_1 \Delta(\hat{k}_x - i\hat{k}_y)/2$. Thus, to the self-consistency condition

$$\frac{\partial\Delta_{\mathbf{k}}}{\partial e_1} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\left[\frac{1}{2E_{\mathbf{k}'}}\frac{\partial\Delta_{\mathbf{k}'}}{\partial e_1} - \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}^2}\left(\frac{\epsilon_{\mathbf{k}'}-\mu}{E_{\mathbf{k}'}}\frac{\partial\epsilon_{\mathbf{k}'}}{\partial e_1} + \frac{|\Delta_{\mathbf{k}'}|}{E_{\mathbf{k}'}}\frac{\partial|\Delta_{\mathbf{k}'}|}{\partial e_1}\right)\right], \tag{A9}$$

we can insert

$$\begin{aligned}\frac{\partial\Delta_{\mathbf{k}}}{\partial e_1} =& \frac{1}{2}\gamma\Delta(\hat{k}_x - i\hat{k}_y),\\ \frac{\partial\epsilon_{\mathbf{k}}}{\partial e_1} =& \epsilon_{\mathbf{k}}(\hat{k}_x^2 - \hat{k}_y^2),\end{aligned} \tag{A10}$$

to obtain

$$\frac{1}{2}\gamma\Delta(\hat{k}_x - i\hat{k}_y) = -\frac{1}{2}\gamma\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta(\hat{k}'_x - i\hat{k}'_y)}{2E_{\mathbf{k}'}} + \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}\epsilon_{\mathbf{k}'}(\epsilon_{\mathbf{k}'}-\mu)}{2E_{\mathbf{k}'}^3}(\hat{k}_x^2 - \hat{k}_y^2) + \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}|\Delta'_{\mathbf{k}}|}{2E_{\mathbf{k}'}^3}\frac{\partial|\Delta_{\mathbf{k}'}|}{\partial e_1} \tag{A11}$$

and solve for γ. For this step, it is convenient to eliminate $\partial|\Delta_{\mathbf{k}'}|/\partial e_1$ through a coordinate transformation argument, which makes use of the fact that in the integral over k', we can do a coordinate transformation $(k'_x, k'_y) \to (k'_x, k'_y) + \gamma e_1(k'_x, -k'_y)/2$ without changing the value of the integral because the Jacobian of this transformation is one, to linear order in e_1. This gives us

$$\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\frac{\Delta_{\mathbf{k}'}|\Delta'_{\mathbf{k}}|}{2E_{\mathbf{k}'}^3}\frac{\partial|\Delta_{\mathbf{k}'}|}{\partial e_1} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\left(\frac{1}{2E_{\mathbf{k}'}}\frac{\partial\Delta_{\mathbf{k}'}}{\partial e_1} - \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}^2}\frac{\epsilon_{\mathbf{k}'}-\mu}{E_{\mathbf{k}'}}\frac{\partial\epsilon_{\mathbf{k}'}}{\partial e_1}\right) - \sum_{\mathbf{k}'}\frac{\partial V_{\mathbf{k}\mathbf{k}'}}{\partial e_1}\frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}. \tag{A12}$$

We can obtain the derivatives on the right-hand side by noting that this coordinate transformation gives us

$$\Delta_{\mathbf{k}'} \to \Delta_{\mathbf{k}'} + \frac{1}{2}\gamma\Delta(\hat{k}'_x - i\hat{k}'_y), \quad \epsilon_{\mathbf{k}'} \to \epsilon_{\mathbf{k}'} + \gamma\epsilon_{\mathbf{k}}(\hat{k}_x^2 - \hat{k}_y^2), \quad V_{\mathbf{k}\mathbf{k}'} \to V_{\mathbf{k}\mathbf{k}'} - \gamma V(\hat{k}_x\hat{k}'_x - \hat{k}_y\hat{k}'_y) \tag{A13}$$

where we used $V_{\mathbf{k}\mathbf{k}'} = -2V(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')$. Since the change in $|\Delta_{\mathbf{k}'}|$ is same, we can insert this result from the coordinate transformation into Eq.(A9).

We find at the weak coupling limit

$$\gamma = -\frac{1-\ln(2\epsilon_c/|\Delta_0|)}{1+\ln(2\epsilon_c/|\Delta_0|)}, \tag{A14}$$

where ϵ_c is the cutoff energy of the pairing and $|\Delta_0|$ is the value of Δ at the Fermi surface. The negative sign of γ is because our kinetic metric deformation increases the density of state around $(0, \pm k_f)$ compared to $(\pm k_f, 0)$ and therefore it is energetically advantageous to have the p_y pairing to be stronger than the p_x pairing.

Appendix B: p+ip BCS Hall viscosity calculation

Now we would like to calculate the Hall viscosity for the $p_x + ip_y$ BdG Hamiltonian. We want:

$$\eta^H = \frac{1}{2}(\eta_{xxxy} + \eta_{xxyx}) = \frac{\hbar}{16\pi^2}\int d^2k\,\hat{\mathbf{d}}\cdot\left(\frac{\partial\hat{\mathbf{d}}}{\partial w_{xx}}\times\frac{\partial\hat{\mathbf{d}}}{\partial u_{xy}}\right) \tag{B1}$$

From $H_{BdG} = \boldsymbol{d}\cdot\boldsymbol{\tau}$, we have

$$\begin{aligned} d_x &= \Delta(\hat{k}_x e_{xx} + \hat{k}_y e_{xy}),\\ d_y &= \Delta(\hat{k}_y e_{yy} + \hat{k}_x e_{yx}),\\ d_z &= \frac{1}{2m^*}k_i k_j g_{ij} - \mu; \end{aligned} \tag{B2}$$

note that in a flat metric, $d_\parallel \equiv \sqrt{d_x^2 + d_y^2}$ and d_z would depend only on k. In this section, since $\mathbf{d}_\parallel$ has explicit dependence only on e_{ij} while d_z has explicit dependence only on g_{ij}, we will treat e_{ij} and g_{ij} as independent variables.

For our calculation, we want to write the derivatives in terms of g_{ij} and e_{ij}, which gives us

$$\begin{aligned}\frac{\partial\mathbf{d}_\parallel}{\partial w_{xx}} =& \frac{1}{2}\left(\tilde{a}\frac{\partial\mathbf{d}_\parallel}{\partial e_{xx}} + \tilde{b}\frac{\partial\mathbf{d}_\parallel}{\partial e_{yy}}\right) = \frac{1}{2}\left(\tilde{a}k_x\frac{\partial\mathbf{d}_\parallel}{\partial k_x} + \tilde{b}k_y\frac{\partial\mathbf{d}_\parallel}{\partial k_y}\right),\\ \frac{\partial d_z}{\partial w_{xx}} =& a\frac{\partial d_z}{\partial g_{xx}} + b\frac{\partial d_z}{\partial g_{yy}} = \frac{1}{2}\left(ak_x\frac{\partial d_z}{\partial k_x} + bk_y\frac{\partial d_z}{\partial k_y}\right),\\ \frac{\partial\mathbf{d}_\parallel}{\partial u_{xy}} =& \tilde{b}\left(\frac{\partial\mathbf{d}_\parallel}{\partial e_{xy}} + \tilde{b}\frac{\partial\mathbf{d}_\parallel}{\partial e_{yx}}\right) = \tilde{b}\left(k_x\frac{\partial\mathbf{d}_\parallel}{\partial k_y} + k_y\frac{\partial\mathbf{d}_\parallel}{\partial k_x}\right),\\ \frac{\partial d_z}{\partial u_{xy}} =& 2b\frac{\partial d_z}{\partial g_{xy}} = b\left(k_x\frac{\partial d_z}{\partial k_y} + k_y\frac{\partial d_z}{\partial k_x}\right),\end{aligned} \tag{B3}$$

where

$$\begin{aligned} a &= m_{eff}\left(t'+\tilde{t}'\right), \\ b &= m_{eff}\tilde{t}', \\ \tilde{a} &= \gamma[m_{eff}(t'+\tilde{t}')] + (1-\gamma)\frac{V'+\tilde{V}'}{V/2+\tilde{V}}, \\ \tilde{b} &= \gamma(m_{eff}\tilde{t}') + (1-\gamma)\frac{\tilde{V}'}{V/2+\tilde{V}}, \end{aligned} \tag{B4}$$

and we used

$$\begin{aligned} \frac{\partial d_z}{\partial g_{ij}} =& \frac{1}{4}\left(k_i\frac{\partial d_z}{\partial k_j} + k_j\frac{\partial d_z}{\partial k_i}\right)_{g_{ij}=\delta_{ij}}, \\ \frac{\partial \mathbf{d}_\parallel}{\partial e_{ij}} =& k_i\left.\frac{\partial \mathbf{d}_\parallel}{\partial k_j}\right|_{e_{ij}=\delta_{ij}} \end{aligned} \tag{B5}$$

To see this, first observe:

$$\begin{aligned} \frac{\partial \mathbf{d}_\parallel}{\partial w_{ij}} &= \sum_{kl}\frac{\partial e_{kl}}{\partial w_{ij}}\frac{\partial \mathbf{d}_\parallel}{\partial e_{kl}}, \\ \frac{\partial d_z}{\partial w_{ij}} &= \sum_{kl}\frac{\partial g_{kl}}{\partial w_{ij}}\frac{\partial d_z}{\partial g_{kl}}. \end{aligned} \tag{B6}$$

Assuming $w_{xy}-w_{yx} = e_{xy}-e_{yx}$, all $\partial g_{kl}/\partial w_{ij}, \partial e_{kl}/\partial w_{ij}$ vanish except for

$$\begin{aligned} a =& \frac{\partial g_{xx}}{\partial w_{xx}} = \frac{\partial g_{yy}}{\partial w_{yy}}, \\ b =& \frac{\partial g_{xx}}{\partial w_{yy}} = \frac{\partial g_{yy}}{\partial w_{xx}} = \frac{\partial g_{xy}}{\partial w_{xy}} = \frac{\partial g_{xy}}{\partial w_{yx}}, \end{aligned} \tag{B7}$$

and

$$\begin{aligned} \tilde{a} =& 2\frac{\partial e_{xx}}{\partial w_{xx}} = 2\frac{\partial e_{yy}}{\partial w_{yy}}, \\ \tilde{b} =& 2\frac{\partial e_{xx}}{\partial w_{yy}} = 2\frac{\partial e_{yy}}{\partial w_{xx}} \\ =& 2\frac{\partial e_{xy}}{\partial w_{xy}} - 1 = 2\frac{\partial e_{xy}}{\partial w_{yx}} + 1 \\ =& 2\frac{\partial e_{yx}}{\partial w_{xy}} + 1 = 2\frac{\partial e_{yx}}{\partial w_{yx}} - 1. \end{aligned} \tag{B8}$$

Now we need to calculate

$$\begin{aligned} \eta^H =& \frac{\hbar}{16\pi^2}\int d^2k\hat{\mathbf{d}}\cdot\left(\frac{\partial\hat{\mathbf{d}}}{\partial w_{xx}}\times\frac{\partial\hat{\mathbf{d}}}{\partial u_{xy}}\right) \\ =& \frac{\hbar}{32\pi^2}ab\int d^2k\frac{1}{d^2}\hat{\mathbf{d}}\cdot\left[k_x\frac{\partial\mathbf{d}''}{\partial k_x}\times\left(k_x\frac{\partial\mathbf{d}'}{\partial k_y}+k_y\frac{\partial\mathbf{d}'}{\partial k_x}\right)\right] \\ &+\frac{\hbar}{32\pi^2}b^2\int d^2k\frac{1}{d^2}\hat{\mathbf{d}}\cdot\left[k_y\frac{\partial\mathbf{d}'}{\partial k_y}\times\left(k_x\frac{\partial\mathbf{d}'}{\partial k_y}+k_y\frac{\partial\mathbf{d}'}{\partial k_x}\right)\right] \\ =& \frac{\hbar}{32\pi^2}ab\int k_x^2d^2k\frac{1}{d^2}\hat{\mathbf{d}}\cdot\left(\frac{\partial\mathbf{d}''}{\partial k_x}\times\frac{\partial\mathbf{d}'}{\partial k_y}\right) \\ &+\frac{\hbar}{32\pi^2}b^2\int k_y^2d^2k\frac{1}{d^2}\hat{\mathbf{d}}\cdot\left(\frac{\partial\mathbf{d}'}{\partial k_y}\times\frac{\partial\mathbf{d}'}{\partial k_x}\right) \end{aligned} \tag{B9}$$

where $\mathbf{d}' = (\tilde{b}\mathbf{d}_\parallel/b, d_z)$ and $\mathbf{d}'' = (\tilde{a}\mathbf{d}_\parallel/a, d_z)$. Then, using

$$\begin{aligned} \frac{\partial}{\partial k_x} =& \cos\phi\frac{\partial}{\partial k} - \frac{\sin\phi}{k}\frac{\partial}{\partial\phi}, \\ \frac{\partial}{\partial k_y} =& \sin\phi\frac{\partial}{\partial k} + \frac{\cos\phi}{k}\frac{\partial}{\partial\phi}, \end{aligned} \tag{B10}$$

we obtain:

$$\begin{aligned} \eta^H =& C_1\int k^2dk\frac{\partial\hat{d}_z}{\partial k} + C_2\int k^2dk\left[\hat{d}_z(1-\hat{d}_z^2)\frac{1}{d}\frac{\partial d}{\partial k} - \hat{d}_z^2\frac{\partial\hat{d}_z}{\partial k}\right] + C_3\int k^3dk\left[(1-\hat{d}_z^2)\frac{1}{d}\frac{\partial d}{\partial k} - \hat{d}_z\frac{\partial\hat{d}_z}{\partial k}\right]\left(\frac{\partial\hat{d}_z}{\partial k} + \hat{d}_z\frac{1}{d}\frac{\partial d}{\partial k}\right) \\ =& 8\pi C_1 n - \frac{2}{3}C_2\int kdk(1-\hat{d}_z^3) + C_2\int k^2dk\hat{d}_z(1-\hat{d}_z^2)\frac{1}{d}\frac{\partial d}{\partial k} + C_3\int k^3dk\left[(1-\hat{d}_z^2)\frac{1}{d}\frac{\partial d}{\partial k} - \hat{d}_z\frac{\partial\hat{d}_z}{\partial k}\right]\left(\frac{\partial\hat{d}_z}{\partial k} + \hat{d}_z\frac{1}{d}\frac{\partial d}{\partial k}\right), \end{aligned} \tag{B11}$$

where

$$\begin{aligned} C_1/\hbar =& \frac{1}{128\pi}(\tilde{a}b + 3a\tilde{b} - 4b\tilde{b}), \\ C_2/\hbar =& C_1/\hbar - \frac{1}{32\pi}\tilde{b}(\tilde{a}-\tilde{b}), \\ C_3/\hbar =& \frac{1}{256\pi}(a\tilde{b} - \tilde{a}b). \end{aligned} \tag{B12}$$

Now note that when the kinetic and interaction metrics are equal, we have $a = \tilde{a}$, $b = \tilde{b}$, which means $C_1/\hbar = b(a-b)/32\pi$ and $C_2 = C_3 = 0$. Inserting these into Eq.(B11) gives us

$$\eta^H = 8\pi C_1 n = b(a-b)\hbar n/4 = (t/2+\tilde{t})^{-2}\frac{t'\tilde{t}'}{4}\frac{\hbar n}{4}. \tag{B13}$$

In the weak pairing limit, even when the kinetic and interaction metrics are different, we have $\eta^H \propto n$ under some reasonable assumptions. First, note that

$$\int kdk(1-\hat{d}_z^3) = \int kdk(1-\hat{d}_z) = 4\pi n,$$
$$\int k^2 dk \frac{d_z d_\parallel^2}{d^5}\left(d_z\frac{\partial d_z}{\partial k} + d_\parallel \frac{\partial d_\parallel}{\partial k}\right) = \int kdk \frac{d_z d_\parallel^2}{d^5} k\frac{\partial d_z}{\partial k} = k_f^2 \int_{-\mu}^{\infty} d\xi \frac{\xi^2\Delta^2}{(\xi^2+\Delta^2)^{5/2}} = \frac{2}{3}k_f^2 = \frac{8\pi}{3}n, \tag{B14}$$

in this limit. Meanwhile, if we assume $|\Delta_{\mathbf{k}}| \propto k$ near the Fermi surface, we have

$$\int k^3 dk \frac{d_\parallel}{d^3}\frac{\partial d_z}{\partial k}\frac{\partial d_\parallel}{\partial k} = \int kdk \frac{d_\parallel^2}{d^3} 2(d_z+\mu) = k_f^2 \int_{-\mu}^{\infty} d\xi \frac{\Delta^2}{(\xi^2+\Delta^2)^{3/2}} = 2k_f^2 = 8\pi n. \tag{B15}$$

At the quantum critical point, the ratio of the phonon Hall viscosity to the gravitational Hall viscosity becomes different from what we have for the weak-pairing limit. We can see this from

$$\int kdk(1-\hat{d}_z) = \int kdk \frac{d_\parallel^2}{d(d+d_z)} = \frac{1}{2}\int dk^2 \frac{\hat{\Delta}^2 k^2}{d(d+d_z)} = \frac{1}{4}\left(\frac{2m^*}{\hbar^2}\right)^2 \int_0^\infty d\xi \frac{\hat{\Delta}^2}{\xi + m^*\hat{\Delta}^2/\hbar^2} = 4\pi n,$$
$$\int k^2 dk \frac{d_z d_\parallel^2}{d^5}\left(d_z\frac{\partial d_z}{\partial k} + d_\parallel \frac{\partial d_\parallel}{\partial k}\right) = \int kdk \frac{d_z d_\parallel^2}{d^5}(2d_z^2+d_\parallel^2) = \left(\frac{2m^*}{\hbar^2}\right)^2 \int_0^\infty d\xi \frac{\hat{\Delta}^2}{\xi+\frac{2m^*\hat{\Delta}^2}{\hbar^2}} \frac{\sqrt{\xi}(\xi+\frac{m^*\hat{\Delta}^2}{\hbar^2})}{(\xi+\frac{2m^*\hat{\Delta}^2}{\hbar^2})^{5/2}} > \frac{8\pi}{3}n,$$
$$\int k^3 dk \frac{d_\parallel}{d^3}\frac{\partial d_z}{\partial k}\frac{\partial d_\parallel}{\partial k} = \int kdk \frac{d_\parallel^2}{d^3} 2d_z = \left(\frac{2m^*}{\hbar^2}\right)^2 \int_0^\infty d\xi \frac{\hat{\Delta}^2}{\xi+\frac{2m^*\hat{\Delta}^2}{\hbar^2}} \sqrt{\frac{\xi}{\xi+\frac{2m^*\hat{\Delta}^2}{\hbar^2}}} > 8\pi n, \tag{B16}$$

where $d_\parallel = \hat{\Delta}k$. We also have

$$4\pi n < \int kdk(1-\hat{d}_z^3) = \int kdk(1-\hat{d}_z)(1+\hat{d}_z+\hat{d}_z^2) < 12\pi n. \tag{B17}$$

In the strong-pairing limit, we do not find asymptotic limit to the ratio of integrals to n that is independent of cutoff.

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