Gravitational anomalies and thermal Hall effect in topological insulators

Michael Stone

Phys. Rev. B **85**, 184503 — Published 3 May 2012

DOI: 10.1103/PhysRevB.85.184503

Gravitational Anomalies and Thermal Hall effect in Topological Insulators

MICHAEL STONE

University of Illinois,

Department of Physics

1110 W. Green St.

Urbana, IL 61801 USA

E-mail: m-stone5@illinois.edu

Abstract

It has been suggested that, after being gapped by small symmetry breaking field, the Majorana quasiparticles localized on the surface of a class DIII topological insulator will exhibit a thermal Hall effect that arises from a gravitational Chern-Simons term. We critically examine this idea, and argue that the thermo-gravitational Hall effect is more complicated than its familiar analogue. A conventional Hall current is generated by a *uniform* electric field, but computing the flux from the gravitational Chern-Simons functional shows that gravitational field *gradients* — *i.e.* tidal forces — are needed to induce a energy-momentum flow. We relate the resulting surface energy-momentum flux to a domain-wall gravitational anomaly *via* the Callan-Harvey inflow mechanism. We stress that the gauge invariance of the combined bulk-plus-boundary theory ensures that the current in the domain wall always experiences a "covariant" rather than "consistent" anomaly. We use this observation to confirm that the tidally induced energy-momentum current exactly accounts for the covariant gravitational anomaly in $(1+1)$ dimensional domain-wall fermions. The same anomaly arises whether we write the Chern-Simons functional in terms of the Christoffel symbol or in terms of the the spin connection.

PACS numbers: 73.43.-f, 11.10.Lm, 04.62+v, 11.15.Yc

I. INTRODUCTION

One of the key properties of topological insulators is the intimate connection between the non-trivial bundle structure of the bulk electronic states and the presence of protected gapless surface modes. The most intuitive way of understanding this connection is that the twisted bundle gives rise to bulk quantum-Hall-like conductivities, and the gapless surface modes need to be present to soak up the corresponding conserved currents where they run into the surface of the sample [1, 2]. In this way the bulk-surface connection is seen to be a manifestation of the Callan-Harvey "anomaly inflow" mechanism [3]. Most of the Altland-Zirnbauer classes [4, 5] of topological insulators possess conserved U(1) charge or SU(2) spin currents, and the necessity of their protected surface modes can be understood *via* ordinary gauge-field anomalies. An important exception is the class DIII, which includes superconductors with spin-orbit interactions, and superfluid ^{3}He-B. Here the only conserved quantities are energy and (in the translation invariant superfluid) momentum. An anomaly-inflow understanding of the electrically neutral (Majorana) surface modes in the DIII systems therefore requires a failure of some edge-mode energy-momentum conservation law — in other words a *gravitational anomaly* [6].

Gravitational anomalies originate in the $\hat{A}$-genus contribution to the Dirac index theorem that is non-zero only in $4k$ space-time dimensions. They descend *via* a parity-violating gravitational Chern-Simons term in $4k-1$ dimensions to an energy-momentum inflow anomaly in $4k-2$ space-time dimensions. For physically realizable topological insulators we are restricted to the $k=1$, and therefore to a gravitational Chern-Simons term in a $(2+1)$-dimensional surface, and a gravitational anomaly in a $(1+1)$-dimensional edge.

Following [7–11] we expect that (after the application of a small symmetry-breaking field that opens a gap) the $(2+1)$-dimensional Majorana fermion surface modes of the DIII systems will include a gravitational Chern-Simons term in their low-energy effective action. It is argued in [12–14] that this term can, in principle, be observed through a thermal Hall (or Leduc-Righi) effect. A key step in the reasoning in [12–14] requires that, in analogy with the conventional Hall effect, a uniform gravitational field induces a surface energy-momentum current. The Leduc-Righi, coefficient is then obtained by means of an Einstein argument. The idea is that thermal equilibrium in the presence of a gravitational field requires the local temperature to vary so as to compensate for the gravitational red shift experienced by radiation as it moves in the potential. The energy flux induced by a thermal gradient is then balanced by an equal and opposite energy flux due to the gravitational potential gradient. The thermal Hall conductance can thus be found from that of the gravitational Hall conductance.

The purpose of this paper is to argue that, although the arguments in [12–14] are very appealing, the gravitational “Hall effect” is a little more complicated than its electromagnetic analogue. While a temperature gradient across a finite $(2 + 1)$-dimensional surface does indeed induce a thermal Hall current whose magnitude is related to the gravitational anomaly [11, 15], the surface-state energy gap exponentially suppress any surface thermal current. The heat must therefore be carried entirely by the gapless $(1 + 1)$-dimensional edge modes. In this respect the thermal current differs from the charge Hall effect, which can flow either at the $(1 + 1)$-dimensional edge, or, in the presence of a uniform electric field, *within* the $(2+1)$-dimensional electron gas. Furthermore, the gravitational Chern-Simons term yields an energy-momentum flux that is proportional to to gradients of the Ricci tensor. Consequently a uniform bulk gravitational field cannot create an energy-momentum flux within the $(2+1)$-dimensional surface. A surface energy flux requires an *inhomogeneous* field — *i.e.* tidal forces. Nonetheless, the tide-induced energy-momentum flow does retain the bulk-boundary connection because it demands an anomalous $(1 + 1)$-dimensional gapless mode to absorb the flux as it runs into an edge or domain wall.

In section II we will describe the thermal Hall effect and show how it can maintain an equilibrium balance between a temperature gradient and a gravitational potential gradient even in the absence of a bulk energy flow. In section III we review the Callan-Harvey anomaly-inflow picture, and stress that this mechanism always leads to the *covariant* form of the associated anomaly. In section IV we explain the origin of the gravitational anomaly in $(1+1)$ dimensional chiral theories. In section V we compute the energy momentum flows arising from a $(2 + 1)$-dimensional gravitational Chern-Simons functional and show that it exactly accounts for the anomaly obtained in section IV. We also show that the same anomaly is obtained from the Chern-Simons functional whther it is written in terms of the Christoffel symbols Γ or the spin connection ω. Finally section VI provides a brief summary of our results.

II. THERMAL HALL CURRENTS

The edge of a $(2+1)$-dimensional quantum Hall system hosts gapless chiral fermions [16], and both the edge of a $p_x + ip_y$ superconductor [11] and domain walls on the surface of a suitably engineered topological insulator [17] host gapless $(1+1)$-dimensional chiral *Majorana* fermions. In superfluid He^3-B, the presence of a small magnetic field causes the $(2+1)$-dimensional Majorana modes on the surface of the fluid to acquire a mass-gap m that can change sign even when the field is uniform [18]. The resulting domains have been detected by NMR [19], and now the domain walls between regions of $\pm m$ host gapless $(1+1)$-dimensional chiral Majorana fermions.

Consider a collection of such $(1+1)$-dimensional edge modes, and suppose for a moment that they can be modelled as a set of n independent conformal fields possessing (positive or negative) propagation velocities v_i, $i = 1, \ldots, n$ and central charges c_i. Then, at temperature T, each independent edge mode contains an energy density [20]

$$\varepsilon_i = c_i \frac{\pi}{12|v_i|} \frac{k_B^2}{\hbar} T^2, \tag{1}$$

where k_B is the Boltzmann constant. Thermal energy is therefore being transported along the edge at a rate [21]

$$\begin{aligned} J_T &= \sum_{i=1}^{n} v_i \varepsilon_i \\ &= \frac{\pi}{12} \sum_{i=1}^{n} \mathrm{sgn}\,(v_i) c_i \frac{k_B^2}{\hbar} T^2 \\ &= \frac{\pi}{12} (c - \bar{c}) \frac{k_B^2}{\hbar} T^2. \end{aligned} \tag{2}$$

Here c and $\bar{c}$ are the total conformal charges of the right and left-moving modes respectively. Although motivated by the model of independent modes, this formula continues to hold for more complicated conformal theories [15]. If we construct a parallel-sided Hall bar and maintain a small temperature gradient ΔT across it, then the difference between the contra-propagating energy fluxes (2) on the two edges gives rise to a net thermal current

$$J_{\mathrm{L-R}} = C_{\mathrm{L-R}} \Delta T \tag{3}$$

that flows along the bar and perpendicular to the temperature gradient. Here

$$C_{\mathrm{L-R}} = (c - \bar{c}) \frac{\pi}{6} \frac{k_B^2}{\hbar} T, \tag{4}$$

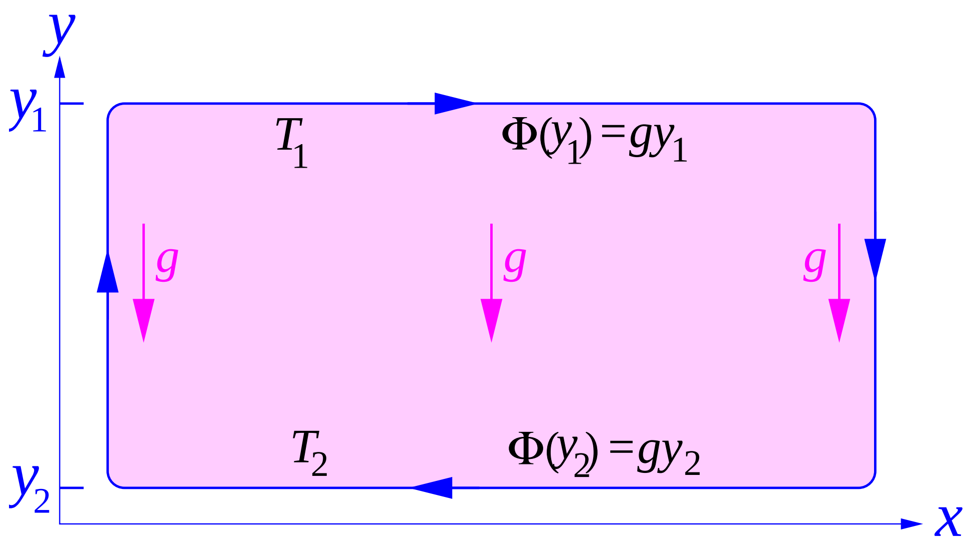

FIG. 1: Chiral edge modes carry thermal energy clockwise around the boundary of a rectangular Hall bar. The system is in equilibrium when the temperature difference between the cooler upper and hotter lower edge is balanced by the gravitational red- and blue-shifts experienced by the energy quanta as they ascend and descend the vertical sides.

is the Leduc-Righi coefficient.

It is remarkable that the non-universal edge-mode velocities have cancelled, leaving in $C_{\mathrm{L-R}}/T$ only fundamental constants and the numbers c, $\bar{c}$ that are characteristic of the quantum Hall phase. It is therefore reasonable to suppose that $C_{\mathrm{L-R}}/T$ may be extracted from topological data, as is the quantum Hall coefficient. It is, however, difficult to provide a direct derivation of thermal conductivities from linear response theory. There is no term that can be added to the Hamiltonian to describe the temperature. An ingenious trick was introduced by Luttinger [22], who instead coupled the system to *gravity* and proceeded indirectly by adopting the method used by Einstein to relate diffusion coefficients to viscosity [23]. Luttinger's idea is that the de-equilibrating effect of a small temperature gradient will be precisely compensated for by the red- or blue-shift induced by gravitational potential Φ when

$$\frac{1}{T}\frac{\partial T}{\partial x} = -\frac{1}{c_{\mathrm{light}}^2}\frac{\partial \Phi}{\partial x}. \tag{5}$$

Consequently, assuming that all currents vanish in equilibrium, and that the effects of the two driving forces are additive, a linear-response derivation of the current induced by gravity allows one to deduce the current induced by the thermal gradient.

Can we use the Luttinger technique to compute the thermal Hall current? — And does it imply that a uniform gravitational field will cause heat to flow not only at the edges, but also in the bulk where the system has a gap? To address these questions consider a rectangular Hall bar (see Figure 1) whose upper, right-propagating, edge at co-ordinate $y = y_1$ is held at temperature T_1 and whose left-propagating lower edge at $y = y_2$ is held at temperature $T_2 > T_1$. If the bar lies in a gravitational field such that the gravitational frequency shift obeys

$$\frac{\omega(y_1)}{\omega(y_2)} \equiv \sqrt{\frac{g_{00}(y_2)}{g_{00}(y_1)}} = \frac{T_1}{T_2}, \tag{6}$$

then, as the thermal excitations from the hotter lower edge rise on the left hand vertical side to the upper edge they will red-shift to the lower temperature. Similarly, as the excitations from the cooler upper edge descend *via* the right hand vertical side to the lower edge they will blue-shift to match the hotter temperature. The system is in equilibrium therefore. Since for weak gravitational fields we have

$$\sqrt{g_{00}(y)} \approx 1 + \frac{\Phi(y)}{c_{\text{light}}^2}, \tag{7}$$

this situation satisfies (5). Observe, however, that in our Hall bar, the currents are *not* zero in equilibrium. Therefore a knowledge of the thermal Hall current at a point does not allow one to deduce the gravitational Hall current at that point, nor *vice versa*. We must distinguish between net transport currents that relocate energy (and to which the Luttinger argument applies) and the local energy-momentum current that acts as a source for gravity (and is the current appearing in [12–14]). See [24] for a detailed discussion of the distinction. Further, the steady-state equilibrium of the Hall bar neither requires nor permits thermal energy to be flowing *within* the gapped surface states. This already suggests that a uniform gravitational field does not induce a surface energy flow.

This suggestion is perhaps not surprising. A mathematical analogy between the conventional Hall effect and gravitation would naturally identify the field strength F with the Riemann curvature R. A uniform field gravitational field does not, however, require space-time curvature. The Rindler metric

$$d\tau^2 = \left(1 + \frac{(r - r_0)g}{c_{\text{light}}^2}\right)^2 dt^2 - \frac{1}{c_{\text{light}}^2} dr^2, \quad g = c_{\text{light}}^2 / r_0 \tag{8}$$

of a uniformly accelerated observer provides a gravitational potential $\Phi(r) = (r - r_0)g$, but is merely a re-parametrization

$$c_{\rm light}T \;=\; r\sinh\left(\frac{c_{\rm light}t}{r_0}\right),$$
$$X \;=\; r\cosh\left(\frac{c_{\rm light}t}{r_0}\right),$$

of a part of Minkowski space with flat metric

$$d\tau^2 = dT^2 - \frac{1}{c_{\rm light}^2}dX^2. \tag{9}$$

It might, therefore, be more physical to identify the thermal-Hall analogue of the electric field with the Christoffel symbols Γ which describe the frame-dependent inertial forces that we perceive as gravity. If this new analogy is to work, the energy-momentum influx into the edge modes would have to be given by the non-covariant "consistent" gravitational anomaly, which contains Γ's, rather that the "covariant" anomaly which contains only R [25]. In the following sections, however, we will argue that anomaly inflow always give rise to the *covariant* anomaly, and not to the consistent anomaly. Moreover, we will see that *gradients* of curvature are needed to produce an energy flow into edge states.

To simplify the argument, we will follow the authors of [12] and argue that since we are interested in topological effects, we can choose non-universal quantities such as propagation velocities as we like. We will therefore from now on make all modes propagate at $c_{\rm light}$, and work with fully relativistic systems. (However, $c_{\rm light}$ does not have to be the actual speed of light.) We will also use natural units, in which $\hbar = c_{\rm light} = 1$.

III. THE CALLAN-HARVEY MECHANISM AND COVARIANT VERSUS CONSISTENT ANOMALIES

Let us recall how the conservation (or non-conservation) of a gauge current is related to the gauge invariance (or the lack of it) of an action functional. Suppose, for example, that $S[A]$ is a functional of an $\mathfrak{su}(N)$ Lie-algebra-valued gauge field $A_\mu = \lambda_a A^a_\mu$, where the matrices λ_a are the generators of $\mathfrak{su}(N)$. We define the matrix-valued gauge current $J^\mu(x) = \lambda_a J^{\mu,a}$ by setting

$$\delta S[A] = \int d^dx\,{\rm tr}\,\{J^\mu \delta A_\mu\}. \tag{10}$$

Under a gauge transformation the field changes as $A_\mu \to A^g_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g$, where $g \in {\rm SU}(N)$. For an infinitesimal transformation $g = 1 - \epsilon$ the transformation becomes

$A_\mu \to A_\mu + \delta_\epsilon A_\mu$, where $\delta_\epsilon A_\mu = -([A_\mu, \epsilon] + \partial_\mu \epsilon) \equiv -\nabla_\mu \epsilon$. The corresponding change in $S[A]$ is

$$\begin{aligned} \delta_\epsilon S &= \int d^d x \, \mathrm{tr} \, \{J^\mu([\epsilon, A_\mu] - \partial_\mu \epsilon)\} \\ &= \int d^d x \, \mathrm{tr} \, \{\epsilon(\partial_\mu J^\mu + [A_\mu, J^\mu])\}. \end{aligned} \tag{11}$$

The covariant divergence

$$\nabla_\mu J^\mu \equiv \partial_\mu J^\mu + [A_\mu, J^\mu] \tag{12}$$

is therefore zero if and only if $S[A]$ is gauge invariant.

We are interested in *effective actions* $S[A]$ that arise as a result integrating out a collection of Fermi fields ψ, $\psi^\dagger$ in the presence of a classical background gauge field A_μ:

$$\exp\{-S[A]\} = \int d[\psi] d[\psi^\dagger] \exp\left\{-S[\psi, \psi^\dagger, A]\right\}. \tag{13}$$

The calculated currents are then the expectation value $J^\mu = \langle \hat{J}_\mu \rangle$ of a quantum operator. The original $S[\psi, \psi^\dagger, A]$ action will be invariant under $A_\mu \to A_\mu^g$, $\psi \to g^{-1}\psi$, $\psi^\dagger \to \psi^\dagger g$, but the invariance may lost during the functional integration. In this case we will have

$$\nabla_\mu J^\mu = G(A), \tag{14}$$

where the anomaly $G(A)$ is a local polynomial in the A_μ and their derivatives. A gauge anomaly provides an obstruction to a subsequent quantization of the A_μ fields, but when the A_μ are simply classical probes it provides a useful source of non-perturbative information.

The Callan-Harvey effect links the non-conservation of gauge and other currents to an inflow of charge from some higher dimensional space in which the anomalous theory is embedded as modes localized on a domain wall or string defect. In the cases we are interested in, the inflow is derived from a Chern-Simons term in one-higher space dimension.

As usual we will think of A as a Lie-algebra-valued one-form $A = \lambda_a A_\mu^a dx^\mu$, and define the field strength as the Lie-algebra-valued two-form

$$F = dA + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu. \tag{15}$$

The Chern-Simons form $\omega_{2n-1}(A)$ is then defined as

$$\omega_{2n-1}(A) = n \int_0^1 \mathrm{tr} \, \{A F_t^{n-1}\}, \tag{16}$$

where $F_t = tF + t(t-1)A^2$. It is constructed so that $d\omega_{2n-1} = \mathrm{tr}\,\{F^n\}$. For example,

$$\begin{aligned}\omega_3(A) &= \mathrm{tr}\,\{AdA + \frac{2}{3}A^3\}, \\ &= \mathrm{tr}\,\{AF - \frac{1}{3}A^3\},\end{aligned} \tag{17}$$

and

$$\begin{aligned}\omega_5(A) &= \mathrm{tr}\,\{A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5\} \\ &= \mathrm{tr}\,\{AF^2 - \frac{1}{2}FA^3 + \frac{1}{10}A^5\}.\end{aligned} \tag{18}$$

The F-free last term $\propto A^{2n-1}$ in the second forms of ω_{2n-1} has coefficient

$$c_n = (-1)^{n-1}\frac{n!(n-1)!}{(2n-1)!}. \tag{19}$$

It is this last term that governs the change in integrals of ω_{2n-1} under large gauge transformations. If A undergoes a finite gauge transformation

$$A \to A^g = g^{-1}Ag + g^{-1}dg, \tag{20}$$

then

$$\omega_{2n-1}(A^g) = \omega_{2n-1}(A) + c_n \mathrm{tr}\,\{(g^{-1}dg)^{2n-1}\} + d\alpha_{2n-2}(A,g), \tag{21}$$

where, for example [29]

$$\alpha_2 = -\mathrm{tr}\,\{dgg^{-1}A\} \tag{22}$$

and

$$\alpha_4(A,g) = -\frac{1}{2}\mathrm{tr}\,\{(dgg^{-1})(AdA + dAA + A^3) - \frac{1}{2}(dgg^{-1})A(dgg^{-1})A - (dgg^{-1})^3 A\}. \tag{23}$$

The Chern-Simons functional $C[A]$ is defined by setting

$$C[A] = 2\pi\left(\frac{i}{2\pi}\right)^n \frac{1}{n!}\int_M \omega_{2n-1}(A), \tag{24}$$

where M is some $2n-1$ dimensional manifold. The coefficient in front of the integral has been chosen so that $\exp\{iC[A]\}$ is single-valued when M is the $(2n-1)$-sphere. In this case

$$C[A^g] - C[A] = 2\pi\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!}\int_{S^{2n-1}} \mathrm{tr}\,\{(g^{-1}dg)^{2n-1}\}, \tag{25}$$

and it is shown in [28] that the right-hand side of (25) is 2π times an integer whenever $g \in \mathrm{GL}(n, \mathbb{C})$ or any of its compact subgroups such as $\mathrm{SU}(N)$. This means that when a Chern-Simons functional appears in a functional integral

$$Z = \int d[A] \exp\{ikC[A] + \cdots\} \tag{26}$$

then gauge invariance demands that k be an integer. This constraint on k need not hold when $C[A]$ appears in an effective action. Indeed k is $1/2$ when we integrate out a massive Dirac fermion in odd dimensional space time.

Given a $(2n-1)$-manifold M possessing a $2n-2$-dimensional boundary ∂M, we can use $C[A]$ to construct an action $S[A, g] \stackrel{\text{def}}{=} C[A^g]$ that is obviously invariant under $A \to A^h$, $g \to h^{-1}g$. In this action, the gauge non-invariance of the bulk Chern-Simons term $C[A]$ is compensated by the complementary gauge non-invariance of the *Wess-Zumino action* [26]

$$\begin{aligned} W[A,g] &\stackrel{\text{def}}{=} C[A^g] - C[A] \\ &= 2\pi \left(\frac{i}{2\pi}\right)^n \frac{1}{n!} \left\{ \int_{\partial M} \alpha_{2n-2}(A, g) + c_n \int_M \mathrm{tr}\left\{(g^{-1}dg)^{2n-1}\right\} \right\}. \end{aligned} \tag{27}$$

Although $W[A, g]$ requires g to be defined on the $2n-1$ dimensional manifold M, the identity

$$\delta \mathrm{tr}\left\{(g^{-1}dg)^{2n-1}\right\} = (2n-1) d\mathrm{tr}\left\{(g^{-1}\delta g)(g^{-1}dg)^{2n-2}\right\} \tag{28}$$

ensures that variation of $W[A, g]$ depends only on the values that δA and δg take on the boundary ∂M. It can therefore serve as an anomaly-capturing non-local effective action for a $2n-2$ dimensional theory [27]. The meaning of the gauge-group element g depends on the context. In a two dimensional boundary $g(x, t)$ could be the dynamical chiral boson equivalent to a chiral fermion. In this case we still have to integrate over g in order to obtain the action $S[A]$ appearing in 10. In two or higher dimensions it might parametrize a Higgs field that gives a left-handed chiral fermion a mass by coupling it to a right handed chiral fermion that does not itself couple to A. In this case a vacuum expectation value for g will explicitly break the gauge symmetry. We will consider only the first of these possibilities.

The gauge anomaly arising from the Wess-Zumino action for a four dimensional theory may be read off from

$$\begin{aligned} \int_{\partial M} d^4x\, \mathrm{tr}\left\{\epsilon \nabla_\mu J^\mu_{\mathrm{WZ}}\right\} &= \delta_\epsilon W[A, g] \\ &= -\delta_\epsilon C[A] \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{24\pi^2}\int_{\partial M} \mathrm{tr}\,\{d\epsilon(AdA + dAA + A^3)\} \\
&= \frac{1}{24\pi^2}\int_{\partial M} \mathrm{tr}\,\{\epsilon\partial_\mu(A_\nu\partial_\sigma A_\tau + \partial_\nu A_\sigma A_\tau + A_\nu A_\sigma A_\tau)\}\varepsilon^{\mu\nu\sigma\tau} d^4x. \quad (29)
\end{aligned}
$$

So

$$
\mathrm{tr}\,\{\epsilon \nabla_\mu J^\mu_{\mathrm{WZ}}\} = \frac{1}{24\pi^2}\mathrm{tr}\,\{\epsilon\partial_\mu(A_\nu\partial_\sigma A_\tau + \partial_\nu A_\sigma A_\tau + A_\nu A_\sigma A_\tau)\}\varepsilon^{\mu\nu\sigma\tau}. \quad (30)
$$

Because this anomaly is found as the variation of the functional $W[A, g]$, it satisfies the Wess-Zumino consistency condition

$$
(\delta_\epsilon\delta_{\epsilon'} - \delta_{\epsilon'}\delta_\epsilon)W = \delta_{[\epsilon,\epsilon']}W.
$$

It is therefore known as a "consistent" anomaly. The right hand side of the (non) conservation equation is not gauge covariant, however, and so neither is the left. The gauge current itself is therefore not covariant, and the physical meaning of the (non) conservation equation is unclear.

In the full bulk-plus-boundary theory, whose gauge-invariant effective action is $C[A^g]$ the non-zero divergence of the boundary current is being supplied by the inflow of gauge current from the higher dimensional bulk. This bulk current *is* covariant,

$$
\mathrm{tr}\,\{\lambda_a J^\lambda\} = \frac{1}{32\pi^2}\mathrm{tr}\,\{\lambda_a F_{\mu\nu}F_{\sigma\tau}\}\varepsilon^{\lambda\mu\nu\sigma\tau}. \quad (31)
$$

It comes from the variation

$$
\begin{aligned}
\delta\int\omega_5 &= 3\int_M \mathrm{tr}\,\{\delta A F^2\} + \int_{\partial M}\mathrm{tr}\,\{\delta A(AdA + dAA + \frac{3}{2}A^3)\} \\
&= 3\int_M \mathrm{tr}\,\{\delta A F^2\} + \int_{\partial M}\mathrm{tr}\,\{\delta A(AF + FA - \frac{1}{2}A^3)\}. \quad (32)
\end{aligned}
$$

We usually ignore the boundary term when computing a bulk current, but in the total bulk-plus-boundary theory we must retain it as it provides a contribution to the current in the boundary of

$$
\mathrm{tr}\,\{\lambda_a X^\mu\} \stackrel{\mathrm{def}}{=} \frac{1}{48\pi^2}\mathrm{tr}\,\{\lambda_a(A_\nu F_{\sigma\tau} + F_{\nu\sigma}A_\tau - A_\nu A_\sigma A_\tau)\}\varepsilon^{\mu\nu\sigma\tau}. \quad (33)
$$

This quantity is exactly the extra current ([25] equation (2.16)) that has to be added to the consistent current to obtain the *covariant* anomaly

$$
\mathrm{tr}\,\{\lambda_a \nabla_\mu(J^\mu_{\mathrm{WZ}} + X^\mu)\} = \frac{1}{32\pi^2}\mathrm{tr}\,\{\lambda_a F_{\mu\nu}F_{\sigma\tau}\}\varepsilon^{5\mu\nu\sigma\tau}. \quad (34)
$$

The new current $J^\mu_{\rm tot} = J^\mu_{\rm WZ} + X^\mu$ is now gauge-covariant, and its anomalous divergence entirely accounted for by the Callan-Havey anomaly inflow [30, 31].

Similarly, in two dimensions we find that

$$\nabla_\mu J^\mu_{\rm WZ} = \frac{1}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu \tag{35}$$

is the consistent anomaly, and

$$X^\mu = \frac{1}{4\pi}\epsilon^{\mu\nu} A_\nu \tag{36}$$

is the Chern-Simons term's contribution to the boundary current. Then

$$\begin{aligned}\nabla_\mu (J^\mu_{\rm WZ} + X^\mu) &= \frac{1}{4\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu + \frac{1}{4\pi}\epsilon^{\mu\nu}(\partial_\mu A_\nu + [A_\mu, A_\nu]) \\ &= \frac{1}{4\pi}\epsilon^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \\ &= \frac{1}{4\pi}\epsilon^{\mu\nu} F_{\mu\nu}, \end{aligned} \tag{37}$$

is the covariant anomaly.

We have seen that Bardeen-Zumino polynomial $X^\mu(A)$ that converts the consistent gauge current to the covariant gauge current is precisely the contribution to the boundary current provided by the boundary variation of the bulk Chern-Simons functional. The analogous conversion of a consistent to a covariant *gravitational* anomaly requires an extra integration by parts, and so is more intricate. Indeed some puzzlement was expressed in [3] about what happened to the inflowing energy-momentum — see the discussion after equation (30) in [3] — but it was later understood that the anomaly inflow is always leads to a covariant current [30, 31].

In the above examples, the Chern-Simons term was defined in the bulk and the lower-dimensional degrees of freedom resided on the boundary. This is, for example, the situation in the ordinary quantum Hall effect. For (3 + 1)-dimensional topological insulators it is the Chern-Simons functional that is defined on the boundary, and the lower-dimensional theory is defined on a domain wall within the boundary. In this case the coefficient of the Chern-Simons functional is multiplied by ${\rm sgn}(m)/2$, where m denotes the mass gap induced by a small symmetry breaking field that changes sign at the domain wall. The resulting domain-wall chiral fermions then experience half of the the usual inflow from each side, but there are *two* sides, and so the resulting edge-theory anomaly is unchanged.

IV. TWO-DIMENSIONAL GRAVITATIONAL ANOMALIES

In this section we will review the origin and possible forms of gravitational anomalies. We start from an effective action $S[g]$ that depends on the space-time metric $g_{\mu\nu}$. The associated Hilbert energy-momentum tensor $T^{\mu\nu}$ is then defined by the variation

$$\begin{aligned} \delta S_{\text{eff}} &= -\frac{1}{2}\int d^dx\sqrt{|g|}\,T^{\mu\nu}\,\delta g_{\mu\nu}, &(38)\\ &= +\frac{1}{2}\int d^dx\sqrt{|g|}\,T_{\mu\nu}\,\delta g^{\mu\nu}. &(39)\end{aligned}$$

Under a change of co-ordinates $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ we have $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, where

$$\begin{aligned} \delta g_{\mu\nu} &= (\mathcal{L}_\epsilon g)_{\mu\nu} \\ &= \nabla_\mu \epsilon_\mu + \nabla_\nu \epsilon_\mu. &(40)\end{aligned}$$

Here $\mathcal{L}_\epsilon g$ denotes the Lie derivative of the metric with respect to ϵ^μ, and ∇_μ is the covariant derivative with respect to the torsion-free Levi-Civita connection. When the effective action is invariant under this reparametrization we find (taking into account that $T^{\mu\nu} = T^{\nu\mu}$) that

$$\begin{aligned} 0 &= -\int d^dx\sqrt{|g|}\,T^{\mu\nu}\,\nabla_\mu\epsilon_\nu \\ &= \int d^dx\sqrt{|g|}\epsilon_\nu\nabla_\mu T^{\mu\nu}. &(41)\end{aligned}$$

Thus a gravitational anomaly — *i.e.* a failure of the covariant conservation law [39] $\nabla_\mu T^{\mu\nu} = 0$ — reflects a failure of reparametrization invariance. While it seems reasonable that any physical system should be independent of how we choose to describe it, co-ordinate dependence can creep into $S[g]$ when we tacitly tie a regularization procedure to the co-ordinate grid rather than to some intrinsic property such as the metric.

An equivalent *Lorentz anomaly* can also occur in theories when we use a frame field e^μ_a rather than the metric to encode the geometry. This anomaly manifests itself as a failure of the energy momentum tensor (now defined in terms of a functional derivative with respect to e^μ_a) to be symmetric.

We will focus on two-dimensional systems expressed in terms of Euclidean signature isothermal co-ordinates x, y, in which $ds^2 = e^\phi(dx^2 + dy^2)$. It is convenient to set $z = x + iy$, $\bar z = x - iy$ so that $ds^2 = e^\phi dz d\bar z$. The non-zero component of the metric tensor and its inverse are then $g_{\bar z z} = g_{z\bar z} = (1/2)e^\phi$, and $g^{\bar z z} = g^{z\bar z} = 2e^{-\phi}$. In these complex isothermal co-ordinates the only non-zero entries in the Levi-Civita connection are

$$\begin{aligned}\Gamma^{z}_{zz} &= \partial_z \phi, \\ \Gamma^{\bar z}_{\bar z \bar z} &= \partial_{\bar z} \phi.\end{aligned} \tag{42}$$

The curvature is completely encoded in the Ricci scalar

$$R = R^{\mu\nu}{}_{\mu\nu} = R^{\bar z z}{}_{\bar z z} + R^{z \bar z}{}_{z \bar z} = -4e^{-\phi}\partial^2_{z\bar z}\phi. \tag{43}$$

In our convention R is twice the Gaussian curvature, and hence positive for a sphere.

The effective action for a left-right symmetric theory with conformal central charge c was obtained by Polyakov [32] as

$$\begin{aligned}S_{\text{Polyakov}}[g] &= -\frac{c}{96\pi}\int d^2x\,(\partial\phi)^2 \\ &= -\frac{c}{24\pi}\int d^2x\,\partial_z\phi\partial_{\bar z}\phi.\end{aligned} \tag{44}$$

Here d^2x denotes $dx \wedge dy = d\bar z \wedge dz/2i$. To evaluate (44) for a given geometry we must select a system of isothermal co-ordinates, and this choice is not unique. It is therefore not immediately obvious that $S_{\text{Polyakov}}[g]$ is co-ordinate independent. To verify that it is so, we must examine the conservation of the energy-momentum tensor.

Now to make use of the Hilbert definition of $T^{\mu\nu}$, we must be free to make an arbitrary infinitesimal variation in the metric. A general variation, however, will take us away from the class of isothermal metrics. We therefore make a variation $\delta g_{\mu\nu}$ and follow if with a change of co-ordinates

$$\begin{aligned}z &\to z' = z + \epsilon(z, \bar z) \\ \bar z &\to \bar z' = \bar z + \bar\epsilon(z, \bar z)\end{aligned} \tag{45}$$

so as to return to the isothermal gauge. Now

$$\begin{aligned}\delta(ds^2) &= [e^{\phi}(\epsilon\partial_z\phi + \bar\epsilon\partial_{\bar z}\phi + \partial_z\epsilon + \partial_{\bar z}\bar\epsilon) + \delta g_{\bar z z} + \delta g_{z\bar z})]d\bar z dz \\ &\qquad +(\delta g_{zz} + e^{\phi}\partial_z\bar\epsilon)dzdz + (\delta g_{\bar z\bar z} + e^{\phi}\partial_{\bar z}\epsilon)d\bar z d\bar z.\end{aligned} \tag{46}$$

The required co-ordinate change is obtained by solving

$$\begin{aligned}e^{\phi}\partial_z\bar\epsilon &= -\delta g_{zz} \\ e^{\phi}\partial_{\bar z}\epsilon &= -\delta g_{\bar z\bar z}.\end{aligned} \tag{47}$$

Let us assume for the moment that given δg_{zz} and $\delta g_{\bar{z}\bar{z}}$ we can always solve these equations for ϵ and $\bar{\epsilon}$. Then, comparing with $\delta(ds^2) = e^{\phi}\delta\phi d\bar{z}dz$ we find that the metric variation leads to

$$\delta\phi = \epsilon\partial_z\phi + \bar{\epsilon}\partial_{\bar{\epsilon}}\phi + \partial_z\epsilon + \partial_{\bar{z}}\bar{\epsilon} + e^{-\phi}(\delta g_{\bar{z}z} + \delta g_{z\bar{z}}). \tag{48}$$

We insert this variation of ϕ into equation (44), and, assuming that integration by parts is legitimate, reduce the terms involving ϵ to

$$\begin{aligned} & -\frac{c}{12\pi}\int d^2x\, \partial_{\bar{z}}\epsilon \left(\frac{1}{2}(\partial_z\phi)^2 - \partial^2_{zz}\phi\right) \\ = & -\frac{c}{12\pi}\int d^2x\, e^{-\phi}\delta g_{\bar{z}\bar{z}} \left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right). \end{aligned} \tag{49}$$

On comparing with

$$\begin{aligned} \delta S_{\text{Polyakov}}[g] &= -\frac{1}{2}\int d^2x\, \sqrt{g}\delta g_{\mu\nu}T^{\mu\nu} \\ &= -\frac{1}{2}\int d^2x\, \sqrt{g}\delta g_{\bar{z}\bar{z}}T^{\bar{z}\bar{z}}, \end{aligned} \tag{50}$$

where $\sqrt{g}\, d^2x = e^{\phi}dxdy$, we read off that

$$\begin{aligned} \frac{c}{6\pi}e^{-2\phi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right) &= T^{\bar{z}\bar{z}} \\ &= g^{\bar{z}z}g^{\bar{z}z}T_{zz} \\ &= 4e^{-2\phi}T_{zz}. \end{aligned} \tag{51}$$

Thus

$$T_{zz} = \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right). \tag{52}$$

Similarly we find that

$$T_{\bar{z}\bar{z}} = \frac{c}{24\pi}\left(\partial^2_{\bar{z}\bar{z}}\phi - \frac{1}{2}(\partial_{\bar{z}}\phi)^2\right). \tag{53}$$

Next, examining the effects of $\delta g_{\bar{z}z} + \delta g_{z\bar{z}}$, we have

$$\delta S_{\text{Polyakov}}[g] = -\frac{c}{12\pi}\int d^2x e^{-\phi}(\delta g_{\bar{z}z} + \delta g_{z\bar{z}})(-\partial^2_{z\bar{z}}\phi). \tag{54}$$

From this we read off that

$$\begin{aligned} T^{z\bar{z}} = T^{\bar{z}z} &= -\frac{c}{6\pi}e^{-2\phi}\partial^2_{z\bar{z}}\phi \\ &= -\frac{c}{24\pi}e^{-2\phi}\partial^2\phi, \end{aligned} \tag{55}$$

and

$$T_{z\bar{z}} = -\frac{c}{24\pi}\partial_{z\bar{z}}\phi. \tag{56}$$

We also recover the well-known trace anomaly [33]

$$T^{\mu}{}_{\mu} = g_{\bar{z}z}T^{\bar{z}z} + g_{z\bar{z}}T^{\bar{z}z} = e^{\phi}T^{z\bar{z}} = \frac{c}{24\pi}R. \tag{57}$$

This is a comforting consistency check, as Polyakov derived (44) by working backwards from (57).

We can now verify that $T_{\mu\nu}$ is covariantly conserved:

$$\begin{aligned}
\frac{1}{2}e^{\phi}(\nabla^{z}T_{zz} + \nabla^{\bar{z}}T_{\bar{z}z}) &= \nabla_{\bar{z}}T_{zz} + \nabla_{z}T_{\bar{z}z} \\
&= \partial_{\bar{z}}T_{zz} + \partial_{z}T_{\bar{z}z} - \Gamma^{z}{}_{zz}T_{\bar{z}z} \\
&= \partial_{\bar{z}}T_{zz} + \partial_{z}T_{\bar{z}z} - \partial_{z}\phi T_{\bar{z}z} \\
&= 0.
\end{aligned} \tag{58}$$

This is evidence that $S_{\mathrm{Polyakov}}[g]$ is indeed co-ordinate independent. There is a problem however: if S_{Polyakov} is co-ordinate independent then its functional derivative $T_{\mu\nu}$ must transform as a tensor. When we make a holomorphic change of variables $z = z(\zeta)$, $\bar{z} = \bar{z}(\bar{\zeta})$, however, we have $ds^2 = e^{\chi}d\zeta d\bar{\zeta} = e^{\phi}dzd\bar{z}$ and so

$$\phi = \chi + \ln\left(\frac{\partial\zeta}{\partial z}\right) + \ln\left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right). \tag{59}$$

Consequently

$$\begin{aligned}
T_{zz} &= \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right) \\
&= \frac{c}{24\pi}\left(\frac{\partial\zeta}{\partial z}\right)^2\left(\partial^2_{\zeta\zeta}\chi - \frac{1}{2}(\partial_\zeta\chi)^2\right) + \frac{c}{24\pi}\left(\frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2\right) \\
&= \left(\frac{\partial\zeta}{\partial z}\right)^2 T_{\zeta\zeta} + \frac{c}{24\pi}\{\zeta, z\},
\end{aligned} \tag{60}$$

where $T_{\zeta\zeta}$ is the energy-momentum tensor component evaluated in the ζ, $\bar{\zeta}$ co-ordinates and $\{\zeta, z\}$ is the Schwarzian derivative in whose definition the primes denote differentiation with respect to z. Our $T_{\mu\nu}$ does *not* transform as a tensor therefore. The paradox is resolved by looking back at the first line in equation (49). We see that if we are allowed to integrate by parts we can take the $\partial_{\bar{z}}$ derivative off of ϵ and onto T_{zz}. Thus any holomorphic addition to T_{zz} is invisible to the variation $\delta g_{\bar z\bar z}$. Another way of saying this is that there can be metric variations $\delta g_{\bar z\bar z}$ that cannot be written in the form the form $\delta g_{\bar z\bar z} = -\partial_{\bar z}\epsilon = -2\nabla_{\bar z}\epsilon_{\bar z}$. (The displacements ϵ and $\bar\epsilon$ should really be written as ϵ^z and $\epsilon^{\bar z}$ as they are the components of a contravariant vector.) The solvability of (47) depends on the global topology or on boundary conditions. On a torus, for example, metric variations due to change in the modular parameter τ are not expressible in this way. On a closed manifold of genus $g \geq 2$, there will be $3(g-1)$ linearly independent unobtainable metric variations.

The addition of a purely holomorphic term is indeed required. The full *operator* energy momentum tensor is

$$\begin{aligned}
\hat T_{zz} &= \hat T(z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right), \\
\hat T_{\bar z\bar z} &= \hat{\bar T}(\bar z) + \frac{c}{24\pi}\left(\partial^2_{zz}\phi - \frac{1}{2}(\partial_z\phi)^2\right), \\
\hat T_{\bar z z} &= -\frac{c}{24\pi}\partial^2_{z\bar z}\phi,
\end{aligned} \tag{61}$$

where, for a free $c = 1$ boson field $\varphi(z,\bar z) = \varphi(z) + \varphi(\bar z)$ for example,

$$\begin{aligned}
\hat T(z) &= :\partial_z\varphi(z)\partial_z\varphi(z): \\
&= \lim_{\delta\to 0}\left(\partial_z\varphi(z+\delta/2)\partial_z\varphi(z-\delta/2) + \frac{1}{4\pi\delta^2}\right).
\end{aligned} \tag{62}$$

(Note that conformal field theory papers often define $\hat T(z)$ to be -2π times (62) so as to simplify the operator product expansion.) The operator $\hat T(z)$ has been constructed to be explicitly holomorphic, but at a price of tying its definition to the z, $\bar z$ coordinate system — both in the mode normal ordering expression in the first line and by the explicit counterterm in the second. It is not surprising, therefore, that under a holomorphic change of co-ordinates the operator $\hat T(z)$ does not transform as a tensor. It is well-known that instead

$$\hat T(z) = \left(\frac{\partial\zeta}{\partial z}\right)^2 \hat T(\zeta) - \frac{c}{24\pi}\{\zeta, z\}. \tag{63}$$

We see that the inhomogeneous Schwarzian derivative terms cancel in the transformation of the energy momentum tensor $\hat T_{\mu\nu}$ defined in (61). Thus $\hat T_{\mu\nu}$ transforms as a tensor and is still covariantly conserved. It is notable that both the covariant conservation and the trace anomaly in $\hat T_{\mu\nu}$ are accounted for by the c-number terms. These properties are therefore independent of the quantum state in which the expectation is taken. This quantum state only influences the holomorphic part of $\langle\hat T_{zz}\rangle$ and an antiholomorphic part of $\langle\hat T_{\bar z\bar z}\rangle$.

In a *chiral* theory we might constrain both $\hat{T}_{\bar{z}\bar{z}}$ and $\hat{T}_{\bar{z}z}$ to be zero, while keeping the covariant form of $\hat{T}_{zz}$ defined in the first line of (61). The term $\nabla^{\bar{z}}\hat{T}_{\bar{z}z}$ needed for the continued mathematical validity of (58) would then be interpreted as

$$\begin{aligned}\nabla^{\bar{z}}T_{\bar{z}z} &\to -\frac{c}{12\pi}\partial_z e^{-\phi}\partial^2_{z\bar{z}}\phi \\ &= \frac{c}{48\pi}\partial_z R, \end{aligned} \tag{64}$$

so that conservation law (58) is reinterpreted as the anomaly equation appearing in [15]

$$\nabla^z\hat{T}_{zz} = -\frac{c}{48\pi}\partial_z R. \tag{65}$$

By adding in an identically zero term we can write this as

$$\nabla^z\hat{T}_{zz} + \nabla^{\bar{z}}\hat{T}_{\bar{z}z} = -\frac{c}{48\pi}\partial_z R, \tag{66}$$

which at first glance looks like a covariant tensor equation. It is is not, however, because replacing the free index z with $\bar{z}$ leads to

$$\nabla^z\hat{T}_{z\bar{z}} + \nabla^{\bar{z}}T_{\bar{z}\bar{z}} \stackrel{?}{=} -\frac{c}{48\pi}\partial_{\bar{z}} R \tag{67}$$

on which the left hand side is identically zero, but the right need not be. Thus (66) is *not* the covariant anomaly.

A more symmetric treatment [37] divides the trace anomaly between the left and right chiral sectors and constrains one of them to zero. Then $\hat{T}_{\bar{z}\bar{z}}$ remains zero, but

$$\hat{T}_{z\bar{z}} \to -\frac{c}{48\pi}\partial^2_{z\bar{z}}\phi, \tag{68}$$

so that

$$\hat{T}^\mu{}_\mu = \frac{c}{48\pi}R. \tag{69}$$

This physical reinterpretation makes the (still *mathematically* valid) equation (58) read

$$\nabla^z\hat{T}_{zz} + \nabla^{\bar{z}}\hat{T}_{\bar{z}z} = -\frac{c}{96\pi}\partial_z R, \tag{70}$$
$$\nabla^z\hat{T}_{z\bar{z}} + \nabla^{\bar{z}}\hat{T}_{\bar{z}\bar{z}} = +\frac{c}{96\pi}\partial_{\bar{z}} R, \tag{71}$$

where the second term on the left hand side of the second equation is constrained to be zero.

In our z, $\bar{z}$ co-ordinates system we have $\sqrt{g} = \sqrt{-g_{\bar{z}z}g_{z\bar{z}}} = -ie^{\phi}/2$, and we can write these last two equations in a covariant manner as

$$\nabla^z \hat{T}_{zz} + \nabla^{\bar{z}} \hat{T}_{\bar{z}z} = i\frac{c}{96\pi}\sqrt{|g|}\epsilon_{z\bar{z}}\partial^{\bar{z}}R, \tag{72}$$

$$\nabla^z \hat{T}_{z\bar{z}} + \nabla^{\bar{z}} \hat{T}_{\bar{z}\bar{z}} = i\frac{c}{96\pi}\sqrt{|g|}\epsilon_{\bar{z}z}\partial^{z}R. \tag{73}$$

In general euclidean co-ordinates we therefore have [38]

$$\nabla^\mu \hat{T}_{\mu\nu} = i\frac{c}{96\pi}\sqrt{g}\epsilon_{\nu\sigma}\partial^\sigma R. \tag{74}$$

The factor "i" appears in (75) because it is only the *imaginary* part of the Euclidean effective action that can be anomalous [6, 34]. It is absent when we write the equation in Minkowksi signature space-time, where

$$\nabla_\mu \hat{T}^{\mu\nu} = \frac{c}{96\pi}\frac{1}{\sqrt{|g|}}\epsilon^{\nu\sigma}\partial_\sigma R. \tag{75}$$

Note that (75) can be rewritten as $\nabla_\mu \widetilde{T}^{\mu\nu} = 0$ where

$$\widetilde{T}^{\mu\nu} = \hat{T}^{\mu\nu} - \frac{c}{96\pi}\frac{1}{\sqrt{|g|}}\epsilon^{\nu\sigma}R. \tag{76}$$

The new tensor $\widetilde{T}^{\mu\nu}$ is conserved, but not symmetric. We have therefore exchanged a reparametrization anomaly for a Lorentz anomaly.

We now show that the manifestly covariant anomaly (75) is that expected from the anomaly inflow.

V. GRAVITATIONAL CHERN-SIMONS TERMS

In this section we will use both the co-ordinate and frame-field (vielbein) description of geometric quantities. Thus e_a^μ are the components of the frame field $\mathbf{e}_a = e_a^\mu \partial_\mu$ and $e_\mu^{\star b}$ the components of the co-frame $\mathbf{e}^{*a} = e_\mu^{*a} dx^\mu$, with $\delta_b^a = e_\mu^{*a} e_b^\mu$. The frame metric

$$\eta_{ab} = g_{\mu\nu} e_a^\mu e_b^\nu \tag{77}$$

is $\mathrm{diag}(1,1,1)$ and $\mathrm{diag}(1,-1,-1)$ in Euclidean and Minkowski space, respectively.

A gravitational $(2+1)$ dimensional Chern-Simons functional can be written either in terms of the Christoffel-symbol form $\Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu\sigma} dx^\sigma$ as

$$C[\Gamma] = \frac{c}{96\pi}\int_M \mathrm{tr}\,\{\Gamma d\Gamma + \frac{2}{3}\Gamma^3\}. \tag{78}$$

or in terms of the spin connection $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$ as

$$C[\omega] = \frac{c}{96\pi} \int_M \mathrm{tr}\,\{\omega d\omega + \frac{2}{3}\omega^3\}. \tag{79}$$

The integrands in these two functionals have the same exterior derivative

$$d\,\mathrm{tr}\,\{\Gamma d\Gamma + \frac{2}{3}\Gamma^3\} = d\,\mathrm{tr}\,\{\omega d\omega + \frac{2}{3}\omega^3\} = \mathrm{tr}\,\{R^2\}, \tag{80}$$

and so they coincide when $M = \partial N$ is a boundary, but they are no longer equal when M itself has a boundary. Their normalization is related to the index

$$\begin{aligned} \mathrm{Index}(D_{\mathrm{Dirac}}) &= \mathrm{DimKer}(D_{\mathrm{Dirac}}) - \mathrm{DimKer}(D^\dagger_{\mathrm{Dirac}}) \\ &= \frac{1}{192\pi^2} \int_N \mathrm{tr}\,\{R^2\} \end{aligned} \tag{81}$$

of the four dimensional Dirac operator. The Dirac index an *even* integer for any four-dimensional manifold possessing a spin structure.

The spin connection is related to the Christoffel form by a GL(3) gauge transformation

$$\omega^i{}_{j\mu} = e^{*i}_\nu \Gamma^\nu{}_{\lambda\mu} e^\lambda_j + e^{*i}_\nu \partial_\mu e^\nu_j, \tag{82}$$

and so

$$C[\omega] = C[\Gamma] - \frac{c}{96\pi} \int_{\partial M} \mathrm{tr}\,\{(dee^*)\Gamma\} - \frac{c}{288\pi} \int_M \mathrm{tr}\,\{(e^* de)^3\}. \tag{83}$$

Here the matrix-valued one-forms dee^* and e^*de are defined by $(dee^*)^\mu{}_\nu \equiv (\partial_\sigma e^\mu_a) e^{*a}_\nu dx^\sigma$ and $(e^*de)^a{}_b \equiv e^{*a}_\mu \partial_\sigma e^\mu_b dx^\sigma$.

The functional $C[\Gamma]$ is invariant under reparametrization $x^\mu \to X^\mu(x)$ up to boundary terms. To obtain an energy-momentum conserving theory it has to be attached to a suitable boundary theory with compensating transformation properties. We do not have to write down the corresponding Wess-Zumino action $W(\Gamma, X)$ to know the boundary theory anomaly. All we need to do is calculate the out-flowing bulk energy-momentum flux by computing the response of $C[\Gamma]$ to a change in the metric.

The variation of the Chern-Simons functional due to a change in Γ is

$$\delta C[\Gamma] = \frac{c}{48\pi} \int_M \mathrm{tr}\,\{\delta\Gamma R\} + \frac{c}{96\pi} \int_{\partial M} \mathrm{tr}\,\{\delta\Gamma d\Gamma\}. \tag{84}$$

To compute the contribution to the energy-momentum tensor we also need

$$\delta\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda}(\nabla_\nu \delta g_{\lambda\sigma} + \nabla_\sigma \delta g_{\sigma\lambda} - \nabla_\lambda \delta g_{\nu\sigma}). \tag{85}$$

Then, making use of properties of the Riemann tensor that are unique to three dimensions (See [35, 36] for more details), we find

$$\delta C[\Gamma] = \frac{c}{48\pi} \int_M d^3x \sqrt{|g|} C^{\mu\nu} \delta g_{\mu\nu} + \text{boundary terms}, \tag{86}$$

where

$$C^{\mu\nu} = -\frac{1}{2\sqrt{|g|}} \left(\epsilon^{\rho\sigma\mu} \nabla_\rho R^\nu_\sigma + \epsilon^{\rho\sigma\nu} \nabla_\rho R^\mu_\sigma \right). \tag{87}$$

is the *Cotton tensor*. We read-off the bulk energy-momentum tensor to be

$$T^{\mu\nu} = -\frac{c}{24\pi} C^{\mu\nu} \tag{88}$$

In deriving this result we have had to integrate by parts a second time so as to remove the derivatives from the metric variations. Consequently the boundary terms are more complicated than the usual ones arising from the variation of gauge field Chern-Simons functionals. We are, however, confident that these boundary terms provide the same conversion of the consistent anomaly of the boundary theory into the covariant anomaly that we saw with the gauge anomalies.

We restrict ourselves to product metrics of the form

$$ds^2 = (dx^2)^2 + g_{ab}(x^0, x^1) dx^a dx^b, \quad a, b = 0, 1 \tag{89}$$

with boundary being at $x^2 = 0$. The Ricci tensor apearing in (87) then coincides with the Ricci tensor of the two-dimensional boundary, and can be written as

$$R^a_b = \frac{1}{2} \delta^a_b R(x^0, x^1), \quad a, b = 0, 1 \tag{90}$$

The flux of the $a = 0, 1$ energy-momentum components into the boundary becomes

$$T^{2a} = \frac{c}{96\pi} \frac{1}{\sqrt{g}} \epsilon^{\rho a 2} \partial_\rho R. \tag{91}$$

The energy momentum inflow into the boundary therefore precisely accounts for the the gravitational anomaly (75). The "suitable boundary theory" is thus exactly the chiral theory whose anomaly we obtained in the previous section.

In contrast to $C[\Gamma]$, the Chern-Simons functional $C[\omega]$ is reparametrization invariant, but it fails by boundary terms to be invariant under rotations (or Lorentz transformations) of the frame field:

$$\begin{aligned} \mathbf{e}_a &\to \mathbf{e}^O_a = \mathbf{e}_b \, O^b{}_a, \\ \omega^a{}_b &\to (\omega^O)^a{}_b = (O^{-1})^a{}_c \omega^c{}_d O^d{}_b + (O^{-1})^a{}_c dO^c{}_b. \end{aligned} \tag{92}$$

To obtain the energy momentum flow associated with $C[\omega]$ we should remember that ω is linked to the metric through the torsion-free condition

$$d\mathbf{e}^{*a} + \omega^a{}_b \wedge \mathbf{e}^{*b} = 0. \tag{93}$$

and through $g_{\mu\nu} = \eta_{ab} e^{*a}_\mu e^{*b}_\nu$. We therefore define a tensor T_{bc} and its contravariant version $T^{da} = \eta^{db}\eta^{ac} T_{bc}$ by varying the vielbein:

$$\begin{aligned} \delta S_{\text{eff}} &= \int d^n x \sqrt{g} \left(\frac{\delta S}{\delta e^\mu_a} \right) \delta e^\mu_a \\ &\equiv \int d^n x \sqrt{g} \left(T_{bc} \eta^{ca} e^{*b}_\mu \right) \delta e^\mu_a. \\ &= \int d^n x \sqrt{g}\, T^{da} \delta e_{da}. \end{aligned} \tag{94}$$

The last line introduces the useful quantity. $\delta e_{da} = \eta_{db} e^{*b}_\mu \delta e^\mu_a$. As defined, there is no immediate reason for T_{bc} to be symmetric. However when the functional S is invariant under an infinitesimal local rotation $\delta e^\mu_a = e^\mu_b \theta^b{}_a$, we have

$$\begin{aligned} 0 &= \delta S_{\text{eff}} \\ &= \int d^n x \sqrt{g}\, T_{bc}\, \eta^{ca} e^{*b}_\mu e^\mu_d \theta^d{}_a \\ &= \int d^n x \sqrt{g}\, T_{bc}\, \eta^{ca} \theta^b{}_a \\ &= \int d^n x \sqrt{g}\, T^{da}\, \theta_{da}. \end{aligned}$$

Since θ_{da} is an arbitrary skew symmetric matrix, we see that $T^{da} = T^{ad}$. Accepting this symmetry, we can now set

$$\begin{aligned} \delta S_{\text{eff}} &= \frac{1}{2} \int d^n x \sqrt{g}\, T_{bc} \left(\eta^{ca} e^{*b}_\mu \delta e^\mu_a + \eta^{ba} e^{*c}_\mu \delta e^\mu_a \right) \\ &= \frac{1}{2} \int d^n x \sqrt{g}\, T_{\alpha\beta} \left(e^\beta_c \eta^{ca} \delta e^\alpha_a + e^\alpha_b \eta^{bc} \delta e^\beta_c \right) \\ &= \frac{1}{2} \int d^n x \sqrt{g}\, T_{\alpha\beta}\, \delta g^{\alpha\beta}. \end{aligned}$$

Here $T_{\alpha\beta} = e^b_\alpha e^c_\beta T_{bc}$. Thus, for rotation invariant actions, the vielbein variation leads to the same energy-momentum tensor as Hilbert's metric variation.

Now we have

$$\delta C[\omega] = \frac{c}{48\pi} \int_M \text{tr}\,\{\delta\omega R\} + \frac{c}{96\pi} \int_{\partial M} \text{tr}\,\{\delta\omega d\omega\}. \tag{95}$$

and we can use

$$
\begin{aligned}
(\delta\omega_{ij\mu})e^{\mu}_{k} \;=\; -\frac{1}{2}\{&(\nabla_j\delta e_{ik}-\nabla_k\delta e_{ij})\\
&+(\nabla_k\delta e_{ji}-\nabla_i\delta e_{jk})\\
&-(\nabla_i\delta e_{kj}-\nabla_j\delta e_{ki})\}
\end{aligned}
$$

to compute T_{ab}. We do not have to perform this rather tedious computation, however. We know that the variations of $C[\Gamma]$ and $C[\omega]$ differ only by boundary terms. The *bulk* energy-momentum tensors for the two actions must therefore coincide. The boundary variations will differ though. Because $C[\omega]$ is reparametrization invariant, the Wess-Zumino term

$$
W[\omega,O] \stackrel{\text{def}}{=} C[\omega^{O}]-C[\omega] \tag{96}
$$

that together with $C[\omega]$ gives the rotation and reparametrization invariant action $C[\omega^{O}]$, must give rise to a *conserved* boundary-theory energy-momentum tensor T^{ab}_{WZ}. This tensor must also be covariant under co-ordinate changes, but will not be symmetric. There is only one possibility — the frame field version of (76):

$$
T^{ab}_{\mathrm{WZ}}=\widetilde{T}^{ab}=\hat{T}^{ab}-\frac{c}{96\pi}\frac{1}{\sqrt{g}}\epsilon^{ab}R. \tag{97}
$$

The contribution X^{ab} that comes from the boundary part of the variation of $C[\omega]$ will then repair the asymmetry. This contribution is easily computed, and is

$$
X^{ab}=\frac{c}{96\pi}\frac{1}{\sqrt{g}}\epsilon^{ab}R. \tag{98}
$$

The net effect is that we get the same boundary-theory energy-momentum tensor $\hat{T}^{\mu\nu}=T^{\mu\nu}_{\mathrm{WZ}}+X^{\mu\nu}$, and the same anomaly equation, independent of whether we write the gravitational Chern-Simons function in terms of Γ or in terms of ω. The only difference between the two formulations lies in the manner in which the boundary energy-momentum is apportioned between the bulk Chern-Simons contribution $X^{\mu\nu}$ and the boundary Wess-Zumino part $T^{\mu\nu}_{\mathrm{WZ}}$.

VI. CONCLUSIONS

We have seen that it is most likely that the thermal Hall currents on the surface of topological insulators are confined to one dimensional domain walls, and cannot flow in the two-dimensional surface. To confirm this idea we computed the energy-momentum flux associated with a gravitational Chern-Simons term in the boundary effective action. We found that the energy-momentum flux is proportional to gradients of the Ricci curvature, and therefore needs tidal forces to be non-zero. We related this flux to the gravitational anomaly experienced by modes localized on one-dimensional domain walls within the surface, and showed that this anomaly takes the same covariant form independently of whether we write the gravitational Chern-Simons functional in terms of the Christoffel symbol Γ or the spin connection ω.

VII. ACKNOWLEDGEMENTS

This work was supported by the National Science Foundation under grant DMR 09-03291. This work was begun during the TopoMat-11 program at KITP, Santa Barbara, and was there supported in part by the National Science Foundation under Grant No. NSF PHY05-51164. I would like to thank Andreas Ludwig, Shinsei Ryu, and Taylor Hughes for discussions, and also Joel Moore and Shoucheng Zhang for comments and feedback.

---

[1] D. Boyanovsky, E. Dagotto, E. Fradkin, Nucl. Phys, **B285** 340-362 (1987).

[2] M. Stone, Ann. Phys. **207** 38-52 (1991).

[3] C. G. Callan, J. A. Harvey, Nucl. Phys. **B250** 427-436 (1985).

[4] M. R. Zirnbauer, J. Math. Phys. **37** (1996), 4986-5018.

[5] A. Altland, M. R. Zirnbauer, Phys Rev **B55** (1997) 1142-1161.

[6] L. Alvarez-Gaume, E. Wittem, Nucl. Phys. **B234** 269-330 (18983).

[7] I. Vurio, Phys. Lett. **B175** 176-178 (1986).

[8] M. A. Goñi, M. A. Valle, Phys. Rev. **D34** 648-649 (1986).

[9] J. J. van der Bij, R D. Pisarski, A. Rao, Phys. Lett. **B179** 87-91 (1986).

[10] G. E. Volovik, JETP Lett. **51**, 125-128, (1990).

[11] N. Read, D. Green, Phys. Rev. **B61** 10267-10297 (2000).

[12] S. Ryu, J. E. Moore, A. W. W. Ludwig. arXiv:1010.0936.

[13] Z. Wang, X-L. Qi, S-C. Zhang, Phys. Rev. **B84** 014527 (2011) [1-5].

[14] K. Nomura, S. Ryu, A. Furusaki, N. Nagaosa, Phys. Rev. Lett. **108** 026802 (2012) [1-5]

[15] A. Cappelli, M. Huerta, G. R. Zemba, Nucl. Phys. **B636** 568-582 (2002).

[16] X. G. Wen, Int. J. Mod. Phys. **B6** 1711-1762 (1992).

[17] L. Fu, C. L. Kane, Phys. Rev. Lett. **102** 216403 (2009) [1-4].

[18] G. E. Volovik, JETP Letters **91** 201-205 (2010).

[19] L. V. Levitin, R. G. Bennett, A. J. Casey, B. Cowan, J. Parpia, J. Saunders, J. Low Temp. Phys. **158**, 159-162 (2010).

[20] I. Affleck, Phys. Rev. Lett. **56** 746-8 (1986); H. W. J. Blöte, J. L. Cardy, M. P. Nightingale, Phys. Rev. Lett. **56** 742-745 (1986).

[21] C. L. Kane, M. P. A. Fisher, Phys. Rev. **B55** 15832-15837 (1997).

[22] J. M. Luttinger, Phys. Rev. **135** A1505-A1504 (1964).

[23] A. Einstein, Ann. Phys. **322** 549-560 (1905).

[24] N. R. Cooper, B. I. Halperin, I. .M. Ruzin, Phys. Rev. **B 55** 2344-2359 (1997).

[25] W. A. Bardeen, B. Zumino, Nucl. Phys. **B244** 421-453 (1984).

[26] . J. Wess, B. Zumino, Physics Letters **37 B** 95-97 (1971)

[27] E. Witten, Nucl. Phys. **B223** 422-432 (1983).

[28] R. Bott, R. Seeley, Comm. Math. Phys. **62** 235-245 (1978).

[29] J. L. Mañes, Nucl. Phys. **B250** 269-384 (1985).

[30] S. G. Nachulich, Nucl. Phys. **B296** 837-867 (1988).

[31] J. A. Harvey, O. Ruchayskiy JHEP06(2001)044 [1-14]; J. Blum, J. A. Harvey, Nucl. Phys. **B416** 119-136 (1994).

[32] A. Polyakov, Phys. Lett. **103B** 207-210 (1981).

[33] M. J. Duff, Class. Quantum Grav. **11** 1387-1403 (1994).

[34] L. Alvarez-Gaume, P. Ginsparg, Nucl. Phys. **161** 423-490 (1985).

[35] P. Krauss, F. Larsen, JHEP01(2006)022 [1-22].

[36] R. F. Perez, arxiv:1004.3161.

[37] R. Banerjee, S. Kulkarni, Phys. Rev. **D79** 084035 (2009) [1-9].

[38] S. A Fulling, Gen. Relativ. Gravit. **18** 609-615 (1986).

[39] The "covariant conservation" of $T^{\mu\nu}$ becomes an actual conservation law only when the manifold possesses a Killing vector field η^μ. Then the Killing equation $\nabla_\mu\eta_\nu + \nabla_\nu\eta_\mu = 0$ leads to

$J^{\mu}_{\mathrm{C}} \equiv T^{\mu\nu}\eta_{\nu}$ obeying

$$\nabla_{\mu} J^{\mu}_{\mathrm{C}} \equiv \frac{1}{\sqrt{|g|}} \frac{\partial \sqrt{|g|} J^{\mu}_{\mathrm{C}}}{\partial x^{\mu}} = 0,$$

and hence to the time-independence of the integral of J^{0}_{C}.