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Scattering theory of chiral Majorana fermion interferometry

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Using scattering theory, we investigate interferometers composed of chiral Majorana fermion modes coupled to normal metal leads. We advance an approach in which also the basis states in the normal leads are written in terms of Majorana modes. Thus each pair of electron-hole states is associated with a pair of Majorana modes. Only one lead Majorana mode couples to the intrinsic Majorana mode whereas its partner is completely reflected. Similarly the remaining Majorana modes are completely reflected but in general mix pair-wise. We demonstrate that the charge current can also be expressed in terms of interference between pairs of Majorana modes. These two basic facts permit a treatment and understanding of current and noise signatures of chiral Majorana fermion interferometry in an especially elegant way. As a particular example of applications, in Fabry-Perot-type interferometers where chiral Majorana modes form loops, resonances (anti-resonances) from such loops always lead to peaked (suppressed) Andreev differential conductances, and negative (positive) cross-correlations that originate purely from two-Majorana-fermion exchange. These investigations are intimately related to current and noise signatures of Majorana bound states.

I. INTRODUCTION

Majorana fermions (MFs) are their own anti-particles. They have been the subject of theories for more than seven decades without an experimental signature. However recently it has been suggested that these exotic particles can be discovered in condensed matters systems, in particular in topological superconductors (TSCs)^{1–8}. TSCs are superconducting systems that admit exceptional boundary states –owing to nontrivial topology associated with bulk spectra– inside the quasi-particle excitation gaps. These exceptional boundary states are coherent superpositions of electrons (particles) and holes (anti-particles) which, upon proper compositions, represent realizations of Majorana states. The presence of Majorana states in TSCs is robust because of their deep roots in the global properties of bulk states. This leads to renewed efforts to reveal MFs in laboratories.

Two examples of TSCs are chiral p-wave superconductors in one dimension (1D) and two dimensions (2D). In 1D, a chiral p-wave superconductor may accommodate a pair of Majorana bound states (MBSs) separately at its two ends¹. The energy of this pair of MBSs at large separation is exponentially close to the chemical potential – by definition the zero energy– of the superconductor, and each MBS hence provides a container for a MF to stay. Recently, systems equivalent to 1D p-wave superconductors have been proposed based on proximity effects in 2D topological insulators^{9,10}, or even more close to experimental reality, in 1D semiconducting quantum wires^{11–13}. To detect MBSs in these systems^{14,15}, and possibly to manipulate the MFs therein, is of great interest in current research^{16–18}. 2D chiral p-wave superconductors², and equivalent systems based on the proximity effect in topological insulators^{19,20} or semiconductors¹¹, are hosts for chiral Majorana modes (χ MMs), which are gapless, charge-neutral edge excitations. χ MMs can serve as co-

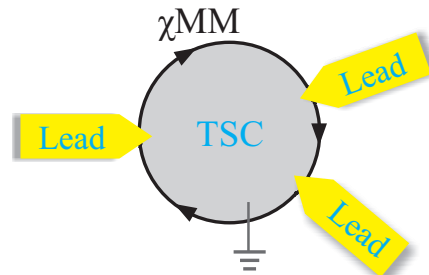


FIG. 1: Schematic picture of a typical chiral Majorana fermion interferometer composed of a chiral Majorana mode (χ MM) coupled to one or several normal metal leads. The chiral Majorana mode lives at the edge of the underlying topological superconductor (TSC), which has a full pairing gap and is grounded. The interior of the topological superconductor may accommodate a number of vortices, each permitting a magnetic flux quantum to penetrate through and containing a Majorana bound state. Majorana fermions traveling in the chiral Majorana mode pick up a phase that encodes (the parity of) the number of vortices, then scatter into normal leads. To generate charge current and noise Majorana fermions interfere pairwise.

herent transmission channels for MFs, and hence are ideal ingredients for building MF interferometers^{21–25} (see, e.g., Fig. 1). Unlike conventional (electronic) interferometers, these MF interferometers do not permit an arbitrary magnetic flux insertion due to the presence of the underlying superconductors. Instead, magnetic flux in the MF interferometers is trapped in superconducting vortex cores and remains quantized in integer multiples of a flux quantum. In addition, each vortex here allows for one MBS^{2,26,27}, and the parity of the number of vortices gives rise to a \mathbb{Z}_2 -type of MF interferometry^{21–23,25}.

Both examples mentioned above belong to the same universality class, labeled D in the Altland-Zirnbauer

classification²⁸. This means that in these systems time-reversal symmetry is broken but particle-hole symmetry (PHS) –as far as the Bogoliubov-de Gennes theory is concerned– is present. Effectively these systems are spinless, and the PHS operator can be simply expressed as $\Xi = \sigma_x \mathcal{K}$ with \mathcal{K} the complex conjugate operator and σ_x the Pauli matrix reversing the electron and hole components. We will limit ourselves to this universality class in this paper, and use the inherent PHS as the only constraint to the scattering theory that will be developed here.

Scattering theory in mesoscopic physics^{29–31} has been proven a very useful tool in dealing with electronic coherent transport phenomena. Recent developments have also shown powerful application of scattering theory to the topological classification of condensed matter^{32–34}, extending an approach known from adiabatic pumping processes³⁵. In this paper we employ scattering theory to investigate interferometers built on χ MMs. We pursue especially a general understanding of how MFs –being exotic (quasi-) particles which are charge-neutral– can be involved in charge transport. For such a purpose we adopt consistently the Majorana basis in our treatment, where the scattering theory of (charge) current and noise takes interestingly an off-diagonal form. Namely, it is the pairwise interference between transmitted MFs, instead of the self-interference of individual particles (as for electrons or holes), that plays the central role in observables like current and noise.

In the following, we will first analyze the scattering at a junction between a χ MM and a normal metal lead, which turns out to be also pairwise in the properly-chosen Majorana basis. In other words, the scattering at the junction can be significantly simplified in terms of both its physical picture and its mathematical parametrization. Next we will apply the simplified scattering picture to compute current and noise in several examples of chiral MF interferometers, including prototypes like Fabry-Perot²³, Mach-Zehnder^{21,22} and Hanbury Brown-Twiss two-particle interferometers²⁵. The emphasis of these calculations is placed on demonstrating the way Majorana scattering amplitudes enter the expressions for current and noise, and interpreting the results in a consistent manner. Most of the examples used in this paper have been discussed in the literature, therefore we will not dwell on detailed discussions about the implications of specific results. Nevertheless our approach shows its elegance in dealing with chiral MF interferometers and in understanding them.

II. JUNCTION BETWEEN A CHIRAL MAJORANA MODE AND A NORMAL LEAD

To address the question how MFs can be involved in charge transport, it is necessary to analyze how they are contacted with a normal (metal) lead which is indispensable for measurements. In the chiral MF interferometry,

where the propagating pathways for MFs can be devised to serve various particular interests, junctions between χ MMs and normal leads become a universal, as well as crucial, ingredient and deserve a dedicated investigation. In this section we will focus on such a contact problem, leaving the propagation of chiral MFs to be handled in the next section with specific examples of interferometers.

To this end we consider a χ MM tunneling-coupled to a normal lead which contains N transmission modes around the Fermi energy and is connected to an electron reservoir with chemical potential μ at its far end⁴². The junction is described by the scattering theory adapted to deal with interfaces between normal and superconducting systems³⁶. Namely, the N transmission modes in the normal lead are artificially doubled into their electron and hole copies, which are then treated independently (but constrained by the PHS; see below) with different occupation functions

$$f_e(E) = \frac{1}{e^{\beta(E-\delta\mu)} + 1}, \quad (1)$$

$$f_h(E) = 1 - f_e(-E) = \frac{1}{e^{\beta(E+\delta\mu)} + 1}, \quad (2)$$

where $\beta = 1/k_B T$ with k_B the Boltzmann constant and T the temperature, $\delta\mu \equiv \mu - \mu_s$ with μ_s the chemical potential in the superconductor. Throughout this paper we assume $\delta\mu$ is much smaller than the superconducting gap, μ_s is fixed by grounding the superconductor, and energy E is always measured relative to μ_s . In order to remove the artificial doubling from actual physical effects, it suffices to count only contributions by states with $E \geq 0$.

Specifically, the scattering matrix, S , for the junction at energy E is defined by

$$\begin{pmatrix} \gamma^{(+)}(E) \\ \psi_e^{(+)}(E) \\ \psi_h^{(+)}(E) \end{pmatrix} = S(E) \begin{pmatrix} \gamma^{(-)}(E) \\ \psi_e^{(-)}(E) \\ \psi_h^{(-)}(E) \end{pmatrix} \quad (3)$$

where γ is the annihilation operator for chiral MFs, $\psi_{e/h} \equiv (\psi_{1e/h}, \psi_{2e/h}, \dots, \psi_{Ne/h})^T$ is the N -vector of electron/hole annihilation operators in the normal lead and the subscripts (+/–) always stand for the outgoing/incoming states. The operators here are each normalized by (the square root of) the velocity of the corresponding mode, such that S is a $(2N + 1)$ -dimensional unitary matrix as a result of quasi-particle probability current conservation. We emphasize that S by itself does not respect charge conservation because of the implicit presence of the TSC upon which the χ MM has to live.

Often it turns out to be more convenient to work with the scattering matrix in an alternative basis –the basis is called the Majorana basis and hence the scattering matrix is denoted by S_M – which is defined as follows

$$\begin{pmatrix} \gamma^{(+)}(E) \\ \boldsymbol{\eta}^{(+)}(E) \end{pmatrix} = S_M(E) \begin{pmatrix} \gamma^{(-)}(E) \\ \boldsymbol{\eta}^{(-)}(E) \end{pmatrix} \quad (4)$$

where

$$\boldsymbol{\eta}^{(+/-)}(E) \equiv U_N \begin{pmatrix} \boldsymbol{\psi}_e^{(+/-)}(E) \\ \boldsymbol{\psi}_h^{(+/-)}(E) \end{pmatrix}, \quad (5)$$

$$U_N \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \otimes \mathbb{1}_N \quad (6)$$

with $\mathbb{1}_N$ the N -dimensional identity matrix acting within the space of electron or hole modes. Evidently S_M is also unitary, and is related to S by a change of basis in the normal lead:

$$S(E) = \begin{pmatrix} 1 & \\ & U_N^\dagger \end{pmatrix} S_M(E) \begin{pmatrix} 1 & \\ & U_N \end{pmatrix}. \quad (7)$$

The inherent PHS in the current problem imposes that, by definition,

$$\gamma^\dagger(E) = \gamma(-E), \quad (8)$$

$$\boldsymbol{\psi}_e^\dagger(E) = \boldsymbol{\psi}_h(-E), \quad (9)$$

both of which apply to outgoing and incoming states separately. The latter of the above equations implies

$$\boldsymbol{\eta}^\dagger(E) = \boldsymbol{\eta}(-E), \quad (10)$$

which allows us to regard the scattering modes represented by η -operators as artificial Majorana modes. The scattering matrices constrained accordingly by the PHS satisfy: in the electron-hole basis,

$$\begin{pmatrix} 1 & \\ & \Sigma_x \end{pmatrix} S^*(E) \begin{pmatrix} 1 & \\ & \Sigma_x \end{pmatrix} = S(-E), \quad (11)$$

where $\Sigma_x \equiv \sigma_x \otimes \mathbb{1}_N$ with the Pauli matrix σ_x interchanges the flavors of electron and hole; in the Majorana basis,

$$S_M^*(E) = S_M(-E). \quad (12)$$

It is clear that at $E = 0$, S_M becomes a real orthogonal matrix⁴³.

Indeed, in the rest of this paper, we consider the low-energy limit where the energy dependence of the scattering matrices in Eqs. (11) and (12) can be dropped (accordingly we will also drop the energy dependence of the operators for the scattering modes). As a result S_M is always taken to be a real orthogonal matrix with determinant 1 (i.e. $S_M \in SO(2N + 1)$). This significantly simplifies the theory that will be developed in the following.

A. General strategy

In this section our goal is to understand the contact between a χ MM and a normal lead by decomposing the scattering matrix at the junction into its physical part and its (physically) irrelevant part. The physical part is

what is relevant to physical observables in the scattering process, and can be taken as the canonical form of the scattering matrix; the irrelevant part by definition does not make physical contributions. To achieve this goal we will work exclusively with scattering matrices in the Majorana basis, S_M , in the current section.

The advantage of working in the Majorana basis is that S_M , being a real orthogonal matrix in the low-energy limit, is both a faithful and a convenient representation of the scattering processes which involve MFs. Such a representation naturally includes all possible scattering events at the junction without additional constraints as Eq. (11) (i.e. the set of S_M in the low-energy limit is precisely $SO(2N + 1)$, while the corresponding set of S is only a subset of $SU(2N + 1)$). This greatly facilitates the decomposition that we will perform in this section. The disadvantage of working in the Majorana basis is that the ‘‘artificial’’ Majorana modes in the normal lead, defined in Eq. (5), are not associated directly with the distribution of electrons/holes in the reservoir. Therefore we need to return to the electron-hole basis in the next section to compute physical observables such as average current and noise. Nevertheless we will see that the decomposition of S_M pays off not only in simplifying the computations, but also in understanding physically the results.

The idea to decompose a scattering matrix into its physical and irrelevant parts is to notice that the physics of a single scattering event is unchanged under $U(N)$ -transformations of the (electron) basis for the N degenerate modes (separately for outgoing and incoming ones) in the normal lead. Decompositions following this idea are conventionally referred to as polar decompositions of a scattering matrix when the physical part contains only the transmission eigenvalues^{37,38}. In the current problem the same idea gets slightly more complicated by the demand of the PHS, which insists that the valid transformations are of the form

$$\begin{pmatrix} \tilde{\boldsymbol{\psi}}_e^{(+)} \\ \tilde{\boldsymbol{\psi}}_h^{(+)} \end{pmatrix} = \begin{pmatrix} V & \\ & V^* \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_e^{(+)} \\ \boldsymbol{\psi}_h^{(+)} \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} \tilde{\boldsymbol{\psi}}_e^{(-)} \\ \tilde{\boldsymbol{\psi}}_h^{(-)} \end{pmatrix} = \begin{pmatrix} W & \\ & W^* \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_e^{(-)} \\ \boldsymbol{\psi}_h^{(-)} \end{pmatrix} \quad (14)$$

with $V, W \in U(N)$. In the Majorana basis, if Eq. (5) is also imposed on the transformed operators (with tildes), this means

$$\tilde{\boldsymbol{\eta}}^{(+)} = \mathcal{V}\boldsymbol{\eta}^{(+)}, \quad \tilde{\boldsymbol{\eta}}^{(-)} = \mathcal{W}\boldsymbol{\eta}^{(-)}, \quad (15)$$

where

$$\mathcal{V} = U_N \begin{pmatrix} V & \\ & V^* \end{pmatrix} U_N^\dagger, \quad \mathcal{W} = U_N \begin{pmatrix} W & \\ & W^* \end{pmatrix} U_N^\dagger \quad (16)$$

$(\mathcal{V}, \mathcal{W} \in SO(2N)).$

Accordingly the decomposition of S_M is given by

$$S_M = \begin{pmatrix} 1 & \\ & \mathcal{V}^T \end{pmatrix} \tilde{S}_M \begin{pmatrix} 1 & \\ & \mathcal{W} \end{pmatrix}. \quad (17)$$

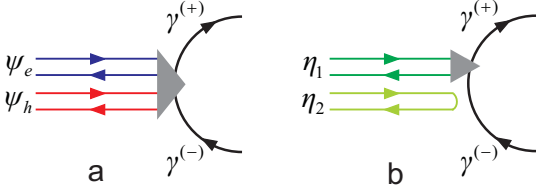


FIG. 2: Illustrations of the scattering at a junction between a single-mode ($N = 1$) normal lead and a χ MM, in the electron-hole basis of the normal lead (a), and in the properly-chosen Majorana basis (b). In the latter basis, only one Majorana mode in the normal lead is coupled to the χ MM, the other is simply reflected with amplitude 1.

When \mathcal{V} and \mathcal{W} are properly chosen, \tilde{S}_M contains a minimal number of parameters⁴⁴ that are relevant to physical observables, and hence is the canonical form of S_M .

As a consequence of the PHS, the decompositions here do not in general bring \tilde{S}_M into a diagonal form as conventional polar decompositions do (the diagonal form is even impossible in the electron-hole basis because the χ MM cannot be coupled only to a single electron/hole mode at all). Instead the decompositions of S_M turn out to take the form of Euler decompositions: mathematically this means \tilde{S}_M appears to be a combination of uncoupled (2D-)planar rotations; physically this means the scattering of many Majorana modes is essentially pairwise. In the rest of this section, we will derive such an Euler decomposition first in the simplest case when $N = 1$, and then in the arbitrary N -channel case.

B. The $N = 1$ case

When $N = 1$, S_M is of dimension 3. Namely, S_M is equivalent to a rotation in three dimensions. Such a rotation can be parametrized by three parameters, known as the Euler angles, as follows

$$S_M = \begin{pmatrix} 1 & 0 \\ 0 & R(\alpha) \end{pmatrix} \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R(\beta) \end{pmatrix}, \quad (18)$$

where

$$R(\xi) \equiv \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix} \quad (\xi = \alpha, \beta, \theta) \quad (19)$$

is a planar rotation by an angle ξ .

The above decomposition is exactly what is dictated by Eq. (17). To see this, simply notice that

$$U_1^\dagger R(\xi) U_1 = \begin{pmatrix} e^{-i\xi} & \\ & e^{i\xi} \end{pmatrix}, \quad (20)$$

thus we can identify $\mathcal{V} = R(-\alpha)$ and $\mathcal{W} = R(\beta)$. Namely, if we redefine the electron and hole operators by trivial

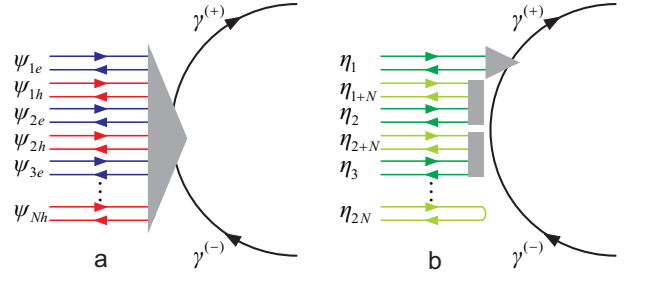


FIG. 3: Illustrations of the scattering at a junction between a multi-mode ($N > 1$) normal lead and a χ MM, in the electron-hole basis for the normal lead (a), and in the properly-chosen Majorana basis (b). In the latter basis, only one Majorana mode in the normal lead is coupled to the χ MM, the others are reflected in pairs and one is reflected with amplitude 1.

gauge transformations (in the normal lead)

$$\begin{pmatrix} \tilde{\psi}_e^{(+)} \\ \tilde{\psi}_h^{(+)} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \psi_e^{(+)} \\ e^{-i\alpha} \psi_h^{(+)} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\psi}_e^{(-)} \\ \tilde{\psi}_h^{(-)} \end{pmatrix} = \begin{pmatrix} e^{-i\beta} \psi_e^{(-)} \\ e^{i\beta} \psi_h^{(-)} \end{pmatrix}, \quad (21)$$

and redefine $\tilde{\eta}$ according to Eq. (5), then the Majorana scattering matrix contains one single parameter and takes the form

$$\tilde{S}_M = \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}. \quad (22)$$

Eq. (22) is the canonical form of the Majorana scattering matrix when $N = 1$, which says that, effectively, the χ MM γ is coupled to one (artificial) Majorana mode $\tilde{\eta}_1$ in the normal lead, leaving the other (artificial) Majorana mode $\tilde{\eta}_2$ to be completely reflected with amplitude 1 (see Fig. 2).

C. The arbitrary N case

When $N > 1$, there is no decomposition like Eq. (18) readily available. Nevertheless one can show that (see Appendix A), the $(2N + 1)$ -dimensional Majorana scattering matrix can always be decomposed as Eq. (17) with

$$\tilde{S}_M = \left[\bigoplus_{i=1}^N R(\theta_i) \right] \oplus 1, \quad (23)$$

where $R(\theta_1)$ couples the modes γ and $\tilde{\eta}_1$, $R(\theta_i)$ with $i > 1$ couples the modes $\tilde{\eta}_i$ and $\tilde{\eta}_{i+N-1}$, and the last diagonal entry 1 signifies a complete reflection of mode $\tilde{\eta}_{2N}$. Note that by definition (see Eq. (5)), $\tilde{\eta}_i$ and $\tilde{\eta}_{i+N}$ are partners derived from the same normal mode $\tilde{\psi}_{ie/h}$, that is,

$$\tilde{\eta}_i = \frac{1}{\sqrt{2}}(\tilde{\psi}_{ie} + \tilde{\psi}_{ih}), \quad (24)$$

$$\tilde{\eta}_{i+N} = \frac{i}{\sqrt{2}}(\tilde{\psi}_{ie} - \tilde{\psi}_{ih}). \quad (25)$$

The concept of such pairs of Majorana modes turns out to be fundamental in relating Majorana scattering matrices to charge transport, as will be revealed in the next section.

The above canonical form of Majorana scattering matrices should be no surprise after we have understood the $N = 1$ case: the χ MM γ effectively couples to a single Majorana mode $\tilde{\eta}_1$ – determined by the specifics of the junction – in the normal lead, which leaves its partner $\tilde{\eta}_{1+N}$ – determined solely by $\tilde{\eta}_1$ – to be coupled to the rest of the Majorana modes in the normal lead; $\tilde{\eta}_{1+N}$ then “picks” another (single) Majorana mode $\tilde{\eta}_2$ to couple to, leaving in turn $\tilde{\eta}_{2+N}$ behind; this process goes on until the last one $\tilde{\eta}_{2N}$ is left completely alone (see Fig. 3). Interestingly, this re-pairing picture is a good analogy to the re-pairing of MFs that happens at the topological phase transition in the Kitaev model for 1D p-wave superconductors¹ – here, the re-pairing of Majorana modes happens when a normal lead makes contact to a 2D p-wave superconductor with a chiral Majorana edge mode.

III. CURRENT AND NOISE FOR SPECIFIC EXAMPLES

In this section we compute the average current and the zero-frequency noise power in various prototypical chiral MF interferometers.

To this end we introduce for convenience two matrices: F , which is a diagonal matrix containing the Fermi distribution functions for all incoming modes, and $(\Sigma_z)_\nu$, which is another diagonal matrix containing the weights for charge current carried by electron and hole modes in lead ν (1 for electron modes and -1 for hole modes; 0 for modes that do not belong to lead ν). Explicitly, we define

$$F \equiv \bigoplus_{\nu=1}^K F_\nu, \quad (26)$$

$$F_\nu \equiv \begin{pmatrix} f_{\nu e} & 0 \\ 0 & f_{\nu h} \end{pmatrix} \otimes \mathbb{1}_{N_\nu}, \quad (27)$$

$$(\Sigma_z)_\nu \equiv \left(\bigoplus_{\nu' \neq \nu} \mathbb{0}_{\nu'} \right) \oplus (\sigma_z \otimes \mathbb{1}_{N_\nu}), \quad (28)$$

where K is the total number of contacts, $f_{\nu e/h}$ is the Fermi distribution function for electrons/holes in contact ν , N_ν is the number of transmission modes (without double counting) in lead ν , $\mathbb{0}_{\nu'}$ is the empty matrix corresponding to the modes in lead ν' .

We will use the following formulas from the scattering theory adapted for normal-superconducting hybrid systems^{31,36}: the time-averaged current at normal contact α , $\langle \hat{I}_\alpha \rangle$, is given by

$$I_\alpha = \frac{e}{h} \int_{E \geq 0} dE \text{Tr}[F A_\alpha]; \quad (29)$$

the average current flowing through the grounded superconducting contact is given, owing to current conservation, by (omitting the minus sign)

$$I_S = \sum_{\alpha \in \text{normal}} I_\alpha; \quad (30)$$

the zero-frequency noise power, defined as $P_{\alpha\beta} = \int_{-\infty}^{\infty} dt \frac{1}{2} \langle \delta \hat{I}_\alpha(t) \delta \hat{I}_\beta(0) + \delta \hat{I}_\beta(0) \delta \hat{I}_\alpha(t) \rangle$ with the current fluctuation $\delta \hat{I}_{\nu=\alpha,\beta}(t) = \hat{I}_\nu(t) - \langle \hat{I}_\nu \rangle$, is given by

$$P_{\alpha\beta} = \frac{e^2}{h} \int_{E \geq 0} dE \text{Tr}[F A_\alpha (\mathbb{1} - F) A_\beta]. \quad (31)$$

In the above formulas we have used a shorthand notation A_ν defined as follows

$$A_\nu \equiv (\Sigma_z)_\nu - S^\dagger (\Sigma_z)_\nu S. \quad (32)$$

Note that S here is the full scattering matrix in the electron-hole basis, which includes K contacts but does not explicitly include the χ MM (see, e.g., Eq. (46) in Sec. III B). The full Majorana scattering matrix S_M is related to S by

$$S = U_\oplus^\dagger S_M U_\oplus, \quad U_\oplus \equiv \bigoplus_{\nu=1}^K U_{N_\nu}. \quad (33)$$

Before we proceed presenting the results, we confirm that the transformations in Eqs. (13) and (14) indeed lead to no change in physical observables. That is, we verify that I_α and $P_{\alpha\beta}$ are unchanged upon replacing S by \tilde{S} , given by

$$S = V_\oplus^\dagger \tilde{S} W_\oplus, \quad (34)$$

$$V_\oplus \equiv \bigoplus_{\nu=1}^K \begin{pmatrix} V_\nu \\ V_\nu^* \end{pmatrix}, \quad W_\oplus \equiv \bigoplus_{\nu=1}^K \begin{pmatrix} W_\nu & \\ & W_\nu^* \end{pmatrix}. \quad (35)$$

The proof is straightforward after noticing that

$$[W_\oplus, F] = 0 \quad \text{and} \quad \forall \nu : [V_\oplus, (\Sigma_z)_\nu] = 0, \quad (36)$$

where $[,]$ stands for the commutator. Therefore we will adopt only the canonical form \tilde{S}_M in the actual calculations, and also suppress the tildes from now on.

A. A single normal lead probing a chiral Majorana loop: $N = 1$

We start with the simplest setup: a single normal lead with a single mode is probing a closed χ MM, or, a chiral Majorana loop; the interior of the TSC – enclosed by the chiral Majorana loop – may accommodate n_ν vortices, each allowing for one MBS at zero energy. This setup, as well as the setup that will be discussed in the next subsection (Sec. III B), has been discussed by Law, Lee and Ng²³. Here we reformulate the problem in a consistent

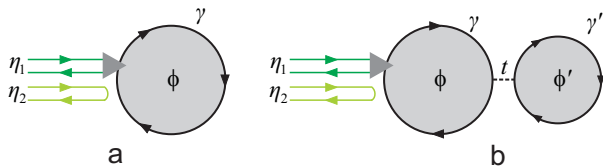


FIG. 4: Effective scattering for setups with a single-mode ($N = 1$) normal lead probing a chiral Majorana loop: the left panel case (a) can be extended into the right panel one (b), to model, e.g., the presence of an additional point contact or scattering at two coupled MBSs (see text for details).

Majorana language, with extensions that are intriguing in their own right.

We assume that the MBSs are all deep inside the TSC, such that the coupling between the MBSs and the χ MM or the normal lead, which exponentially decays with respect to their spatial separation, is negligible. With this assumption the vortices are taken into account only as an additional phase $\phi = n_v\pi$ that the chiral MF picks up after moving in a full circle, and the acquired total phase hence is $\varphi = n_v\pi + \pi + 2\pi E/E_l$, where the single π comes from the Berry phase, and $E_l = \hbar v_M/L$, with L the circumference of the loop, v_M the group velocity of the χ MM, is the level spacing for the chiral Majorana loop.

Following our discussion in Section II B, the effective scattering process for the current example in the Majorana language is pictured in Fig. 4a, and the Majorana scattering matrix is given by

$$S_M = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad (37)$$

where $r_2 = 1$, and

$$r_1 = \frac{r_0 - e^{i\varphi}}{1 - r_0 e^{i\varphi}} \quad (38)$$

with r_0 the (local) reflection amplitude of η_1 at the junction connecting to the still unclosed γ (namely, $\cos\theta$ in Eq. (22)). Note that the above Majorana scattering matrix is implicitly energy-dependent because of the energy-dependence of φ ; it is unitary, but not necessarily real except at $E = 0$.

Inserting S_M into Eqs. (29) and (31), together with (33), we find the average current and the zero-frequency noise power (auto-correlator) to be

$$I = \frac{e}{\hbar} \int_{E \geq 0} dE [1 - \Re(r_1 r_2^*)] \delta f(E), \quad (39)$$

$$P = \frac{e^2}{\hbar} \int_{E \geq 0} dE \left\{ 2[1 - \Re(r_1 r_2^*)] \Theta(E) + \Im(r_1 r_2^*)^2 \delta f(E)^2 \right\}, \quad (40)$$

where

$$\delta f(E) \equiv f_e(E) - f_h(E) = \frac{\sinh \beta \delta \mu}{\cosh \beta E + \cosh \beta \delta \mu}, \quad (41)$$

$$\begin{aligned} \Theta(E) &\equiv f_e(E)[1 - f_e(E)] + f_h(E)[1 - f_h(E)] \\ &= -k_B T \frac{\partial}{\partial E} [f_e(E) + f_h(E)]. \end{aligned} \quad (42)$$

Note that $\delta f(E)$ is positive (negative) definite if $\delta \mu > 0$ ($\delta \mu < 0$), zero at equilibrium, while $\Theta(E)$ is positive definite at finite temperature. As a convention, we shall intentionally keep the formal complex conjugate of real variables (e.g. r_2^*) for a reason that will be clear in a moment.

1. Current as interference of Majorana modes

Let us now examine the expressions for the current Eq. (39) and noise Eq. (40) closely. The net current in Eq. (39) is evidently attributed to the Andreev process: an incoming electron or hole, weighted effectively by the difference of the Fermi distribution functions δf , is reflected with “probability” $\Re(r_1 r_2^*) \in [-1, 1]$, and the missing charges are absorbed by the superconducting condensate as Cooper pairs. Here, negative $\Re(r_1 r_2^*)$ means that a particle with an opposite charge with respect to the incoming particle is reflected with probability $|\Re(r_1 r_2^*)|$. The noise in Eq. (40) has two contributions: the first term in the curly brackets gives the thermal noise which persists at equilibrium at finite temperature; the second term gives the shot noise which vanishes either at equilibrium, where $f_e = f_h$, or upon total normal or Andreev reflections, where $\Im(r_1 r_2^*)^2 = 1 - \Re(r_1 r_2^*)^2 = 0$.

The remarkable expression for the reflection “probability”, $\Re(r_1 r_2^*)$, highlights one essential feature of MF transport, namely, charge is always “carried” by the interference between (a pair of) Majorana modes⁴⁵, even though an individual Majorana mode is charge-neutral. This can be heuristically understood as follows. Take one (quasi-particle) scattering state given by, in the Majorana basis, $|\Psi(E)\rangle = \frac{1}{\sqrt{2}}[r_1|\eta_1(E)\rangle + ir_2|\eta_2(E)\rangle]$; in the electron-hole basis, the same state reads $|\Psi(E)\rangle = a_e|\psi_e(E)\rangle + a_h|\psi_h(E)\rangle$ with $a_e = (r_1 + r_2)/2$ and $a_h = (r_1 - r_2)/2$; we immediately see that the current density carried by this state is $I(E) = (e/\hbar)(|a_e|^2 - |a_h|^2) = (e/\hbar)\Re(r_1 r_2^*)$, which is precisely the interference term between the two Majorana modes. When r_1 and r_2 are identified with the reflection amplitudes in S_M , $\Re(r_1 r_2^*)$ becomes the reflection “probability” that appears in Eqs. (39) and (40). In order to stress the central importance of interference between Majorana modes in this work, we will on most occasions keep formal expressions similar to $\Re(r_1 r_2^*)$ without simplifying expressions written with formally complex conjugate but in reality real variables.

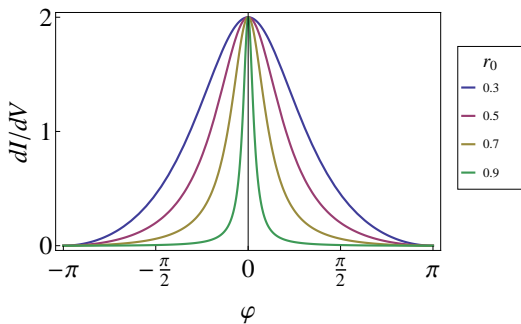


FIG. 5: Differential conductance, dI/dV in units of e^2/h , as a function of $\varphi = 2\pi eV/E_l + (n_v + 1)\pi$ for the setup in Fig. 4a. The resonance peak for φ equal to an even multiple of π and the (anti-resonance) valley for φ equal to an odd multiple of π is a robust feature regardless of the coupling strength, parametrized by r_0 , between the normal lead and the χ MM. This plot is essentially a reproduction of the results of Law et al.²³. The purpose of this reproduction is to facilitate comparisons with our later results.

2. The zero-temperature low-bias limit

At zero-temperature and low bias ($\delta\mu = eV \ll E_l$), $\varphi \simeq (n_v + 1)\pi$, $r_1 \simeq \pm 1$, and Eqs. (39) and (40) can be reduced to²³:

$$\text{for even } n_v: \quad I = P = 0; \quad (43)$$

$$\text{for odd } n_v: \quad I = \frac{2e^2}{h} V, P = 0. \quad (44)$$

The Fano factor, defined as $F = P/(eI)$, is given by $1 + r_1$ and takes the value 2 if n_v is even, or 0 if n_v is odd.

Physically the two situations here are realized exactly when the chiral MF self-interferes maximally destructive or constructive upon reflection, hence a stationary state is disallowed or allowed at zero energy. $r_1 = \pm 1$ therefore corresponds to the reflection of a Majorana mode that is completely off-resonance ($r_1 = +1$) or on-resonance ($r_1 = -1$) with such a stationary state²³. As a consequence, the scattering process is a total normal reflection ($I = 0$) or a total Andreev reflection ($I = 2e^2V/h$) – in either case, the shot noise vanishes. It deserves to be mentioned that the zero-energy eigenstate of the chiral Majorana loop can be effectively identified as a MBS, therefore the above picture also accounts for the zero-energy conductance peak that is regarded as an important signature to detect MBSs.

For more general cases beyond the low-bias limit, where φ is allowed to vary with energy continuously between an even and an odd multiple of π , we plot the zero-temperature differential conductance dI/dV , as a function of $\varphi = 2\pi eV/E_l + (n_v + 1)\pi$ and in units of e^2/h , in Fig. 5 with different r_0 . It is clearly seen that a quantized resonance peak is established at $\varphi = 0 \pmod{2\pi}$ and completely suppressed at $\varphi = \pi \pmod{2\pi}$ ²³; decreasing r_0 broadens the resonance peak but does not remove such a general feature.

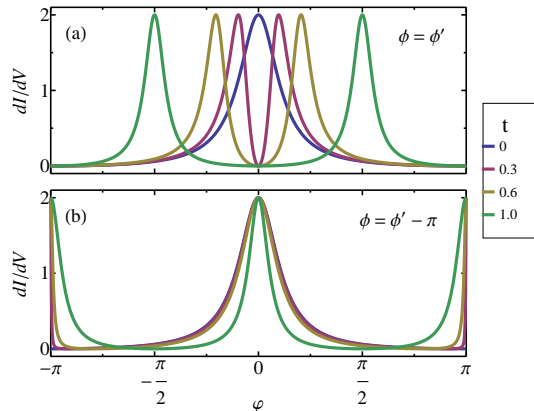


FIG. 6: Differential conductance, dI/dV in units of e^2/h , as a function of $\varphi = 2\pi eV/E_l + \pi + \phi$ with $\phi \equiv n_v\pi$ for the setup in Fig. 4b. We set $\phi = \phi'$ in the upper panel and set $\phi = \phi' - \pi$ in the lower panel; for both panels $L = L'$ is assumed and r_0 is fixed to be 0.7. In the upper panel the resonance peak splits into two with their distance proportional to the energy splitting of the two coupled chiral Majorana loops; in the lower panel the resonance peak persists whenever φ is an integer multiple of π as long as the coupling between the two chiral Majorana loops is finite.

3. Extended setups

The present simple setup can be slightly extended by including another chiral Majorana loop, labeled by γ' , which couples only to the original loop γ with tunneling amplitude t (see Fig. 4b). This extended setup can be used to model, for example, the effect of an additional point contact formed by confining the path of the original χ MM, or scattering upon two coupled MBSs. The computation for the extended setup requires a minimal effort, which amounts to replacing φ in Eq. (38) with φ_{ext} , defined as

$$\varphi_{\text{ext}} = \varphi + \arg\left(\frac{r - e^{i\varphi'}}{1 - r e^{i\varphi'}}\right), \quad (45)$$

where $\varphi' = EL'/\hbar v_M + \pi + \phi'$ with ϕ' the magnetic flux inside γ' , and $r = \sqrt{1 - t^2}$ is the local reflection amplitude between the two chiral Majorana loops. The expressions for the average current and the noise power are unchanged from Eqs. (39) and (40).

For simplicity we consider $\phi = \phi' = \pi$ and $L = L'$. When the coupling between the two loops are turned off, we know from previous discussions that a resonance-led total Andreev reflection occurs at $E = 0$ (more generally, at $E = nE_l$ with n an integer; we will focus on the energy range around zero). Upon turning on the coupling – even with infinitesimal t – the resonance at zero energy is immediately switched off because φ_{ext} has a π shift from φ and hence r_1 changes sign. Indeed, since the energy spectrum of the coupled loops is given by $e^{i\varphi_{\text{ext}}} = 1$ (which is also equivalent to the resonance condition $r_1 = -1$), we

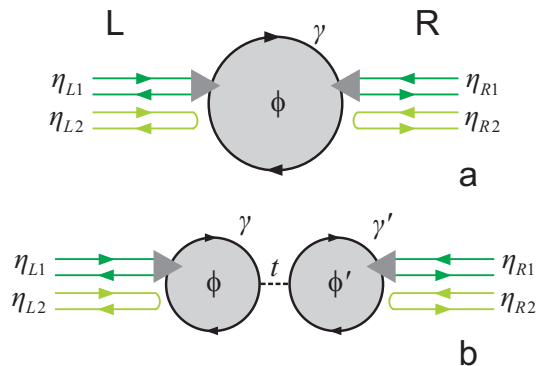


FIG. 7: Majorana scattering channels for Fabry-Perot interferometers considered in Sec. III B: the upper panel case (a) can be extended to the lower panel case (b) to model the presence of an additional point contact or two coupled MBSs (see text for details).

find that the resonance shifts to $E \simeq \pm(E_l/2\pi)t$ when $t \ll 1$. This is in close analogy to a system in which two MBSs (e.g. at the two ends of a chiral p-wave superconducting wire) are coupled and lifted from zero energy³⁹. It deserves to be mentioned that if the current case is modified by setting one of the phases ϕ or ϕ' to be 0 and the other π , then the resonance occurs at zero energy regardless of the coupling strength between the loops as long as it is finite. Differential conductances for both situations, $\phi' - \phi = 0$ or π , are plotted in Fig. 6.

B. Fabry-Perot interferometer: $N = 1$

Our second example deals with the Fabry-Perot interferometer composed of two single-mode normal leads (labeled by L and R) probing a chiral Majorana loop simultaneously. We assume that the two normal leads are mediated only by the χ MM and other tunneling processes are negligible. Fig. 7a illustrates the effective scattering in the Majorana language, and the corresponding Majorana scattering matrix is given by

$$S_M = \begin{pmatrix} r_{L1} & 0 & t_{LR} & 0 \\ 0 & r_{L2} & 0 & 0 \\ t_{RL} & 0 & r_{R1} & 0 \\ 0 & 0 & 0 & r_{R2} \end{pmatrix}, \quad (46)$$

where $r_{L2} = r_{R2} = 1$, and

$$r_{L1} = Z(r_{L0} - r_{R0}e^{i\varphi}), \quad (47)$$

$$r_{R1} = Z(r_{R0} - r_{L0}e^{i\varphi}), \quad (48)$$

$$t_{LR} = -Zt_{L0}t_{R0}e^{i\varphi}, \quad (49)$$

$$t_{RL} = -Zt_{L0}t_{R0}, \quad (50)$$

$$Z \equiv (1 - r_{L0}r_{R0}e^{i\varphi})^{-1}, \quad (51)$$

with the subscript 0 meaning that the reflection/transmission amplitude is local at the junction (corresponding to Eq. (22), $r_0 = \cos\theta$, $t_0 = \sin\theta$, for two

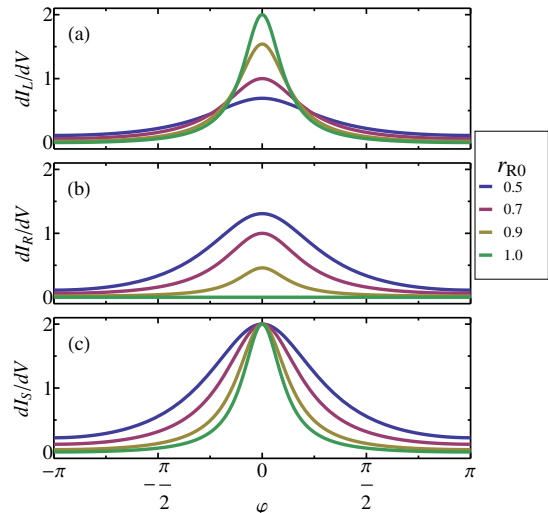


FIG. 8: Differential conductance measured at different contacts (L, R, S) and in units of e^2/h , as a function of $\varphi = 2\pi eV/E_l + (n_v + 1)\pi$ for the Fabry-Perot interferometer in Fig. 7a, where r_{L0} is fixed to be 0.7 and the two normal contacts are equally biased. In general, resonance peaks (anti-resonance valleys) of the differential conductances at $\varphi = 0$ ($\varphi = \pm\pi$) persist as a robust feature. Asymmetric coupling at the two junctions ($r_{L0} \neq r_{R0}$) results in unbalanced resonance peaks in dI_L/dV and dI_R/dV , while the resonance peak in dI_S/dV remains quantized at $2e^2/h$. Particularly when one of the junctions is shut off (e.g. $r_{R0} = 1$), the result returns to what is shown in Fig. 5, where the resonant-Andreev-reflection is completely local. This plot is again essentially a reproduction of the results in Ref.²³, presented here for the purpose of comparison with later results.

leads separately). Note that we have chosen a gauge such that φ belongs entirely to the lower arm.

The average current and the zero-frequency noise power are given by

$$I_L = \frac{e}{h} \int_{E \geq 0} dE [1 - \Re(r_{L1}r_{L2}^*)] \delta f_L, \quad (52)$$

$$P_{LL} = \frac{e^2}{h} \int_{E \geq 0} dE \left\{ 2[1 - \Re(r_{L1}r_{L2}^*)] \Theta_L + \Im(r_{L1}r_{L2}^*)^2 \delta f_L^2 + \frac{1}{2} T_{LR} [\Theta_R - \Theta_L + \frac{1}{2} \sum_{a,b=e,h} (f_{La} - f_{Rb})^2] \right\}, \quad (53)$$

$$P_{LR} = P_{RL} = \frac{e^2}{h} \int_{E \geq 0} dE \left(-\frac{1}{2}\right) \Re(t_{LR}r_{L2}^* t_{RL}r_{R2}^*) \delta f_L \delta f_R, \quad (54)$$

where $T_{LR} = t_{LR}^* t_{LR}$. I_R and P_{RR} can be obtained by interchanging subscripts L and R in the expressions for I_L and P_{LL} .

In the presence of multiple normal contacts, it is particularly interesting to look at the average current I_S

flowing through the grounded superconducting contact. Assuming the two normal contacts here are equally biased ($\delta\mu_L = \delta\mu_R = eV$), I_S is simply given by

$$\begin{aligned} I_S &= I_L + I_R \\ &= \frac{e}{h} \int_{E \geq 0} dE \left[2 - (r_{L0} + r_{R0}) \Re \left(\frac{1 - e^{i\varphi}}{1 - r_{L0} r_{R0} e^{i\varphi}} \right) \right] \delta f, \end{aligned} \quad (55)$$

where we have used the explicit expressions of the reflection amplitudes. An immediate observation based on the above result is that the resonance peak in the differential conductance dI_S/dV at $\varphi = 0$ remains quantized at $2e^2/h$ regardless of the coupling of the two normal leads to the χ MM (see Fig. 8).

1. Cross-correlation as two-MF-exchange effect

Formally the expression (52) for I_L (or I_R) is the same as Eq. (39) in the single lead case, meaning that one contact does not contribute to the average current at the other contact regardless of the electron/hole occupation in either contact. Naively this can be seen as a result of the charge neutrality of the χ MM that bridges two contacts. A more precise interpretation, however, is that the Majorana scattering states sourced from different contacts (e.g. η_{L1} transmitted from contact R and η_{L2}) are not phase coherent when only single-particle scattering is considered, therefore cannot lead to finite average current according to our discussion in Sec. III A. This statement will be clear after we further examine the cross-correlation between the two contacts.

The charge-current cross-correlator P_{LR} (or P_{RL}), given by Eq. (54), is manifestly non-vanishing at low temperature and low bias if, and only if, both contacts are biased ($\delta f_{L,R}(E) \neq 0$) and exchange of Majorana states is allowed ($t_{LR} t_{RL} \neq 0$). This is despite the fact that the mediating χ MM carries no charge current by itself. Indeed, the non-vanishing cross-correlation here is entirely due to those scattering events that involve two-particle exchange and permit coherence between scattering states from different sources, and hence is entirely the ‘‘exchange noise’’. Such an exchange contribution is irrelevant for the average current but present in noise. Meanwhile, the absence of other types of noise (e.g. thermal noise, partition noise) in the cross-correlator again highlights the essential role that the interference between Majorana modes plays in charge transport.

In contrast to the cross-correlation, the auto-correlation for contact L , given by Eq. (53), contains both the equilibrium thermal contribution (the first term in the kernel) and nonequilibrium contributions (the rest two terms). Contact R manifests itself in P_{LL} only away from equilibrium (except for modifying r_{L0} to r_{L1}), which is clearly different from a conventional normal-metal-superconductor-normal-metal device³⁶.

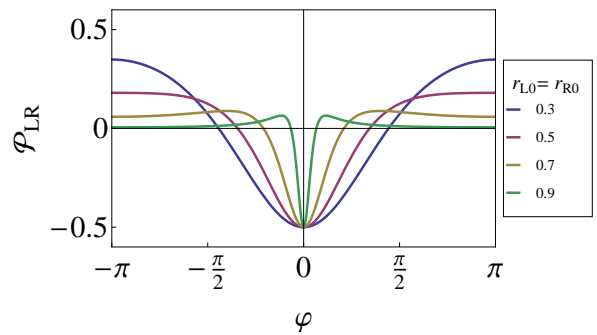


FIG. 9: The kernel $\mathcal{P}_{LR}(E)$ of the integral for the cross-correlator P_{LR} in Eq. (54), excluding the factor due to Fermi distribution functions, as a function of $\varphi(E)$. The coupling between the normal leads and the χ MM is assumed to be symmetric. Situations with various coupling strength are plotted. Between resonance ($\varphi = 0$) and anti-resonance ($\varphi = \pm\pi$) points (cf. Fig. 8), \mathcal{P}_{LR} changes sign.

2. The zero-temperature low-bias limit

At zero-temperature and low bias ($eV \ll E_l$, equal for both normal contacts), the average current and the zero-frequency noise power can be reduced to²³:

for even n_v :

$$I_L = I_R = \frac{1}{2} I_S = \frac{e^2 V}{h} (1 - r_1), \quad (56)$$

$$P_{LL} = P_{RR} = P_{LR} = P_{RL} = \frac{e^3 V}{2h} (1 - r_1^2), \quad (57)$$

$$r_1 \equiv r_{L1} = r_{R1} = \frac{r_{L0} + r_{R0}}{1 + r_{L0} r_{R0}}; \quad (58)$$

for odd n_v :

$$I_L = \frac{e^2 V}{h} (1 - r_1), \quad I_R = \frac{e^2 V}{h} (1 + r_1), \quad I_S = \frac{2e^2}{h} V, \quad (59)$$

$$P_{LL} = P_{RR} = -P_{LR} = -P_{RL} = \frac{e^3 V}{2h} (1 - r_1^2), \quad (60)$$

$$r_1 \equiv r_{L1} = -r_{R1} = \frac{r_{L0} - r_{R0}}{1 - r_{L0} r_{R0}}. \quad (61)$$

The Fano factors, for individual normal leads defined as $F_\alpha = P_{\alpha\alpha}/(eI_\alpha)$, and for the total noise power (or equivalently, for the superconducting contact) defined as $F = (\sum_{\alpha,\beta} P_{\alpha\beta})/(e \sum_\alpha I_\alpha)$, where $\alpha, \beta = L, R$, are given by

$$\text{for even } n_v: \quad F_L = F_R = \frac{1 + r_1}{2}, \quad F = 1 + r_1; \quad (62)$$

$$\text{for odd } n_v: \quad F_L = \frac{1 + r_1}{2}, \quad F_R = \frac{1 - r_1}{2}, \quad F = 0. \quad (63)$$

We see that the even/odd n_v cases represent rather distinct processes. For even n_v the conductance and Fano factors of both contacts L and R are equal. This indicates that the statistics of the electron transport is in fact identical at these leads and the process consists of a random sequence of crossed Andreev reflection processes²³.

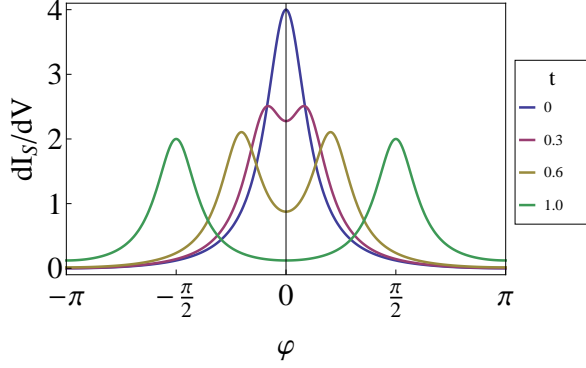


FIG. 10: Differential conductance measured at the superconducting contact, dI_S/dV in units of e^2/h , as a function of $\varphi = 2\pi eV/E_l + \pi + \phi$ with $\phi \equiv n_v\pi$ for the extended Fabry-Perot interferometer in Fig. 7b. We assume $\phi = \phi'$, $L = L'$ and $r_{L0} = r_{R0} = 0.7$. Situations with various t between the two loops are plotted. When $t = 0$, this setup is simply two copies of the single-lead setup in Fig. 4; when $t = 1$, this setup reduces to the original Fabry-Perot interferometer in Fig. 7a, with $\varphi + \phi' - \pi$ playing the role of φ in the original setup. For the case of $\phi = \phi' - \pi$ and $L = L'$, I_S does not depend on t at symmetric coupling ($r_{L0} = r_{R0}$), and its expression reduces to Eq. (55) (see also Fig. 8c) with φ substituted by 2φ . The plot shows that quite generally, dI_S/dV peaks at resonance (upon eigenstates of the coupled loops) points, and is suppressed at anti-resonance points.

In contrast for n_v odd, electrons are drawn at different rates from both contacts but such as to generate a perfect stream of Cooper pairs entering the superconductor noiselessly ($F = 0$). Meanwhile, the cross-correlators change from being positive for even n_v to being negative for odd n_v , indicating increasing importance of local Andreev reflections.

For a more general picture of the scattering processes beyond the low-bias limit, we plot in Fig. 9 the cross-correlation noise power spectrum (the kernel of the integral in Eq. (54) excluding the factor due to Fermi distribution functions) $\mathcal{P}_{LR} \equiv (-\frac{1}{2})\Re(t_{LR}r_{L2}^*t_{RL}r_{R2}^*)$ as a function of φ . By using the explicit expressions of the scattering amplitudes, one finds clearly that \mathcal{P}_{LR} changes sign between resonance ($\varphi = 0 \bmod 2\pi$) and anti-resonance ($\varphi = \pi \bmod 2\pi$) points (cf. Fig. 8). The interpretation for this sign change is in fact the same as our preceding discussion for the low-bias cases.

3. Extended setups

As in the single lead case discussed in Sec. III A, the present setup can be extended in the same way by including another chiral Majorana loop which couples to the original one (see Fig. 7b). For the extended setup, the following substitutions for the scattering amplitudes

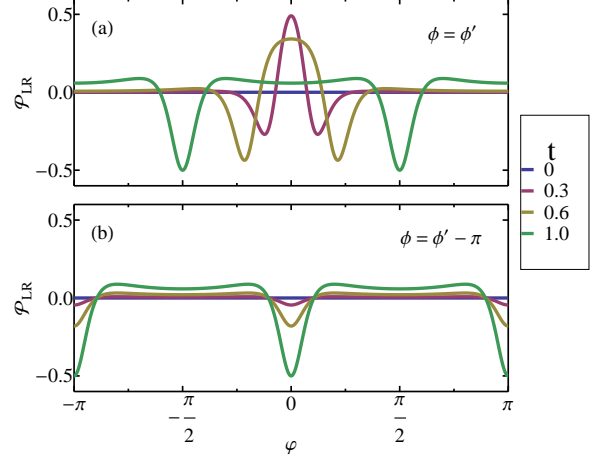


FIG. 11: Cross-correlation noise power spectrum \mathcal{P}_{LR} as a function of $\varphi(E)$ for the extended Fabry-Perot interferometer in Fig. 7b. In the upper panel we assume $\phi = \phi'$, and in the lower panel $\phi = \phi' - \pi$; for both panels we take $L = L'$ and $r_{L0} = r_{R0} = 0.7$. Situations with various transmission amplitudes t between the two loops are plotted. Clearly between resonance and anti-resonance points (cf. Fig. 10), \mathcal{P}_{LR} changes sign.

need to be made

$$r_{L1} = Z[r_{L0} + r_{R0}e^{i(\varphi+\varphi')} - r(e^{i\varphi} + r_{L0}r_{R0}e^{i\varphi'})], \quad (64)$$

$$r_{R1} = Z[r_{R0} + r_{L0}e^{i(\varphi+\varphi')} - r(e^{i\varphi'} + r_{L0}r_{R0}e^{i\varphi})], \quad (65)$$

$$t_{LR} = Zt_{L0}t_{R0}te^{i(\varphi+\varphi')}, \quad (66)$$

$$t_{RL} = -Zt_{L0}t_{R0}t, \quad (67)$$

$$Z \equiv [t^2 + (r - r_{L0}e^{i\varphi})(r - r_{R0}e^{i\varphi'})]^{-1}. \quad (68)$$

Note that the above amplitudes reduce to Eq. (38) at $r = \sqrt{1-t^2} = 1$, and to Eqs. (47)-(50) at $r = 0$, $t = 1$. We plot the differential conductance dI_S/dV and the cross-correlation noise power spectrum \mathcal{P}_{LR} for the present case in Fig. 10 and 11, respectively. Similar to our previous discussions, the coupling between the two loops (assumed again to be of the same circumferences) leads to split resonance peaks if $\phi = \phi'$, or a halved period if $\phi = \phi' - \pi$, for the differential conductance dI_S/dV ; the sign of \mathcal{P}_{LR} shows consistently changes between resonance and anti-resonance points upon eigenstates of the coupled loops.

C. A single normal lead probing a chiral Majorana loop: $N > 1$

Our previous two examples deal with single-mode ($N = 1$) leads, which are simple but nevertheless reveal most of the interesting physics. The real strength of the decompositions discussed in Sec. II, however, becomes obvious only in dealing with multi-mode ($N > 1$) leads.

It follows that by employing the same strategy as presented in Sec. II A, the canonical form of the scattering matrix, in the properly chosen Majorana basis, is simply an identity matrix. In other words, the junction effectively splits or joins, without mixing, two χ MMs. This certainly simplifies significantly our discussions which we present below in terms of specific examples.

1. Mach-Zehnder interferometer

The Mach-Zehnder interferometer illustrated in Fig. 14a has been investigated by Fu and Kane²¹, and Akhmerov, Nilsson and Beenakker²². Here we simply reformulate the problem in the Majorana language we have been speaking so far, to echo some general features of Majorana transport that have been discussed in the previous part of this paper.

The Majorana scattering matrix for the Mach-Zehnder interferometer (see Fig. 14b), defined as

$$\begin{pmatrix} \eta_{R1} \\ \eta_{R2} \end{pmatrix} = S_M \begin{pmatrix} \eta_{L1} \\ \eta_{L2} \end{pmatrix}, \quad (81)$$

is effectively given by

$$S_M = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad (82)$$

where we can choose a specific gauge such that $t_1 = e^{i\varphi}$ with $\varphi = (n_v + 1)\pi + 2\pi E/E_L$, and $t_2 = 1$.

The average current and zero-frequency noise power can be easily obtained:

$$I_L = \frac{e}{h} \int_{E \geq 0} dE \delta f_L, \quad (83)$$

$$I_R = -\frac{e}{h} \int_{E \geq 0} dE \Re(t_1 t_2^*) \delta f_L, \quad (84)$$

$$I_S = \frac{e}{h} \int_{E \geq 0} dE [1 - \Re(t_1 t_2^*)] \delta f_L, \quad (85)$$

$$P_{LL} = \frac{e^2}{h} \int_{E \geq 0} dE \Theta_L, \quad (86)$$

$$P_{RR} = \frac{e^2}{h} \int_{E \geq 0} dE [\Theta_L + \Im(t_1 t_2^*)^2 \delta f_L^2], \quad (87)$$

$$P_{LR} = P_{RL} = -\frac{e^2}{h} \int_{E \geq 0} dE \Re(t_1 t_2^*) \Theta_L, \quad (88)$$

where $\Re(t_1 t_2^*) = \cos \varphi$ and $\Im(t_1 t_2^*) = \sin \varphi$ when the explicit expressions of t_1 and t_2 are taken.

Clearly the outgoing current in Eq. (84) appears to be the interference between two χ MMs – the transmission “probability” $\Re(t_1 t_2^*)$ has a physically identical interpretation as the reflection “probability” $\Re(r_1 r_2^*)$ appearing in Eq. (39). An obvious difference, however, between the present setup and a Fabry-Perot-type interferometer is that contact L here indeed provides coherent sources for the MF pairs – instead of single ones – that are drained by

contact R . As a direct consequence, the cross-correlator in Eq. (88) involves only the thermal fluctuation in contact L , while all the other contributions (e.g. partition or exchange noise) are absent due to the one-way nature of the present setup.

2. Hanbury Brown-Twiss interferometer

The Hanbury Brown-Twiss interferometer shown in Fig. 14c has recently been investigated by Strübi, Belzig, Choi, and Bruder²⁵. Here we reformulate the problem, again in the Majorana language.

The Majorana scattering matrix for the Hanbury Brown-Twiss interferometer (see Fig. 14d), defined as

$$\begin{pmatrix} \eta_{U1} \\ \eta_{U2} \\ \eta_{D1} \\ \eta_{D2} \end{pmatrix} = S_M \begin{pmatrix} \eta_{L1} \\ \eta_{L2} \\ \eta_{R1} \\ \eta_{R2} \end{pmatrix}, \quad (89)$$

is effectively given by

$$S_M = \begin{pmatrix} t_{UL} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{UR} \\ 0 & 0 & t_{DR} & 0 \\ 0 & t_{DL} & 0 & 0 \end{pmatrix}. \quad (90)$$

In this case we can choose a specific gauge such that $t_{UL} = e^{i\varphi}$, and $t_{UR} = t_{DR} = t_{DL} = 1$.

The average current and zero-frequency noise power are given by

$$I_\alpha = \frac{e}{h} \int_{E \geq 0} dE \delta f_\alpha, \quad (\alpha = L, R) \quad (91)$$

$$I_\beta = 0, \quad (\beta = U, D) \quad (92)$$

$$P_{\alpha\alpha} = \frac{e^2}{h} \int_{E \geq 0} dE \Theta_\alpha, \quad (\alpha = L, R) \quad (93)$$

$$P_{\beta\beta} = \frac{e^2}{h} \int_{E \geq 0} dE \left[\frac{1}{2} (\Theta_L + \Theta_R) + \frac{1}{4} \sum_{a,b=e,h} (f_{La} - f_{Rb})^2 \right], \quad (\beta = U, D) \quad (94)$$

$$\begin{aligned} P_{UD} &= P_{DU} \\ &= \frac{e^2}{h} \int_{E \geq 0} dE \left(-\frac{1}{2} \right) \Re(t_{UL} t_{UR}^* t_{DR} t_{DL}^*) \delta f_L \delta f_R, \end{aligned} \quad (95)$$

and all other cross-correlators are zero. Taking the explicit expressions, we have $\Re(t_{UL} t_{UR}^* t_{DR} t_{DL}^*) = \cos \varphi$.

As pointed out by Strübi *et al.*²⁵, a remarkable fact in the present setup is that although the average current at the two drains U and D vanishes identically, the cross-correlation between the two contacts is in general non-zero. This fact, in the Majorana language, can be interpreted as follows: the MF pair in either drain are necessarily from different sources, and are therefore phase incoherent for single-particle scattering which the average current measures – the incoherence between MF pairs

implies vanishing average current; two-particle scattering allows coherence between MF pairs in both drains, and is included in the measurement of noise power – the coherent injection of two electrons/holes from the two sources leads to non-vanishing cross-correlation between current in the two drains.

The scenario here bears close analogy to the $N = 1$ Fabry-Perot interferometer discussed in Sec. III B, where the average current contributed by a foreign contact is always zero but the cross-correlation between two contacts is in general non-vanishing. The underlying physics in both scenarios, as it has been for essentially all our examples, boils down to the central importance of interference in the charge transport with Majorana modes.

IV. SUMMARY

In this work we emphasized a scattering approach using a global Majorana basis. Both the normal state leads and the system which intrinsically carries chiral Majorana states are described in the same basis. As a consequence, from a purely technical particle scattering point of view, Majorana scattering problems are rather similar to scattering problems in normal mesoscopic structures. However, Majorana modes are neutral and carry neither charge nor (electric) current. In contrast to normal scattering problems the charge and current operators are not diagonal in the Majorana scattering states. Therefore, unlike in normal systems, current is not determined by transmission probabilities of Majorana fermions. We have shown that charge and current appear due to interference of pairs of Majorana modes. For averages of single particle quantities, like charge and current, to be non-vanishing, interfering Majorana modes must necessarily have originated in the same contact. In contrast, in two particle quantities, like the current noise, two-particle exchange permits interference of two Majorana particles even if they originate from different leads. The physics of neutral excitations is certainly an important field of future research. It can be expected that the theory of Majorana particles represents an instructive example of this development.

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Appendix A: Decomposition of a $(2N + 1)$ -dimensional orthogonal matrix

We prove in this appendix that for an arbitrary matrix $\mathcal{O}_N \in SO(2N + 1)$, one can find a decomposition

$$\mathcal{O}_N = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{V}^T \end{pmatrix} \begin{pmatrix} R(\theta_1) & & & 0 \\ & \ddots & & \\ & & R(\theta_N) & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{W} \end{pmatrix}, \quad (\text{A1})$$

such that

$$R(\theta_i) = \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \quad (i = 1, \dots, N) \quad (\text{A2})$$

and $\mathcal{V}, \mathcal{W} \in SO(2N)$ satisfy

$$\mathcal{V}_{2i} = \mathcal{V}_{2i-1} \Lambda, \quad (i = 1, \dots, N) \quad (\text{A3})$$

$$\mathcal{W}_{2i} = \mathcal{W}_{2i-1} \Lambda, \quad (i = 1, \dots, N) \quad (\text{A4})$$

where \mathcal{V}_n (\mathcal{W}_n) stands for the n -th row of \mathcal{V} (\mathcal{W}), and the $2N \times 2N$ matrix Λ is defined as

$$\Lambda = \mathbb{1}_N \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}. \quad (\text{A5})$$

Before we start the proof, we point out that \mathcal{V} (\mathcal{W}) satisfying Eq. (A3) (Eq. (A4)) is nothing but \mathcal{V} (\mathcal{W}) given in Eq. (16) written in a differently ordered basis (the purpose of such reordering is to make the central matrix at the right-hand-side of Eq. (A1) taking a compact block-diagonal form). This can be shown by using explicitly

$$U_N = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_N & \mathbb{1}_N \\ i\mathbb{1}_N & -i\mathbb{1}_N \end{pmatrix}, \quad (\text{A6})$$

hence we have, from Eq. (16),

$$\mathcal{V} = \begin{pmatrix} \Re(V) & \Im(V) \\ -\Im(V) & \Re(V) \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} \Re(W) & \Im(W) \\ -\Im(W) & \Re(W) \end{pmatrix}, \quad (\text{A7})$$

where V and W are N -dimensional unitary matrices (cf. Eqs. (13) and (14)). It is then straightforward to see that by rearranging, for $i = 1, \dots, N$, the i -th row and column to the $(2i - 1)$ -th and the $(i + N)$ -th row and column to the $(2i)$ -th, the rearranged \mathcal{V} and \mathcal{W} satisfy Eqs. (A3) and (A4).

Now we prove our main claim by construction.

First we notice that \mathcal{O}_N can always be written as

$$\mathcal{O}_N = \begin{pmatrix} \cos \theta_1 & (-\sin \theta_1)w \\ (\sin \theta_1)v^T & X_0 \end{pmatrix} \quad (\text{A8})$$

where both v and w are normalized $2N$ -dimensional row vectors and X_0 is a $2N \times 2N$ matrix. We assume $|\cos \theta_1| \neq 1$ (namely, the upper-left corner element of the matrix \mathcal{O}_N is not ± 1), such that both v and w are well-defined up to an irrelevant sign.

Then we construct two matrices $\mathcal{V}^{(1)}, \mathcal{W}^{(1)} \in SO(2N)$ by setting

$$\mathcal{V}_1^{(1)} = v, \quad \mathcal{V}_2^{(1)} = \mathcal{V}_1^{(1)} \Lambda, \quad (\text{A9})$$

$$\mathcal{W}_1^{(1)} = w, \quad \mathcal{W}_2^{(1)} = \mathcal{W}_1^{(1)} \Lambda, \quad (\text{A10})$$

and choosing $\mathcal{V}_i^{(1)}$ and $\mathcal{W}_i^{(1)}$ with $2 < i \leq 2N$ so that the orthogonality and Eqs. (A3) and (A4) are all satisfied (which can be easily shown to be always possible). We find

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{V}^{(1)} \end{pmatrix} \mathcal{O}_N \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{W}^{(1)T} \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & \dots & 0 \\ \sin \theta_1 & & & & \\ 0 & & & & \\ \vdots & & X_1 & & \\ 0 & & & & \end{pmatrix}, \quad (\text{A11}) \end{aligned}$$

where $X_1 = \mathcal{V}^{(1)} X_0 \mathcal{W}^{(1)T}$.

Since all three factor matrices at the left-hand-side of the above equation (A11) belong to $SO(2N+1)$, so must do their product. It follows that

$$X_1 = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \mathcal{O}_{N-1} \end{pmatrix}, \quad (\text{A12})$$

where $\mathcal{O}_{N-1} \in SO(2N-1)$. In other words, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{V}^{(1)} \end{pmatrix} \mathcal{O}_N \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{W}^{(1)T} \end{pmatrix} = \begin{pmatrix} R(\theta_1) & 0 \\ 0 & \mathcal{O}_{N-1} \end{pmatrix}. \quad (\text{A13})$$

Obviously this procedure can be continued for \mathcal{O}_{N-1} , and then for \mathcal{O}_{N-2} etc., until we arrive at \mathcal{O}_0 , which is 1. Meanwhile we obtain Eq. (A1) with

$$\mathcal{V} = \begin{pmatrix} \mathbb{1}_{2(N-1)} & 0 \\ 0 & \mathcal{V}^{(N)} \end{pmatrix} \cdots \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathcal{V}^{(2)} \end{pmatrix} \mathcal{V}^{(1)}, \quad (\text{A14})$$

$$\mathcal{W} = \begin{pmatrix} \mathbb{1}_{2(N-1)} & 0 \\ 0 & \mathcal{W}^{(N)} \end{pmatrix} \cdots \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathcal{W}^{(2)} \end{pmatrix} \mathcal{W}^{(1)}. \quad (\text{A15})$$

It remains to show that \mathcal{V} and \mathcal{W} defined above satisfy Eqs. (A3) and (A4). To this end we observe $\Lambda^2 = -\mathbb{1}$, hence Eqs. (A3) and (A4) are equivalent to

$$[\mathcal{V}, \Lambda] = 0, \quad [\mathcal{W}, \Lambda] = 0, \quad (\text{A16})$$

where $[\cdot, \cdot]$ stands for the commutator. We also keep in mind that, by construction, all factor matrices at the right-hand-sides of Eqs. (A14) and (A15) satisfy the above commutation relations. This immediately verifies the validity of our construction, and thus completes the proof.

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- ⁴² In the literature there is another type of junctions consisting of one unidirectional “normal” mode and two Majorana modes with opposite chirality^{21,22}. The scattering theory for this type of junctions will be discussed in Sec. III E.
- ⁴³ Ref. 40 suggests to call an ensemble of such random matrices CRE (circular real ensemble) to distinguish it from the more familiar CUE (circular unitary ensemble).
- ⁴⁴ Following our strategy, the lower bound of the number of physical parameters is given by $\text{Dim}[SO(2N + 1)] - 2\text{Dim}[U(N)] = N$, which turns out to be the actual number of parameters in \tilde{S}_M .
- ⁴⁵ Of course, the real container of charges here is the superconducting condensate, the interference between Majorana modes simply determines how normal leads draw electrons/holes from the container.
- ⁴⁶ Indeed, such robustness is of a more fundamental origin, namely the Béri degeneracy⁴¹, which we will discuss in details in our following work, using the Majorana picture similar to this paper.