Symmetry-protected topological phases in noninteracting fermion systems

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Symmetry protected topological (SPT) phases are gapped quantum phases with a certain symmetry, which can all be smoothly connected to the same trivial product state if we break the symmetry. For non-interacting fermion systems with time reversal ($\hat{T}$), charge conjugation ($\hat{C}$), and/or $U(1)$ ($\hat{N}$) symmetries, the total symmetry group can depend on the relations between those symmetry operations, such as $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$ or $\hat{T}\hat{N}\hat{T}^{-1} = -\hat{N}$. As a result, the SPT phases of those fermion systems with different symmetry groups have different classifications. In this paper, we use Kitaev's K-theory approach to classify the gapped free fermion phases for those possible symmetry groups. In particular, we can view the $U(1)$ as a spin rotation. We find that superconductors with the S_z spin rotation symmetry are classified by $\mathbb{Z}$ in even dimensions, while superconductors with the time reversal plus the S_z spin rotation symmetries are classified by $\mathbb{Z}$ in odd dimensions. We show that all 10 classes of gapped free fermion phases can be realized by electron systems with certain symmetries. We also point out that to properly describe the symmetry of a fermionic system, we need to specify its full symmetry group that includes the fermion number parity transformation $(-)^{\hat{N}}$. The full symmetry group is actually a projective symmetry group.

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I. INTRODUCTION

We used to believe that all possible phases and phase transitions are described by Landau symmetry breaking theory.[1–3] However, the experimental discovery of fractional quantum Hall states[4,5] and the theoretical discovery of chiral spin liquids[6,7] indicate that new states of quantum matter without symmetry breaking and without long range order can exist. Such new kind of orders is called topological order,[8,9] since their low energy effective theories are topological quantum field theories.[10] At first, the theory of topological order was developed based on their robust ground state degeneracy on compact spaces and the associated robust non-Abelian Berry's phases.[8,9] Later, it was realized that topological order can be characterized by the boundary excitations,[11,12] which can be directly probed by experiments. One can develop a theory of topological order based on the boundary theory.[13]

Since its introduction, we have been trying to obtain a systematic understanding of topological orders. Some progresses have been made for certain simple cases. We found that all 2D Abelian topological orders can be classified by integer K-matrices.[14–16] The 2D nonchiral topological orders (which can be smoothly connected to time reversal and parity symmetric states) are classified by spherical fusion category.[17–20] The recent realization of the relation between topological order and long range entanglement[19] (defined through local unitary transformations[21,22]) allows us to separate another simple class of gapped quantum phases – symmetry protected topological (SPT) phases. SPT phases are gapped quantum phases with a certain symmetry, which can all be smoothly connected to the same trivial product state if we break the symmetry. A generic construction of bosonic SPT phases in any dimension using the group cohomology of the symmetry group was obtained in Ref. 23,24. The constructed SPT phases include interacting bosonic topological insulators and topological superconductors (and much more).

Another type of simple systems are free fermion systems, for which a classification of gapped quantum phases can be obtained through K-theory[25–27] or nonlinear σ-model of disordered fermions.[28] They include the non-interacting topological insulators[29–35] and the non-interacting topological superconductors.[36–40] Most gapped quantum phases of free fermion systems are SPT phases protected by some symmetries, such as topological insulators protected by the time reversal symmetry. While some others have intrinsic topological orders (*ie* stable even without any symmetry), such as topological superconductors with no symmetry. Just like the interacting topological ordered phases, the topological phases for free fermions are also characterized by their gapless boundary excitations. The boundary excitations play a key role in the theory and experiments of free fermion topological phases.

For non-interacting fermion systems with time reversal (generated by $\hat{T}$), charge conjugation (generated by $\hat{C}$), and/or $U(1)$ (generated by $\hat{N}$) symmetries, the total symmetry group may not simply be $Z_2^T \times Z_2^C \times U(1)$. The group can take different forms, depending on the different relations between those symmetry operations, such as $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$ or $\hat{T}\hat{N}\hat{T}^{-1} = -\hat{N}$. As a result, the gapped phases of those fermion systems with different symmetry groups have different classifications. In this paper, we use Kitaev's K-theory approach to classify the gapped free fermion phases for those different symmetry groups. In Table I, we list some electron systems and their full symmetry group G_f. In Table II and Table III the 10 classes[25,28] of gapped free fermion phases protected by those many-body symmetry groups (and many other symmetry groups) are listed. Here we have assumed that the fermions form one irreducible rep-

Electron systems	Full symm. group G_f
Insulators with spin-orbital coupling and spin order (or non-coplanar spin order) $(\mathrm{i}\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_{i'} + \mathrm{i}\hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_{j'} + \mathrm{i}\hat{c}_k^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_{k'}) + (\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_i + \hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_j + \hat{c}_k^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_k)$	$U(1)$
Superconductors with spin-orbital coupling and spin order (or non-coplanar spin order) $\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_i + \hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_j + \hat{c}_k^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_k + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$	"none"$= Z_2^f$
Insulators with spin-orbital coupling $\mathrm{i}\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_{i'} + \mathrm{i}\hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_{j'} + \mathrm{i}\hat{c}_k^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_{k'}$ (*symmetry*: charge conservation and time reversal symmetries)	$G_-^-(U,T)$
Superconductors with spin-orbital coupling and real pairing $\mathrm{i}\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_{i'} + \mathrm{i}\hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_{j'} + \mathrm{i}\hat{c}_k^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_{k'} + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: time reversal symmetry)	$G_-(T) = Z_4$
Superconductors with S_z conserving spin-orbital coupling and real pairing $\mathrm{i}\hat{c}_i^\dagger \sigma^z\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: time reversal and S_z spin rotation symmetries)	$G_-^+(U,T) = U(1)\times Z_2^T$
Superconductors with coplanar spin order and real pairing $\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_i + \hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: a combined time reversal and 180° spin rotation symmetry)	$G_+(T) = Z_2^T \times Z_2^f$
Superconductors with real pairing and collinear spin order $\hat{c}_i^\dagger \sigma^z\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: S_z spin rotation and a combined time reversal and 180° S_y spin rotation symmetry)	$G_+^-(U,T) = U(1)\rtimes Z_2^T$
Insulators with coplanar spin order $\hat{c}_i^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_i + \hat{c}_j^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_j$ (*symmetry*: charge conservation and a combined time reversal and 180° spin rotation symmetries)	$G_+^-(U,T) = U(1)\rtimes Z_2^T$
Superconductors with real triplet $S_z = 0$ paring $\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} + \hat{c}_{\downarrow i}\hat{c}_{\uparrow j}$ (*symmetry*: a combined 180° S_y spin rotation and time reversal symmetry, a combined 180° S_y spin rotation and charge rotation symmetry, and S_z-spin rotation symmetry)	$G_{++}^{--}(U,T,C)$
Superconductors with time reversal, 180° S_y-spin rotation, and S_z-spin rotation symmetries	$G_{--}^{++}(U,T,C)$
Superconductors with real singlet pairing $\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j}$ (*symmetry*: time reversal and $SU(2)$ spin rotation symmetries)	$G[SU(2),T]$
Superconductors with 180° S_y-spin rotation and S_z-spin rotation symmetries	$G_-(U,C)$
Superconductors with complex singlet pairing $\mathrm{e}^{\mathrm{i}\theta_{ij}}(\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: $SU(2)$-spin rotation symmetry)	$SU(2)$
Insulators with spin-orbital coupling and inter-sublattice hopping $\mathrm{i}\hat{c}_{i_A}^\dagger \boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat{c}_{i_B} + \mathrm{i}\hat{c}_{j_A}^\dagger \boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat{c}_{j_B} + \mathrm{i}\hat{c}_{k_A}^\dagger \boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat{c}_{k_B}$ (*symmetry*: charge conservation, time reversal and charge conjugation symmetries)	$G_{--}^{-+}(U,T,C)$

TABLE I: (Color online) Electron systems and their full symmetry groups G_f. The groups are defined in Table IV. The symmetry group symbols have the following meaning: for example, $G_{--}^{++}(U,T,C)$ is a symmetry group generated by $\hat{N}$ (the $U(1)$ fermion number conservation or spin rotation), $\hat{T}$ (the time reversal), and $\hat{C}$ (the charge conjugation or 180° spin rotation). The $\pm$ subscripts/superscripts describe the relations between the transformations $\hat{N}$, $\hat{T}$, and/or $\hat{C}$ (see Table IV). Some times, when we describe the symmetry of a fermion system, we do not include the fermion number parity transformation $(-)^{\hat{N}}$ in the symmetry group G. Here the full symmetry group G_f does include the fermion number parity transformation $(-)^{\hat{N}}$. So the full symmetry group of a fermion system with no symmetry is $G_f = Z_2^f$ generated by the fermion number parity transformation. G_f is a Z_2^f extension of G: $G = G_f/Z_2^f$. Free electron systems with symmetry $G_{--}^{++}(U,T,C)$ actually have a higher symmetry $G[SU(2),T]$. Similarly, free electron systems with symmetry $G_-(U,C)$ actually have a higher symmetry $SU(2)$. The groups in blue are Abelian and electron operators form their 1D representations.

resentation of the full symmetry group. The result will differ if the fermions contain several distinct irreducible representations of the full symmetry group (see section III E). In Ref. 25,28, the 10 classes of gapped free fermion phases are already associated with many different many-body symmetries of electron systems. In this paper, we generalize the results in Ref. 25,28 to more symmetry groups.

We note that electron systems, with $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$, only realize a subset of the possible symmetry groups. The emergent fermion (such as the spinon in spin liquid) may realize other possible symmetry groups, since their symmetries are described by projective symmetry groups (PSG) which can be different for different topologically ordered states.[41]

The $p = 0$ line in Table II classifies two types of electron systems: (1) Insulators with only fermion number conservation (which includes integer quantum Hall states). (2) Superconductors with only S_z spin rotation symmetry which can be realized by superconductors with collinear spin order. The $p = 1$ line in Table II classifies superconductors with only time reversal and S_z spin rotation symmetry (full symmetry group $G_-^+(U,T)$), which can be realized by superconductors with real pairing and S_z conserving spin-orbital coupling.

In Table III, the $p = 0$ column classifies electronic insulators with coplanar spin order (full symmetry group $G_+^-(U,T)$ which contains the charge conservation and a time reversal symmetry). The $p = 1$ column classifies electronic superconductors with coplanar spin order and real pairing (full symmetry group $G_+(T)$ which contains a time reversal symmetry). The $p = 2$ column classifies electronic superconductors with non-coplanar spin order (full symmetry group "none"). The $p = 3$

Symmetry	$C_p\|_{\text{for } d=0}$	$p\backslash d$	0	1	2	3	4	5	6	7	example
$U(1)$ $G_-(C)$	$\frac{U(l+m)}{U(l)\times U(m)}\times \mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	(Chern) insulator; supercond. with collinear spin order
$G^+_\pm(U,T)$ $G^+_{--}(T,C)$ $G^+_{+-}(T,C)$	$U(n)$	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	supercond. w/ real pairing and S_z conserving spin-orbital coupling

TABLE II: (Color online) Classification of the gapped phases of non-interacting fermions in d-dimensional space, for some symmetries. The space of the gapped states is given by $C_{p+d \bmod 2}$, where p depends on the symmetry group. The distinct phases are given by $\pi_0(C_{p+d \bmod 2})$. "0" means that only trivial phase exist. $\mathbb{Z}$ means that non-trivial phases are labeled by non-zero integers and the trivial phase is labeled by 0. The groups in red can be realized by electron systems (see Table I).

Symm.	$G^-_+(U,T)$ $G^-_{+-}(T,C)$	$G_+(T)$ $G^+_{++}(T,C)$ $G^{--}_{++}(U,T,C)$ $G^{-+}_{++}(U,T,C)$ $G^{+-}_{-+}(U,T,C)$ $G^{++}_{++}(U,T,C)$	"none" $G_+(C)$ $G^-_{++}(T,C)$ $G^-_{-+}(T,C)$ $G_+(U,C)$	$G_-(T)$ $G^+_{-+}(T,C)$ $G^{--}_{-+}(U,T,C)$ $G^{-+}_{-+}(U,T,C)$ $G^{+-}_{++}(U,T,C)$ $G^{++}_{-+}(U,T,C)$	$G^-_-(U,T)$ $G^-_{--}(T,C)$	$G^{--}_{--}(U,T,C)$ $G^{-+}_{--}(U,T,C)$ $G^{+-}_{--}(U,T,C)$ $G^{++}_{+-}(U,T,C)$	$G_-(U,C)$ $SU(2)$	$G^{--}_{+-}(U,T,C)$ $G^{-+}_{+-}(U,T,C)$ $G^{+-}_{+-}(U,T,C)$ $G^{++}_{--}(U,T,C)$ $G[SU(2),T]$
$R_p\|_{\text{for } d=0}$	$\frac{O(l+m)}{O(l)\times O(m)}\times \mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times \mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
$d=0$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$d=1$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$d=2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$d=3$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$d=4$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$d=5$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$d=6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$d=7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Example	insulator w/ coplanar spin order	supercond. w/ coplanar spin order	supercond.	supercond. w/ time reversal	insulator w/ time reversal	insulator w/ time reversal and inter-sublattice hopping	spin singlet supercond.	spin singlet supercond. w/ time reversal

TABLE III: (Color online) Classification of gapped phases of non-interacting fermions in d spatial dimensions, for some symmetries. The space of the gapped states is given by $R_{p-d \bmod 8}$, where p depends on the symmetry group. The phases are classified by $\pi_0(R_{p-d \bmod 8})$. Here $\mathbb{Z}_2$ means that there is one non-trivial phase and one trivial phase labeled by 1 and 0. The groups in red can be realized by electron systems (see Table I).

column classifies electronic superconductors with spin-orbital coupling and real pairing (full symmetry group $G_-(T)$ which contains the time reversal symmetry). The $p=4$ column classifies electronic insulators with spin-orbital coupling (full symmetry group $G^-_-(U,T)$ which contains the charge conservation and the time reversal symmetry). The $p=5$ column classifies electronic insulators on bipartite lattices with spin-orbital coupling and only inter-sublattice hopping (full symmetry group $G^{-+}_{--}(U,T,C)$ which contains the charge conservation, the time reversal symmetry, and a charge conjugation symmetry). The $p=6$ column classifies electronic spin singlet superconductors with complex pairing (full symmetry group $SU(2)$). The $p=7$ column classifies electronic spin-singlet superconductors with real pairing (full symmetry group $G[SU(2),T]$ which contains the $SU(2)$ spin rotation and the time reversal symmetry).[42]

In this paper, we will first discuss a simpler case where fermion systems have only the $U(1)$ symmetry. Then we will discuss a more complicated case where fermion systems can have time reversal, charge conjugation, and/or $U(1)$ symmetries. The classification of the gapped phases with translation symmetry and the classification of non-trivial defects with protected gapless excitations will also be studied.

II. GAPPED FREE FERMION PHASES – THE COMPLEX CLASSES

A. The $d = 0$ case

Let us first consider a 0-dimension free fermion system with 1 orbital. How many different gapped phases do we have for such a system? The answer is 2. The 2 different gapped phases are labeled by $m = 0, 1$: the $m = 0$ gapped phase correspond to the empty orbital, while the $m = 1$ gapped phase correspond to the occupied orbital. But 2 is not the complete answer. We can alway add occupied and empty orbitals to the system and still regard the extended system as in the same gapped phase. So we should consider a system with n orbitals in $n \to \infty$ limit. In this case, the 0-dimension gapped phases are labeled by an integer m in $\mathbb{Z}$, where m (with a possible constant shift) still corresponds to the number of occupied orbitals.

Now let us obtain the above result using a fancier mathematical set up. The single-body Hamiltonian of the n-orbital system is given by $n \times n$ hermitian matrix H. If the orbitals below a certain energy are filled, we can deform the energies of those orbitals to -1 and deform the energies of other orbitals to $+1$ without closing the energy gap. So, without losing the generality, we can assume the H to satisfy

$$H^2 = 1 \tag{1}$$

Such a hermitian matrix has a form

$$H = U_{n\times n} \begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m} \end{pmatrix} U^\dagger_{n\times n} \tag{2}$$

where $n = l + m$ and $U_{n\times n} in U(n)$ is an $n \times n$ unitary matrix. But $U_{n\times n}$ is not an one-to-one labeling of the hermitian matrix satisfying $H^2 = 1$. To obtain an one-to-one labeling, we note that $\begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m} \end{pmatrix}$ is invariant under the unitary transformation $\begin{pmatrix} V_{l\times l} & 0 \\ 0 & W_{m\times m} \end{pmatrix}$ with $V_{l\times l} \in U(l)$ and $W_{m\times m} \in U(m)$. Thus the space C_0 of the hermitian matrix satisfying $H^2 = 1$ is given by $\bigcup_m U(l+m)/U(l) \times U(m)$, which, in $n \to \infty$ limit, has a form

$$C_0 \equiv \frac{U(l+m)}{U(l) \times U(m)} \times \mathbb{Z} \tag{3}$$

Clearly $\pi_0(C_0) = \mathbb{Z}$, which recovers the result obtained above using a simple argument: the 0-dimension gapped phases of free conserved fermions are labeled by integers $\mathbb{Z}$.

B. The properties of classifying spaces

The space C_0 is the complex Grassmannian – the space formed by the subspaces of (infinity dimensional) complex vector space. It is also the space of the hermitian matrix satisfying $H^2 = 1$. Actually, C_0 is a part of a sequence. More generally, a space C_p can be defined by first picking p fixed hermitian matrices γ_i, $i = 1, 2, ..., p$, satisfying

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{4}$$

Then C_p is the space of the hermitian matrix satisfying

$$H^2 = 1, \quad \gamma_i H = -H\gamma_i, \quad i = 1, ..., p. \tag{5}$$

To see what is the C_1 space, let us choose $\gamma_1 = I_{n\times n}$ that satisfy $\gamma_1^2 = 1$. But for such a choice, we cannot find any H that satisfy $\gamma_1 H = -H\gamma_1$. Actually, we have a condition on the choice of γ_1. We must choose a γ_1 such that $\gamma_1 H = -H\gamma_1$ and $H^2 = 1$ has a solution. So we should choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$. We note that γ_1 is invariant under the following unitary transformations $e^{i\sigma^x\otimes A_{n\times n}} e^{i\sigma^0\otimes B_{n\times n}} \in U(n) \times U(n)$ (where $A_{n\times n}$ and $B_{n\times n}$ are hermitian matrices). Then H satisfying $H^2 = 1$ and $\gamma_1 H + H\gamma_1 = 0$ has a form

$$H = e^{i\sigma^x\otimes A_{n\times n}} e^{i\sigma^0\otimes B_{n\times n}} (\sigma^z \otimes I_{n\times n}) e^{-i\sigma^0\otimes B_{n\times n}} e^{-i\sigma^x\otimes A_{n\times n}}, \tag{6}$$

whose positive and negative eigenvalues are paired. We see that the space C_1 is $U(n) \times U(n)/U(n) = U(n)$.

To construct the C_2 space, we can choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$ and $\gamma_2 = \sigma^y \otimes I_{n\times n}$. Then H satisfying $H^2 = 1$ and $\gamma_i H + H\gamma_i = 0$, $i = 1, 2$, has a form

$$H = \sigma^0 \otimes U_{n\times n} \left[\sigma^z \otimes \begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m} \end{pmatrix} \right] \sigma^0 \otimes U^\dagger_{n\times n} \tag{7}$$

where $n = l + m$ and $U_{n\times n} \in U(n)$. We see that the space $C_2 = C_0$.

To construct the C_3 space, we can choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$, $\gamma_2 = \sigma^y \otimes I_{n\times n}$, and $\gamma_3 = \sigma^z \otimes I_{n\times n}$. But for such a choice, the equations $\gamma_i H + H\gamma_i = 0$, $i = 1, 2, 3$, and $H^2 = 1$ has no solution for H. So we need to impose the following condition on γ_i's:

$$\text{The equations } \gamma_i H + H\gamma_i = 0,\ H^2 = 1 \text{ have a solution for } H. \tag{8}$$

(Later, we will see that such an condition has an amazing geometric origin.) Let us choose $\gamma_1 = \sigma^x \otimes \sigma^x \otimes I_{n\times n}$, $\gamma_2 = \sigma^y \otimes \sigma^x \otimes I_{n\times n}$, and $\gamma_3 = \sigma^z \otimes \sigma^x \otimes I_{n\times n}$ instead. Then H satisfying $H^2 = 1$ and $\gamma_i H + H\gamma_i = 0$, $i = 1, 2, 3$, has a form

$$H = e^{i\sigma^0\otimes\sigma^x\otimes A_{n\times n}} e^{i\sigma^0\otimes\sigma^0\otimes B_{n\times n}} (\sigma^0 \otimes \sigma^z \otimes I_{n\times n}) \times e^{-i\sigma^0\otimes\sigma^0\otimes B_{n\times n}} e^{-i\sigma^0\otimes\sigma^x\otimes A_{n\times n}}, \tag{9}$$

We find that $C_3 = C_1$.

Now, it is not hard to see that $C_p = C_{p+2}$ which leads to $\pi_d(C_p) = \pi_d(C_{p+2})$. Thus

$$\pi_0(C_p) = \begin{cases} \mathbb{Z}, & p = 0 \text{ mod } 2, \\ \{0\}, & p = 1 \text{ mod } 2. \end{cases} \tag{10}$$

C. The $d \neq 0$ cases

Next we consider a d-dimension free conserved fermion systems and their gapped ground states. Note that the only symmetry that we have is the $U(1)$ symmetry associated with the fermion number conservation. We do not have translation symmetry and other symmetries.

To be more precise, our d-dimension space is a ball with no non-trivial topology. Since the systems have a boundary, here we can only require that the "bulk" gap of the fermion systems are non-zero. The free fermion system may have protected gapless excited at the boundary. (Requiring the fermion systems to be even gapped at boundary will only give us trivial gapped phases.) We will call the free fermion systems that are gapped only inside of the d dimensional ball as "bulk" gapped fermion systems. A "bulk" gapped fermion system may or may not be gapped at the boundary.

Kitaev has shown that the space C_d^H of such "bulk" gapped free fermion systems is homotopically equivalent of the space C_d^M of mass matrices of a d-dimensional Dirac equation: $\pi_n(C_d^H) = \pi_n(C_d^M)$.[25] In the following, we will give a hand-waving explanation of the result.

To start, let us first assume that the fermion system has the translation symmetry and the charge conjugation symmetry. We also assume that its energy bands have some Dirac points at zero energy and there are no other zero energy states in the Brillouin zone. So if we fill the negative energy bands, the single-body gapless excitations in the system are described by the hermitian matrix H, whose continuous limit has a form:

$$H = \sum_{i=1}^{d} \gamma_i \, \mathrm{i}\partial_i, \tag{11}$$

where we have folded all the Dirac points to the $\boldsymbol{k} = 0$ point. Without losing generality, we have also assumed that all the Dirac points have the same velocity. Since $\mathrm{i}\partial_i$ is hermitian, thus γ_i, $i = 1, ..., d$ are the hermitian γ-matrices (of infinity dimension) that satisfy eqn. (4).

When $d = 1$, do we have a system that has $\gamma_1 = I_{n\times n}$? The answer is no. Such a system will have n right-moving chiral modes that cannot be realized by any pure 1-dimensional systems with short-ranged hopping. In fact γ_1 must have a form $\gamma_1 = \begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m} \end{pmatrix}$ with $l = m$ (the same number of right-moving and left-moving modes). So the allowed γ_1 always satisfy the condition that $H^2 = 1$ and $\gamma_1 H + H\gamma_1 = 0$ has a solution for H. We see that the extra condition eqn. (8) on γ_i has a very physical meaning.

Now we add perturbations that may break the translation symmetry. We like to know how many different ways are there to gap the Dirac point. The Dirac points can be fully gapped by the "mass" matrix M that satisfies

$$\gamma_i M + M\gamma_i = 0, \qquad M^\dagger = M. \tag{12}$$

To fully gap the Dirac points, M must have no zero eigenvalues. Without losing generality, we may also assume that $M^2 = 1$. (Since the Dirac points may have different crystal momenta before the folding to $\boldsymbol{k} = 0$, we need perturbations that break the translation symmetry to generate a generic mass matrix that may mix Dirac points at different crystal momenta.) The space C_d^M of such mass matrices is nothing but C_d introduced before: $C_d^M = C_d$.

So the different ways to gap the Dirac point form a space C_d. The different disconnected components of C_d represent different gapped phases of the free fermions. Thus, the gapped phases of the conserved free fermions in d-dimensions are classified by $\pi_0(C_d)$, which is $\mathbb{Z}$ for even d and 0 for odd d. The non-trivial phases at $d = 2$ are labeled by $\mathbb{Z}$, which are the integer quantum Hall states. The results is summarized in Table II.

III. GAPPED FREE FERMION PHASES – THE REAL CLASSES

When fermion number is not conserved and/or when there is a time reversal symmetry, the gapped phases of non-interacting fermions are classified differently. However, using the idea and the approaches similar to the above discussion, we can also obtain a classification. Instead of considering hermitian matrices that satisfy certain conditions, we just need to consider real antisymmetric matrices that satisfy certain conditions.

A. The $d = 0$ case – the symmetry groups

Again, we will start with $d = 0$ dimension. In this case, a free fermion system with n orbitals is described by the following quadratic Hamiltonian

$$\hat{H} = \sum_{ij} H_{ij} \hat{c}_i^\dagger \hat{c}_j + \sum_{ij} [G_{ij} \hat{c}_i \hat{c}_j + h.c.], \qquad i, j = 1, ..., n \tag{13}$$

Introducing Majorana fermion operator $\hat{\eta}_I$, $I = 1, ..., 2n$,

$$\{\hat{\eta}_I, \hat{\eta}_J\} = 2\delta_{IJ}, \qquad \hat{\eta}_I^\dagger = \hat{\eta}_I, \tag{14}$$

to express the complex fermion operator $\hat{c}_i$:

$$\hat{c}_i = \frac{1}{2}(\hat{\eta}_{2i} + \mathrm{i}\eta_{2i+1}), \tag{15}$$

we can rewrite $\hat{H}$ as

$$\hat{H} = \frac{\mathrm{i}}{4}\sum_{IJ} A_{IJ}\hat{\eta}_I\hat{\eta}_J. \tag{16}$$

where A is a real antisymmetric matrix. For example, for a 1-orbital Hamiltonian $\hat{H} = \epsilon(\hat{c}^\dagger\hat{c} - \frac{1}{2})$, we get $A = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}$.

If the fermion number is conserved, $\hat{H}$ commutes with the fermion number operator

$$\hat{N} \equiv \sum_i (\hat{c}_i^\dagger\hat{c}_i - \frac{1}{2}) = \frac{\mathrm{i}}{4}\sum_{IJ} Q_{IJ}\hat{\eta}_I\hat{\eta}_J \tag{17}$$

where

$$Q = \varepsilon \otimes I, \quad Q^2 = -1, \quad \varepsilon \equiv -\mathrm{i}\sigma^y. \tag{18}$$

$[\hat{H}, \hat{N}] = 0$ requires that

$$[A, Q] = 0. \tag{19}$$

Such matrix A has a form $A = \sigma^0 \otimes H_a + \varepsilon \otimes H_s$, where H_s is symmetric and H_a antisymmetric. We can convert such an antisymmetric matrix A into a hermitian matrix $H = H_s + \mathrm{i}H_a$, and reduce the problem to the one discussed before.

The time reversal transformation $\hat{T}$ is antiunitary: $\hat{T}\mathrm{i}\hat{T}^{-1} = -\mathrm{i}$. Since $\hat{T}$ does not change the fermion numbers, thus $\hat{T}\hat{c}_i\hat{T}^{-1} = U_{ij}\hat{c}_j$, where U is an unitary matrix. In terms of the Majorana fermions, we have

$$\begin{aligned} \hat{T}\hat{\eta}_{2i}\hat{T}^{-1} &= \mathrm{Re}U_{ij}\hat{\eta}_{2j} - \mathrm{Im}U_{ij}\hat{\eta}_{2j+1}, \\ \hat{T}\hat{\eta}_{2i+1}\hat{T}^{-1} &= -\mathrm{Re}U_{ij}\hat{\eta}_{2j+1} - \mathrm{Im}U_{ij}\hat{\eta}_{2j}. \end{aligned} \tag{20}$$

Therefore, we have

$$\hat{T}\hat{\eta}_i\hat{T}^{-1} = T_{ij}\hat{\eta}_j, \quad T = \sigma^3 \otimes \mathrm{Re}U - \sigma^1 \otimes \mathrm{Im}U. \tag{21}$$

We see that in the Majorana fermion basis, $U \to \sigma^3 \otimes \mathrm{Re}U - \sigma^1 \otimes \mathrm{Im}U = T$ and $\mathrm{i} \to \epsilon \otimes I$. We indeed have $T(\epsilon \otimes I) = -T(\epsilon \otimes I)$.

For fermion systems, we may have $\hat{T}^2 = s_T^{\hat{N}}$ $s_T = \pm$. In fact $s_T = -$ for electron systems. This implies that $\hat{T}^2\hat{c}_i\hat{T}^{-2} = s_T\hat{c}_i$ and $T^2 = s_T$. The time reversal invariance $\hat{T}\hat{H}\hat{T}^{-1} = \hat{H}$ implies that $T^\top AT = -A$, where $T^\top$ is the transpose of T. We can show that

$$\begin{aligned} T^\top T &= (\sigma^3 \otimes \mathrm{Re}U^\dagger - \sigma^1 \otimes \mathrm{Im}U^\dagger)(\sigma^3 \otimes \mathrm{Re}U + \sigma^1 \otimes \mathrm{Im}U) \\ &= \sigma^0 \otimes (\mathrm{Re}U^\dagger\mathrm{Re}U - \mathrm{Im}U^\dagger\mathrm{Im}U) \\ &\quad - \epsilon \otimes (\mathrm{Re}U^\dagger\mathrm{Im}U + \mathrm{Im}U^\dagger\mathrm{Re}U) \\ &= \sigma^0 \otimes I, \end{aligned} \tag{22}$$

where we have used $\mathrm{Re}U^\dagger\mathrm{Re}U - \mathrm{Im}U^\dagger\mathrm{Im}U = I$ and $\mathrm{Re}U^\dagger\mathrm{Im}U + \mathrm{Im}U^\dagger\mathrm{Re}U = 0$ for unitary matrix U. Therefore $T^\top = T^{-1}$ and

$$AT = -TA, \quad T^2 = s_T. \tag{23}$$

Also, for fermion systems, the time reversal transformation $\hat{T}$ and the $U(1)$ transformation $\hat{N}$ may have a non-trivial relation: $\hat{T}\mathrm{e}^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = \mathrm{e}^{s_{UT}\mathrm{i}\theta\hat{N}}$, $s_{UT} = \pm$, or $\hat{T}\hat{N}\hat{T}^{-1} = -s_{UT}\hat{N}$. This gives us

$$TQ = s_{UT}QT. \tag{24}$$

The charge conjugation transformation $\hat{C}$ is unitary. Since $\hat{C}$ changes $\hat{c}_i$ to $\hat{c}_i^\dagger$, thus $\hat{C}\hat{c}_i\hat{C}^{-1} = U_{ij}\hat{c}_j^\dagger$, where U is an unitary matrix. In terms of the Majorana fermions, we have

$$\begin{aligned} \hat{C}\hat{\eta}_{2i}\hat{C}^{-1} &= \mathrm{Re}U_{ij}\hat{\eta}_{2j} + \mathrm{Im}U_{ij}\hat{\eta}_{2j+1}, \\ \hat{C}\hat{\eta}_{2i+1}\hat{C}^{-1} &= -\mathrm{Re}U_{ij}\hat{\eta}_{2j+1} + \mathrm{Im}U_{ij}\hat{\eta}_{2j}. \end{aligned} \tag{25}$$

Therefore, we have

$$\hat{C}\hat{\eta}_i\hat{C}^{-1} = C_{ij}\hat{\eta}_j, \quad C = \sigma^3 \otimes \mathrm{Re}U + \sigma^1 \otimes \mathrm{Im}U. \tag{26}$$

Again, we can show that $C^\top = C^{-1}$.

For fermion systems, we may have $\hat{C}^2 = s_C^{\hat{N}}$, $s_C = \pm$, which implies that $\hat{C}^2\hat{c}_i\hat{C}^{-2} = s_C\hat{c}_i$ and $C^2 = s_C$. The charge conjugation invariance $\hat{C}\hat{H}\hat{C}^{-1} = \hat{H}$ implies that A satisfies

$$CA = CA, \quad C^2 = s_C. \tag{27}$$

Since $\hat{C}\hat{N}\hat{C}^{-1} = -\hat{N}$, we have

$$CQ = -QC. \tag{28}$$

However, the commutation relation between $\hat{T}$ and $\hat{C}$ has two choices: $\hat{T}\hat{C} = s_{TC}^{\hat{N}}\hat{C}\hat{T}$, $s_{TC} = \pm$, we have

$$CT = s_{TC}TC. \tag{29}$$

We see that when we say a system has $U(1)$, time reversal, and/or charge conjugation symmetries, we still do not know what is the actual symmetry group of the system, since those symmetry operations may have different relations as described by the signs s_T, s_C, s_{UT}, s_{TC}, which lead to different full symmetry groups. Because symmetry plays a key role in our classification, we cannot obtain a classification without specifying the symmetry groups. We have discussed the possible relations among various symmetry operations. In the Table IV, we list the corresponding symmetry groups.

We like to point that that some times, when we describe the symmetry of a fermion system, we do not include the fermion number parity transformation $(-)^{\hat{N}}$ in the symmetry group G. However, in this paper, we will use the full symmetry group G_f to describe the symmetry of a fermion system. The full symmetry group G_f does include the fermion number parity transformation $(-)^{\hat{N}}$. So the full symmetry group of a fermion system with no symmetry is $G_f = Z_2^f$ generated by the fermion number parity transformation. G_f is actually a Z_2^f extension of G: $G = G_f/Z_2^f$. It is a projective symmetry group discussed in Ref. 41.

Symmetry groups	Relations
$U(1),\ SU(2)$	
$G_{s_C}(C)$	$\hat C^2 = s_C^{\hat N}, \quad s_C = \pm.$
$G_{s_C}(U,C)$	$\hat C^2 = s_C^{\hat N},\ \hat C e^{i\theta \hat N}\hat C^{-1} = e^{-i\theta\hat N},\ s_C = \pm.$
$G_{s_T}(T)$	$\hat T^2 = s_T^{\hat N}, \quad s_T = \pm.$
$G_{s_T}^{s_{UT}}(U,T)$	$\hat T e^{i\theta\hat N}\hat T^{-1} = e^{s_{UT}\, i\theta\hat N},\ \hat T^2 = s_T^{\hat N}, \quad s_{UT}, s_T = \pm.$
$G_{s_T s_C}^{s_{TC}}(T,C)$	$\hat T^2 = s_T^{\hat N},\ \hat C^2 = s_C^{\hat N},\ \hat C\hat T = (s_{TC}^{\hat N})\hat T\hat C, \quad s_{TC}, s_T, s_C = \pm.$
$G_{s_T s_C}^{s_{UT} s_{TC}}(U,T,C)$	$\hat C e^{i\theta\hat N}\hat C^{-1} = e^{-i\theta\hat N},\ \hat T e^{i\theta\hat N}\hat T^{-1} = e^{s_{UT}\, i\theta\hat N},$ $\hat T^2 = s_T^{\hat N},\ \hat C^2 = s_C^{\hat N},\ \hat C\hat T = (s_{TC}^{\hat N})\hat T\hat C, \quad s_T, s_C, s_{UT}, s_{TC} = \pm.$

TABLE IV: Different relations between symmetry transformations gives rise to 36 different groups that contain $U(1)$ (represented by U), time reversal T, and/or charge conjugation C symmetries.

In the following, we will study the symmetries of various electron systems, to see which symmetry groups listed in Table IV can be realized by electron systems.

For insulators with non-coplanar spin order $\delta H = \hat c_i^\dagger \boldsymbol n_1\cdot\boldsymbol\sigma \hat c_i + \hat c_j^\dagger \boldsymbol n_2\cdot\boldsymbol\sigma \hat c_j + \hat c_k^\dagger \boldsymbol n_3\cdot\boldsymbol\sigma \hat c_k$, the full symmetry group is $G_f = U(1)$ generated by the total charge $\hat N_C$.

For superconductors with non-coplanar spin order $\delta H = \hat c_i^\dagger \boldsymbol n_1\cdot\boldsymbol\sigma \hat c_i + \hat c_j^\dagger \boldsymbol n_2\cdot\boldsymbol\sigma \hat c_j + \hat c_k^\dagger \boldsymbol n_3\cdot\boldsymbol\sigma \hat c_k + (\hat c_{\uparrow i}\hat c_{\downarrow j} - \hat c_{\downarrow i}\hat c_{\uparrow j})$, the full symmetry group is reduced to $G_f = Z_2^f$ generated by the fermion number parity operator $P_f = (-)^{\hat N_C}$. We note that the full symmetry group of any fermion system contain Z_2^f as a subgroup. So we usually, use the group G_f/Z_2^f to describe the symmetry of fermion system, and we say there is no symmetry for superconductors with non-coplanar spin order. But in this paper, we will use the full symmetry group G_f to describe the symmetry of fermion systems.

For insulators with spin-orbital coupling $\delta H = i\hat c_i^\dagger \boldsymbol n_1 \cdot \boldsymbol\sigma \hat c_{i'} + i\hat c_j^\dagger \boldsymbol n_2\cdot\boldsymbol\sigma\hat c_{j'} + i\hat c_k^\dagger \boldsymbol n_3\cdot\boldsymbol\sigma \hat c_{k'}$, they have the charge conservation $\hat N_C$ and the time reversal $\hat T_{\text{phy}}$ symmetries. The time reversal symmetry is defined by

$$\hat T_{\text{phy}} \hat c_{\alpha,i}\hat T_{\text{phy}}^{-1} = \epsilon_{\alpha\beta}\hat c_{\beta,i}, \quad \hat T_{\text{phy}}\hat c_{\alpha,i}^\dagger \hat T_{\text{phy}}^{-1} = \epsilon_{\alpha\beta}\hat c_{\beta,i}^\dagger. \tag{30}$$

We can show that

$$\begin{aligned} &\hat T_{\text{phy}} \hat c_i^\dagger \boldsymbol\sigma \hat c_j \hat T_{\text{phy}}^{-1} = -\hat c_i^\dagger\boldsymbol\sigma\hat c_j, \\ &\hat T_{\text{phy}}\hat N_C \hat T_{\text{phy}}^{-1} = \hat N_C, \quad \hat T_{\text{phy}}^2 = (-)^{\hat N_C} \end{aligned} \tag{31}$$

Thus δH is invariant under $\hat T_{\text{phy}}$ and $e^{i\theta\hat N_C}$. Let $\hat T = \hat T_{\text{phy}}$ and $\hat N = \hat N_C$, we find that

$$\hat T e^{i\theta\hat N}\hat T^{-1} = e^{-i\theta\hat N}, \quad \hat T^2 = (-)^{\hat N}, \tag{32}$$

which define the full symmetry group $G_-^-(U,T)$ of an electron insulator with spin-orbital coupling.

For superconductors with spin-orbital coupling and real pairing $\delta H = i\hat c_i^\dagger \boldsymbol n_1\cdot\boldsymbol\sigma\hat c_{i'} + i\hat c_j^\dagger\boldsymbol n_2\cdot\boldsymbol\sigma\hat c_{j'} + i\hat c_k^\dagger\boldsymbol n_3\cdot\boldsymbol\sigma\hat c_{k'} + (\hat c_{\uparrow i}\hat c_{\downarrow j} - \hat c_{\downarrow i}\hat c_{\uparrow j})$, they have the time reversal symmetry $\hat T_{\text{phy}}$. Setting $\hat T = \hat T_{\text{phy}}$ and $\hat N = \hat N_C$, we find

$$\hat T^2 = (-)^{\hat N}, \tag{33}$$

which defines the full symmetry group $G_-^+(U,T) = Z_4$ of superconductors with spin-orbital coupling and real pairing.

For superconductors with S_z conserving spin-orbital coupling and real pairing $\delta H = i\hat c_i^\dagger \sigma^z \hat c_j + (\hat c_{\uparrow i}\hat c_{\downarrow j} - \hat c_{\downarrow i}\hat c_{\uparrow j})$, they have the time reversal $\hat T_{\text{phy}}$ and S_z spin rotation symmetries. Setting $\hat T = \hat T_{\text{phy}}$ and $\hat N = 2\hat S_z$, we find

$$\hat T e^{i\theta\hat N}\hat T^{-1} = e^{i\theta\hat N}, \quad \hat T^2 = (-)^{\hat N}, \tag{34}$$

which defines the full symmetry group $G_-^+(U,T) = U(1)\times Z_2$ of superconductors with S_z conserving spin-orbital coupling and real pairing.

For superconductors with real pairing and coplanar spin order $\delta H = \hat c_i^\dagger \boldsymbol n_1\cdot\boldsymbol\sigma\hat c_i + \hat c_j^\dagger\boldsymbol n_2\cdot\boldsymbol\sigma\hat c_j + (\hat c_{\uparrow i}\hat c_{\downarrow j} - \hat c_{\downarrow i}\hat c_{\uparrow j})$, they have a combined time reversal and $180°$ spin rotation symmetry. The spin rotation is generated by $S_a = \sum_i \frac12 c_i^\dagger\sigma^a c_i$, $a = x, y, z$. We have

$$\hat T_{\text{phy}}\hat S_a \hat T_{\text{phy}}^{-1} = -\hat S_a. \tag{35}$$

The Hamiltonian δH is invariant under $\hat T = e^{i\pi\hat S^y}\hat T_{\text{phy}}$. Since $\hat T^2 = 1$, the full symmetry group of superconductors real pairing and coplanar spin order is $G_+(T) = Z_2\times Z_2^f$.

For superconductors with real pairing and collinear spin order $\delta H = \hat c_i^\dagger\sigma^z\hat c_j + (\hat c_{\uparrow i}\hat c_{\downarrow j} - \hat c_{\downarrow i}\hat c_{\uparrow j})$, they have the S_z spin rotation and a combined time reversal and $180°$ S_y spin rotation symmetry. The Hamiltonian δH is invariant under $\hat T = e^{i\pi\hat S^y}\hat T_{\text{phy}}$ and S_z spin rotation $\hat N = 2\hat S_z$. We find that

$$\hat T e^{i\theta\hat N}\hat T^{-1} = e^{-i\theta\hat N}, \quad \hat T^2 = (-)^{\hat N}, \tag{36}$$

which define the full symmetry group $G_+^-(U,T)$ of superconductors with real pairing and collinear spin order.

For insulators with coplanar spin order $\hat c_i^\dagger\boldsymbol n_1\cdot\boldsymbol\sigma\hat c_i + \hat c_j^\dagger\boldsymbol n_2\cdot\boldsymbol\sigma\hat c_j$, they have the charge conservation and a combined time reversal and $180°$ spin rotation symmetries. The Hamiltonian δH is invariant under $\hat T = e^{i\pi\hat S^y}\hat T_{\text{phy}}$ and the charge rotation $\hat N = \hat N_C$. We find that

$$\hat T e^{i\theta\hat N}\hat T^{-1} = e^{-i\theta\hat N}, \quad \hat T^2 = (-)^{\hat N}, \tag{37}$$

which define the full symmetry group $G_+^-(U,T)$ of insulators with coplanar spin order.

For superconductors with real triplet $S_z=0$ paring $\delta H=\hat c_{\uparrow i}\hat c_{\downarrow j}+\hat c_{\downarrow i}\hat c_{\uparrow j}$, they have a combined time reversal and charge rotation symmetry, a combined 180° S_y spin rotation and charge rotation symmetry, and the S_z-spin rotation symmetry. The Hamiltonian δH is invariant under $\hat T=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\hat N_C}\hat T_{\text{phy}}$, $\hat C=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\hat N_C}\mathrm{e}^{\mathrm{i}\pi\hat S_y}$, and the S_z spin rotation $\hat N=2\hat S_z$. We find that

$$\begin{aligned}&\hat C\mathrm{e}^{\mathrm{i}\theta\hat N}\hat C^{-1}=\mathrm{e}^{-\mathrm{i}\theta\hat N},\\&\hat T\mathrm{e}^{\mathrm{i}\theta\hat N}\hat T^{-1}=\mathrm{e}^{\mathrm{i}\theta\hat N},\\&\hat T^2=1,\quad \hat C^2=1,\quad \hat C\hat T=(-)^{\hat N}\hat T\hat C\end{aligned}\tag{38}$$

which define the full symmetry group $G_{++}^{--}(U,T,C)$ of superconductors with real triplet $S_z=0$ paring.

For superconductors with real triplet $S_z=0$ paring and collinear spin order $\delta H=\hat c_i\sigma^z c_i+(\hat c_{\uparrow i}\hat c_{\downarrow j}+\hat c_{\downarrow i}\hat c_{\uparrow j})$, they have a combined time reversal and 180° S_y spin rotation symmetry, and the S_z-spin rotation symmetry. The Hamiltonian δH is invariant under $\hat T=\mathrm{e}^{\mathrm{i}\pi\hat S_y}\hat T_{\text{phy}}$, and the S_z spin rotation $\hat N=2\hat S_z$. We find that

$$\begin{aligned}&\hat T\mathrm{e}^{\mathrm{i}\theta\hat N}\hat T^{-1}=\mathrm{e}^{-\mathrm{i}\theta\hat N},\\&\hat T^2=1,\end{aligned}\tag{39}$$

which define the full symmetry group $G_+^-(U,T)$ of superconductors with real triplet $S_z=0$ paring and collinear spin order.

For superconductors with the time reversal, the 180° S_y-spin rotation, and the S_z-spin rotation symmetries, the Hamiltonian is invariant under $\hat T=\hat T_{\text{phy}}$, $\hat C=\mathrm{e}^{\mathrm{i}\pi\hat S_y}$, and $\hat N=2\hat S_z$. We find that

$$\begin{aligned}&\hat C\mathrm{e}^{\mathrm{i}\theta\hat N}\hat C^{-1}=\mathrm{e}^{-\mathrm{i}\theta\hat N},\ \ \hat T\mathrm{e}^{\mathrm{i}\theta\hat N}\hat T^{-1}=\mathrm{e}^{\mathrm{i}\theta\hat N},\\&\hat T^2=\hat C^2=(-)^{\hat N},\ \ \hat C\hat T=\hat T\hat C.\end{aligned}\tag{40}$$

which define the full symmetry group $G_{--}^{++}(U,T,C)$ of superconductors with the time reversal, the 180° S_y-spin rotation, and the S_z-spin rotation symmetries. For free electrons with the 180° S_y-spin rotation, and the S_z-spin rotation symmetries, they actually have the full $SU(2)$ spin rotation symmetry. So the above systems are also superconductors with real pairing and the $SU(2)$ spin rotation symmetry. Similarly, for superconductors with complex pairing and the $SU(2)$ spin rotation symmetry, the symmetry group is $SU(2)$, or $G_-(U,C)$.

For insulators with spin-orbital coupling and only inter-sublattice hopping $H=\mathrm{i}\hat c_{i_A}^\dagger\boldsymbol{n}_1\cdot\boldsymbol{\sigma}\hat c_{i_B}+\mathrm{i}\hat c_{j_A}^\dagger\boldsymbol{n}_2\cdot\boldsymbol{\sigma}\hat c_{j_B}+\mathrm{i}\hat c_{k_A}^\dagger\boldsymbol{n}_3\cdot\boldsymbol{\sigma}\hat c_{k_B}$, they have the charge conservation, the time reversal and a deformed charge conjugation symmetries. The charge conjugation transformation $\hat C_{\text{phy}}$ is defined as

$$\hat C_{\text{phy}}\hat c_{\alpha,i}\hat C_{\text{phy}}^{-1}=\epsilon_{\alpha\beta}\hat c_{\beta,i}^\dagger,\quad \hat C_{\text{phy}}\hat c_{\alpha,i}^\dagger\hat C_{\text{phy}}^{-1}=\epsilon_{\alpha\beta}\hat c_{\beta,i}.\tag{41}$$

We find that

$$\begin{aligned}&\hat C_{\text{phy}}\hat c_i^\dagger\boldsymbol{\sigma}\hat c_j\hat C_{\text{phy}}^{-1}=\hat c_i^\dagger\boldsymbol{\sigma}\hat c_j,\quad \hat C_{\text{phy}}\hat T_{\text{phy}}=\hat T_{\text{phy}}\hat C_{\text{phy}},\\&\hat C_{\text{phy}}\hat N_C\hat C_{\text{phy}}^{-1}=-\hat N_C,\quad \hat C_{\text{phy}}^2=(-)^{\hat N_C}\end{aligned}\tag{42}$$

The above Hamiltonian is invariant under $\hat T=\hat T_{\text{phy}}$, $\hat N=\hat N_C$, and $\hat C=(-)^{\hat N_B}\hat C_{\text{phy}}$, where $\hat N_B$ is the number of electrons on the B-sublattice. We find

$$\begin{aligned}&\hat C\mathrm{e}^{\mathrm{i}\theta\hat N}\hat C^{-1}=\mathrm{e}^{-\mathrm{i}\theta\hat N},\\&\hat T\mathrm{e}^{\mathrm{i}\theta\hat N}\hat T^{-1}=\mathrm{e}^{-\mathrm{i}\theta\hat N},\\&\hat T^2=(-)^{\hat N},\quad \hat C^2=(-)^{\hat N},\quad \hat C\hat T=\hat T\hat C\end{aligned}\tag{43}$$

which define the full symmetry group $G_{--}^{-+}(U,T,C)$ of insulators with spin-orbital coupling and only inter-sublattice hopping. The above results for electron systems and their full symmetry groups G_f are summarized in Table I.

B. The $d=0$ case – the classifying spaces

The hermitian matrix $\mathrm{i}A$ describes single-body excitations above the free fermion ground state. We note that the eigenvalues of $\mathrm{i}A$ are $\pm\epsilon_i$. The positive eigenvalues $|\epsilon_i|$ correspond to the single-body excitation energies above the many-body ground state. The minimal $|\epsilon_i|$ represents the excitation energy gap of $\hat H$ above the ground state (the ground state is the lowest energy state of $\hat H$). So, if we are considering gapped systems, $|\epsilon_i|$ is alway non-zero. We can shift all ϵ_i to ±1 without closing the gap and change the (matter) phase of the state. Thus, we can set $A^2=-1$.

In presence of symmetry, A should also satisfy some other conditions. The space formed by all those A's is called the classifying space. Clearly, the classifying space is determined by the full symmetry group G_f. In this section, we will calculate the classifying spaces for some simple groups.

If there is no symmetry, then the real antisymmetric matrix A satisfies

$$A^2=-1.\tag{44}$$

The space of those matrices is denoted as R_0^0, which is the classifying space for trivial symmetry group.

If there is only the charge conjugation symmetry (full symmetry group $=G_{s_C}(C)$), then the real antisymmetric matrix A satisfies

$$A^2=-1,\quad AC=CA,\quad C^2=s_C.\tag{45}$$

For $s_C=+$, since C commute with A and C is symmetric, we can always restrict ourselves in an eigenspace of C and C can be dropped. Thus the space of the matrices is R_0^0, the same as before. For $s_C=-$, we can assume $C=\varepsilon\otimes I$. In this case, A has a form $A=\sigma^0\otimes H_a+\varepsilon\otimes H_s$, where $H_s = H_s^T$ and $H_a = -H_a^T$. Thus, we can convert A into a hermitian matrix $H = H_s + \mathrm{i} H_a$, and the space of the matrices is C_0.

If there are $U(1)$ and charge conjugation symmetries (full symmetry group $= G_{s_C}(U,C)$), then the real anti-symmetric matrix A satisfies

$$A^2 = -1, \quad AQ = QA, \quad AC = CA, \quad QC = -CQ, \\ Q^2 = -1, \quad C^2 = s_C. \tag{46}$$

For $s_C = +$, we can assume $C = \sigma^z \otimes I$ and $Q = \varepsilon \otimes I$. Since Q and C commute with A, we find that A must have a form $A = \sigma^0 \otimes \tilde{A}$, with $\tilde{A}^2 = -1$. Thus the space of the matrices $\tilde{A}$, and hence A, is R_0^0.

For $s_C = -$, we can assume $C = \varepsilon \otimes \sigma^x \otimes I$ and $Q = \varepsilon \otimes \sigma^z \otimes I$. We find that A must have a form $A = \sigma^0 \otimes \sigma^0 \otimes H_0 + \varepsilon \otimes \sigma^0 \otimes H_1 + \sigma^z \otimes \varepsilon \otimes H_2 + \sigma^x \otimes \varepsilon \otimes H_3$, where $H_0 = -H_0^T$ and $H_i = H_i^T$, $i = 1,2,3$. Now, we can view $\sigma^0 \otimes \sigma^0$ as 1, $\varepsilon \otimes \sigma^0$ as i, $\sigma^z \otimes \varepsilon$ as j, and $\sigma^x \otimes \varepsilon$ as k. We find that i,j,k satisfy the quaternion algebra. Thus A can be mapped into a quaternion matrix $H = H_0 + \mathrm{i}H_1 + \mathrm{j}H_2 + \mathrm{k}H_3$ satisfying $H^\dagger = -H$ and $H^2 = -1$. The quaternion matrices that satisfy the above two conditions has a form

$$H = \mathrm{e}^{X_{n\times n}}\, \mathrm{i}\, I_{n\times n}\, \mathrm{e}^{-X_{n\times n}} \tag{47}$$

where $X_{n\times n}^\dagger = -X_{n\times n}$ is a quaternion matrix. $\mathrm{e}^{X_{n\times n}}$ form the group $Sp(n)$. However, the transformations $\mathrm{e}^{A_{n\times n} + \mathrm{i} B_{n\times n}}$ keeps $\mathrm{i} I_{n\times n}$ unchanged, where $A_{n\times n}$ is a real antisymmetric matrix and $B_{n\times n}$ is a real symmetric matrix. $\mathrm{e}^{A_{n\times n} + \mathrm{i} B_{n\times n}}$ form the group $U(n)$. Thus, the space of the quaternion matrices that satisfy the above two conditions is given by $Sp(n)/U(n)$. Such a space is the space R_6 which will be introduced later.

When $s_C = -1$, we can view Q as the generator of $\hat{S}_z$ spin rotation, and C as the generator of $\hat{S}_x$ spin rotation acting on spin-1/2 fermions. In fact $\mathrm{e}^{\theta_z Q}$ and $\mathrm{e}^{\theta_x C}$ in this case generate the full $SU(2)$ group. So when $s_C = -$, the free spin-1/2 fermions with $U(1) \rtimes Z_2^C$ symmetry actually have the full $SU(2)$ spin rotation symmetry. Therefore $G_-(U,C) \sim SU(2)$.

If there is only the time reversal symmetry (full symmetry group $= G_{s_T}(T)$), then A satisfies

$$A^2 = -1, \quad A\rho_1 + \rho_1 A = 0, \quad \rho_1^2 = s_T, \quad \rho_1 = T. \tag{48}$$

The space of those matrices is denoted as R_0^1 for $s_T = -1$ and R_1^0 for $s_T = 1$.

If there are time reversal and $U(1)$ symmetries (full symmetry group $= G_{s_T}^{s_{UT}}(U,T)$), then for $s_{UT} = -$, A satisfies

$$A^2 = -1, \quad A\rho_i + \rho_i A = 0, \quad \rho_1^2 = s_T, \quad \rho_2^2 = s_T, \\ \rho_1 = T, \quad \rho_2 = TQ. \tag{49}$$

The space of those matrices is denoted as R_2^0 for $s_T = +$ and R_0^2 for $s_T = -$.

For $s_{UT} = +$, Q commute with both A and T. Since $Q^2 = -1$, we can treat Q as the imaginary number i and convert both A and T to complex matrices. To see this, let us choose a basis in which Q has a form $Q = \epsilon \otimes I$. In this basis A and T become $A = \sigma^0 \otimes A_2 + \epsilon \otimes A_1$ and $T = \sigma^0 \otimes T_1 + \epsilon \otimes T_2$, where A_1 is symmetric and A_2 is antisymmetric. Let us introduce complex matrices $H = -A_1 + \mathrm{i} A_2$ and $\tilde{T} = T_1 + \mathrm{i} T_2$ for $s_T = +$ or $\tilde{T} = -T_2 + \mathrm{i} T_1$ for $s_T = -$. From $A^2 = -1$, $T^2 = s_T$, and $AT = -TA$, we find

$$H^2 = 1, \quad H\tilde{T} + \tilde{T}H = 0, \quad \tilde{T}^2 = 1. \tag{50}$$

Also $A^T = -A$ allows us to show $H^\dagger = H$. For a fixed $\tilde{T}$, the space formed by H's that satisfy the above conditions is C_1 introduced before. This allows us to show that the space of the corresponding matrices A is C_1 for $s_T = \pm$, $s_{UT} = +$.

If there are time reversal and charge conjugation symmetries (full symmetry group $= G_{s_T s_C}^{s_{TC}}(T,C)$), then for $s_{TC} = -$, A satisfies

$$A^2 = -1, \quad A\rho_i + \rho_i A = 0, \quad \rho_1^2 = s_T, \quad \rho_2^2 = -s_T s_C, \\ \rho_1 = T, \quad \rho_2 = TC. \tag{51}$$

The space of the matrices A is R_1^1 for $s_T = +$, $s_C = +$; R_2^0 for $s_T = +$, $s_C = -$; R_1^1 for $s_T = -$, $s_C = +$; and R_0^2 for $s_T = -$, $s_C = -$. For $s_{TC} = +$, C will commute with both A and T. We find space of the matrices A to be R_1^0 for $s_T = +$, $s_C = +$; R_0^1 for $s_T = -$, $s_C = +$; and C_1 for $s_T = \pm$, $s_C = -$.

If there are $U(1)$, time reversal, and charge conjugation symmetries (full symmetry group $= G_{s_T s_C}^{s_{UT} s_{TC}}(U,T,C)$), then for $s_{TC} = s_{UT} = -$, A satisfies

$$A^2 = -1, \ A\rho_i + \rho_i A = 0, \ \rho_1^2 = \rho_2^2 = s_T, \ \rho_3^2 = -s_T s_C, \\ \rho_1 = T, \quad \rho_2 = TQ, \quad \rho_3 = TC. \tag{52}$$

The space of the matrices A is R_2^1 for $s_T = +$, $s_C = +$; R_3^0 for $s_T = +$, $s_C = -$; R_1^2 for $s_T = -$, $s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT} = -$, $s_{TC} = +$, A satisfies

$$A^2 = -1, \ A\rho_i + \rho_i A = 0, \ \rho_1^2 = \rho_2^2 = s_T, \ \rho_3^2 = -s_T s_C, \\ \rho_1 = T, \quad \rho_2 = TQ, \quad \rho_3 = TQC. \tag{53}$$

The space of the matrices A is R_2^1 for $s_T = +$, $s_C = +$; R_3^0 for $s_T = +$, $s_C = -$; R_1^2 for $s_T = -$, $s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT} = +$, $s_{TC} = -$, A satisfies

$$A^2 = -1, \ A\rho_i + \rho_i A = 0, \ \rho_1^2 = s_T, \ \rho_2^2 = \rho_3^2 = -s_T s_C, \\ \rho_1 = T, \ \rho_2 = TC, \ \rho_3 = TCQ. \tag{54}$$

The space of the matrices A is R_1^2 for $s_T = +$, $s_C = +$; R_3^0 for $s_T = +$, $s_C = -$; R_2^1 for $s_T = -$, $s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT}=+,\ s_{TC}=+$, we find that A satisfy

$$A^2=-1,\ A\rho_i+\rho_i A=0,\ \rho_1^2=-s_T,\ \rho_2^2=\rho_3^2=s_Ts_C,$$
$$\rho_1=TQ,\ \rho_2=TC,\ \rho_3=TCQ. \qquad (55)$$

We see that the matrices A form a space R_2^1 for $s_T=+,\ s_C=+$; R_0^3 for $s_T=+,\ s_C=-$; R_1^2 for $s_T=-,\ s_C=+$; and R_3^0 for $s_T=-,\ s_C=-$;

C. The properties of classifying spaces

In general, we can consider a real antisymmetric matrix A that satisfies (for fixed real matrices ρ_i, $i=1,...,p+q$)

$$A=\rho_{p+q+1},\quad \rho_j\rho_i+\rho_i\rho_j=\Big|_{i\neq j}0,$$
$$\rho_i^2=\Big|_{i=1,\dots,p}1,\quad \rho_i^2=\Big|_{i=p+1,\dots,p+q+1}-1. \qquad (56)$$

The space of those A matrices is denoted as R_p^q.

Let us show that

$$R_p^q=R_{p+1}^{q+1}. \qquad (57)$$

From $\tilde A\in R_p^q$ that satisfies the following Clifford algebra $Cl(p,q+1)$

$$\tilde A=\tilde\rho_{p+q+1},\ \ \tilde\rho_j\tilde\rho_i+\tilde\rho_i\tilde\rho_j=\Big|_{i\neq j}0,$$
$$\tilde\rho_i^2=\Big|_{i=1,\dots,p}1,\ \ \tilde\rho_i^2=\Big|_{i=p+1,\dots,p+q+1}-1, \qquad (58)$$

we can define

$$\rho_i=\Big|_{i=1,\dots,p}\tilde\rho_i\otimes\sigma^z,\ \ \rho_{p+1}=I\otimes\sigma^x,$$
$$\rho_i=\Big|_{i=p+2,\dots,p+q+2}\tilde\rho_{i-1}\otimes\sigma^z,\ \ \rho_{p+q+3}=I\otimes\varepsilon. \qquad (59)$$

We can check that such ρ_i satisfy the following Clifford algebra $Cl(p+1,q+2)$

$$\rho_j\rho_i+\rho_i\rho_j=\Big|_{i\neq j}0,$$
$$\rho_i^2=\Big|_{i=1,\dots,p+1}1,\ \ \rho_i^2=\Big|_{i=p+2,\dots,p+q+3}-1. \qquad (60)$$

If we fix ρ_i, $i\neq p+q+2$, then the space formed by $A=\rho_{p+q+2}$ satisfying the above condition is given by R_{p+1}^{q+1}. The above construction gives rise to a map from $R_p^q\to R_{p+1}^{q+1}$. Since $A=\rho_{p+q+2}$ satisfying eqn. (60) must has a form $\tilde A\otimes\sigma^z$, with $\tilde A$ satisfying eqn. (58). This gives us a map $R_{p+1}^{q+1}\to R_p^q$. Thus $R_{p+1}^{q+1}=R_p^q$.

We can also consider real symmetric matrix A that satisfies (for fixed real matrices ρ_i, $i=1,...,p$)

$$A=\rho_{p+1},\quad \rho_j\rho_i+\rho_i\rho_j=\Big|_{i\neq j}0,\quad \rho_i^2=\Big|_{i=1,\dots,p+1}1. \qquad (61)$$

The space of those matrices is denoted as R_p.

In the following, we are going to show that

$$R_0^q=R_{q+2}. \qquad (62)$$

From $\tilde A\in R_0^q$ that satisfies the Clifford algebra $Cl(0,q+1)$

$$\tilde A=\tilde\rho_{q+1},\quad \tilde\rho_j\tilde\rho_i+\tilde\rho_i\tilde\rho_j=\Big|_{i\neq j}0,\quad \tilde\rho_i^2=\Big|_{i=1,\dots,q+1}-1, \qquad (63)$$

we can define

$$\rho_i=\Big|_{i=1,\dots,q+1}\tilde\rho_i\otimes\varepsilon,\quad \rho_{q+2}=I\otimes\sigma^z,\quad \rho_{q+3}=I\otimes\sigma^x. \qquad (64)$$

we can check that ρ_i form the Clifford algebra $Cl(q+3,0)$

$$\rho_j\rho_i+\rho_i\rho_j=\Big|_{i\neq j}0,\quad \rho_i^2=\Big|_{i=1,\dots,q+3}1. \qquad (65)$$

If we fix ρ_i, $i\neq q+1$, then the space formed by $A=\rho_{q+1}$ satisfying the above condition is given by R_{q+2}. The above construction gives rise to a map from $R_0^q\to R_{q+2}$. Since $A=\rho_{q+1}$ satisfying eqn. (65) must has a form $\tilde A\otimes\varepsilon$, with $\tilde A$ satisfying eqn. (63). This gives us a map $R_{q+2}\to R_0^q$. Thus $R_0^q=R_{q+2}$.

In addition we also have the following periodic relations

$$R_p^q=R_p^{q+8}=R_{p+8}^q,\qquad R_p=R_{p+8}. \qquad (66)$$

This can be shown by noticing the following 16 dimensional real symmetric representation of Clifford algebra $Cl(0,8)$:

$$\theta_1=\varepsilon\otimes\sigma^z\otimes\sigma^0\otimes\varepsilon,\quad \theta_2=\varepsilon\otimes\sigma^z\otimes\varepsilon\otimes\sigma^x,$$
$$\theta_3=\varepsilon\otimes\sigma^z\otimes\varepsilon\otimes\sigma^z,\quad \theta_4=\varepsilon\otimes\sigma^x\otimes\varepsilon\otimes\sigma^0,$$
$$\theta_5=\varepsilon\otimes\sigma^x\otimes\sigma^x\otimes\varepsilon,\quad \theta_6=\varepsilon\otimes\sigma^x\otimes\sigma^z\otimes\varepsilon,$$
$$\theta_7=\varepsilon\otimes\varepsilon\otimes\sigma^0\otimes\sigma^0,\quad \theta_8=\sigma^x\otimes\sigma^0\otimes\sigma^0\otimes\sigma^0, \qquad (67)$$

which satisfy

$$\theta_i\theta_j+\theta_j\theta_i=\Big|_{i\neq j}0,\qquad \theta_i^2=\Big|_{i=0,\dots,8}1. \qquad (68)$$

We find that $\theta=\theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7\theta_8=\sigma^z\otimes\sigma^0\otimes\sigma^0\otimes\sigma^0$ anticommute with θ_i. From $\tilde A\in R_p^q$ that satisfies eqn. (58), we can define

$$\rho_i=\Big|_{i=1,\dots,p}\tilde\rho_i\otimes\theta,\quad \rho_{p+i}=\Big|_{i=1,\dots,8}I\otimes\theta_i,$$
$$\rho_i=\Big|_{i=p+9,\dots,p+q+9}\tilde\rho_{i-8}\otimes\sigma^z. \qquad (69)$$

We can check that such ρ_i satisfy

$$\rho_j\rho_i+\rho_i\rho_j=\Big|_{i\neq j}0,\quad \rho_i^2=\Big|_{i=1,\dots,p+8}1,$$
$$\rho_i^2=\Big|_{i=p+9,\dots,p+q+9}-1. \qquad (70)$$

p mod 8	0	1	2	3	4	5	6	7
R_p	$\frac{O(l+m)}{O(l)\times O(m)}\times \mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times \mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
$\pi_0(R_p)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$\pi_1(R_p)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_2(R_p)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_3(R_p)$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_4(R_p)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$\pi_5(R_p)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\pi_6(R_p)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$\pi_7(R_p)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0

TABLE V: The spaces R_p and their homotopy groups $\pi_d(R_p)$.

If we fix ρ_i, $i \neq p+q+9$, then the space formed by $A=\rho_{p+q+9}$ satisfying the above condition is given by R^q_{p+8}. The above construction gives rise to a map from $R^q_p \to R^q_{p+8}$. On the other hand, the matrix that anticommute with all θ_i's must be proportional to θ. Thus $A=\rho_{p+q+9}$ satisfying eqn. (70) must has a form $\tilde A\otimes\theta$, with $\tilde A$ satisfying eqn. (58). This gives us a map $R^q_{p+8} \to R^q_p$. Thus $R^q_{p+8} = R^q_p$. Using a similar approach, we can show $R_p = R_{p+8}$. The relation eqn. (57), eqn. (62), and eqn. (66) allow us show

$$R^q_p = R_{q-p+2 \text{ mod } 8}. \tag{71}$$

So we can study the space R^q_p via the space $R_{q-p+2 \text{ mod } 8}$.

Let us construct some of the R_p spaces. R_0 is formed by real symmetric matrices A that satisfy $A^2=1$. Thus A has a form $O\begin{pmatrix} I_{l\times l} & 0\\ 0 & -I_{m\times m}\end{pmatrix}O^{-1}$, $O\in O(l+m)$. We see that $R_0 = \bigcup_m O(l+m)/O(l)\times O(m) = \frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$.

R_1 is formed by real symmetric matrices A that satisfy $A^2=1$ and $A\rho_1 = -\rho_1 A$ with $\rho_1 = \sigma^z\otimes I_{n\times n}$. Thus A has a form

$$\begin{aligned} A &= \mathrm{e}^{\sigma^z\otimes M_{n\times n}}\mathrm{e}^{\sigma^0\otimes L_{n\times n}}[\sigma^x\otimes I_{n\times n}]\mathrm{e}^{-\sigma^0\otimes L_{n\times n}}\mathrm{e}^{-\sigma^z\otimes M_{n\times n}}\\ &= \mathrm{e}^{\sigma^z\otimes M_{n\times n}}[\sigma^x\otimes I_{n\times n}]\mathrm{e}^{-\sigma^z\otimes M_{n\times n}}, \end{aligned} \tag{72}$$

where $\mathrm{e}^{\sigma^0\otimes L_{n\times n}}\in O(n)$ and $\mathrm{e}^{\sigma^z\otimes M_{n\times n}}\in O(n)$ are the transformations that leave ρ_1 unchanged. We see that $R_1 = O(n)$. The other spaces R_p and $\pi_0(R_p)$ are listed in Table V. Note that for space $S\times\mathbb{Z}$, we have $\pi_0(S\times Z) = \pi_0(S)\times\mathbb{Z}$. Also $O(n)$ in dividend usually leads to the $\mathbb{Z}_2$ in π_0. Otherwise $\pi_0=\{0\}$. $O(n)$ in dividend can give rise to $\mathbb{Z}_2$ because $O\in O(n)$ with $\det(O)=1$ and $\det(O)=-1$ cannot be smoothly connected. $O(l+m)$ in $\frac{O(l+m)}{O(l)\times O(m)}$ does not lead to $\mathbb{Z}_2$ because for $O\in O(l+m)$, we can change the sign of $\det(O)$ by multiplying O with an element in $O(l)$ [or $O(m)$].

For free fermion systems in 0-dimension with no symmetry and no fermion number conservation, the classifying space is R^0_0. Since $\pi_0(R^0_0)=\pi_0(R_2)=\mathbb{Z}_2$, such free fermion systems has two possible gapped phases. One phase has even numbers of fermions in the ground state and the other phase has odd numbers of fermions in the ground state. (Note that the fermion number mod 2 is still conserved even without any symmetry.)

For free electron systems in 0-dimension with time reversal symmetry and electron number conservation (the symmetry group $G^-_-(U,T)$), the classifying space is R^2_0. Since $\pi_0(R^2_0)=\pi_0(R_4)=\mathbb{Z}$, the possible gapped phases are labeled by an integer n. The ground state has $2n$ fermions. The electron number in the ground state is always even due to the Kramer degeneracy.

If we drop the electron number conservation (the symmetry group becomes $G_-(T)$), then the ground state will have uncertain but even numbers of electrons. The ground state cannot have odd numbers of electrons. This implies that free electron systems with only time reversal symmetry in 0-dimension has only one possible gapped phase. This agrees with $\pi_0(R^1_0)=\pi_0(R_3)=\{0\}$, where R^1_0 is the classifying space for symmetry group $G_-(T)$.

D. The $d\neq 0$ cases

Now let us consider the $d\neq 0$ cases. Again, let us first assume that the fermion system described by $\hat H = \frac{\mathrm{i}}{4}\sum_{IJ}A_{IJ}\hat\eta_I\hat\eta_J$ has the translation symmetry, as well as the time reversal symmetry and fermion number conservation. We also assume that the single-body energy bands of antisymmetric hermitian matrix $\mathrm{i}A$ have some Dirac points at zero energy and there are no other zero energy states in the Brillouin zone. The gapless single-body excitations in the system are described by the continuum limit of $\mathrm{i}A$:

$$\mathrm{i}A = \mathrm{i}\sum_{i=1}^d \gamma_i\partial_i, \tag{73}$$

where we have folded all the Dirac points to the $\boldsymbol{k}=0$ point. Without losing generality, we have also assumed that all the Dirac points have the same velocity. Since ∂_i is real and antisymmetric, thus γ_i, $i=1,...,d$ are real symmetric γ-matrices (of infinity dimension) that satisfy:

$$\gamma_i\gamma_j+\gamma_j\gamma_i = 2\delta_{ij}, \qquad \gamma_i^* = \gamma_i. \tag{74}$$

Again, the allowed γ_i always satisfy the condition that $M^2 = -1$ and $\gamma_i M + M\gamma_i = 0$ has a solution for M. Since the time reversal and the $U(1)$ transformations do not affect ∂_i, therefore the symmetry conditions on A, $AT + TA = 0$ and $AQ - QA = 0$, become the symmetry conditions on the γ-matrices:

$$\gamma_i T + T\gamma_i = 0, \quad \gamma_i Q - Q\gamma_i = 0. \tag{75}$$

Now we add perturbations that may break the translation, time reversal and the $U(1)$ symmetries, and ask: how many different ways are there to gap the Dirac points. The Dirac points can be fully gapped by real antisymmetric mass matrices M that satisfy

$$\gamma_i M + M\gamma_i = 0, \tag{76}$$

The resulting single-body Hamiltonian becomes $\mathrm{i}A = \mathrm{i}\sum_{i=1}^{d}[\gamma_i\partial_i + M]$.

If there is no symmetry, we only require the real antisymmetric mass matrix M to be invertible (in addition to eqn. (76)). Without losing generality, we can choose the mass matrix to also satisfy

$$M^2 = -1. \tag{77}$$

The space of those mass matrices is given by R_d^0.

If there are some symmetries, the real antisymmetric mass matrix M also satisfy some additional condition, as discussed before: M anticommutes with a set of $p+q$ matrices ρ_i that anticommute among themselves with p of them square to 1 and q of them square to -1. The number of p, q depend on full symmetry group G_f. Since γ_i do not break the symmetry, so, just like M, γ_i also anticommute ρ_i. So, in total, M anticommutes with a set of $p+q+d$ matrices ρ_i and γ_i that anticommute among themselves with $p+d$ of them square to 1 and q of them square to -1. Those mass matrices from a space R_{p+d}^q.

The different disconnected components of R_{p+d}^q represent different "bulk" gapped phases of the free fermions. Thus, the "bulk" gapped phases of the free fermions in d-dimensions are classified by $\pi_0(R_{p+d}^q) = \pi_0(R_{q-p-d+2 \bmod 8})$, with (p,q) depending on the symmetry. The results are summarized in Table III.

E. A general discussion

Now, let us give a general discussion of the classifying problem of free fermion systems. To classify the gapped phases of free fermion Hamiltonian we need to construct the space of antisymmetric mass matrix M that satisfy

$$M^2 = -1. \tag{78}$$

The mass matrices A always anti commute with γ-matrices γ_i, $i = 1, 2, ..., d$. When the mass matrices M has some symmetries, then the mass matrices satisfy more linear conditions. Let us assume that all those conditions can be expressed in the following form

$$M\rho_i = -\rho_i M, \quad \rho_i\rho_j = -\rho_j\rho_i,$$
$$MU_I = U_I M, \quad U_I\rho_i = \rho_i U_I, \tag{79}$$

where ρ_i and U_I are real matrices labeled by i and I, and $\gamma_1, ..., \gamma_d$ are included in ρ_i's. If we have another symmetry condition W such that $MW = -WM$ and $W\rho_{i_0} = \rho_{i_0}W$ for a particular i_0, Then $U = W\rho_{i_0}$ will commute with M and ρ_i, and will be part of U_I.

U_I will form some algebra. Let us use α to label the irreducible representations of the algebra. Then the one fermion Hilbert space has a form $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha \otimes \mathcal{H}_\alpha^0$, where the space $\mathcal{H}_\alpha^0$ forms the α^{th} irreducible representations of the algebra. For such a decomposition of the Hilbert space, M has the following block diagonal form

$$M = \oplus_\alpha (M^\alpha \otimes I^\alpha), \tag{80}$$

where I^α acts within $\mathcal{H}_\alpha^0$ as an identity operator, and M^α acts within $\mathcal{H}_\alpha$. ρ_i's have a similar form

$$\rho_i = \oplus_\alpha (\rho_i^\alpha \otimes I_\alpha), \tag{81}$$

where ρ_i^α act within $\mathcal{H}_\alpha$. So within the Hilbert space $\mathcal{H}_\alpha$, we have

$$M^\alpha \rho_i^\alpha = -\rho_i^\alpha M^\alpha, \quad \rho_i^\alpha \rho_j^\alpha = -\rho_j^\alpha \rho_i^\alpha. \tag{82}$$

What we are trying to do in this paper is actually to construct the space of M^α matrices that satisfy the condition eqn. (82).

If fermions only form one irreducible representation of the U_I algebra, then the classifying space of M_α and M will be the same. The results of this paper (such as Tables II, III) are obtained under such an assumption.

If fermions form n distinct irreducible representations of the U_I algebra, then the classifying space of M will be R^n, where R is the classifying of M^α constructed in this paper. Note that the classifying spaces of M^α are the same for different irreducible representations and hence R is independent of α. So if M_α's are classified by $\mathbb{Z}_k$, $k = 1, 2$, or ∞, then M's are classified by $\mathbb{Z}_k^n$.

To illustrate the above result, let us use the symmetry $G_+(C) = Z_2^C \times Z_2^f$ as an example. If the fermions form one irreducible representation of Z_2^C, for example, $\hat{C}c_i\hat{C}^{-1} = -c_i$, then the non-interacting symmetric gapped phases are classified by

$$\begin{array}{rcccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{gapped phases}: & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 \end{array} \tag{83}$$

which is the result in Table III. If the fermions form both the irreducible representations of Z_2^C, $\hat{C}c_{i+}\hat{C}^{-1} = +c_{i+}$ and $\hat{C}c_{i-}\hat{C}^{-1} = -c_{i-}$ (*ie* one type of fermions carries Z_2^C-charge 0 and another type of fermions carries Z_2^C-charge 1), then the non-interacting symmetric gapped phases are classified by

$$\begin{array}{rcccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{gapped phases}: & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 & \mathbb{Z}^2 & 0 & 0 & 0 & \mathbb{Z}^2 & 0 \end{array} \quad (84)$$

The four $d = 0$ phases correspond to the ground state with even or odd Z_2-charge-0 fermions and even or odd Z_2-charge-1 fermions. The four $d = 1$ phases correspond to the phases where the Z_2-charge-0 fermions are in the trivial or non-trivial phases of Majorana chain and the Z_2-charge-1 fermions are in the trivial or non-trivial phases of Majorana chain. The $d = 2$ phase labeled by two integers $(m, n) \in \mathbb{Z}^2$ corresponds to the phase where the Z_2-charge-0 fermions have m right moving Majorana chiral modes and the Z_2-charge-1 fermions have n right moving Majorana chiral modes. (if m and/or n are negative, we then have the corresponding number of left moving Majorana chiral modes.)

Some of the above gapped phases have intrinsic fermionic topological orders. So only a subset of them are non-interacting fermionic SPT phases:

$$\begin{array}{rcccccccc} d_{sp}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{SPT phases}: & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 \end{array} \quad (85)$$

The two $d_{sp} = 0$ phases correspond to the ground states with even numbers of fermions and 0 or 1 Z_2-charges. The two $d_{sp} = 1$ phases correspond to the phases where the Z_2-charge-0 fermions and the Z_2-charge-1 fermions are both in the trivial or non-trivial phases of Majorana chain. The $d_{sp} = 2$ phase labeled by one integers $n \in \mathbb{Z}$ corresponds to the phase where the Z_2-charge-0 fermions have n right moving Majorana chiral modes and the Z_2-charge-1 fermions have n left moving Majorana chiral modes.

IV. CLASSIFICATION WITH TRANSLATION SYMMETRY

For the free fermion systems with certain internal symmetry G_f, we have shown that their gapped phases are classified by $\pi_0(R_{p_G-d})$ or $\pi_0(C_{p_G-d})$ in d dimensions, where the value of p_G is determined from the full symmetry group G_f. We know that π_0 is an Abelian group with commuting group multiplication "+": $a, b \in \pi_0$ implies that $a + b \in \pi_0$. The "+" operation has a physical meaning. If two "bulk" gapped fermion systems are labeled by a and b in π_0, then stacking the two systems on together will give us a new "bulk" gapped fermion system labeled by $a + b \in \pi_0$.

We note that the classification by $\pi_0(R_{p_G-d})$ or $\pi_0(C_{p_G-d})$ is obtained by assume there is no translation symmetry. In the presence of translation symmetry, the gapped phases are classified differently.[32–34,43] However, the new classification can be obtained from $\pi_0(R_{p_G-d})$. For the free fermion systems with internal symmetry G_f and translation symmetry, their gapped phases are classified by[25]

$$\prod_{k=0}^{d} [\pi_0(R_{p_G-d+k})]^{\binom{d}{k}}, \quad (86)$$

where $\binom{d}{k}$ is the binomial coefficient. The above is for the real classes. For the complex classes, we have a similar classification:

$$\prod_{k=0}^{d} [\pi_0(C_{p_G-d+k})]^{\binom{d}{k}}. \quad (87)$$

Such a result is obtained by stacking the lower dimensional topological phases to obtain higher dimensional ones. For 1-dimensional free fermion systems with internal symmetry G_f, their gapped phases are classified by $\pi_0(R_{p_G-1})$. We can also have a 0-dimensional gapped phase on each unit cell of the 1-dimensional system if there is a translation symmetry. The 0-dimensional gapped phases are classified by $\pi_0(R_{p_G})$. Thus the combined gapped phases (with translation symmetry) are classified by $\pi_0(R_{p_G-1}) \times \pi_0(R_{p_G})$. In 2-dimensions, the gapped phases are classified by $\pi_0(R_{p_G-2})$. The gapped phases on each unit cell are classified by $\pi_0(R_{p_G})$. Now we can also have 1-dimensional gapped phases on the lines in x-direction, which are classified by $\pi_0(R_{p_G-1})$. The same thing for the lines in y-direction. So the combined gapped phases (with translation symmetry) are classified by $\pi_0(R_{p_G-2}) \times [\pi_0(R_{p_G-1})]^2 \times \pi_0(R_{p_G})$. In three dimensions, the translation symmetric gapped phases are classified by $\pi_0(R_{p_G-3}) \times [\pi_0(R_{p_G-2})]^3 \times [\pi_0(R_{p_G-1})]^3 \times \pi_0(R_{p_G})$.

V. DEFECTS IN d-DIMENSIONAL GAPPED FREE FERMION PHASES WITH SYMMETRY G_f

For the d-dimensional free fermion systems with internal symmetry G_f, we have shown that their "bulk" gapped Hamiltonians (or the mass matrices) form a space R_{p_G-d} or C_{p_G-d}. (More precisely, the gapped Hamiltonians or the mass matrices form a space that is homotopically equivalent to R_{p_G-d} or C_{p_G-d}). From the space R_{p_G-d} or C_{p_G-d}, we find that the point defects that have the symmetry G_f are classified by $\pi_{d-1}(R_{p_G-d})$ or $\pi_{d-1}(C_{p_G-d})$.

Physically, there is another way to classify point defects: we can simply add a segment of 1D "bulk" gapped free fermion hopping system with the same symmetry to the d-dimensional system. Since the translation symmetry is not required, the new d-dimensional system still belong to the same symmetry class. There are finite bulk gap away from the two ends of the added 1D segment. So the new d-dimensional system may contain two non-trivial defects. The classes of defect is classified by the classes of the added 1D "bulk" gapped free fermion hopping system. So we find that the point defects that

Symmetry	C_{p_G}	p_G	point defect	line defect	example phases
$U(1)$ $G_-(C)$	$\frac{U(l+m)}{U(l)\times U(m)}\times \mathbb{Z}$	0	0	$\mathbb{Z}$	(Chern) insulator; supercond. with collinear spin order
$G^+_\pm(U,T)$ $G^+_{--}(T,C)$ $G^+_{+-}(T,C)$	$U(n)$	1	$\mathbb{Z}$	0	supercond. w/ real pairing and S_z conserving spin-orbital coupling

TABLE VI: (Color online) Classification of point defects and line defects that have some symmetries in gapped phases of non-interacting fermions. "0" means that there is no non-trivial topological defects. $\mathbb{Z}$ means that topological non-trivial defects plus the topological trivial defect are labeled by the elements in $\mathbb{Z}$. Non-trivial topological defects have protected gapless excitations, while trivial topological defects have no protected gapless excitations.

Symm.	$G^-_+(U,T)$ $G^-_{+-}(T,C)$	$G_+(T)$ $G^+_{++}(T,C)$ $G^{--}_{++}(U,T,C)$ $G^{-+}_{++}(U,T,C)$ $G^{+-}_{-+}(U,T,C)$ $G^{++}_{++}(U,T,C)$	"none" $G_+(C)$ $G^-_{++}(T,C)$ $G^-_{-+}(T,C)$ $G_+(C)$ $G_+(U,C)$	$G_-(T)$ $G^+_{-+}(T,C)$ $G^{--}_{-+}(U,T,C)$ $G^{-+}_{-+}(U,T,C)$ $G^{+-}_{++}(U,T,C)$ $G^{++}_{-+}(U,T,C)$	$G^-_-(U,T)$ $G^-_{--}(T,C)$	$G^{--}_{--}(U,T,C)$ $G^{-+}_{--}(U,T,C)$ $G^{+-}_{--}(U,T,C)$ $G^{++}_{+-}(U,T,C)$	$G_-(U,C)$ $SU(2)$	$G^{--}_{+-}(U,T,C)$ $G^{-+}_{+-}(U,T,C)$ $G^{+-}_{+-}(U,T,C)$ $G^{++}_{--}(U,T,C)$ $G[SU(2),T]$
R_{p_G}	$\frac{O(l+m)}{O(l)\times O(m)}\times \mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times \mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
p_G:	0	1	2	3	4	5	6	7
point defect	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
line defect	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
example phases	insulator w/ coplanar spin order	supercond. w/ coplanar spin order	supercond.	supercond. w/ time reversal	insulator w/ time reversal	insulator w/ time reversal and inter-sublattice hopping	spin singlet supercond.	spin singlet supercond. w/ time reversal

TABLE VII: (Color online) Classification of point defects and line defects that have some symmetries in gapped phases of non-interacting fermions. "0" means that there is no non-trivial topological defects. $\mathbb{Z}_n$ means that topological non-trivial defects plus the topological trivial defect are labeled by the elements in $\mathbb{Z}_n$. Non-trivial defects have protected gapless excitations in them.

have the symmetry G_f are also classified by $\pi_0(R_{p_G-1})$ or $\pi_0(C_{p_G-1})$.

Similarly, the line defects that have the symmetry G_f are classified by $\pi_{d-2}(R_{p_G-d})$ or $\pi_{d-2}(C_{p_G-d})$. Again, we can also create line defects by adding a disk of 2D "bulk" gapped free fermion hopping system to the original d-dimensional system. This way, we find that the line defects that have the symmetry G_f are also classified by $\pi_0(R_{p_G-2})$ or $\pi_0(C_{p_G-2})$.

In general, the defects with dimension d_0 are classified by $\pi_{d-d_0-1}(R_{p_G-d})$ or $\pi_{d-d_0-1}(C_{p_G-d})$, or equivalently by $\pi_0(R_{p_G-d_0-1})$ or $\pi_0(C_{p_G-d_0-1})$. In order for the above physical picture to be consistent, we require that

$$\pi_n(R_{p_G}) = \pi_0(R_{p_G+n}), \qquad \pi_n(C_{p_G}) = \pi_0(C_{p_G+n}). \tag{88}$$

The classifying spaces indeed satisfy the above highly non-trivial relation. This is the Bott periodicity theorem. The theorem is obtained by the following observation: the space C_{p+1} can be viewed (in a homotopic sense) as ΩC_p – the space of loops in C_p. So we have $\pi_1(C_p) = \pi_0(\Omega C_p) = \pi_0(C_{p+1})$. Similarly, the space R_{p+1} can be viewed (in a homotopic sense) as ΩR_p – the space of loops in R_p. So we have $\pi_1(R_p) = \pi_0(\Omega R_p) = \pi_0(R_{p+1})$. As a result of Bott periodicity theorem, the classification of defects is independent of spatial dimensions. It only depends on the dimension and the symmetry of the defects. If the defects lower the symmetry, then, we should use the reduced symmetry to classify the defects. In Tables VI and VII, we list the classifications of those symmetric point and line defects for gapped free fermion systems with various symmetries. We would like to point out that the line defects classified by $\mathbb{Z}$ in superconductors without symmetry *do not* correspond to the vortex lines (which usually belong to the trivial class under our classification). The non-trivial line defect here should carry chiral modes that only move in one direction along the defect line. In general, non-trivial defects have protected gapless excitations in them.

VI. SUMMERY

In this paper, we study different possible full symmetry groups G_f of fermion systems that contain $U(1)$, time reversal T, and/or charge conjugation C symmetry. We show that each symmetry group G_f is associated with a classifying space C_{p_G} or R_{p_G} (see Tables II and III). We classify d-dimensional gapped phases of free fermion systems that have those full symmetry groups. We find that the different gapped phases are described by $\pi_0(C_{p_G-d})$ or $\pi_0(R_{p_G-d})$. Those results, obtained using the K-theory approach, generalize the results in Ref. 25,28

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