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# Equilibration of a spinless Luttinger liquid

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We study how a Luttinger liquid of spinless particles in one dimension approaches thermal equilibrium. Full equilibration requires processes of backscattering of excitations which occur at energies of order of the bandwidth. Such processes are not accounted for by the Luttinger liquid theory. We treat the high-energy excitations as mobile impurities and derive an expression for the equilibration rate in terms of their spectrum. Our results apply at any interaction strength.

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The concept of Luttinger liquid was proposed by Haldane as an effective low-energy description of one-dimensional systems of interacting fermions<sup>1</sup> or bosons<sup>2</sup>. The main feature of this theory is that regardless of the statistics of the particles, the low-energy excitations of the system are bosons. The latter propagate at a fixed velocity  $v$  in either left or right direction and have the meaning of the waves of particle density, analogous to phonons in solids.

In its simplest form the Luttinger liquid is described by a Hamiltonian quadratic in boson variables, resulting in excitations with infinite life time. Once excited, such a system will never reach thermal equilibrium. Absence of equilibration is the physical reason<sup>3</sup> for perfect quantization of conductance of a quantum wire connected to ideal leads, when the electronic system in the wire is treated as a Luttinger liquid<sup>4-6</sup>. Of course, real systems do equilibrate, possibly explaining the experimentally observed corrections to quantized conductance<sup>7-9</sup>. Equilibration of one-dimensional boson systems was recently studied in atomic traps<sup>10,11</sup>.

A finite life time of excitations in the Luttinger liquid can be understood if small anharmonic corrections are added to the Hamiltonian. Such perturbations are irrelevant in the sense that their effect rapidly decreases as the temperature approaches zero. However, they are responsible for the interaction of bosonic excitations and therefore for their equilibration. Scattering of the excitations caused by the anharmonic coupling terms preserves not only their total energy but also momentum. Thus the resulting equilibrium distribution of the bosonic excitations

$$N_q = \frac{1}{e^{\hbar(v|q|-uq)/T} - 1} \quad (1)$$

is controlled by two parameters, temperature  $T$  and velocity  $u$ . Here  $q$  is the wave vector of the excitation.

It is important to note that translation invariance of the problem ensures conservation of the total momentum of the system, rather than that of its elementary excitations. This subtle distinction can be understood

by considering the expression

$$P = \frac{\pi \hbar N}{L} J + \sum_q \hbar q b_q^\dagger b_q \quad (2)$$

for the momentum of a Luttinger liquid<sup>1,2</sup>. Here  $b_q$  is the boson annihilation operator,  $N$  is the total number of particles,  $L$  is the system size. Periodic boundary conditions require that  $J$  be an even number if the underlying physical particles are bosons, while for fermions  $J + N$  must be odd. The first term in Eq. (2) accounts for the momentum associated with the motion of the system as a whole, which is possible even in the absence of excitations.

Unless additional conservation laws are present, one should expect the existence of scattering processes which transfer momentum between the excitations and the system as a whole. The minimum momentum transfer  $\Delta p = 2\pi \hbar N/L$  corresponds to  $J$  changing by 2. Because the typical momentum of an excitation  $\hbar q \sim T/v$  is small at  $T \rightarrow 0$ , such processes involve a large number of excitations. They are not included in the standard Luttinger liquid theory. Although their rate is small, these processes are required for the full equilibration of the Luttinger liquid. Physically one expects them to lead to relaxation of the velocity  $u$  of the gas of excitations in Eq. (1) towards an equilibrium value  $v_d$ ,

$$\dot{u} = -\frac{u - v_d}{\tau}. \quad (3)$$

We limit our consideration to Galilean invariant systems of particles, whose mass is denoted by  $m$ . In this case the system must be at rest in a reference frame moving with the center of mass, and  $v_d = P/mN$ .

Equilibration of the electron liquid in a quantum wire gives rise to a correction to the quantized conductance<sup>3,12-14</sup>, which can be expressed in terms of  $v$  and the equilibration time  $\tau$ . The latter has been studied in the limits of weakly and strongly interacting electrons<sup>13,14</sup>. The goal of this paper is to develop a theory of equilibration of spinless Luttinger liquids valid for any interaction strength. Although we are primarily interested in one-dimensional electrons, our result for  $\tau$  applies to both fermionic and bosonic Luttinger liquids.

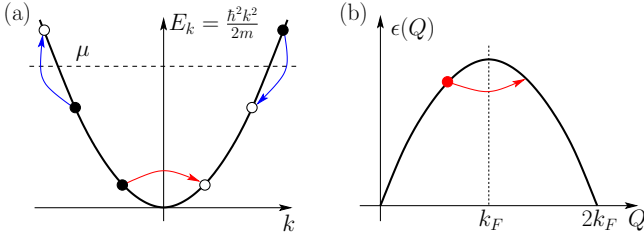


FIG. 1: (a) The simplest backscattering process in a weakly interacting Fermi gas involves three particles, including one near the bottom of the band. (b) Backscattering of a fermion at the bottom of the band can be interpreted as a hole excitation overcoming a barrier at  $Q = k_F$ .

We start by reviewing the simplest case of a system with Luttinger liquid behavior at low energies, namely, the weakly interacting Fermi gas. In fermionic Luttinger liquids the integer  $J$  can be interpreted<sup>1</sup> as the difference of the numbers of right- and left-moving fermions,  $J = N^R - N^L$ . Clearly, the scattering process changing  $J$  by 2 involves backscattering of a fermion,  $\Delta N^R = -\Delta N^L = \pm 1$ . Because of conservation of energy and momentum, two-particle scattering in one dimension results only in particles exchanging their momenta, and the distribution function remains unchanged. Thus the simplest scattering process involves three particles, see Fig. 1(a). Simultaneous conservation of momentum and energy requires involvement of hole states below the Fermi level. At  $T \rightarrow 0$  the most efficient process involves a hole near the bottom of the band, whose scattering is accompanied by creation and collapse of particle-hole pairs with energies of order  $T$  near the two Fermi points<sup>15</sup>. Since the occupation probability of a hole state near  $k = 0$  is exponentially small, one finds a small equilibration rate  $\tau^{-1} \propto e^{-E_F/T}$ , where  $E_F = \hbar^2 k_F^2 / 2m$  is the Fermi energy<sup>13</sup>.

It is instructive to rephrase the above argument in the language of a hole excitation with wave vector  $Q$  and energy  $\epsilon(Q) = \hbar v_F Q (1 - Q/2k_F)$  constructed by moving a fermion from state  $k_F - Q$  to the Fermi level state  $k_F$ . (Here  $v_F$  is the Fermi velocity.) The hole is scattered off of particles near the Fermi level, with its momentum changing in steps of  $\Delta Q \sim T/v_F$ . Backscattering occurs when such a hole crosses the point  $Q = k_F$ , Fig. 1(b).

This picture can now be generalized to the case of arbitrary interaction strength. The particle hole pairs with momenta  $\hbar q \sim T/v$  near the two Fermi points transform into the bosonic excitations in the Luttinger liquid<sup>1</sup>. On the other hand, the hole with the large wave vector  $Q \sim k_F$  is not accounted for by the Luttinger liquid theory and should be treated as a mobile impurity<sup>16–19</sup>. In the presence of interactions its energy  $\epsilon(Q)$  is defined as that of the lowest energy state of momentum  $\hbar Q$ , measured from the ground state. Throughout this paper we assume that  $\epsilon(Q)$  remains concave. Then the equilibration rate shows activated temperature dependence  $\tau^{-1} \propto e^{-\epsilon(k_F)/T}$ , where  $k_F = \pi n_0$  is determined by the

average particle density  $n_0 = N/L$ .

To obtain a full expression for the equilibration rate, the distribution function of the holes should be considered carefully. To first approximation it can be obtained by noticing that the holes are scattered by the bosonic excitations, distributed according to Eq. (1). These scattering events involve exchange of both energy and momentum between the hole and the bosons, leading to the equilibrium distribution

$$f(Q) \simeq \begin{cases} e^{-\epsilon_u(Q)/T}, & Q < k_F, \\ e^{-[\epsilon_u(Q) + 2\hbar k_F u]/T}, & Q > k_F, \end{cases} \quad (4)$$

where  $\epsilon_u(Q) = \epsilon(Q) - \hbar u Q$ . The apparent asymmetry between the cases of right- and left-moving holes,  $Q < k_F$  and  $Q > k_F$ , is caused by our convention to measure the momentum  $Q$  of the hole from the right Fermi point,  $k = +k_F$ .

The discontinuity of the hole distribution function (4) at  $Q = k_F$  originates from the implicit assumption that the right- and left-moving holes are distinct particles. In reality, the backscattering processes shown in Fig. 1 convert right-moving holes into left-moving ones, thereby smearing the discontinuity of the distribution function  $f(Q)$ . Because the hole moves in momentum space via random small steps of  $\Delta Q \sim T/\hbar v$ , this motion is diffusive. Such diffusion was considered previously for the cases of weakly-interacting<sup>13</sup> and strongly-interacting<sup>14</sup> electrons. It is described by the Fokker-Planck equation

$$\partial_t f = -\partial_Q J, \quad J = -\frac{B(Q)}{2} \left[ \frac{\epsilon'_u(Q)}{T} + \partial_Q \right] f, \quad (5)$$

where the expression for the probability current  $J$  assumes that the system as a whole is at rest,  $v_d = 0$ , and prime denotes the derivative with respect to  $Q$ . The diffusion constant in momentum space

$$B(Q) = \sum_{\delta Q} [\delta Q]^2 W_{Q, Q+\delta Q} \quad (6)$$

is defined in terms of the rate  $W_{Q, Q+\delta Q}$  of scattering events changing the wave vector of the hole from  $Q$  to  $Q + \delta Q$ .

We now find a stationary solution of the Fokker-Planck equation with the boundary conditions (4), which gives a uniform in  $Q$ -space probability current

$$J = u B(k_F) \frac{\hbar k_F}{T} \left( \frac{|\epsilon''(k_F)|}{2\pi T} \right)^{1/2} e^{-\epsilon(k_F)/T}. \quad (7)$$

Here to obtain the expression for  $J$  to leading order in  $u$  we neglected the difference between  $\epsilon(Q)$  and  $\epsilon_u(Q)$ .

A non-zero probability current  $J$  means that the holes backscatter at a rate  $JL/2\pi$ . With each backscattering event transferring momentum  $\Delta p = 2\hbar k_F$  from excitations to the motion of the system as a whole, we find  $\dot{P}_{\text{ex}} = -JL\hbar k_F/\pi$ . Comparing this result with the expression  $P_{\text{ex}} = (\pi L T^2 / 3\hbar v^3) u$  for the total momentum

of the excitations obtained using the distribution (1), we find the relaxation law  $\dot{u} = -u/\tau$  with the rate

$$\tau^{-1} = \frac{3\hbar k_F^2 B}{\pi^2 \sqrt{2\pi m^* T}} \left(\frac{\hbar v}{T}\right)^3 e^{-\Delta/T}. \quad (8)$$

Here  $\Delta = \epsilon(p_F)$ , the effective mass of the hole  $m^* = -\hbar^2/\epsilon''(k_F)$ , and the diffusion constant  $B = B(k_F)$  remains to be determined.

Following Refs.<sup>16–19</sup>, we treat the hole in a Luttinger liquid as a mobile impurity. The Fokker-Planck equation for such an impurity was discussed in Ref.<sup>20</sup>. The parameter  $B$  was found to scale as

$$B = \chi T^5 \quad (9)$$

at  $T \rightarrow 0$ . The approach of Ref.<sup>20</sup> does not allow for the determination of the coefficient  $\chi$ . The latter is controlled by the interactions between the physical particles forming the Luttinger liquid. In the limit of strong Coulomb repulsion it was calculated in Ref.<sup>14</sup>. A related calculation was performed in the context of decay of dark solitons in weakly-interacting one-dimensional Bose systems<sup>21</sup>.

Our next goal is to obtain an exact expression for the coefficient  $\chi$  in Eq. (9) for arbitrary interactions between the particles forming the Luttinger liquid. Microscopically the case of arbitrary interaction strength can be approached only for integrable systems, where an infinite number of conservation laws allows one to diagonalize the Hamiltonian exactly. However, the same conservation laws ensure that the excitations have infinite life times and  $B = 0$ . We thus develop a phenomenological theory and express  $B$  in terms of hole spectrum  $\epsilon(Q)$ .

We describe the system in terms of the displacement  $u(y)$  of a small element of the liquid from its reference position  $y$  in a state of uniform particle density  $n_0$ , and the conjugate momentum density  $p(y)$  such that  $[u(y), p(y')] = i\hbar\delta(y - y')$ . In the absence of the hole excitations the Hamiltonian of the liquid can be written as

$$H_L = \int \left[ \frac{p^2}{2mn_0} + n_0 U(n) \right] dy, \quad (10)$$

where  $U(n)$  is the internal energy per particle, determined by the fluctuating density  $n(y) = n_0/[1 + u'(y)]$ . Expanding (10) up to the third order in small deformation  $u'$  one finds

$$H_L = \int \left( \frac{p^2}{2mn_0} + \frac{mn_0 v^2}{2} u'^2 - \alpha u'^3 \right) dy. \quad (11)$$

Here the sound velocity  $v = [(2n_0 U' + n_0^2 U'')/m]^{1/2}$  and  $\alpha = n_0^2 U' + n_0^3 U'' + n_0^4 U'''/6$ . The quadratic part of Eq. (11) is the Hamiltonian of the Luttinger liquid, which can be brought to the form  $\sum \hbar v |q| b_q^\dagger b_q$  by introducing

the boson operators  $b_q$  via the standard procedure

$$u(y) = \sum_q \sqrt{\frac{\hbar}{2mn_0 L v |q|}} (b_q e^{iqy} + b_q^\dagger e^{-iqy}), \quad (12a)$$

$$p(y) = -i \sum_q \sqrt{\frac{\hbar mn_0 v |q|}{2L}} (b_q e^{iqy} - b_q^\dagger e^{-iqy}). \quad (12b)$$

The presence of a hole excitation at the point in the liquid with reference position  $Y$  is accounted for by adding a term  $H_h = \epsilon(Q) = \epsilon(-i\partial_Y)$  to the Hamiltonian (11). Since our goal is to evaluate  $B = B(k_F)$ , we assume that  $Q$  is near  $k_F = \pi n_0$  and use the expansion

$$H_h = \Delta(n(Y)) - \frac{\hbar^2}{2m^*} (-i\partial_Y - \pi n_0)^2. \quad (13)$$

It is worth mentioning that our Hamiltonian is written in terms of the Lagrangian variable  $u(y)$  defined as function of reference position  $y$ , rather than Eulerian variable  $n(x)$  at the physical position  $x = y + u(y)$ . The two approaches are, of course, equivalent and lead to the same results<sup>22</sup>. Although the use of Eulerian variables is more common in the Luttinger liquid theory, our method has the advantage of more simply accounting for the Galilean invariance of the problem. In addition, since  $Y$  is the position of the hole in the reference state of uniform density  $n_0$ , the maximum of  $\epsilon(Q)$  is located at  $Q = \pi n_0$ , regardless of the physical density  $n$ . On the other hand, the maximum value  $\Delta$  is a function of  $n = n_0/(1 + u')$ . This dependence gives rise to interaction of the hole with the Luttinger liquid. Expanding (13) in  $u'$ , we obtain

$$H_h = -\beta_1 u'(Y) + \beta_2 [u'(Y)]^2 - \frac{\hbar^2}{2m^*} (-i\partial_Y - \pi n_0)^2, \quad (14)$$

where  $\beta_1 = n_0 \Delta'$ ,  $\beta_2 = n_0 \Delta' + n_0^2 \Delta''/2$ , and we omitted the constant  $\Delta(n_0)$ . Note that unlike the calculation of the dynamic correlation functions<sup>16–19</sup>, evaluation of the diffusion constant requires expansion of  $H_h$  to second order in bosonic fields.

In order to find the diffusion constant in momentum space  $B(k_F)$ , Eq. (6), we evaluate the scattering rate  $W_{Q, Q+\delta Q}$ . The momentum of the hole changes as it interacts with the bosonic excitations, see Eqs. (14) and (12). The processes involving one boson cannot simultaneously conserve both energy and momentum of the system. The simplest allowed process for a hole near  $Q = \pi n_0$  involves absorption of a boson  $q_1$  and simultaneous emission of a boson  $q_2$  such that  $q_2 \approx -q_1$ <sup>14,20,21</sup>. The scattering rate is then found from the Fermi golden rule expression

$$W_{Q, Q+\delta Q} = \frac{2\pi}{\hbar} \sum_{q_1, q_2} |t_{q_1, q_2}|^2 N_{q_1} (N_{q_2} + 1) \delta_{q_1 - q_2, \delta Q} \times \delta(\epsilon(Q) - \epsilon(Q + \delta Q) + \hbar v |q_1| - \hbar v |q_2|).$$

The matrix element  $t_{q_1, q_2}$  accounts for all processes that destroy boson  $q_1$  and create boson  $q_2$ . For example, a

contribution proportional to  $\beta_2 b_{q_2}^\dagger b_{q_1}$  is found in the second term in Eq. (14). Identical scattering processes can be obtained in the second-order perturbation theory with amplitudes proportional to  $\beta_1^2$  or  $\alpha\beta_1$ . The calculation is simplified considerably by applying to the Hamiltonian the unitary transformation  $U^\dagger(H_L + H_h)U$  with

$$U = \exp\left(\frac{i\beta_1}{\hbar m n_0 v^2} \int_{-\infty}^Y p(y) dy\right). \quad (15)$$

This removes the  $-\beta_1 u'(Y)$  term in (14) and generates a correction to  $\beta_2$  proportional to  $\alpha\beta_1$ . In addition, a new term  $p^2(Y)$  is generated with the coefficient proportional to  $\beta_1^2$ . Both the  $[u'(Y)]^2$  and  $p^2(Y)$  terms contain contributions of the form  $b_{q_2}^\dagger b_{q_1}$  and give rise to the matrix element

$$t_{q_1, q_2} = -\frac{\hbar\sqrt{|q_1 q_2|}}{m n_0 L v} \left( \beta_2 - \frac{3\alpha\beta_1}{m n_0 v^2} + \frac{\beta_1^2}{2m^* v^2} \right). \quad (16)$$

In addition to the amplitude  $\beta_2$  describing the coupling of the hole to two bosons there are two additional contributions in parentheses. The second term accounts for the possibility of the hole coupling to a single boson which then decays into two bosons. Such processes are possible due to the non-linearity of the excitation spectrum near the Fermi level which gives rise to the cubic term  $-\alpha u'^3$  in Eq. (11). The last term accounts for coupling to one boson in second order, and its magnitude is controlled by the curvature  $1/m^*$  of the hole spectrum near the maximum.

Using the expression (16) we recover the temperature dependence (9) with the coefficient  $\chi$  given by

$$\chi = \frac{4\pi^3 n_0^2}{15\hbar^5 m^2 v^8} \left( \Delta'' - \frac{2v'}{v} \Delta' + \frac{\Delta'^2}{m^* v^2} \right)^2, \quad (17)$$

where prime denotes the derivative with respect to the particle density  $n_0$ . The above result completes our evaluation of the relaxation rate of a Luttinger liquid, given by Eqs. (8), (9), and (17). The rate has activated temperature dependence with both the activation temperature  $\Delta$  and the prefactor determined by the spectrum of holes  $\epsilon(Q)$ .

Although our result is applicable at any interaction strength, the spectrum  $\epsilon(Q)$  is known analytically only in a few special cases. For non-interacting spinless fermions  $\Delta$  is given by the Fermi energy  $(\pi\hbar n_0)^2/2m$ ,  $v$  is the Fermi velocity  $\pi\hbar n_0/m$ , and  $m^* = m$ . This results in  $\chi = 0$ , as there is no scattering of holes in the absence of interactions. In the limit of weak interactions, the spectrum  $\epsilon(Q)$  should be evaluated up to second order in interaction strength. This gives rise to a result<sup>22</sup> for  $\chi$  consistent with the rather complicated expression for the three-particle scattering amplitude<sup>15</sup> that controls the scattering of holes, Fig. 1(a). In the limit of strong long-range repulsion, the system forms a Wigner crystal, and the hole spectrum coincides with that of phonons in the crystal. Assuming the interactions fall off faster than

$1/|x|$  at large distances, the Luttinger liquid theory applies to the Wigner crystal. We have verified that in this regime our expression (17) recovers the results of Ref.<sup>14</sup>. We have also found that in the case of weakly interacting bosons Eq. (17) is consistent with the expression for the mobility of the so-called dark soliton<sup>21</sup>.

At arbitrary interaction strength the spectrum of holes is known only for integrable models. As we already mentioned, integrability means absence of scattering of excitations,  $B = 0$ . We have verified that our expression (17) vanishes for the Calogero-Sutherland model of particles with inverse-square repulsion<sup>23</sup>, and for the Lieb-Liniger model of bosons with point-like repulsion<sup>24,25</sup>.

Comparison of measured equilibration rates with the prediction (8) should be performed at temperatures somewhat lower than  $\Delta$ , so that the exponential  $e^{-\Delta/T}$  is small but the rate (8) is still measurable. It is worth mentioning that in quantum wires  $\Delta$  and the shape of the hole spectrum  $\epsilon(Q)$  can be measured by studying momentum resolved tunneling between two wires<sup>26</sup>.

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