Planar approximation for spin transfer systems with application to tilted polarizer devices

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Planar approximation for spin transfer systems with application to tilted polarizer devices

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Planar spin-transfer devices with dominating easy-plane anisotropy can be described by an effective one-dimensional equation for the in-plane angle. Such a description provides an intuitive qualitative understanding of the magnetic dynamics. We give a detailed derivation of the effective planar equation and use it to describe magnetic switching in devices with tilted polarizer.

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I. INTRODUCTION

Spin-transfer effect is a non-equilibrium interaction that arises when a current of electrons flows through a non-collinear magnetic texture[1–3]. Spin-transfer torque can lead to current induced magnetic switching in multilayer devices or domain wall motion in devices with continuous change of magnetization.[4–6] Both phenomena serve as an underlying mechanism for a number of suggested memory and logic applications.[7,8]

Magnetic dynamics in spin-transfer devices is described by the Landau-Lifshitz-Gilbert (LLG) equation with added current-induced torques.[2,6] Analytic solutions of LLG can be readily found in the simplest case of an easy axis magnetic anisotropy. However, when the form of anisotropy energy becomes more complicated, investigations of the static equilibria stability become much more involved.[9] A study of precession cycles is even more complicated[10] and often makes it necessary to resort to numeric simulations.[11,12] Due to the complexity of the LLG equation it is always interesting to consider cases where some simplifying approximations can be made.

In many devices the easy plane anisotropy energy is much larger than the other anisotropy energies, and the system is in the planar spintronic device regime[13] (Fig. 1). The limit of dominating easy plane energy is characterized by a simplification of the dynamic equations[14], which comes not from the high symmetry of the problem, but from the existence of a small parameter: the ratio of the energy modulation within the plane to the easy plane energy. Strong easy plane anisotropy forces the deviations of the magnetization from the plane to be small, making the motion effectively one dimensional. As a result, an effective description in terms of just one azimuthal angle becomes possible.

Our publications[15–19] extended the planar approximation to systems with spin-transfer torques and presented a number of results, highlighting the practical use of the method. In this paper we give a detailed derivation the effective planar equation for a macrospin free layer in the presence of spin transfer torques (Sec. III). We then show how this equation can be applied to a system with the tilted polarizer and obtain a qualitative picture of the device dynamics (Sec. IV).

II. MAGNETIC DYNAMICS OF THE FREE LAYER

We consider a conventional spin-transfer device consisting of a a magnetic polarizer (fixed layer) and a small magnet (free layer) with electric current flowing from one to another (Fig. 1). The free layer is influenced by spin-transfer torque, while the polarizer is too large to feel it. It is assumed that in the limit of large exchange stiffness the free layer can be described by a macrospin model, where its state is characterized by just one vector, the total magnetic moment $\mathbf{M} = M\mathbf{n}$ with a constant absolute value M and a direction given by a unit vector $\mathbf{n}(t)$. The LLG equation[2,9] reads:

$$\dot{\mathbf{n}} = \left[-\frac{\delta \varepsilon}{\delta \mathbf{n}} \times \mathbf{n} \right] + u(\mathbf{n})[\mathbf{n} \times [\mathbf{s} \times \mathbf{n}]] + \alpha[\mathbf{n} \times \dot{\mathbf{n}}] \ . \quad (1)$$

Here the re-scaled energy $\varepsilon = (\gamma/M)E$ has the dimensions of frequency and is expressed through the total magnetic energy $E(\mathbf{n})$ of the free layer; γ is the gyromagnetic ratio, and α is the Gilbert damping constant. The second term on the right hand side of the equation is the spin transfer torque. Unit vector $\mathbf{s}$ points along the polarizer direction. We do not impose any restrictions on the fixed layer (e.g., it is not required to have a strong

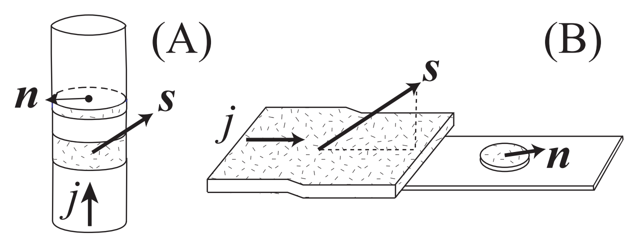

FIG. 1: Planar spin-transfer systems driven by current j. Hashed parts of the devices are ferromagnetic and clear parts are made from a non-magnetic metal. Spin polarizers have arbitrary magnetization directions $\mathbf{s}$, while the free layer magnetization $\mathbf{n}$ is directed in the easy plane.

planar anisotropy) and assume that the direction of $\mathbf{s}$ can be arbitrary (Fig. 1). The spin transfer strength $u(\mathbf{n})$ is proportional to the electric current I,[9] and in general is a function of the angle between the polarizer and the free layer $u(\mathbf{n}) = g[(\mathbf{n}\cdot\mathbf{s})]\, I$. The spin current efficiency factor $g[(\mathbf{n}\cdot\mathbf{s})]$ is a material and device specific function.[20]

In standard polar coordinates θ and ϕ (see Fig. 8 in Appendix A) equation (1) reads

$$\begin{aligned} \dot\theta + \alpha\dot\phi\sin\theta &= -\frac{1}{\sin\theta}\frac{\partial\varepsilon}{\partial\phi} + u(\mathbf{n})(\mathbf{s}\cdot\mathbf{e}_\theta)\ , \\ \dot\phi\sin\theta - \alpha\dot\theta &= \frac{\partial\varepsilon}{\partial\theta} + u(\mathbf{n})(\mathbf{s}\cdot\mathbf{e}_\phi)\ , \end{aligned} \tag{2}$$

where the unit tangent vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are also defined in Appendix A.

We choose the easy plane to be defined by $\theta = \pi/2$, and write the re-scaled magnetic energy in the form

$$\varepsilon = \frac{\omega_p}{2}\cos^2\theta + \varepsilon_r(\theta,\phi)\ , \tag{3}$$

where the first term represents the easy plane anisotropy, with the planar frequency ω_p related to the easy plane constant K_p as $\omega_p = \gamma K_p/M$. The remainder ε_r is the "residual" energy. The planar limit is characterized by $\omega_p \to \infty$. Large easy plane constant forces the energy minima to be very close to the easy plane and the low energy solutions of LLG to have the property $\theta(t) = \pi/2 + \delta\theta$ with $\delta\theta \to 0$. Equations (2) can then be expanded in small parameters

$$\frac{|\varepsilon_r|}{\omega_p} \ll 1 \quad \text{and} \quad \frac{|u(\mathbf{n})|}{\omega_p} \ll 1\ .$$

By truncating this expansion one obtains the effective planar approximation.

III. DERIVATION OF THE EFFECTIVE PLANAR EQUATION

Explicitly separating the large easy plane terms, we rewrite equations (2) as

$$\begin{aligned} \dot\theta + \alpha\dot\phi\sin\theta &= f_\theta + u_\theta \\ \dot\phi\sin\theta - \alpha\dot\theta &= -\omega_p\cos\theta\sin\theta + f_\phi + u_\phi \end{aligned}$$

where the residual energy is responsible for the terms

$$\begin{aligned} f_\theta(\theta,\phi) &= -\frac{1}{\sin\theta}\frac{\partial\varepsilon_r}{\partial\phi}\ , \\ f_\phi(\theta,\phi) &= \frac{\partial\varepsilon_r}{\partial\theta}\ , \end{aligned} \tag{4}$$

and the spin transfer torque produces the terms

$$\begin{aligned} u_\theta(\theta,\phi) &= u(\mathbf{n})(\mathbf{s}\cdot\mathbf{e}_\theta)\ , \\ u_\phi(\theta,\phi) &= u(\mathbf{n})(\mathbf{s}\cdot\mathbf{e}_\phi)\ . \end{aligned} \tag{5}$$

We also introduce a notation $F_{\theta,\phi} = f_{\theta,\phi} + u_{\theta,\phi}$, and re-write the LLG system as

$$\begin{aligned} \dot\theta &= \frac{F_\theta - \alpha(-\omega_p\cos\theta\sin\theta + F_\phi)}{1+\alpha^2} \\ \dot\phi &= \frac{-\omega_p\cos\theta\sin\theta + F_\phi + \alpha F_\theta}{(1+\alpha^2)\sin\theta} \end{aligned} \tag{6}$$

Next, we make approximations. In the $\omega_p \to \infty$ limit the solution is expected to have a property $\theta(t) = \pi/2 + \delta\theta(t)$ with $\delta\theta \to 0$. Expanding all quantities on the r.h.s. of (6) in small $\delta\theta$ up to the first order we get

$$\delta\dot\theta = \frac{F^0_\theta + F^1_\theta\delta\theta - \alpha(\omega_p\delta\theta + F^0_\phi + F^1_\phi\delta\theta)}{1+\alpha^2}\ , \tag{7}$$

$$\dot\phi = \frac{\omega_p\delta\theta + F^0_\phi + F^1_\phi\delta\theta + \alpha(F^0_\theta + F^1_\theta\delta\theta)}{1+\alpha^2}\ , \tag{8}$$

where we have used the notation

$$\begin{aligned} F^0 &= F\left(\frac{\pi}{2},\phi\right)\ , \\ F^1 &= \frac{\partial F}{\partial\theta}\left(\frac{\pi}{2},\phi\right)\ . \end{aligned}$$

In the approximation (7,8) equations are linear with respect to the unknown function $\delta\theta(t)$, but still fully nonlinear with respect to $\phi(t)$.

Eq. (8) can be solved with respect to $\delta\theta$

$$\delta\theta = \frac{(1+\alpha^2)\dot\phi - F^0_\phi - \alpha F^0_\theta}{\omega_p + F^1_\phi + \alpha F^1_\theta} = q(\phi,\dot\phi)\ , \tag{9}$$

so that the out-of-plane deviation becomes a "slave" of the in-plane motion.[14] The presence of the large ω_p in the denominator ensures the smallness of $\delta\theta$. Substituting the resulting expression $\delta\theta = q(\phi,\dot\phi)$ back into equation (7) one obtains a second order differential equation for a single unknown function $\phi(t)$

$$\frac{\partial q}{\partial\dot\phi}\,\ddot\phi + \frac{\partial q}{\partial\phi}\,\dot\phi = \frac{F^0_\theta - \alpha F^0_\phi}{1+\alpha^2} + \left(\frac{F^1_\theta - \alpha\omega_p - \alpha F^1_\phi}{1+\alpha^2}\right) q\ .$$

Denoting $\Omega(\phi) = \omega_p + F^1_\phi + \alpha F^1_\theta$ and simplifying the terms we get

$$\begin{aligned} \frac{1+\alpha^2}{\Omega}\ddot\phi \ + \ \left(\alpha + \frac{\partial q}{\partial\phi} - (1+\alpha^2)\frac{F^1_\theta}{\Omega}\right)\dot\phi = \\ = F^0_\theta - \frac{F^1_\theta(F^0_\phi + \alpha F^0_\theta)}{\Omega} \end{aligned} \tag{10}$$

In this form the equation is still rather complicated but since it was obtained from the approximate system (7,8) we are allowed to drop any terms below the approximation accuracy. The terms neglected in Eqs. (7,8) were the second order terms $F''\delta\theta^2$ in the expansion of F, and the third order terms $\omega_p\delta\theta^3$ in the easy plane energy expansion. While formally the latter terms are of higher order

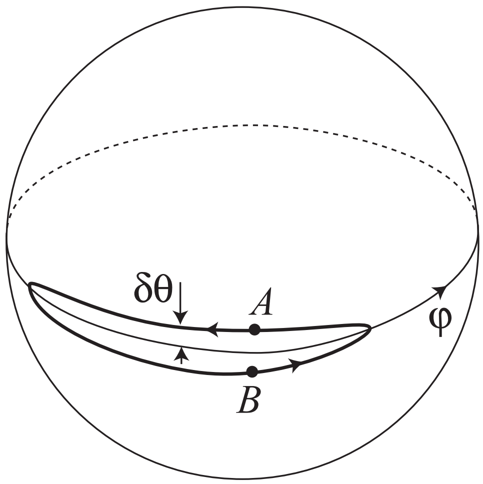

FIG. 2: Typical trajectory of $\mathbf{n}(t)$ on a unit sphere in a system with a dominating easy plane anisotropy. The azimuthal angle ϕ is measured along the equator in the direction shown by the arrow. The deviation $\delta\theta$ from the equator is small. In the planar picture vectors $\mathbf{n}$ ending in the points A and B correspond to effective particles with the same coordinates ϕ but opposite velocities $\dot{\phi}$.

in the $\delta\theta$ expansion, the presence of a large coefficient ω_p causes them to be comparable to the former terms. To compare the orders of magnitude of the terms consistently, we need to know the order of magnitude of $\delta\theta$. At the present stage we know that $\delta\theta$ is small but its exact order of magnitude is not known because we do not have an estimate for the $\dot{\phi}$ term in the numerator of Eq. (9).

A quick way to estimate $\delta\theta$ is to consider a typical trajectory $\mathbf{n}(t)$ in a planar system with negligible dissipation (Fig. 2). In the absence of dissipation the trajectory is an equipotential line $\varepsilon(\mathbf{n}) = \text{const}$. Using the energy expression (3) and denoting the change of the residual energy on the trajectory as $\Delta\varepsilon_r$, one can find the maximum deviation from the equator as $\delta\theta_{\max} = \sqrt{2\Delta\epsilon_r/\omega_p}$. Below we will show how the same result can be observed in the framework of effective planar description.

A. Simple residual energy in the absence of spin torque

Let us first consider the problem in the absence of spin transfer,[14] assuming a simple form of residual energy $\varepsilon_r = \varepsilon_r(\phi)$. In this case we find $f^0_\phi = f^1_\phi = f^1_\theta = 0$ and get

$$q(\phi, \dot{\phi}) = \frac{(1+\alpha^2)\dot{\phi} - \alpha f^0_\theta}{\omega_p} \; . \tag{11}$$

Equation (10) takes a form

$$\frac{1+\alpha^2}{\omega_p}\ddot{\phi} + \alpha\left(1 - \frac{1}{\omega_p}\frac{\partial f^0_\theta}{\partial\phi}\right)\dot{\phi} = f^0_\theta \; .$$

Assuming that the residual energy $\varepsilon_r(\phi)$ does not have any special points of fast change we can estimate $F^0_\theta \sim \varepsilon \ll \omega_p$. Then

$$1 - \frac{1}{\omega_p}\frac{\partial f^0_\theta}{\partial\phi} \approx 1$$

and we can approximate the equation by

$$\frac{1+\alpha^2}{\omega_p}\ddot{\phi} + \alpha\dot{\phi} = f^0_\theta = -\frac{\partial\varepsilon_r}{\partial\phi} \; .$$

The equation above has the form of the Newton's equation for a particle of mass $(1+\alpha^2)/\omega_p$ moving in a one-dimensional potential $\varepsilon_r(\phi)$, subject to a viscous friction force with a friction coefficient α.

Our goal is to estimate the value of $\dot{\phi}$, i.e., of the speed of the "effective particle". The characteristic speed depends on the total energy of the particle

$$\varepsilon_{tot} = \dot{\phi}^2/2\omega_p + \varepsilon_r(\phi)$$

and on the relative strength of the friction forces. We will assume that the total energy is of the order of ε_r (this is the mathematical equivalent of our original assumption about the low-energy dynamics). Furthermore, in the present paper we will concentrate on the case of $\alpha \to 0$ that corresponds to an almost frictionless motion of the particle. Then one can use the approximate conservation of the total energy and obtain the maximum speed from $\dot{\phi}^2/2\omega_p = \varepsilon_r$, which gives

$$\dot{\phi} \sim \sqrt{\varepsilon_r\omega_p} \; .$$

Using similar arguments one can estimate the maximum acceleration as

$$\ddot{\phi} \sim \varepsilon_r\omega_p \; .$$

Note that the viscous friction can be approximately neglected when $\alpha\dot{\phi} \ll F^0_\theta \sim \varepsilon_r$. Thus the Gilbert damping constant α has to be not just small compared to unity but satisfy a more stringent inequality

$$\alpha \ll \sqrt{\frac{\varepsilon_r}{\omega_p}} \ll 1 \; . \tag{12}$$

We see now that $\dot{\phi}$ is the largest term in the denominator of (11) and hence get an estimate

$$\delta\theta \sim \sqrt{\frac{\varepsilon_r}{\omega_p}} \; , \tag{13}$$

in accord with the result obtained by considering a trajectory on the unit sphere.

B. Arbitrary residual energy in the absence of spin torque

Let us return to the approximation (7,8) and assume a general form of the residual energy $\varepsilon_r = \varepsilon_r(\theta, \phi)$ but still keep the current equal to zero ($u = 0$, $F = f$). It is now possible to use the *a posteriori* estimate (13) for $\delta\theta$ to consider the orders of magnitude of the terms. We start the discussion from the "slave" equation (8). Here

$$\begin{aligned} \omega_p \delta\theta &\sim \sqrt{\varepsilon_r \omega_p} \\ f^0 &\sim \varepsilon_r \\ f^1 \delta\theta &\sim \varepsilon_r \sqrt{\frac{\varepsilon_r}{\omega_p}} \\ \alpha f^0 &\ll \varepsilon_r \sqrt{\frac{\varepsilon_r}{\omega_p}} \\ \alpha f^1 \delta\theta &\ll \varepsilon_r \frac{\varepsilon_r}{\omega_p} \end{aligned}$$

As we see, the orders of magnitude of the terms form a series

$$\dots \quad \varepsilon_r \frac{\varepsilon_r}{\omega_p}, \quad \varepsilon_r \sqrt{\frac{\varepsilon_r}{\omega_p}}, \quad \varepsilon_r, \quad \sqrt{\varepsilon_r \omega_p} \quad \dots \tag{14}$$

where each term is given by an expression $\varepsilon_r (\varepsilon_r/\omega_p)^{n/2}$.

The terms neglected in transition from (6) to (8) were

$$\begin{aligned} \omega_p \delta\theta^3 &\sim \varepsilon_r \sqrt{\frac{\varepsilon_r}{\omega_p}} \ , \\ \frac{\partial^2 f}{\partial \theta^2} \delta\theta^2 &\sim \varepsilon_r \frac{\varepsilon_r}{\omega_p} \ . \end{aligned}$$

This means that in (8) one should only keep the terms of the order ε_r and higher. Lower order terms would be comparable to some of the discarded ones. Using this argument we discard $f^1\delta\theta$, αf^0 and $\alpha f^1 \delta\theta$. The $1/(1+\alpha^2)$ factor in (8) can be expanded using Eq. (12)

$$\frac{1}{1+\alpha^2} = 1 + \delta, \quad \delta \sim \alpha^2 \ll \frac{\varepsilon_r}{\omega_p} \ .$$

This shows that $1/(1+\alpha^2)$ can can be approximated by unity in (8) without changing the accuracy. After all those simplifications equation (9) takes the form

$$q(\phi, \dot{\phi}) = \frac{\dot{\phi} - f^0_\phi}{\omega_p} \ .$$

As for the equation (7), the terms discarded in going from (6) to (7) were

$$\begin{aligned} \alpha \omega_p \delta\theta^3 &\ll \varepsilon_r \frac{\varepsilon_r}{\omega_p} \ , \\ \frac{\partial^2 f}{\partial \theta^2} \delta\theta^2 &\sim \varepsilon_r \frac{\varepsilon_r}{\omega_p} \ , \end{aligned}$$

and therefore we have to keep the terms of the order $\varepsilon_r \sqrt{\varepsilon_r/\omega_p}$ and higher. Thus the $f^1\delta\theta$ and αf^0 terms should be kept in (7) but the $\alpha f^1 \delta\theta$ terms should be discarded. One can also conclude that it is safe to replace the factor $1/(1+\alpha^2)$ by unity. Equation (7) is now replaced by

$$\delta\dot{\theta} = f^0_\theta + f^1_\theta \delta\theta - \alpha(\omega_p \delta\theta + f^0_\phi) \ ,$$

where one should use

$$f^1_\theta \delta\theta = f^1_\theta \frac{\dot{\phi} - f^0_\phi}{\omega_p} \approx \frac{f^1_\theta \dot{\phi}}{\omega_p} \ ,$$

since the term $f^1_\theta f^0_\phi/\omega_p \sim \varepsilon_r^2/\omega_p$ is of the same order as the already discarded terms.

Without the discarded terms Eq. (10) reads

$$\frac{\ddot{\phi}}{\omega_p} + \left(\alpha - \frac{1}{\omega_p} \left[\frac{\partial f^0_\phi}{\partial \phi} + f^1_\theta \right] \right) \dot{\phi} = f^0_\theta = -\frac{\partial \varepsilon_r(\pi/2, \phi)}{\partial \phi} \ .$$

This is the effective particle equation discussed in the previous section, except that the viscous friction coefficient seems to acquire a correction. While this correction is a small quantity of the order $\varepsilon_r/\omega_p \ll 1$, it is added to a small number $\alpha \ll 1$ and therefore can potentially change the sign of the dissipation term, leading to a significant effect. However, one finds

$$\frac{\partial f^0_\phi}{\partial \phi} + f^1_\theta = \frac{\partial}{\partial \phi} \frac{\partial \varepsilon_r}{\partial \theta} + \frac{\partial}{\partial \theta} \left(-\frac{\partial \varepsilon_r}{\partial \phi} \right) = 0 \ , \tag{15}$$

so the correction actually vanishes. We come back to the effective equation

$$\frac{\ddot{\phi}}{\omega_p} + \alpha \dot{\phi} = -\frac{\partial \varepsilon_r(\pi/2, \phi)}{\partial \phi} \ ,$$

which corresponds to the most natural generalization of the equation derived in the previous section. The positive effective friction coefficient ensures that the effective particle always stops at an energy minimum point, as expected for a closed system with dissipation.

C. Effective equation in the presence of spin torque

Finally, we proceed to the derivation of the effective equation in the presence of spin torque. Consider approximations (7,8) with $\varepsilon_r = \varepsilon_r(\theta, \phi)$ and $u \neq 0$.

The orders of magnitude of the extra terms produced by the current will depend on the value of u. For the planar approximation to be valid, the spin torque terms certainly have to be small compared to the torques produced by the easy plane anisotropy. The latter are responsible for the terms of the order $\sqrt{\varepsilon_r \omega_p}$ in Eqs. (7) and (8). Thus it seems that u should not exceed ε_r, which is the largest term before $\sqrt{\varepsilon_r \omega_p}$ in the series (14). Such a conclusion is correct for a general situation. We will, however, see below that in some special cases the current can be increased up to $u \sim \sqrt{\varepsilon_r \omega_p}$ without violating the dominance of the easy plane anisotropy torque.

To include those special cases we assume $u \lesssim \sqrt{\varepsilon_r \omega_p}$ and revisit Eqs. (7,8) discarding the terms smaller than $\varepsilon_r \sqrt{\varepsilon_r/\omega_p}$ in Eq. (7), and smaller than ε_r in Eq. (8).

Eq. (8) acquires the form

$$\dot{\phi} = (\omega_p + u^1_\phi)\delta\theta + f^0_\phi + u^0_\phi \ ,$$

where, just like in the previous section, the factor $1/(1+\alpha^2)$ was approximated by unity without loss of accuracy. By solving for $\delta\theta$ and expanding the denominator up to the same accuracy we find the form of the slave condition (9)

$$\delta\theta = \left(1 - \frac{u_\phi^1}{\omega_p}\right)\frac{\dot\phi}{\omega_p} - \frac{f_\phi^0 + u_\phi^0}{\omega_p} + \frac{u_\phi^1 u_\phi^0}{\omega_p^2} \ . \qquad (16)$$

Differentiating both sides one gets a formula

$$\begin{aligned} \delta\dot\theta \ &= \ \left(1 - \frac{u_\phi^1}{\omega_p}\right)\frac{\ddot\phi}{\omega_p} - \frac{\partial u_\phi^1}{\partial\phi}\frac{\dot\phi^2}{\omega_p^2} - \\ &- \left(\frac{\partial f_\phi^0}{\partial\phi} + \frac{\partial u_\phi^0}{\partial\phi} - \frac{1}{\omega_p}\frac{\partial\left[u_\phi^1 u_\phi^0\right]}{\partial\phi}\right)\frac{\dot\phi}{\omega_p} \qquad (17) \end{aligned}$$

for the time derivative of the out-of-plane angle.

Returning now to Eq. (7), we find that with the declared accuracy it can be rewritten as

$$\delta\dot\theta = f_\theta^0 + u_\theta^0 - \alpha u_\phi^0 + \left(f_\theta^1 + u_\theta^1 - \alpha\omega_p\right)\delta\theta \ .$$

Substituting $\delta\theta$ from (16) and discarding any terms that are smaller than $\varepsilon_r\sqrt{\varepsilon_r/\omega_p}$, we get

$$\begin{aligned} \delta\dot\theta \ &= \ f_\theta^0 + u_\theta^0 - \left(\alpha - \frac{f_\theta^1 + u_\theta^1}{\omega_p} + \frac{u_\theta^1 u_\phi^1}{\omega_p^2}\right)\dot\phi - \\ &- \frac{f_\theta^1 u_\phi^0 + u_\theta^1 f_\phi^0 + u_\theta^1 u_\phi^0}{\omega_p} + \frac{u_\theta^1 u_\phi^1 u_\phi^0}{\omega_p^2} \ . \end{aligned}$$

The last step is to use Eq. (17) to express $\delta\dot\theta$ on the left hand side. This gives the form of the effective equation (10) without the terms below our accuracy

$$\begin{aligned} &\left(1 - \frac{u_\phi^1}{\omega_p}\right)\frac{\ddot\phi}{\omega_p} + \\ &+ \left(\alpha - \frac{1}{\omega_p}\left[\frac{\partial f_\phi^0}{\partial\phi} + f_\theta^1 + \frac{\partial u_\phi^0}{\partial\phi} + u_\theta^1\right] - \right. \\ &\left. + \frac{1}{\omega_p^2}\left[\frac{\partial\left[u_\phi^1 u_\phi^0\right]}{\partial\phi} + u_\theta^1 u_\phi^1\right]\right)\dot\phi - \frac{\partial u_\phi^1}{\partial\phi}\frac{\dot\phi^2}{\omega_p^2} = \\ &= f_\theta^0 + u_\theta^0 - \frac{f_\theta^1 u_\phi^0 + u_\theta^1 f_\phi^0 + u_\theta^1 u_\phi^0}{\omega_p} + \frac{u_\theta^1 u_\phi^1 u_\phi^0}{\omega_p^2} \ . \end{aligned}$$

The bracketed expression on the second line can be further simplified using the identity (15).

We now cast the effective planar equation in its final form

$$m\ddot\phi + \alpha_{eff}\dot\phi - \frac{(u_\phi^1)'}{\omega_p^2}\dot\phi^2 = -(\varepsilon_{eff})' \ . \qquad (18)$$

Here primes denote differentiation with respect to ϕ, and the parameters are given by

$$\begin{aligned} m \ &= \ \frac{1}{\omega_p}\left(1 - \frac{u_\phi^1}{\omega_p}\right) \ , \\ \alpha_{eff} \ &= \ \alpha - \frac{(u_\phi^0)' + u_\theta^1}{\omega_p} + \frac{\left(u_\phi^1 u_\phi^0\right)' + u_\phi^1 u_\theta^1}{\omega_p^2} \ , \qquad (19) \\ \varepsilon_{eff} \ &= \ \varepsilon_r\left(\frac{\pi}{2}, \phi\right) + U \ , \\ -U' \ &= \ u_\theta^0 - \frac{f_\theta^1 u_\phi^0 + u_\theta^1 f_\phi^0 + u_\theta^1 u_\phi^0}{\omega_p} + \frac{u_\theta^1 u_\phi^1 u_\phi^0}{\omega_p^2} \ . \end{aligned}$$

Equations (18) and (19) constitute the first main result of this paper.

In the presence of the current the right hand side of (18) contains additional "effective forces" added to the zero current term $-\varepsilon_r{}'$. All these forces can be represented as derivatives of an additional energy U due to the fact that we are dealing with the functions of one variable.

One of the effective forces, namely the u_θ^0 term, requires a special discussion. When $u \sim \sqrt{\varepsilon_r\omega_p}$ this term becomes larger than $\varepsilon_r{}'$ on the right hand side of Eq. (18), and the estimates for $\dot\phi$ and $\ddot\phi$ made in Sec. III A become invalid. As discussed above, this means that in a general case with non-zero u_θ^0 the effective equations (18, 19) can be only used for currents $u \lesssim \varepsilon_r$. However, if u_θ^0 is identically equal to zero, while the other spin torque terms in U and α_{eff} remain non-zero, one can apply Eqs. (18, 19) for currents up to $u \sim \sqrt{\varepsilon_r\omega_p}$. We will see an example of such a situation in Sec. IV A below.

In the absence of current, corrections to the friction coefficient vanish, giving $\alpha_{eff} = \alpha > 0$. When the current is turned on, the sign of the friction coefficient may change[15–19], reflecting the possible influx of the energy from the current source into the system.

IV. TILTED POLARIZER DEVICE

We now present the application of the effective planar equation approach to devices with a "tilted polarizer" geometry, which recently became a subject of a number of investigations.[21–30]

The general discussion of Sec. III C is applicable to a polarizer with arbitrary direction, $\mathbf{s} = (\sin\theta_s\cos\phi_s, \sin\theta_s\sin\phi_s, \cos\theta_s)$. In this section we will consider a special case of $\mathbf{s}$ lying in the (x,z) plane, i.e., $\phi_s = 0$ (see Fig. 8). Vector $\mathbf{s}$ constitutes an angle of $\pi/2 - \theta_s$ with the easy plane. To calculate u_θ and u_ϕ from Eq. (5) one needs to know the function $u(\mathbf{n})$. In many cases[2,20] it has a form

$$u(\mathbf{n}) = \frac{g_0 I}{1 + g_1(\mathbf{n}\cdot\mathbf{s})} \ .$$

We will consider the case of small $g_1 \ll 1$ and approximate

$$u(\mathbf{n}) = g_0 I(1 - g_1(\mathbf{n}\cdot\mathbf{s})) \ . \qquad (20)$$

Using the expressions in Appendix A, we find

$$\begin{aligned} u_\theta &= g_0 I \left[1 - g_1(\sin\theta_s \sin\theta\cos\phi + \cos\theta_s\cos\theta)\right] \\ &\quad \times(\sin\theta_s\cos\theta\cos\phi - \cos\theta_s\sin\theta) \ , \\ u_\phi &= -g_0 I\left[1 - g_1(\sin\theta_s\sin\theta\cos\phi + \cos\theta_s\cos\theta)\right] \\ &\quad \times\sin\theta_s\sin\phi \ . \end{aligned}$$

Therefore

$$\begin{aligned} u^0_\theta &= -g_0 I\left[1 - g_1\sin\theta_s\cos\phi\right]\cos\theta_s \ , \\ u^0_\phi &= -g_0 I\left[1 - g_1\sin\theta_s\cos\phi\right]\sin\theta_s\sin\phi \ , \end{aligned} \quad (21)$$

and

$$\begin{aligned} u^1_\theta &= -g_0 I[\sin\theta_s\cos\phi - g_1(\sin^2\theta_s\cos^2\phi - \cos^2\theta_s)] \ , \\ u^1_\phi &= -g_0 I g_1 \sin\theta_s\cos\theta_s\sin\phi \ . \end{aligned} \quad (22)$$

A. In-plane polarizer

In the case of in-plane polarizer ($\theta_s = \pi/2$) further simplifications happen:

$$\begin{aligned} u^0_\theta &= 0 \ , \\ u^0_\phi &= -g_0 I(1 - g_1\cos\phi)\sin\phi \ , \\ u^1_\theta &= -g_0 I(1 - g_1\cos\phi)\cos\phi \ , \\ u^1_\phi &= 0 \ . \end{aligned}$$

As we see, the in-plane polarizer happens to be one of the special cases with $u^0_\theta \equiv 0$ discussed at the end of Sec. III C. Consequently, the effective equation can be used up to the rescaled currents $u \sim \sqrt{\varepsilon_r\omega_p}$. Coefficients (19) acquire the form

$$\begin{aligned} m &= \frac{1}{\omega_p} \ , \\ \alpha_{eff} &= \alpha + \frac{g_0 I\left[2\cos\phi - g_1(3\cos^2\phi - 1)\right]}{\omega_p} \ , \quad (23) \\ -U' &= -\frac{g_0 I(1 - g_1\cos\phi)}{\omega_p} \times \quad (24) \\ &\quad \times \ [g_0 I(1 - g_1\cos\phi)\sin\phi\cos\phi - \\ &\qquad -(f^1_\theta\sin\phi + f^0_\phi\cos\phi)] \ . \end{aligned}$$

Importantly, the $\dot\phi^2$ term in Eq. (18) vanishes identically.

In Refs. 15,16 the in-plane polarizer was considered in the case of $g_1 = 0$ and a residual energy

$$\varepsilon_r = -\frac{\omega_a}{2}\sin^2\theta\cos^2\phi - h\sin\theta\cos\phi \ , \quad (25)$$

describing a free layer with a small easy axis anisotropy $\omega_a \ll \omega_p$, and an external magnetic field h, both directed along the x-axis. In this case one finds $f^1_\theta = f^0_\phi = 0$ and expressions (24) reproduce the results obtained in Refs. 15 and 16.

B. General case of a tilted polarizer

When the polarizer magnetization **s** points at an arbitrary angle θ_s the term u^0_θ is nonzero and we have to limit the current magnitudes to $g_0 I \lesssim \varepsilon_r$ to maintain the validity of Eq. (18). With smaller currents more terms can be discarded from the effecitve equation without changing its accuracy. Expressions (19) reduce to

$$\begin{aligned} m &= \frac{1}{\omega_p} \ , \\ \alpha_{eff} &= \alpha - \frac{(u^0_\phi)' + u^1_\theta}{\omega_p} \ , \\ -U' &= u^0_\theta \ . \end{aligned} \quad (26)$$

Moreover, for $g_0 I \lesssim \varepsilon_r$ the nonlinear term with $\dot\phi^2$ becomes small enough to be dropped from Eq. (18).

The effective planar equation now reads

$$\frac{\ddot\phi}{\omega_p} + \alpha_{eff}\dot\phi = -\frac{\partial\varepsilon_{eff}}{\partial\phi} \ , \quad (27)$$

where the effective damping and the effective energy can be expressed through the polarizer tilting angle θ_s using Eqs. (21), (22), and (26)

$$\begin{aligned} \alpha_{eff} &= \alpha + \frac{g_0 I}{\omega_p}(2\sin\theta_s\cos\phi - g_1[3\sin^2\theta_s\cos^2\phi - 1]) \ , \\ \varepsilon_{eff} &= \varepsilon_r\left(\frac{\pi}{2},\phi\right) + g_0 I(\cos\theta_s\cdot\phi - g_1\sin\theta_s\cos\theta_s\sin\phi) \ . \end{aligned} \quad (28)$$

Note how in the presence of spin torque the effective energy acquires a term that is linear in ϕ.

C. Switching diagram of the tilted polarizer device

Let us now discuss the consequences of the spin torque induced modifications $\alpha \to \alpha_{eff}$ and $\varepsilon_r \to \varepsilon_{eff}$. The advantage of the effective planar approximation is the possibility of using the analogy with the particle motion. The latter enables one to utilize mechanical intuition to predict the behavior of the solutions of Eq. (27) and qualitatively understand the dynamics of the spin-transfer device. The mechanical analogy makes it clear that the modifications of the effective damping qualitatively transform the particle motion when $\alpha_{eff}(\phi)$ changes its sign, and the modifications of $\varepsilon_{eff}(\phi)$ become qualitatively important when equilibrium points appear or disappear as the energy profile is deformed.

We will consider the standard nanopillar device described by the residual energy (25). In the special case of an in-plane polarizer this problem was discussed in our earlier publications.[15,16,18] Our goal here is to generalize these results to the case of a non-zero polarizer tilt and show that the effective planar approach allows one

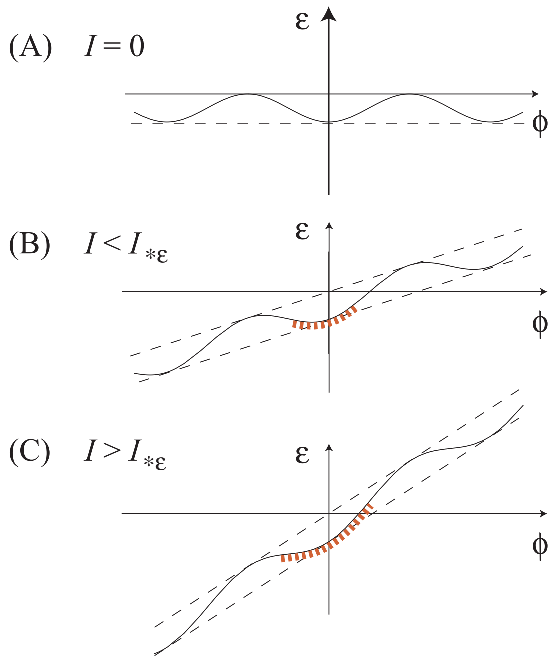

FIG. 3: (Color online) Effective energy profiles at different current magnitudes. Red (gray) dashed lines show the regions of negative friction.

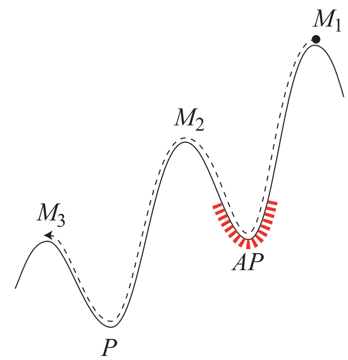

FIG. 4: (Color online) Effective particle performing a 360° motion, starting from the energy maximum M_1 and reaching an equivalent maximum M_3. The red (gray) dashed line denotes the interval of $\alpha_{eff} < 0$.

to understand the qualitative picture of the motion without doing the detailed calculations. Switching diagrams for devices with arbitrary θ_s and $\phi_s = 0$, $h = 0$ were recently studied by conventional methods.[26,27] We will use the same assumptions and compare the results.

For a generic value of the tilt angle θ_s the terms entering expressions (28) with a small factor g_1 produce negligible corrections and can be discarded. Those terms can be important only when θ_s approaches zero or $\pi/2$ and the main terms vanish. We will assume that $\mathbf{s}$ is not too close to either the in-plane or the perpendicular directions and inequalities $\cos\theta_s, \sin\theta_s \gg g_1$ hold. Finding the critical currents in the narrow bands of angles $\theta_s \approx 0$ or $\theta_s \approx \pi/2$ where the g_1 terms are important would require a more careful consideration.

Discarding the g_1 terms one gets a simplified form of Eqs. (28)

$$\begin{aligned} \alpha_{eff} &= \alpha + \frac{2g_0 I \sin\theta_s}{\omega_p} \cos\phi \ , \\ \varepsilon_{eff} &= -\frac{\omega_a}{2}\cos^2\phi + (g_0 I \cos\theta_s)\,\phi \ . \end{aligned} \tag{29}$$

As already mentioned in Sec. IV B, the energy $\varepsilon_{eff}(\phi)$ contains a term linear in ϕ: Spin torque produces a tilted washboard potential for the effective particle (Fig. 3). The washboard tilt reflects the fact that the total magnetic energy of the free layer can change due to the energy transfer from the current source.

In the presence of current the effective energy minima are shifted from their zero current positions $\phi = 0$ (parallel, or P state) and $\phi = \pi$, (antiparallel, or AP state) to the new positions $\phi_m(I)$ given by

$$\sin 2\phi_m = -\frac{2g_0 I\cos\theta_s}{\omega_a} \ .$$

The energy minima are located at the angles $\phi_{min} + \pi n$, and are separated by the energy maxima located at at $\phi_{max} = \pi/2 - \phi_{min} + \pi n$ (Fig. 3B). All minimum points $\phi_{min}(I)$ are equivalent from the point of view of effective energy (29) but the minima that had evolved from the P and AP points can differ in effective friction. To be concise, we will continue calling the minima satisfying $\phi_{min}(0) = 0$ the P points, and those satisfying $\phi_{min}(0) = \pi$ the AP points.

As the current grows, the washboard tilts more and more, until the extrema of the energy $\varepsilon_{eff}(\phi)$ disappear altogether (Fig. 3C). A short calculations shows that this occurs at a critical current

$$I_{*\varepsilon} = \frac{\omega_a}{2g_0\cos\theta_s} \ . \tag{30}$$

For $|I| > I_{*\varepsilon}$ the effective particle slides down the slope of the potential energy profile (either left or right, depending on the current direction), regardless of the sign and magnitude of α_{eff}. This motion corresponds to the full 360° rotations of vector $\mathbf{n}$ in the azimuthal angle ϕ. In the spin transfer literature such a regime is called the out-of-plane precession (OPP).

Importantly, the OPP precession can exist even at $|I| < I_{*\varepsilon}$. When the particle moves down the tilted washboard, the drop of its potential energy during one spatial period may be large enough to overcome the frictional energy loss (Fig. 4). Therefore there must be a second critical current $I_{OPP} < I_{*\varepsilon}$, such that the OPP precession is possible for $|I| > I_{OPP}$. In the interval $I_{OPP} < |I| < I_{*\varepsilon}$ the stable equilibrium state at the energy minimum coexists with the stable OPP state. The functional form of $I_{OPP}(\theta_s)$ depends on the energy profile and the friction coefficient. Our goal here is not to find the expression

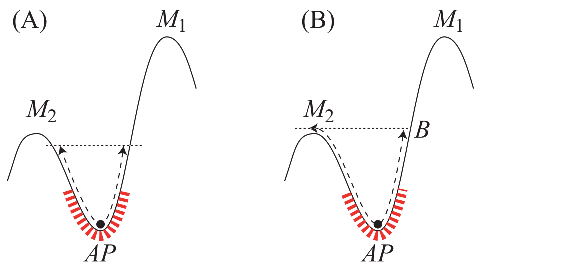

FIG. 5: (Color online) (A) Effective particle performing finite amplitude oscillations (*IPP* precession) near the AP minimum. The red (gray) dashed line denotes the interval of $\alpha_{eff} < 0$. (B) With growing current the amplitude of the oscillations increases and the particle reaches the M_2 maximum point. Above this threshold the *IPP* precession is unstable.

for it, but to see how far can we proceed in qualitative understanding of the device dynamics without doing the actual calculations.

The possibility of energy transfer from the current source to the spin torque device can also manifest itself in the form of negative effective friction. The regions of negative friction $\alpha_{eff}(\phi)$ first appear in the vicinity of the angles $\phi_{+n} = 2\pi(n+1/2)$ for $I > 0$ and $\phi_{-n} = 2\pi n$ for $I < 0$ when the current magnitude exceeds the threshold

$$I_{*\alpha} = \frac{\alpha\omega_p}{2g_0 \sin\theta_s} \ . \tag{31}$$

The intervals of $\alpha_{eff} < 0$ are shown in Figs. 3,4 by red (gray) dashed lines. The first consequence of their presence is that on parts of the trajectory the friction force increases the energy of the system instead of decreasing it in a usual fashion. This, in particular, makes it easier for the particle to achieve the state of the *OPP* precession and thus the actual calculation of the I_{OPP} threshold must take into account the energy gain due to both the tilt of the potential and the presence of negative friction intervals. Both features mathematically represent the ability of the spin torques to transfer energy from the current source to the system.

The second important effect of negative α_{eff} is the local destabilization of the energy minima. A minimum point $\phi_{min}(I)$ lying within the interval of $\alpha_{eff} < 0$ is unstable, and finite amplitude stationary oscillations around it are developed. Fig. 5A shows such oscillations near the AP minimum destabilized by a sufficiently large positive current (see Eq. 29). These oscillations correspond to the motion of vector $\mathbf{n}$ around the equilibrium point, which is called an in-plane precession (*IPP*) in the spin transfer literature.

The critical current of energy minimum destabilization is determined from the equation $\alpha_{eff}(\phi_{min}(I)) = 0$ which can be rewritten for P and AP minima as

$$\begin{aligned} I_{AP}\cos[\phi_{min}(I_{AP})] &= I_{*\alpha} \ , \\ I_{P}\cos[\phi_{min}(I_{P})] &= -I_{*\alpha} \ . \end{aligned} \tag{32}$$

Both threshold currents satisfy $|I_{P,AP}| > I_{*\alpha}$. This result can be naturally understood as follows. The friction first becomes negative at the $\phi = 0$ or $\phi = \pi$ points at $I = \pm I_{*\alpha}$. But the P and AP minima are shifted from the $0, \pi$ points to the $\phi_{min}(I)$ points. In order to destabilize them, the negative friction interval has to grow large enough to cover the actual minima positions.

The amplitude of stationary oscillations is determined by the balance of energy influx and energy dissipation on the intervals of negative and positive friction.[15,16,18,19] Two possible scenarios can be realized at the local destabilization threshold[31] (1) Soft generation. Stationary oscillations with an infinitesimally small amplitude are developed. Their amplitude grows with the further current increase; (2) Hard generation. Stationary oscillations immediately develop a finite amplitude.

Appendix B shows that in our situation the choice between the soft and hard generation scenarios is controlled by the second derivative $d^2\alpha_{eff}/d\phi^2$ at the position of the energy minimum, and a soft scenario is realized for the function α_{eff} given by Eq. (29), i.e., the amplitude of stationary oscillations is zero at the threshold. As the current is increased beyond the threshold, the amplitude grows and eventually becomes so large that the particle reaches the crest of the potential, as shown in Fig. 5B, and falls down into the neighboring valley. This process leads to the destruction of the *IPP* state. The latter therefore exists between the two threshold currents. For the AP equilibrium these are the I_{AP} threshold, where the AP point becomes unstable, and the I_{IPP} threshold, where the oscillation amplitude becomes too large to be contained in the AP valley.

The critical current I_{IPP} depends on the shape of the potential in the entire interval traveled by the particle in Fig. 5B. Its actual calculation is not the goal of our qualitative approach. However, we can make two general statements about I_{IPP}. First, due to the soft character of generation the destabilization of the *IPP* state certainly happens at a current that is larger than the local destabilization threshold: For example, $I_{IPP} > I_{AP}$.

To set the stage for the second observation, we proceed to the discussion of the switching diagram. In our case the experimental parameters are the current I and the tilting angle θ_s. Various critical currents are represented as lines dividing the (I, θ_s) plane into domains with different sets of stable states. The lines for the four thresholds discussed above are sketched in Fig. 6. The $I_{*\varepsilon}(\theta_s)$ and $I_{*\alpha}(\theta_s)$ lines intersect according to Eqs. (30) and (31). Due to the inequality $I_{AP} > I_{*\alpha}$ the $I_{AP}(\theta_s)$ line has to be located to the right of the $I_{*\alpha}(\theta_s)$ line on the diagram. The $I_{IPP}(\theta_s) > I_{AP}(\theta_s)$ line has to be located even further to the right. Both $I_{AP}(\theta_s)$ and $I_{IPP}(\theta_s)$ cross the $I_{*\varepsilon}(\theta_s)$ line, and one can prove that they do it at the same point T. This is the second general property of the I_{IPP} threshold.

The uniqueness of the point T can be proven by considering a hypothetical switching diagram shown in the inset to Fig. 6 where it is assumed that the AP and *IPP*

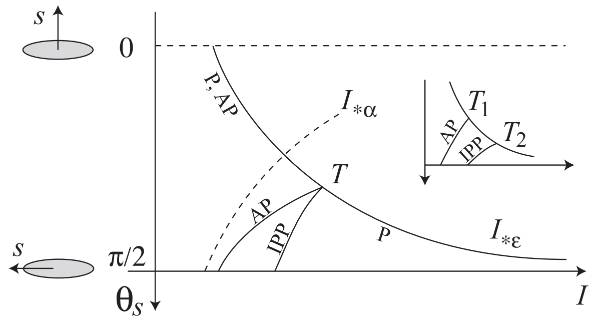

FIG. 6: Preliminary discussion of the critical lines on the (I, θ_s) switching diagrams. Each line is marked by the name of the state which gets destabilized on it. The name is put on the side of the line where the state is stable. Inset: An impossible arrangement of the AP and IPP lines.

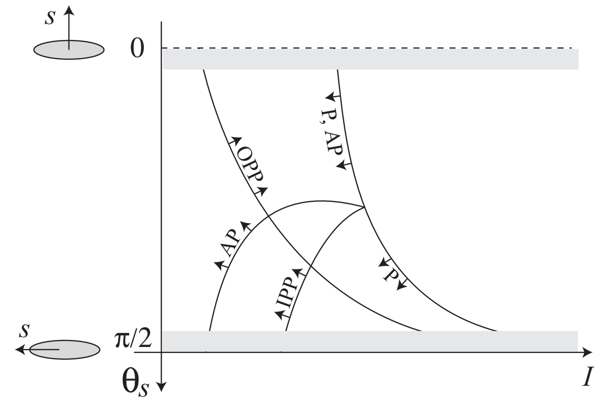

FIG. 7: Switching diagram of a spin-transfer device with a tilted polarizer: a qualitative sketch. Each line is marked by the name of the state which gets destabilized on it. The arrows next to the name point to the side of the line where this state is stable. Horizontal gray bands denote regions where approximations (29) may fail and Eqs. (28) should be used.

lines cross $I_{*\varepsilon}(\theta_s)$ at different points T_1 and T_2. Consider the point T_2. On the one hand, approaching it from within the domain of existence of the IPP precession one should observe a decreasing amplitude of oscillations around the AP minimum, because the size of the valley around AP shrinks to zero with the approach to the $I_{*\varepsilon}$ line. At T_2 the amplitude of sustained oscillations around AP should be infinitesimally small. On the other hand, close to T_2 there is a finite region of negative friction around the AP point because the current is larger than the I_{AP} threshold. Thus infinitesimally small stationary oscillations are impossible since they would be entirely contained in the negative friction region, and their amplitude would grow due to the constant increase of effective particle's energy.[16] This contradiction proves that the assumption about the existence of two different crossing points $T_{1,2}$ was inconsistent. The point T_2 cannot lie to the right of T_1. But we already know that it cannot lie to the left of T_1 either, since that would be incompatible with the soft generation scenario. We conclude that both AP and IPP lines cross $I_{*\varepsilon}$ at the same point T.

The relationships between I_{AP} and I_{IPP} discussed above follow from the fact that both currents are determined by the energy and friction near the same minimum. The I_{OPP} current depends on the details of ε_{eff} and α_{eff} on the whole 2π interval of ϕ and thus no general relationships for it can be found, except for the already mentioned inequality $I_{OPP} < I_{*\varepsilon}(\theta_s)$. A qualitative sketch of the full switching diagram is given in Fig. 7 for $I > 0$. The full diagram is symmetric with respect to the $I \to -I$ transformation, which is a consequence of two symmetries built into the energy and friction functions (29): the π-periodicity of $\varepsilon_r(\phi)$, and the $\alpha_{eff}(\pi - \phi, -I) = \alpha_{eff}(\phi, I)$ symmetry of the effective friction. The latter depends on vector $\mathbf{s}$ lying in the (x, z) plane and the fact that the g_1 terms were dropped. If either of the two symmetries were violated, the P and AP states would no longer be equivalent in all respects. Note also, that had the effective friction function allowed for a hard generation of the IPP states, the diagram would be more complex.

Notably, Fig. 7 reproduces all qualitative features of the switching diagrams obtained by conventional methods.[26,27] In addition, it brings important qualitative understanding of the behavior of critical currents as a function of system parameters and approximations used for the spin torque efficiency factor $g(\mathbf{n})$. For example, it warns that in the two most frequently considered limiting cases of the perpendicular ($\theta_s = 0$) and in-plane ($\theta_s = \pi/2$) polarizers the threshold currents will be most sensitive to the $g(\mathbf{n})$ function used for the calculations.

Finally, we should mention that the results obtainable in the effective planar approach are not limited to the qualitative conclusions. The method also allows one to calculate the critical currents, often being the only one providing analytical expressions in the case of precession states, where the results are traditionally obtained from numeric simulations.[11] In the limit of small friction $\alpha \ll \sqrt{\omega_a/\omega_p}$ considered here, the critical currents $I_{OPP}(\theta_s)$ and $I_{IPP}(\theta_s)$, and the frequencies of the precession states can be analytically found using the methods introduced in Refs. 16 and 19. In the present paper we have found analytic formulas for the $I_{*\varepsilon}$ threshold (30), and the I_P and I_{AP} thresholds (32). The crossing point T of the AP, IPP and $I_{*\varepsilon}$ lines can be found by solving the equation $I_{*\varepsilon}(\theta_s) = I_{AP}(\theta_s)$, which gives

$$\tan\theta_s(T) = \frac{\sqrt{2}\alpha\omega_p}{\omega_a} \; .$$

V. CONCLUSIONS

We have given the detailed derivation of the effective planar equation for spin-transfer devices with dominating easy plane anisotropy and illustrated its application by performing a qualitative study of tilted polarizer devices. Once the parameters of the effective equation are found, the approach allows one to understand the dynamics qualitatively without performing detailed calculations. This is especially important in the case of precession cycles which are usually studied numerically. The method also elucidates the role of approximations used to model the spin-transfer efficiency factor and shows the limits of their applicability.

The obtained switching diagram demonstrates a competition between the two types of switching. For small θ_s the destabilization of the AP minimum results from the merging and disappearance of the minimum and maximum points of ε_{eff}. For θ_s close to $\pi/2$ the destabilization happens locally, changing the nature of the AP equilibrium from stable to unstable. This type of competition is not unique to the systems with strong easy plane anisotropy — it was shown in Ref. 32 that it may happen in any spin-transfer device.

VI. ACKNOWLEDGMENTS

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Appendix A: Vector definitions

We use the standard definitions of polar coordinates and tangent vectors (see Fig. 8):

$$\begin{aligned} \mathbf{n} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0) \end{aligned} \tag{A1}$$

For the polarizer unit vector $\mathbf{s}$ with polar angles (θ_s, ϕ_s) the scalar product expressions are

$$\begin{aligned} (\mathbf{s}\cdot\mathbf{e}_\theta) &= \sin\theta_s\cos\theta\cos(\phi_s-\phi) - \cos\theta_s\sin\theta \\ (\mathbf{s}\cdot\mathbf{e}_\phi) &= \sin\theta_s\sin(\phi_s-\phi) \end{aligned} \tag{A2}$$

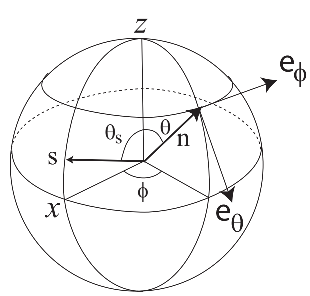

FIG. 8: Definitions of the tangent vectors and polar angles.

Appendix B: Soft and hard generation

Suppose the effective energy has a minimum at the point ϕ_{min} and the effective friction is negative in the interval $[\phi_-(I), \phi_+(I)]$, such that the right end of the interval reaches the equilibrium at the critical current I_c and the minimum remains completely covered by the negative friction region after that

$$\begin{aligned} &\phi_- < \phi_+ < \phi_{min}\ , \quad I < I_c\ , \\ &\phi_- < \phi_+ = \phi_{min}\ , \quad I = I_c\ , \\ &\phi_- < \phi_{min} < \phi_+\ , \quad I > I_c\ . \end{aligned}$$

In order to understand the character of generation that occurs for currents exceeding I_c by a small increment, we will use the simplest Taylor approximations for the energy ε_{eff} and effective friction α_{eff} near the equilibrium point ϕ_{min}. In the case of soft generation the stationary oscillations' amplitude is small. Thus both ε_{eff} and α_{eff} will be accurately approximated by just a few terms of the Taylor series. Solution of an approximate equation using these Taylor expansions instead of the exact functions ε_{eff} and α_{eff} will be close to the actual one. For hard generation there is no solution with small amplitude which will be reflected in the absence of a stationary solution for the approximate equation.

To implement this program we start with Eq. (27)

$$\frac{\ddot{\phi}}{\omega_p} + \alpha_{eff}(\phi)\dot{\phi} = -\frac{\partial\varepsilon_{eff}}{\partial\phi}$$

and approximate

$$\begin{aligned} \varepsilon(\phi) &= \frac{\omega_{min}(\phi-\phi_{min})^2}{2} \\ \alpha_{eff}(\phi) &= \alpha_0 + \alpha'(\phi-\phi_{min}) + \frac{1}{2}\alpha''(\phi-\phi_{min})^2 \end{aligned} \tag{B1}$$

The exact value of the positive number $\omega_{min} = \partial^2\varepsilon/\partial\phi^2 > 0$ is not important. The numbers α_0, α' and α'' are the value and derivatives of the function $\alpha_{eff}(\phi)$ at $\phi = \phi_{min}$. For a current slightly higher than critical α_0 is a small negative number. Condition $\alpha_0(I_c) = 0$ reflects the fact that the negative friction interval touches ϕ_{min} at the critical current. The derivative α' is positive to ensure $\phi_+ > \phi_{min}$ for $I > I_c$.

We will investigate the stationary oscillations by using the condition of zero total dissipation.[16,19]. The change of the total effective particle energy during one period T equals

$$\Delta\varepsilon_{tot} = -\int_0^T \alpha_{eff}(\phi)\dot{\phi}^2 dt\ .$$

In the limit of small friction coefficient one can substitute the zero-friction solution

$$\phi(t) = \phi_{min} + A\cos\Omega t\ , \quad \Omega = \sqrt{\omega_p\,\omega_{min}}$$

into the integral above. The stationary oscillations condition $\Delta\varepsilon_{tot} = 0$ then reads

$$\int_0^{\tilde{T}} (\alpha_0 + \alpha' A \cos\Omega t + \frac{\alpha''}{2} A^2 \cos^2\Omega t) A^2 \Omega^2 \sin^2\Omega t \; dt = 0$$

with the approximate period $\tilde{T} = 2\pi/\Omega$. After the integrals are taken, one gets an expression for the amplitude

$$A^2 = \frac{-8\alpha_0}{\alpha''} \ .$$

Since $\alpha_0 < 0$ for $I > I_c$, the equation for the stationary amplitude can be solved if $\alpha'' > 0$. The solution $A \sim \sqrt{-\alpha_0}$ describes a soft generation of oscillations: Their amplitude is equal to zero at the critical current and then continuously grows.

It is easy to check that the effective friction given by Eq. (29) indeed has positive second derivative at ϕ_{min}. In addition it has a property $(-\alpha_0) \sim I - I_c$, so the oscillation's amplitude obeys the law

$$A \sim \sqrt{I - I_c} \quad (I > I_c) \ .$$

In the case of $\alpha'' \leq 0$ there is no stationary solution. The energy change $\Delta\varepsilon_{tot} > 0$ is always positive and the amplitude would grow, until limited by the properties of the functions ε_{eff} and α_{eff} far away from the equilibrium where the truncated Taylor expansions (B1) are not valid. This would be the case of hard generation.

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