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Finite size effects on the statistical angle of an electron induced dyon in proximity to a topological insulator

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A pointlike electric charge close to the surface of a three dimensional topological insulator induces a magnetic monopole mirror charge. We study the distance dependence of the statistical angle describing this induced dyon system. We find that the total angular momentum, which sometimes is used as signature of the statistical angle, for an electron outside a finite size spherical or tube shaped topological insulator is zero for any finite distance between the electron and the surface. However, we show that in the 2-electron system one can indeed isolate a non-trivial statistical angle for intermediate size loops, that is loops much larger than the distance of the charge to the sample but much smaller than the size of the sample. The necessity for this limit confirms the 2+1 dimensional nature of the non-trivial statistical angle found in previous work. Our results clarify the conditions under which the statistical angle of this system could be measured in real experiments.

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I. INTRODUCTION

The low energy effective theory describing three dimensional topological insulators (TIs) is given by Maxwell electromagnetism augmented by an axion like $\vec{E}\cdot\vec{B}$ term[1] leading to modified constitutive relations. For the bulk material this effective theory is valid at energy scales below the gap. In the presence of interfaces the massless modes on the surface would have to be included in the effective description of the material and the low energy effective description in terms of Maxwell theory with modified constitutive relations only applies if the surface modes are gapped by an external (time reversal breaking) perturbation. Furthermore the effective description is only valid at energy scales below this induced surface gap. Such an external breaking can easily be set up experimentally, e.g. by a magnetic field. Several potential experimental consequences follow from this effective theory, such as, for an example, a non-trivial Faraday and Kerr rotation[1]. By scanning the external field, the topological contribution to the Faraday effect from the time reversal breaking field can be cleanly separated from the topological contribution. One of the most spectacular predictions of this effective theory is the appearance of a magnetic monopole mirror charge when solving for the static electromagnetic fields sourced by a single point charge (located inside a topological trivial insulator such as e.g. vacuum) in the presence of a TI interface[2].

For an infinitely extended planar interface, the corresponding magnetic mirror charge is a pointlike magnetic monopole (also carrying some electric charge). As always with mirror charges, this monopole of course is not physical but simply a mathematical tool to calculate the magnetic fields in the physical region (that is for calculating the fields above the interface, the monopole appears to be located below the interface and vice versa). Microscopically it is surface currents on the interface that source a magnetic field with a $1/r^2$ fall-off where r is the distance to the mirror monopole. Nevertheless, the magnetic fields generated this way are, in the physical region, indistinguishable from the ones generated by a genuine monopole and so share some of its properties. In particular, it is well known that the electromagnetic fields generated by an electric point charge e and a spatially separated monopole of magnetic charge g carry a net angular momentum which has several interesting properties: it is independent of the distance between charge and monopole, pointing in the direction from the charge to the monopole and proportional to eg (see e.g. Ref. 3). This total angular momentum of the composite dyon formed by the charge-monopole pair is the sum of the angular momenta of the two point particles and this angular momentum stored in their fields. Via the spin-statistics theorem this shift in the angular momentum of the dyon is often interpreted as a shift in the statistical angle that determines the behavior of the multi-dyon wavefunction under the exchange of two dyons. For a genuine charge/monopole pair, the Dirac quantization of magnetic charges ensures that the resulting angular momentum is an integer multiple of $\hbar/2$. So while the statistics of the dyonic system can be changed due to angular momentum stored in the fields, the net angular momentum is still properly quantized and so the overall statistical angle is always an integer multiple of π. The full dyonic system is either a fermion or a boson.

As the field of the mirror monopole is indistinguishable from the field of a real monopole, the same calculation implies that the electromagnetic fields generated by a single electric point charge e above the interface of a TI carry a non-trivial angular momentum as well. As the electric as well as the magnetic field below the interface (that is inside the TI) appear to be sourced by coinci-

dent charges located at the location of the actual physical charge, the contribution to the angular momentum from that region of space vanishes. By symmetry, the actual angular momentum in the system obtained from integrating over the electric and magnetic fields above the interface, sourced by the physical electric charge as well as the mirror charge, gives exactly half of the angular momentum one would get from a genuine charge/monopole pair with the same values of e and g (for details see the appendix). However the magnitude of the induced mirror charge is proportional to the finestructure constant α and furthermore depends continuously on the material properties μ and ϵ. Consequently, it generically does not obey Dirac quantization conditions. The resulting angular momentum is not quantized. As a result, the statistics one associates with these charge carriers based on their angular momentum is no longer simply fermionic or bosonic. Instead they seem to behave as anyons[2] with a statistical angle given by[6]

$$\theta_S = \pi \frac{L}{\hbar} = \frac{4\pi\alpha^2 P_3}{(\frac{\epsilon_1}{\epsilon_0} + \frac{\epsilon_2}{\epsilon_0})(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}) + 4\alpha^2 P_3^2}. \qquad (1)$$

$P_3 = \theta/(2\pi)$ is the electromagnetic polarization. It is 0 in a topologically trivial material and 1/2 inside a TI. In order to ensure that our expression for θ_S can be compared between different unit systems, we explicitly displayed factors of ϵ_0 and μ_0. For the bulk of this work we'll work with units where $\epsilon_0 = \mu_0 = 1$.

The non-trivial statistical angle was interpreted in Ref. 2 as a result of the two dimensional nature of the TI surface. It is well known that in two spatial dimensions, anyonic statistic is allowed. The authors of Ref. 2 proposed that in the vicinity of a TI surface an electric point charge indeed turns into an anyon with statistical angle θ_S and proposed an explicit experimental setup that would allow its measurement. This proposal raises one important conceptual puzzle: the angular momentum in the system and hence the inferred θ_S is entirely independent of the distance z_0 between the point charge and the interface. While it is reasonable to assume that a point charge in close vicinity of a TI surface has anyonic character, the calculation of θ_S via the induced angular momentum seems to predict that any point charge moving freely in three dimensional space would pick up a statistical angle θ_S provided there is a planar TI interface somewhere in the universe at arbitrary large distance z_0, which is very counter-intuitive and also seems to indicate that the anyon is truly 3+1 dimensional in character, contradicting the fact that 3+1 dimensional anyons should be impossible (the exception recently proposed in Ref. 4 can readily be understood in terms of a more complicated topology of configuration space in this case[5]).

One could expect the z_0-independence of the angular momentum to be an artifact of the special example of an infinite planar interface. After all, $L/\hbar$ is dimensionless and so could only depend on z_0 in the form of the ratio z_0/a, where a is another geometric scale in the problem. For the infinite plane, no such other scale is present. With this puzzle in mind, we analyze the angular momentum associated with an electric point charge in the vicinity of a TI interface for two different geometries: a TI in the shape of a sphere and a TI with a semi-infinite tube like shape inside a perfectly conducting cavity. We find that in *both* cases the angular momentum vanishes identically for any charge separated from the surface even by an infinitesimal amount. These two examples make us suspect that the angular momentum will in fact vanish for a charge close to (but not right on top) the surface of any finite size TI. In hind-sight, this result is not too surprising. After all, the microscopic description of the topological insulator is in terms of a system of electrons and protons with properly quantized charges, obeying the standard rules of quantum mechanics. Any state described by this microscopic system has to have a properly quantized angular momentum. As long as the effective theory correctly captures the long distance behavior of the system, it has to obey the quantization conditions obeyed by the microscopic constituents. So the angular momentum L for a charge close to any compact TI has to be an integer multiple of $\hbar/2$. As a consequence, L can not continuously vary as a function of z_0/a. As we expect $L \to 0$ for $z_0/a \to 0$, it should have been expected that $L = 0$ is indeed the correct answer for all finite values of z_0/a.

At first this result may indicate that the non-trivial θ_S identified in the planar case does not carry over to any compact sample and hence would not be measurable. But this is too naive. After all, total angular momentum of the one-particle system was only taken as a stand-in for the statistical angle of the excitations. A more careful analysis should directly analyze the two-particle system and study the change in action associated with a non-trivial loop in configuration space. Performing this analysis we find in the case of the planar interface that θ_S as inferred from the angular momentum in the one-particle system only describes the exchange of two particles in the limit that the size of the loop l is much larger than the separations $z_0^{(1)}$ and $z_0^{(2)}$ of the charges to the surface of the TI. This is consistent with the interpretation of θ_S as a topological effect. For $l \sim z_0^{(1),(2)}$ short distance effects become important. In this limit details of the path matter. But for $l \gg d_{1,2}$ the only effect surviving is the topological phase (which in this limit is independent of the shape of the loop as it should be). So the statistical angle governs large loops in configuration space. However, for the realistic case of a compact TI of linear size a, we should clearly expect significant finite size effects in the case that loops are of order the sample size, $l \sim a$. Indeed this expectation is born out. For generic $z_0^{(1),(2)}$ and a the change in action associated to taking particle 2 around particle 1 depends crucially on the path and has no relation to the θ_S obtained from the planar case. However, in the intermediate loop size regime

$$z_0^{(1),(2)} \ll l \ll a$$

we once more are able to show that the phase is topolog-

ical (independent of shape) and is given by the flat space value θ_S. While not surprising, this analysis clearly lays out that any experimental attempt at measuring θ_S e.g. as proposed in Ref. 2 has to be set up as to operate in this intermediate loop size regime.

This note is organized as follows: In the next section we'll review the solution for the potentials in the case of a spherical TI and construct the solution of the potentials in the case of a tube-like TI. In section 3 we analyze the distance dependence of the statistical angle. We first show that in the one particle system angular momentum vanishes identically in both examples. We then analyze explicitly the two-particle system and show, in the case of the sphere, that θ_S obtained from the flat space analysis does govern the intermediate loop size regime. We discuss the relevance of these results in section 4.

II. ELECTRIC AND MAGNETIC POTENTIALS IN TWO COMPACT EXAMPLES

A. Spherical TI

First let us analyze the electric and magnetic fields for a spherical topological insulator and a point-like electric charge outside the sphere. The corresponding potentials have been worked out in the supplementary material of Ref. 2. As shown in Fig. 1, a spherical topological insulator of a radius a and a magneto-electric polarization P_3 is centered at the origin, and a point-like electric charge is at $(0,0,d)$. ϵ_1 and μ_1 are the dielectric constant and the magnetic permeability outside the sphere, ϵ_2 and μ_2 the corresponding quantities inside the sphere.
Both inside and outside the sphere, the curl of electric and magnetic fields is zero, thus we can find a scalar potentials in both regions:

$$\begin{aligned} E^{(i)} &= -\nabla\Phi_E^{(i)} \\ B^{(i)} &= -\nabla\Phi_M^{(i)} \end{aligned} \tag{2}$$

where $i = 1, 2$ stand for inside and outside region. The most general solution for the potentials in eq. (2) can be written in terms of Legendre polynomials:

$$\begin{aligned} \Phi_E^{(1)} &= \tfrac{q}{\epsilon_1}\textstyle\sum \frac{r^l}{d^{l+1}} P_l(\cos\theta) + \sum A_l (\frac{a}{r})^{l+1} P_l(\cos\theta) \\ \Phi_E^{(2)} &= \textstyle\sum B_l (\frac{r}{a})^l P_l(\cos\theta) \\ \Phi_M^{(1)} &= \textstyle\sum C_l (\frac{a}{r})^{l+1} P_l(\cos\theta) \\ \Phi_M^{(2)} &= \textstyle\sum D_l (\frac{r}{a})^l P_l(\cos\theta) \end{aligned} \tag{3}$$

Solving boundary condition for the interface between trivial and topological insulator (that is continuity of the perpendicular components of $\vec{D}$ and $\vec{B}$ as well as the parallel components of $\vec{H}$ and $\vec{E}$), one arrives at:

FIG. 1: A charge q outside a spherical TI.

$$\begin{aligned} A_l &= \tfrac{q}{\epsilon_1}\tfrac{a^l}{d^{l+1}}\big[\tfrac{(\epsilon_1 l-\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]-(2\alpha P_3)^2 l(l+1)}{(2\alpha P_3)^2 l(l+1)+(\epsilon_1(l+1)+\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]}\big] \\ B_l &= \tfrac{q}{\epsilon_1}\tfrac{a^l}{d^{l+1}}\big[\tfrac{(\epsilon_1 l-\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]-(2\alpha P_3)^2 l(l+1)}{(2\alpha P_3)^2 l(l+1)+(\epsilon_1(l+1)+\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]} + 1\big] \\ C_l &= q\tfrac{a^l}{d^{l+1}}\tfrac{(2\alpha P_3)^2 l(2l+1)}{(2\alpha P_3)^2 l(l+1)+(\epsilon_1(l+1)+\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]} \\ D_l &= q\tfrac{a^l}{d^{l+1}}\tfrac{-(2\alpha P_3)^2 (l+1)(2l+1)}{(2\alpha P_3)^2 l(l+1)+(\epsilon_1(l+1)+\epsilon_2 l)[l/\mu_1+(l+1)/\mu_2]} \end{aligned} \tag{4}$$

The fields here could be considered to be generated by a point image electric charge, magnetic monopole, and a line of image electric or magnetic charges[2].

B. Semi-infinite rectangular TI-tube in conducting cavity

Next we want to consider the case of a semi-infinite topological insulator (the TI being at $z < 0$) tube with rectangular cross section inside a conducting wall, see Fig. 2. An electric point-like charge is located at (x', y', z'). As in the spherical case, we can write down general forms for electric and magnetic potential:

$$\begin{aligned} \Phi_E^{(1)} &= \tfrac{q}{\epsilon_1}\textstyle\sum O_{mn}\sin(k_m x)\sin(k_n y)e^{\gamma_{mn}(z-z')} \\ &\quad + \textstyle\sum A_{mn}\sin(k_m x)\sin(k_n y)e^{-\gamma_{mn}z} \\ \Phi_E^{(2)} &= \textstyle\sum B_{mn}\sin(k_m x)\sin(k_n y)e^{\gamma_{mn}z} \\ \Phi_M^{(1)} &= \textstyle\sum C_{mn}\sin(k_m x)\sin(k_n y)e^{-\gamma_{mn}z} \\ \Phi_M^{(2)} &= \textstyle\sum D_{mn}\sin(k_m x)\sin(k_n y)e^{\gamma_{mn}z} \end{aligned} \tag{5}$$

with O_{mn} being the coefficients of the Green's function of a point-like charge in this system:

$$O_{mn} = \frac{2}{ab\gamma_{mn}}\sin(k_m x')\sin(k_n y'). \tag{6}$$

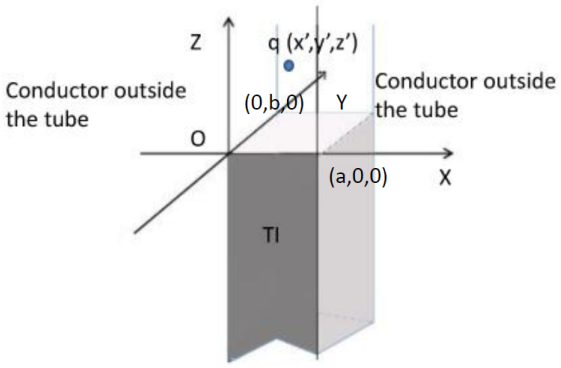

FIG. 2: An electric charge q inside the positive half-tube $z > 0$ with dielectric constant ϵ_1 and magnetic permeability μ_1 in the vicinity of a TI with ϵ_2 and μ_2 filling the negative half-tube $z < 0$. The tube has a width a, and length b. We assume that the walls of the tube are formed by a perfect conductor.

Solving boundary condition, we arrive at:

$$\begin{aligned} D_{mn} &= -\frac{2qO_{mn}e^{-\gamma_{mn}z'}(2\alpha P_3)}{(\epsilon_1+\epsilon_2)(1/\mu_1+1/\mu_2)+(2\alpha P_3)^2} \\ C_{mn} &= -D_{mn} \\ B_{mn} &= \frac{2qO_{mn}e^{-\gamma_{mn}z'}(1/\mu_1+1/\mu_2)}{(\epsilon_1+\epsilon_2)(1/\mu_1+1/\mu_2)+(2\alpha P_3)^2} \\ A_{mn} &= qO_{mn}e^{-\gamma_{mn}z'} \\ &\quad (-\frac{1}{\epsilon_1}+\frac{2(1/\mu_1+1/\mu_2)}{(\epsilon_1+\epsilon_2)(1/\mu_1+1/\mu_2)+(2\alpha P_3)^2}) \end{aligned} \tag{7}$$

In this case the geometry of the TI is not really compact. However, due to the Dirichlet boundary conditions on the conducting walls from the 1d point of view all our gauge fields are massive and exponentially decay at large $|z|$. For all practical purposes, the TI is finite in extend in the z-direction.

III. DISTANCE DEPENDENCE OF STATISTICAL ANGLE

A. Total angular momentum as a global probe of the system

The total angular momentum is given by:

$$L = \int \vec{x} \times \frac{(\vec{E}\times\vec{H})}{4\pi c^2} d^3x = \frac{\epsilon}{4\pi}\int \vec{x}\times(\vec{E}\times\vec{B})d^3x. \tag{8}$$

The extra factor of $1/(4\pi)$ compared to Ref. 3 is due to the fact that we follow Gaussian units as in Ref. 2. In a normal system without a topological insulator, there would not be any non-zero angular-momentum of the system in the presence of only a static point-like electric charge. In the presence of a topological insulator, $\vec{L}$ can be non-zero. In order to address the question about distance dependence of the angular momentum and the statistical angle in the case of TIs with a compact geometry, we want to calculate the angular momentum in the two examples described in the previous section.

Let us start with the case of the rectangular tube. Plugging our answers from eqs. (2), (5) and (7) into the expression for the angular momentum eq. (8), we see that the integral over x and y can easily be done analytically. For example, starting with the x momentum density

$$\begin{aligned} l_x &= y(\partial_x\Phi_E\partial_y\phi_M - \partial_y\Phi_E\partial_x\phi_M) \\ &\quad -z(\partial_z\Phi_e\partial_x\phi_M - \partial_x\phi_E\partial_y\phi_M) \end{aligned} \tag{9}$$

we see that integrating over y first gives an expression proportional to

$$\begin{aligned} \int_0^b dy\, l_x &\sim m_1\cos(m_1\pi x/a)\sin(m\pi x/a) \\ &\quad +m\cos(m\pi x/a)\sin(m_1\pi x/a) \end{aligned} \tag{10}$$

for integers m, m_1 still to be summed over (as well as n and n_1 appearing in the coefficients). Further integration of this expression over dx then vanishes identically. We have confirmed that L_y and L_z similarly vanish after doing the x and y integrals using Mathematica.

The calculation in the spherical case is a little more cumbersome. Plugging eqs. (2), (3) and (4) into eq. (8), we can use the recurrence relation of Legendre polynomials to simplify the integrand. After performing the θ integral, the double sums occurring in $\vec{E}\times\vec{B}$ collapse into a single one. Performing the integral in spherical coordinates, we once more find that the angular momentum vanishes identically as long as the electric charge is separated even an infinitesimal amount from the surface of the sphere, irrespective of the distance d. If we set the distance d to zero to begin with and then perform the integrals and sums, we get back to the result of eq. (1) valid for the infinite half-plane. The fact that for $d = 0$ the sums give back this non-trivial result is a non-trivial check. For $d/a = 0$ one should clearly recover the result of the plane, which can also be thought of as $a \to \infty$. What is surprising is that this limit is not smooth. The angular momentum vanishes for any finite d.
As discussed in the introduction, this result should have been expected based on microscopic considerations. For a real finite system made of electrons and protons total angular momentum has to be quantized. So it can not continuously depend on d/a. For it to vanish at infinity it has to be zero for all d/a. Even for the topological insulator, genuine fractional angular momentum should only exist in the infinite system. With the electron charge being on the surface, one can get back the infinite system result.

B. Finite size loop path in the two particle system

While the angular momentum for a genuine charge/monopole pair correctly captures the statis-

tical angle, our zero result from the previous subsection strongly suggests that in the case of a spatially compact TI one should be more careful. To directly obtain the statistical angle, the natural thing to do is to look at two particles moving adiabatically around each other and study the phase change of the action. The action of two point particles at a fixed distance z_0 above a TI/insulator interface located at $z = 0$ contains the standard terms coupling the point particles to the gauge fields:

$$S = e \int A_\mu^{(2)} \frac{dx^\mu_{(1)}}{d\tau} + (1 \leftrightarrow 2) \tag{11}$$

where the super/sub-scripts (1) and (2) refer to the two particles, $A_\mu^{(i)}$ being the field sourced by particle i, $x^\mu_{(i)}$ its position. Let us for simplicity look at the case where particle 1 is kept at the origin in the x-y plane and particle 2 is taken around a non-trivial loop.

1. *Planar interface*

It is easy to work out the effect of this coupling for the planar interface in detail. Note that we can write the magnetic field of the mirror monopole of magnetic charge g at $z = -z_0$ induced by an electric charge e at $z = z_0$ (and bothat at $x = y = 0$) in terms of a vector potential:

$$A_\phi^{(1)} = \frac{g}{4\pi}(1 - \cos\theta)$$

where ϕ and θ are the angles in a spherical coordinate system centered on the location of the mirror monopole at $(0, 0, -z_0)$. That is the 2nd particle at $(x_{(2)}, y_{(2)}, z_0)$ in this coordinate system is located at $\tan\theta_{(2)} = \rho_{(2)}/z_0$ where $\rho^2 = x^2 + y^2$. This is the standard form of the vector potential of a monopole with a Dirac string running along the negative z-axis. This is the appropriate form to use for a mirror charge located below the interface, as this mirror charge is only supposed to be used when calculating fields above the interface (for the magnetic field below the interface one would similarly use the $\vec{A}$ associated to a monopole with a Dirac string running along the positive z-axis for a monopole located at $z = +z_0$. As there is no physical charge located below the interface and so no contribution to eq. (11) we will not need $\vec{A}$ in this region).

If we keep one particle fixed at $(x = 0, y = 0, z = z_0)$ and take the other particle around a loop at a fixed z_0 ($\phi_{(2)}$ goes to $\phi_{(2)} + 2\pi$ at fixed $\theta_{(2)}$), e^{iS} picks up a phase:

$$e^{iS} \to e^{iS} e^{e \int d\tau \dot{\phi}_{(2)} A_\phi^{(1)}} = e^{i\frac{e\,g}{2}(1-\cos\theta_{(2)})}$$

We see that due to the $\theta_{(2)}$ dependence already in the case of the planar interface, the resulting statistical angle depends on path. As $\tan\theta_{(2)} = \rho_{(2)}/z_0$, we get different answers depending on the size of the loop. When the charge is a finite distance z_0 above the interface and taken around a very small loop of size $l \ll z_0$ we have $\theta_{(2)} = 0$ along the whole path and correspondingly do not pick up any phase. With a loop large compared to z_0 (that is $l \gg z_0$), $\theta_{(2)} \to \pi/2$. In this limit we get a phase $e^{ie\,g/2}$ in the action independent of the detailed shape of the path in the x-y plane[7]. This change in the action corresponds to $e^{2i\theta_S}$ where θ_S is the statistical angle describing the exchange of two particles. Exchanging particle 1 and 2 twice should correspond to a closed loop in configuration space as the one we have been analyzing. So $\theta_S = eg/4$. Plugging in the g obtained for the mirror charge in Ref. 2 (rederived in SI units in the appendix here) we do get back to eq. (1).

In conclusion, we find that for a planar interface in the realistic case of a finite z_0 (so that our effective theory applies), only a large loop (compared to z_0) gives a shape independent answer governed by a topological statistical angle. For these large loops the statistical angle is half of that of a dyon made of a real electron/monopole pair in complete agreement with the angular momentum calculation in previous section. The modification for small loops presumably should be understood as a result of short range interactions.

2. *Spherical TI:*

For the case of a spherical TI of radius a we can also obtain the vector potential associated with a single point charge at $(0, 0, d)$ with $d = a + z_0$. It again only has a ϕ component with

$$A_\phi = -C_0 a \cos\theta - \sum_l^\infty \frac{C_l}{l} \frac{a^{l+1}}{r^l} \sin(\theta) P_l^1(\cos\theta).$$

The C_l are given in eq. (4). Most importantly, $C_0 = 0$ (this would be a net magnetic charge, so it has to vanish).

We use this expression for A_ϕ and study the phase change in spherical coordinates when two particles located at the same $r = d = a + z_0$ but different θ are exchanged. We take one particle fixed at $(0, 0, d)$ while the other one goes from ϕ to $\phi + 2\pi$ at a given θ. The action picks up a phase $2\pi A_\phi$. We plot this value for different z_0 together with the universal θ_S in Fig. 3.

There are a couple of interesting features about this plot. First, A_ϕ always vanishes for $\theta \to 0$ and $\theta \to \pi$, as expected after considering the planar case.

Second, for intermediate θ between 0 and $\pi/2$, there is a certain non-zero plateau region. For small $z_0 = d - a$ (small compared to a), this plateau region repeats the θ behavior of planar case before finite size effect kicks in. In this limit the plateau value of A_ϕ goes back to planar case result, the universal θ_S.

This can also be seen analytically from the large a asymptotic behavior of A_ϕ. To analyze the large radius limit it is convenient to rewrite the C_l in a way that makes

it explicit that A_ϕ is being sourced by a point mirror charge together with a mirror line charge extending from the mirror point charge to the origin:

$$C_l = \frac{a^l}{d^{l+1}}\left[g_2 + \frac{c_1}{-t_1+l+1} + \frac{c_2}{-t_2+l+1}\right]. \quad (12)$$

Explicit formulas for point charge g_2 and the parameters $t_{1,2}$ and $c_{1,2}$ characterizing the line charge appear in the supplemental material of Ref. 2 for $\epsilon = \mu = 1$. Most importantly, the point charge g_2 goes to the planar value g in the $d \to a$ limit. We have confirmed with Mathematica that the same decomposition holds for arbitrary ϵ and μ. The general formulas for g_2, $t_{1,2}$ and $c_{1,2}$ are too lengthy to reproduce here. Taking a large at fixed d, one sees that every term in the potential scales as $1/a$ and so the potential seems to vanish in the large a limit. This conclusion is too fast as the infinite sum can alter the behavior. At $\theta = 0$ (where $P_l(1) = 1$ for all l) the sum can be performed analytically and we see that the $r = a$ contribution for the point charge sums up to the expected $g_2/(d-a) = g_2/z_0$, that is it remains finite in the large a limit. As the sum over Legendre polynomials by construction just represents a standard $1/r$ Coulomb potential, the point charge term automatically reproduces the planar contribution to θ_S. At $\theta = 0$ the contribution of the line charge can also be summed up. The resulting Hypergeometric function vanishes as $1/a$ in the large a limit. As $P_l < 1$ for $\theta \neq 0$ it is clear that at non-zero θ the line charge contribution has to vanish at least as fast as for $\theta = 0$. So in the $a \to \infty$ limit, the line charge contribution can be neglected compared to the point-like one. This result has also been confirmed numerically in Ref. 2.

This shows that for loops of size $l \ll a$ we recover the planar result. However in the planar case we found that for $l \gg z_0$ the exchange is governed by the universal θ_S. From this we conclude that for a compact geometry it is the intermediate size loops with $z_0 \ll l \ll a$ which are governed by the universal topological phase of eq. (1). While we only explicitly demonstrated this result in the case of a sphere, we believe it to be true in general.

IV. DISCUSSION

We find zero angular momentum for a charge outside a spatially compact TI in two examples. While obvious from the microscopic point of view of electrons and protons, the result looks somewhat surprising starting from Maxwell's equations of a topological insulator; in fact we haven not been able to give an analytic proof based on this effective theory that the total angular momentum always vanishes in a finite system even though the microscopic point of view strongly suggest that this is true. By looking at the two particle system we find that the universal statistical phase governs the behavior of intermediate size loops. In realistic experiments, our results indicate that to observe such an anyon in the topological insulator set-up as suggested in Ref. 2, one should use a large bulk sample with a small superconductor loop close to it, to get non-zero flux. The size of the superconductor loop has to be in the intermediate regime we identified, that is much smaller than the size of the sample but much large than the distance between loop and sample. Our results are consistent with the 2+1 dimensional nature of the anyons, as the universal statistical angle ceases to accurately describe the two particle systems once the particles are removed from the surface of the TI beyond a distance of order the sample size.

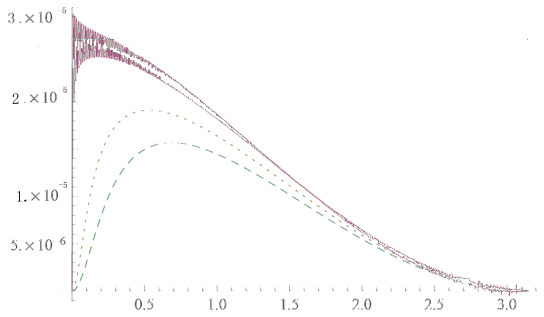

FIG. 3: A_ϕ versus θ with different z_0's for the mirror charge on an electron. $z_0 = 0.05a$ for the dotted line and $z_0 = 0.1a$ for the dashed line and $z_0 = 0.0005a$ for the solid curved line. The horizontal straight line stands for the universal angle θ_S. The thickness of the $z_0 = 0.0005a$ line represents the numerical uncertainty due to the slow convergence of the large l truncation. In this plot we chose $a = 1$ and all permittivities and permeabilities equal 1. The small numerical value of θ_S is due to the appearance of α in the expression for the mirror charge.

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Appendix A: Derivation of the statistical angle in SI units

Consider a planar interface between a topological insulator (with non-trivial μ_2 and ϵ_2 as well as $\theta = \pi$, that is $P_3 = 1/2$) and a trivial insulator (with ϵ_1 and μ_1). Consider a single static point charge q inside the trivial material a distance z_0 away from the surface of the TI.

We want to find the mirror charges for general μ and ϵ. Let the TI occupy the $z < 0$ region of space. Writing Maxwell's equation as usual as

$$\vec{\nabla}\cdot\vec{D} = \rho_e, \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j}_e, \vec{\nabla}\cdot\vec{B} = \rho_m, \vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t} + \vec{j}_m.$$

The constitutive relations in SI unites are

$$\vec{D} = \epsilon\vec{E} - \epsilon_0\alpha\frac{\theta}{\pi}(c_0\vec{B}), \qquad c_0\vec{H} = \frac{c_0\vec{B}}{\mu} + \alpha\frac{\theta}{\pi}\frac{\vec{E}}{\mu_0}.$$

We can introduce potentials $\Phi_{e,m}$ with $\vec{E} = -\vec{\nabla}\Phi_e$ and $\vec{B} = -\vec{\nabla}\Phi_m$. Above the interface they are given by (note that we weight all electric mirror charges by ϵ_0 for convenience; we also use ϵ_1 for q_e on both sides of the interface; these are just definitions of our mirror charges)

$$\Phi_e^I = \frac{q_e}{4\pi\epsilon_1 R_1} + \frac{q_e^{(2)}}{4\pi\epsilon_0 R_2}, \quad \Phi_m^I = \frac{q_m^{(2)}}{4\pi R_2}$$

and below by

$$\Phi_e^{II} = \frac{q_e}{4\pi\epsilon_1 R_1} + \frac{q_e^{(1)}}{4\pi\epsilon_0 R_1}, \quad \Phi_m^{II} = \frac{q_m^{(1)}}{4\pi R_1}$$

where $q_{e,m}^{(1,2)}$ are the mirror charges locate a distance d above (for $q_{e,m}^{(1)}$) or below (for $q_{e,m}^{(2)}$) the interface. $R_1^2 = x^2+y^2+(z_0-z)^2$ and $R_2^2 = x^2+y^2+(z_0+z)^2$. Maxwell's equations in the absence of surface currents or charges as usual demand continuity of $D_\perp$, $B_\perp$, $H_\parallel$ and $E_\parallel$. As, at $z=0$, $R_1 = R_2$, $\partial_z R_1 = -\partial_z R_2$ this demands:

$$\begin{aligned}
(q_e - \frac{\epsilon_1}{\epsilon_0}q_e^{(2)}) &= (\frac{\epsilon_2}{\epsilon_1}q_e + \frac{\epsilon_2}{\epsilon_0}q_e^{(1)}) - \epsilon_0\alpha\frac{\theta}{\pi}(c_0 q_m^{(1)}) \\
q_m^{(1)} &= -q_m^{(2)} \\
\frac{q_m^{(2)}}{\mu_1} &= \frac{q_m^{(1)}}{\mu_2} + \alpha\frac{\theta}{\pi}\frac{q_e/\epsilon_1 + q_e^{(1)}/\epsilon_0}{\mu_0 c_0} \\
q_e^{(2)} &= q_e^{(1)}.
\end{aligned}$$

From this the mirror charges can easily be found:

$$q_m^{(2)} = -q_m^{(1)} = \frac{1}{c_0}\frac{2\alpha\frac{\theta}{\pi}q}{(\epsilon_1+\epsilon_2)(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2}) + \epsilon_0\alpha^2\frac{\theta^2}{\pi^2}}$$

and

$$q_e^{(2)} = q_e^{(1)} = \frac{\epsilon_0}{\epsilon_1}\frac{(\epsilon_1-\epsilon_2)(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2}) - \epsilon_0\alpha^2\frac{\theta^2}{\pi^2}}{(\epsilon_1+\epsilon_2)(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2}) + \epsilon_0\alpha^2\frac{\theta^2}{\pi^2}}\,q$$

The system consisting a the charge q and a magnetic charge q_m gives rise to an angular momentum

$$\vec{L} = \frac{qq_m}{4\pi}\hat{r}$$

where $\hat{r}$ is the unit vector pointing from the electric to the magnetic charge. Clearly $\vec{L}$ vanishes when the two charges are sitting on top of each other (as in this case $\vec{E}$ and $\vec{B}$ are parallel, so the momentum density and hence the angular momentum density vanish identically).

For the interface, we need to calculate the contributions to the angular momentum in the two regions independently. Inside the TI both electric and magnetic fields are pointing radially outward from the point at $z = +z_0$, so the angular momentum vanishes (again, $\vec{E}$ and $\vec{B}$ are parallel and so the Poynting vector vanishes identically). For the region above the interface, we get a non-zero contribution to the angular momentum due to the charge/monopole system formed by the original charge q at $z = +z_0$ and the mirror magnetic charge $q_m^{(2)}$ at $z = -z_0$ (the electric mirror charge $q_e^{(2)}$ at $z = -z_0$ does not contribute, as the charge at $z = +z_0$ is purely electric). If we calculate the angular momentum with respect to the origin $x = y = z = 0$, we see that the integrand

$$\begin{aligned}
\vec{r}\times(\vec{E}\times\vec{B}) &\sim \frac{\vec{r}\times[(\vec{r}-z_0\hat{e}_z)\times(\vec{r}+z_0\hat{e}_z)]}{|\vec{r}-z_0\hat{e}_z|^2|\vec{r}+z_0\hat{e}_z|^2} \\
&= z_0\,\frac{\vec{r}\times[\vec{r}\times\hat{e}_z]}{|\vec{r}-z_0\hat{e}_z|^2|\vec{r}+z_0\hat{e}_z|^2}
\end{aligned} \tag{A1}$$

is symmetric under $\vec{r}\to-\vec{r}$. So we get equal contributions to the angular momentum from the lower and the upper half plane. As in our case we only get a contribution from the upper half plane, the angular momentum to the charge/mirror-charge system is exactly *half* of what it would be for a genuine charge/monopole pair, that is (setting $\theta = \pi$, $q = -e$ in the expressions above)

$$\begin{aligned}
L_z = -\frac{qq_m^{(2)}}{8\pi} &= -\frac{1}{4\pi c_0}\frac{\alpha e^2}{(\epsilon_1+\epsilon_2)(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2})+\epsilon_0\alpha^2} \\
&= -\frac{\alpha^2\hbar}{(\frac{\epsilon_1}{\epsilon_0}+\frac{\epsilon_2}{\epsilon_0})(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2})+\alpha^2}
\end{aligned} \tag{A2}$$

where in the last step we used the standard SI definition for α,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c_0}.$$

Note that for a given q, say $q = -e$, but for general μ and ϵ this is certainly not going to obey the Dirac quantization condition. The statistical angle one would want to associate with a dyon with angular momentum L_z is $\theta_S = 2\pi L_z/\hbar$, so that the statistical angle θ_S is 0 for integer spins (bosons) and π for half-integer spins (fermions). For a charge $-e$ in the presence of a TI surface we therefore obtain

$$|\theta_S| = 2\pi\frac{\alpha^2}{(\frac{\epsilon_1}{\epsilon_0}+\frac{\epsilon_2}{\epsilon_0})(\frac{\mu_0}{\mu_1}+\frac{\mu_0}{\mu_2})+\alpha^2}.$$

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[6] Our expression differs by a factor of 2π from the one quoted in^2. A factor of 4π seems to be due to the fact that in^2, which works in Gaussian units, the formula for the statistical angle has been normalized to yield $\theta_S = \pi$ for a monopole of flux hc/e. However latter has been taken to correspond to $g = hc/e$, which would be correct if $\vec{\nabla} \cdot \vec{B} = g\delta(\vec{r})$ as it is in SI units. However in Gaussian units $\vec{\nabla} \cdot \vec{B} = 4\pi g\delta(\vec{r})$ and so the formula for θ_S in^2 in terms of general e and g should be modified by a factor of 4π. The remaining factor of 2 mismatch seems to be due to the fact that in^2 the factor of $1/2$ from the fact that only the half of space above the interface contributes to the angular momentum has not been taken into account. For the reader's convenience we rederive the expression of the statistical angle in SI units in the appendix, where intermediate steps can be readily compared to standard textbook expressions from e.g.[3]. This confirms the factor of 2π as we have it here. For the bulk of the paper we follow[2] and use Gaussian units. The final expression for the statistical angle has to be independent of choices of units when only expressed in terms of the fine structure constant α as well as the ratios ϵ/ϵ_0 and μ/μ_0.

[7] These answers also obvious when one considers that $\vec{\nabla} \times \vec{A} = \vec{B}$, so the line integral of $\vec{A}$ we are doing is equal to the total magnetic flux through the surface enclosed by the loop. For $\theta_{(2)} = 0$ the loop has vanishing area and no flux. For $\theta_{(2)} = \pi/2$ the loop captures all the flux in the northern hemisphere.