Scaling of the anomalous Hall effect in the insulating regime

Xiong-Jun Liu, Xin Liu, and Jairo Sinova

Phys. Rev. B **84**, 165304 — Published 4 October 2011

DOI: 10.1103/PhysRevB.84.165304

The Scaling of the Anomalous Hall Effect in the Insulating Regime

Xiong-Jun Liu[1], Xin Liu[1] and Jairo Sinova[1,2]

[1]*Department of Physics, Texas A&M University, College Station, Texas 77843-4242, USA*
[2]*Institute of Physics ASCR, Cukrovarnick 10, 162 53 Praha 6, Czech Republic*

(Dated: September 12, 2011)

We develop the theory of the scaling of the anomalous Hall effect (AHE) in the insulating regime, which is observed in experiments to relate the anomalous Hall and diagonal conductivities by $\sigma_{xy}^{AH} \propto \sigma_{xx}^{1.40\sim1.75}$ for a large range of materials. This scaling is qualitatively different from the ones observed in metals. Basing our theory on the phonon-assisted hopping mechanism and percolation theory in random networks, we derive a general formula for the anomalous Hall conductivity (AHC), in which the percolation theory averaging of the random linked triad clusters is a key aspect that captures the correct observed physics. We show that it scales with the longitudinal conductivity as $\sigma_{xy}^{AH} \sim \sigma_{xx}^{\gamma}$ with γ predicted to be $1.33 \leq \gamma \leq 1.76$, quantitatively in agreement with the experimental observations. Our theory predicts that this scaling remains similar regardless of whether the hopping process is long range type (variable range hopping) or short range type (activation hopping), or is influenced by interactions, i.e. Efros-Shklovskii (E-S) regime. Our theory completes the understanding of the AHE phase diagram in the insulating regime.

PACS numbers: 75.50.Pp, 72.20.Ee, 72.20.My

I. INTRODUCTION

The anomalous Hall effect (AHE) is a central topic in the study of Ferromagnetic materials[1]. It exhibits the empirical relation $\rho_{xy} = R_0 B_z + R_S M_z$ between the total Hall resistivity and the magnetization M_z and external magnetic field B_z. Here R_0 and R_S are respectively the ordinary and anomalous Hall coefficients. In experiment the anomalous Hall resistivity (AHR) is usually observed to follow a power law form versus the longitudinal resistivity $\rho_{xy}^{AH} \sim \rho_{xx}^{\beta}$, with ρ_{xx} varied by changing the temperature T, disorder scattering or density of states (DOS) around Fermi surface. When transformed to the anomalous Hall conductivity (AHC), σ_{xy}^{AH}, the scaling relation takes the form $\sigma_{xy}^{AH} \approx \rho_{xy}^{AH}/\sigma_{xx}^2 \sim \sigma_{xx}^{2-\beta}$. Three regimes are observed with respect to its dependence on the diagonal conductivity, σ_{xx}[1]. In the metallic regime the AHC σ_{xy}^{AH} is observed to be linearly proportional to σ_{xx} for the highest metallic systems ($\sigma_{xx} > 10^6 \Omega^{-1}$ cm^{-1}) and roughly constant for the rest of the metallic regime. This dependence indicates the different dominant mechanisms in ferromagnetic metals. These are understood to be the skew scattering, side jump scattering, and intrinsic deflection mechanisms. The intrinsic contribution is induced by a momentum-space Berry phase linked to the electronic structure of the multi-band spin-orbit (SO) coupled system[1,2]. The side jump scattering mechanism gives the same scaling relation as the intrinsic contribution, i.e. $\sigma_{xy}^{AH-sj} \propto \sigma_{xx}^0$, and the skew scattering is linear in the longitudinal conductivity, $\sigma_{xy}^{AH-sk} \propto \sigma_{xx}$. While these mechanisms are now better understood, the maximum scaling exponent of the AHC cannot exceed unity in the metallic regime[1].

On the other hand, experiments in the insulating regime exhibit an unexpected scaling relation of the AHC: $\sigma_{xy}^{AH} \propto \sigma_{xx}^{1.40\sim1.75}$ with the scaling exponent generically larger than unity[3–14]. Earlier experiments on AHE in this regime was done in magnetite Fe_3O_4[3], and the recent experimental observations of this scaling are reported in a large range of materials including granular Fe/SiO_2 films, magnetite epitaxial thin films, dilute magnetic semiconductor (DMS) $Ga_{1-x}Mn_xAs$, and ferromagnetic semiconductor anatas $Ti_{1-x}Co_xO_{2-\delta}$. The observed scaling in the insulating regime has remained unexplained and a major challenge in understanding fully the phase diagram of the AHE

The hopping transport regime prevails when a system is in the disordered insulating regime, with the impurity on-site energies randomly distributed. At low temperature ($T \ll E_0/k_B$ with E_0 the ionization energy of the bound states), the charge transport in such system will be dominated by the phonon-assisted hopping of electrons/holes between impurity sites[16,17].

To capture the Hall effect one requires the hopping process between impurity sites (Fig. 1) to break the time-reversal (TR) symmetry. The two-site direct hopping preserves TR symmetry, and contribute only to the longitudinal charge transport. The hopping through triads (three sites) is the minimum requirement to model theoretically the Hall effect[18]. The total hopping amplitude is obtained by adding the direct and indirect (through the intermediate k-site) hopping terms from i to j sites. The two hopping paths give rise to an interference term for the transition rate which breaks TR symmetry and is responsible for the Hall current in the hopping regime. For the ordinary Hall effect (OHE), the interference is a reflection of the Aharonov-Bohm phase, and for the AHE it reflects the Berry phase due to SO coupling.

While the hopping through triads reveals the minimum element contributing to the AHE in the hopping conduction regime, *the crucial step to understand the insulating AHE and the observed scaling relation is hidden in the evaluation of the AHC within percolation theory.* In the

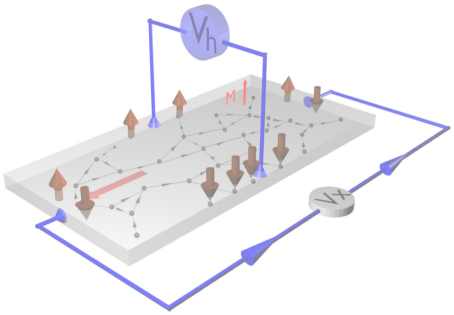

FIG. 1: AHE in the insulating regime. In this regime charge transport occurs via hopping between impurity sites.

hopping conduction regime, the charge transport is *not* dominated by the whole impurity system but by specific percolation clusters, which span the whole material but cover only part of the impurity sites. To correctly evaluate the AHC, one must address a highly nontrivial issue: *how to average the AHC over the percolation clusters with triads.*

The few previous studies of the AHE in this regime have been focused on manganites and $Ga_{1-x}Mn_xAs$ by employing Holstein's theory in the anomalous Hall system. In manganites, a non-uniform magnetic system, the AHC is determined only by optimal triads, and therefore this material does not exhibit the scaling observed in the typical disordered insulators[19,20]. On the other hand, the studies on insulating $Ga_{1-x}Mn_xAs$ did not appreciate the central aspect of the hopping conduction that the AHC should be averaged over percolation cluster, and thus failed in explaining the observed scaling[21,22]. A numerical study of the AHE using metallic theory observed a scaling in a disordered metallic regime but both the AHC and σ_{xx} remained metallic in this study while the scaling was only present for a particular sign of the impurities[1,15]. The AHE theory in the metallic regime is generally based on the perturbation expansion in terms of small parameter $1/(k_F l)$. Here k_F is the magnitude of the Fermi wave vector, and l is the length of mean free path. Hence, the available microscopic theories of metals fail in the insulating regime since the condition $k_F l \gg 1$ is no longer satisfied for disordered insulators by its own construction[1,15].

In this paper we develop a theoretical approach to study the scaling of the AHE in the insulating strongly disordered amorphous regime. It is the first time that the AHC in this regime is correctly formulated and exactly calculated by averaging over the percolation cluster. In light of the fact that the charge transport is dominated by the percolation cluster, we derive rigorously a new configuration averaging formula for the AHC, with the key physics that the Hall currents are averaged over percolation cluster containing triads completely considered. With our formalism we calculate the upper and lower limits of the AHC which correspond to different extreme situations for the triad spatial distribution, and show it scales with σ_{xx} as $\sigma_{xy}^{AH} \sim \sigma_{xx}^{\gamma}$, where γ is predicted to be $1.33 \leq \gamma \leq 1.76$ with only a slightly quantitative dependence on the specific hopping types. Namely, the scaling remains similar in the Mott variable range hopping (VRH), Efros-Shkolvskii (E-S) regime, and in activation E_3 hopping regime. This matches the experimental observation that the scaling is seen in many types of insulators with different hopping types dominating.

II. THE MODEL

Our theory is based on a minimal tight-binding Hamiltonian. With the particle-phonon coupling considered, the total Hamiltonian $H = H_p + H_c + H_{ph}$, with

$$\begin{aligned} H_p &= \sum_{i\alpha} \epsilon_i \hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\alpha} - \sum_{i\alpha,j\beta} t_{i\alpha,j\beta}\hat{c}_{i\alpha}^{\dagger}\hat{c}_{j\beta} + \sum_{i\alpha\beta} \mathbf{M}\cdot\tau_{\alpha\beta}\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\beta} \\ H_c &= i\eta \sum_{i\alpha\lambda} (\vec{q}_\lambda \cdot \vec{e}_\lambda)\omega_\lambda^{-1/2}(b_\lambda e^{i\vec{q}_\lambda\cdot\vec{r}} - b_\lambda^{\dagger}e^{-i\vec{q}_\lambda\cdot\vec{r}})\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\alpha} \\ H_{ph} &= \sum_{\lambda} \omega_\lambda b_\lambda^{\dagger} b_\lambda. \end{aligned}$$

Here H_p describes localized states, H_c gives the particle-phonon coupling with η the coupling constant, H_{ph} is the phonon Hamiltonian, α is the local on-site total angular momentum index, and ϵ_i is the energy measured from the fermi level. Here we consider that the magnetization is saturated and thus assume $\mathbf{M} = M\hat{e}_z$. We rewrite the Hamiltonian H_p in the diagonal basis of the exchange term and obtain

$$H_p = \sum_{\alpha} \epsilon_{i\alpha}\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\alpha} - \sum_{i\alpha,j\beta} t_{i\alpha,j\beta}\hat{c}_{i\alpha}^{\dagger}\hat{c}_{j\beta}, \qquad (1)$$

where $\epsilon_{i\alpha} = \epsilon_i + M\tau_{\alpha\alpha}$. The hopping matrix t_{ij} is generally off-diagonal due to SO coupling. For example, for the dilute $Ga_{1-x}Mn_xAs$, the matrix $t_{i\alpha,j\beta}$ describes the hopping of the holes localized on the Mn impurities. Under the spherical approximation $t_{i\alpha,j\beta}$ can be obtained based on a unitary rotation $U(\mathbf{R}_{ij})$ from the $\hat{e}_z$ direction to the hopping direction $i \to j$[23]. We thus have $t_{i\alpha,j\beta} = [U^{\dagger}(\mathbf{R}_{ij})t_{diag}U(\mathbf{R}_{ij})]_{\alpha\beta}$ with $t_{diag} = \mathrm{diag}[t_{3/2}, t_{1/2}, t_{-1/2}, t_{-3/2}]$ representing the situation that the hopping direction is along the z axis. Another example is for the localized s-orbital electrons. In this case, the hopping matrix is given by $t_{ij} = U^{\dagger}(\mathbf{R}_{ij})[\tilde{t}_{ij}(1 + i\vec{v}_{ij}\cdot\vec{\sigma})]U(\mathbf{R}_{ij})$[20]. Here $\tilde{t}_{ij} = \mathrm{diag}[t_{1/2}, t_{-1/2}]$ and $\vec{v}_{ij} = \frac{\alpha}{\hbar}\int_{\vec{r}_i}^{\vec{r}_j}(\nabla V(\mathbf{r}) \times d\vec{r}')$ with $V(\mathbf{r})$ including the ion and external potentials, the SO coupling coefficient $\alpha = \hbar/(4m^2c^2)$ and m the effective mass of the electron. The localization regime has the condition $|t_{i\alpha,j\beta}| \ll |\epsilon_i - \epsilon_j|$ in average. The specific form of the relevant parameters (t_{ij}, M, spin operator $\tau_{\alpha\beta}$) are material dependent and do not affect the scaling relation between σ_{xy}^{AH} and σ_{xx}.

Considering the dominant contributions to the longitudinal and Hall transports, which include the one- and

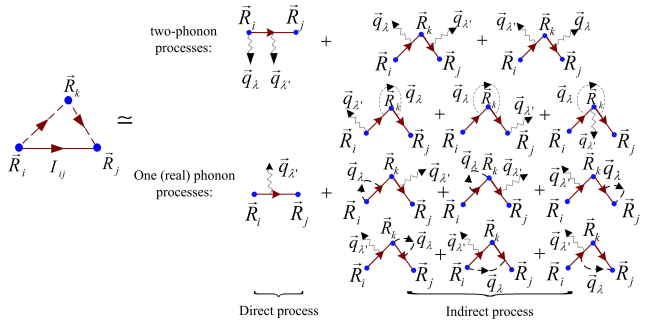

FIG. 2: (Color online) The hopping processes through triads with up to two real phonons absorbed or emitted. (Top) Typical diagrams of the two-phonon direct and indirect hopping processes. (Bottom) One-phonon direct process and typical three-phonon (one real phonon) indirect hopping processes.

two-real-phonon processes through triads (Fig. 2)[18], we obtain the charge current between i and j sites in a single triad with applied voltages[21]:

$$I_{ij} = G_{ij}V_{ij} + \mathcal{G}_{ijk}(V_{ik} + V_{jk}), \tag{2}$$

where the direct conductance

$$G_{ij} = \frac{2\lambda_0 e^2}{k_B T}\sum_{\alpha\beta}|t_{i\alpha,j\beta}|^2 T_{ij}^{(2)}, \tag{3}$$

with $T_{ij}^{(2)} = |\Delta_{ij}|e^{-\frac{1}{2}\beta(|\epsilon_{i\alpha}|+|\epsilon_{j\beta}|+|\epsilon_{i\alpha}-\epsilon_{j\beta}|)}$, $\Delta_{ij} = \epsilon_{i\alpha} - \epsilon_{j\beta}$, $\epsilon_{i\alpha} = \epsilon_i + M\tau_{\alpha\alpha}$, and the constant $\lambda_0 \propto \eta^2$[18]. The second term in I_{ij} is responsible for the Hall transport and

$$\mathcal{G}_{ijk} = \frac{4\lambda_0^2 e^2}{k_B T}\sum_{\alpha\beta\gamma}[\mathrm{Im}(t_{i\alpha,j\beta}t_{j\beta,k\gamma}t_{k\gamma,i\alpha})T_{ijk}^{(3)}], \tag{4}$$

where $T_{ijk}^{(3)} = T_{i,jk}^{(3)} + T_{k,ij}^{(3)} + T_{j,ki}^{(3)}$, with $T_{i,jk}^{(3)} = |\Delta_{ij}\Delta_{ik}|e^{-\frac{1}{2}\beta(|\epsilon_{j\beta}|+|\epsilon_{k\gamma}|+|\epsilon_{i\alpha}-\epsilon_{k\gamma}|+|\epsilon_{i\alpha}-\epsilon_{j\beta}|)}$. The function $\mathrm{Im}(t_{i\alpha,j\beta}t_{j\beta,k\gamma}t_{k\gamma,i\alpha})$ gives a geometric phase term corresponding to the closed path hopping $i \to j \to k \to i$. The formula of I_{ij} gives the microscopic conductance in any single triad. To evaluate the macroscopic AHC, one shall properly average it over all triads in the random system. This is achieved with the aid of percolation theory, a fundamental tool to understand the hopping transport.

III. PERCOLATION THEORY

We first map the random impurity system to a random resistor network by introducing the connectivity between impurity sites with the help of a cut-off $G_c(T)$. When the conductance between two impurity sites satisfies $G_{ij} \geq G_c$, we consider the i, j sites are connected with a finite resistor $Z_{ij} = 1/G_{ij}$. Otherwise, they are treated as disconnected, i.e. $G_{ij} \to 0$. The Hall effect will be treated as a perturbation to the obtained resistor network. The cut-off G_c should be properly chosen so that the long-range critical percolation paths/clusters appear and span the whole material, and dominate the charge transport in the hopping regime. The macroscopic physical quantities will finally be obtained by averaging over the percolation path/cluter.

The hopping coefficient generally has the form $t_{i\alpha,j\beta} = t_{i\alpha,j\beta}^{(0)}e^{-aR_{ij}}$, with a^{-1} the localization length and $R_{ij} = |\mathbf{R}_i - \mathbf{R}_j|$. The direct conductance holds the form $G_{ij} = G_0(T)e^{-2aR_{ij}-\frac{1}{2}\beta(|\epsilon_{i\alpha}|+|\epsilon_{j\beta}|+|\epsilon_{i\alpha}-\epsilon_{j\beta}|)}$, and then the cut-off can be introduced by $G_c = G_0 e^{-\beta\xi_c(T)}$[24]. Here $\beta\xi_c$ is a decreasing function of T, indicating the material in the insulating regime. The number of impurity sites connected to a specific site i with energy ϵ_i can be calculated by

$$n(\epsilon_i, \xi_c) = \int d\epsilon_j \int d^3\vec{R}_{ij}\rho(\epsilon_j, \vec{R}_i)\Theta\big(G_{ij} - G_c\big). \tag{5}$$

Here $\Theta(x)$ is the step function and the DOS $\rho(\epsilon, \vec{R}_i) \approx \frac{1}{V}\sum_i \delta(\epsilon - \epsilon_i)$ is approximated to be spatially homogeneous. The number $n(\epsilon_i, \xi_c)$ can also be given by $n(\epsilon_i, \xi_c) = \sum_n P_n(\epsilon_i, \xi_c)$, with $P_n(\epsilon_i, \xi_c)$ being the probability that the n-th smallest resistor connected to the site i has the resistance less than $1/G_c$. The function P_n reads[25]

$$P_n(\epsilon_i, \xi_c) = \frac{1}{(n-1)!}\int_0^{n(\epsilon_i)} e^{-x}x^{n-1}dx, \tag{6}$$

which can be derived according to the Poisson distribution. The percolation path/cluster appears when the average connections per impurity site $\bar{n} = \langle n(\epsilon_i)\rangle_c$ reaches the critical value $\bar{n}_c$, where the definition of $\langle ...\rangle_c$ is given in Eq. (7). Suppose a physical quantity $F(\epsilon_1, ..., \epsilon_m; \vec{r}_1, ..., \vec{r}_m)$ being a m-site function, requiring the i-th site to have at least η_i sites connected to it. The averaging of $F(\epsilon; \vec{r})$ reads

$$\begin{aligned}\langle F(\epsilon;\vec{r})\rangle_c &= \frac{1}{\mathcal{N}_F}\int d\epsilon_1 ... \int d\epsilon_m \int d^3\vec{r}_{12} ... \int d^3\vec{r}_{m-1,m} \\ &\times \prod_{i=1}^{m}\mathcal{P}_{\eta_i}(\epsilon_i)F(\epsilon_1, ..., \epsilon_m; \vec{r}_1, ..., \vec{r}_m),\end{aligned} \tag{7}$$

where $\mathcal{N}_F$ is a normalization factor and the probability function $\mathcal{P}_{\eta_i}(\epsilon_i) = \rho(\epsilon_i)\sum_{k\geq\eta_i}P_k(\epsilon_i)$. The term $\sum_{k\geq\eta_i}P_k(\epsilon_i)$ entering the probability function has important physical reason. The configuration averaging is not conducted over the whole impurity system, but over the percolation cluster which covers only portion of the impurity sites. Therefore the probability that an impurity site belonging to the percolation cluster must be taken into account for probability function. Moreover, this probability function also distinguishes the physical origins of the AHC and σ_{xx}. For σ_{xy}^{AH} one has $\eta_i = 3$, and for σ_{xx} one has $\eta_i = 2$. This indicates the averaging of σ_{xx} is performed along the one dimensional (1D)

percolation path, while for AHE which is a two dimensional (2D) effect, one shall evaluate AHC over all triads connected in the 2D percolation cluster.

IV. CONFIGURATION AVERAGING OF THE ANOMALOUS HALL CONDUCTIVITY

According to the formula (7), the average value of the 1-site function $n(\epsilon_i, \xi_c)$ in the percolation cluster is calculated by

$$\bar{n} = \frac{\int d\epsilon_i n(\epsilon_i)\rho(\epsilon_i)n(\epsilon_i)}{\int d\epsilon_i n(\epsilon_i)\rho(\epsilon_i)}. \tag{8}$$

When the DOS $\rho(\epsilon_i) = \rho_0$ is a constant, the number $n(\epsilon_i)$ is given by $n(\epsilon_i) = \frac{2\pi}{3}\frac{\rho_0}{(2ak_BT)^3}(\xi_c - |\epsilon_i|)^2(\xi_c^2 - |\epsilon_i|^2)$. Then we have $\bar{n} = 0.406\pi\frac{\rho_0}{(2ak_BT)^3}\xi_c^4$. The hopping conduction occurs when the average value $\bar{n}$ reaches the critical value $\bar{n}_c$. We obtain then the cut-off value ξ_c by $\xi_c(T) = \left[\frac{(2ak_BT)^3\bar{n}_c}{0.406\pi\rho_0}\right]^{1/4}$. Thus it gives

$$\beta\xi_c = \left(\frac{T_0}{T}\right)^{1/4}, \quad T_0 = 16\frac{a^3\bar{n}_c}{k_B\rho_0}, \tag{9}$$

which is the Mott law[17]. Accordingly, if we assume the density of states $\rho(\epsilon) \sim \epsilon^2$, we obtain straightforwardly the E-S law $\beta\xi_c = \left(\frac{T_0}{T}\right)^{1/2}$[26,27].

Numerical solutions show the critical site connectivity is $\bar{n}_c = 2.6 \sim 2.7$ for the appearance of a percolation path/cluster in three dimensional (3D) materials[28,29]. This indicates the triads are sparsely distributed in the percolation cluster, as shown in Fig. 3. The AHC can be derived by examining the transverse voltage V_y^H (along the y-axis) induced by the applied longitudinal current I_0. Denote by $N(x)$ the number of triads distributed

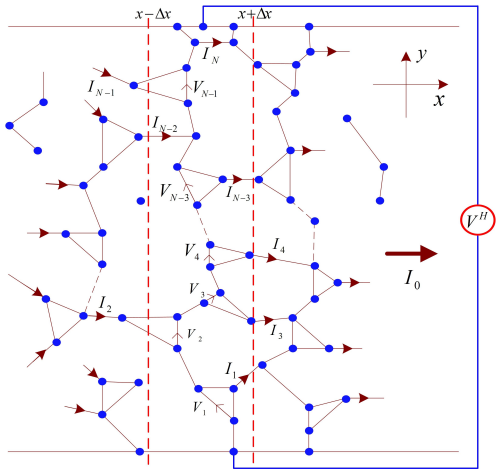

FIG. 3: (Color online) Typical resistor network in the material. The present situation indicates V_{N-2}^H and V_N^H in the region from $x - \Delta x$ to $x + \Delta x$ are zero, where no triads form.

along the y-axis in the region around position x (Note **M** is along the z-axis, hence we assume the system in this direction to be uniform). The transverse voltage equals the summation over the voltage drops of the $N(x)$ triads:

$$V_y^H(x) = \sum_{l=1}^{N(x)} V_l^H. \tag{10}$$

For the general situation we allow some V_i^H's to be zero (see Fig. 3). In that case no triad forms for the incoming current I_i under the condition that all direct conductances in a triad must be no less than G_c. To calculate V_i^H, the voltage contributed by the i-th triad, we employ perturbation theory to the equation[30] $I_{ij} = G_{ij}V_{ij} + \sum_k \mathcal{G}_{ijk}(V_{ik} + V_{jk})$. First, in the zeroth order, we consider only the normal current, namely, the Hall current is zero and thus $\sum_j I_{ij} = \sum_j G_{ij}V_{ij}^{(0)} = 0$, with which one can determine the voltage $V_i^{(0)}$ at each site. Then, for the first-order perturbation, we have $\sum_j I_{ij} = \sum_j G_{ij}V_{ij} + \sum_j J_{ij}^{(H)} = 0$, which leads to $J_i^{(H)} = \sum_j J_{ij}^{(H)} = \sum_j\sum_k \mathcal{G}_{ijk}(V_{jk}^{(0)} + V_{ik}^{(0)}) = -\sum_j G_{ij}V_{ij}$. The current $J_i^{(H)}$ can also be written as

$$J_i^{(H)} = \frac{3}{2}\sum_{jk}\mathcal{G}_{ijk}V_{jk}^{(0)}. \tag{11}$$

For the hopping regime, the triads are dilutedly distributed and the Hall voltages induced by different triads are considered to be uncorrelated. Therefore, we obtain the Hall voltage of the i-th triad from the transformation indicated in Fig. 4 that

$$V_i^{(H)} = \frac{3I_i\mathcal{G}_{i_1i_2i_3}^{(i)}}{G_{i_1i_2}G_{i_2i_3} + G_{i_1i_3}G_{i_2i_3} + G_{i_3i_1}G_{i_1i_2}}. \tag{12}$$

From the resistor network configuration one can see

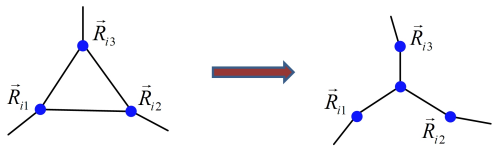

FIG. 4: (Color online) Resistor network transformation.

$\sum_i^{N(x)} I_i = 2I_0$. For convenience, we denote $I_i = 2I_0\lambda_i(x)$ with $\sum_i \lambda_i = 1$. For a macroscopic system, one has $N(x) \to \infty$. Furthermore, we consider at the position x, for each λ_i there are $n_i(x)$ triads that have such same current fraction λ_i. Then the average transverse voltage reads

$$\bar{V}_y^H = 6I_0 \frac{1}{L_x} \int dx \sum_{\{n_i\}} \lambda_i \sum_{j=1}^{n_i \gg 1} \frac{\mathcal{G}^{(j)}_{j_1 j_2 j_3}}{G_{j_1 j_2} G_{j_2 j_3} + G_{j_1 j_3} G_{j_2 j_3} + G_{j_3 j_1} G_{j_1 j_2}}. \quad (13)$$

To simplify this formula we extend the current distribution $\{\lambda_i\}$ for the region between $x - \Delta x$ and $x + \Delta x$ to the whole space along x direction, and then we can exchange the order of the integral and the first summation: $\frac{1}{L_x}\int dx \sum_{\{n_i\}} \lambda_i \sum_{j=1}^{n_i \gg 1} \rightarrow \sum_{\{\lambda_i\}} \lambda_i \frac{1}{L_x} \int dx \sum_{j=1}^{n_i(x)}$. In the limit $N(x) \rightarrow \infty$ and when the length L_x is much larger than the typical length L of the triad, the calculation $\frac{1}{L_x}\int dx \sum_{j=1}^{n_i(x)}$ gives the average of all possible configurations of the triads through the percolating cluster. This leads to

$$\bar{V}_y^H = 6I_0 \sum_{\{\lambda_i\}} \bar{n}_i \lambda_i \langle \frac{\mathcal{G}^{(i)}_{i_1 i_2 i_3}}{G_{i_1 i_2} G_{i_2 i_3} + G_{i_1 i_3} G_{i_2 i_3} + G_{i_3 i_1} G_{i_1 i_2}} \rangle_c,$$

with $\bar{n}_i = (1/L_x)\int dx n_i(x)$ the average number of triads with in/outgoing current I_i. Note the identity $\sum_i n_i \lambda_i = 1$ is independent of position x, and therefore we have also $\sum_i \bar{n}_i \lambda_i = 1$. For this we obtain the AHC

$$\sigma_{xy}^{AH} = 3L\sigma_{xx}^2 \frac{k_B T}{e^2} \langle \frac{\sum_{\alpha\beta\gamma} \left[\mathrm{Im}(t_{i\alpha, j\beta} t_{j\beta, k\gamma} t_{k\gamma, i\alpha}) T^{(3)}_{ijk} \right]}{\sum_{i \leftrightarrow j \leftrightarrow k} |t_{ij} t_{jk}|^2 T^{(2)}_{ij} T^{(2)}_{jk}} \rangle_c, \quad (14)$$

where L is the correlation length of the network. Note the configuration integral given by Eq. (7) is first derived for the AHC in this work. This is an essential difference from the former theory by Burkov *et al*[21], where the configuration averaging applies to the whole system rather than to 2D percolation cluster. With our formalism the key physics that Hall currents are averaged over percolation clusters can be studied, which is a crucial step to understand the insulating regime of the AHE phase diagram. The above configuration integral cannot be solved analytically. In the following we study the upper and lower limits of the AHC with Eq. (14) by considering different extreme situations for the triad distribution, with which the range of the scaling relation between σ_{xy}^{AH} and σ_{xx} can be determined.

Before proceeding further we would like to present a few remarks on Eq. (14). First of all, this formula is generally valid for the disordered insulating regime, as long as the triads are sparsely distributed in the percolation cluster. Second, for different types of hopping regimes (Mott, ES, and activation E_3 hopping regimes), the functions of the DOS $\rho(\epsilon)$ and connectivity $n(\epsilon_i)$ in the configuration integral are different.

V. SCALING RELATION BETWEEN ANOMALOUS HALL CONDUCTIVITY AND LONGITUDINAL CONDUCTIVITY

The lower (upper) limit of the AHC can be formulated by keeping only the maximum (minimum) term in the denominator and the minimum (maximum) term in the numerator. In this section we shall study the scaling relation in the Mott, ES, and activation E_3 hopping regimes, respectively.

A. Mott variable range hopping regime

In this regime we first approximate the DOS to be constant although this approximation is relaxed later. Under this condition one obtains straightforwardly the probability $P_n(\epsilon_i)$ from the number $n(\epsilon_i)$. Note in the hopping conduction mechanism temperature dependence of the conductivities are dominated by exponential functions. It can then be expected that the scaling relation between σ_{xy}^{AH} and σ_{xx} will be governed by the exponential functions in $\mathcal{G}_{ijk}$ and G_{ij}. To focus on the scaling relation, we first drop off the summation of the spin states. This procedure ignores an important physical consequence that the summation over spin-up and spin-down states contribute oppositely to the AHE (we shall return to this discussion later), but keeps the central result of the scaling relation unchanged between σ_{xy}^{AH} and σ_{xx}. As a result, with further simplification we find

$$\{\sigma_{xy}^{AH}\}^{min}_{max} \simeq 3L\sigma_{xx}^2 \frac{k_B T}{e^2 t^{(0)}_{max/min}} \langle R^{min}_{ijk} \rangle_c \langle \epsilon^{min}_{ijk} \rangle_c, \quad (15)$$

where $\langle R^{min}_{ijk} \rangle_c = e^{a\langle R_{ij} + R_{jk} - R_{ik} \rangle_c}|_{R_{ij}, R_{jk} < R_{ik}}$, $\langle \epsilon^{min}_{ijk} \rangle_c = e^{0.5\beta \langle |\epsilon_i| + |\epsilon_j| + |\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| \rangle_c}|_{|\epsilon_i| < |\epsilon_j| < |\epsilon_k|}$, $\langle R^{max}_{ijk} \rangle_c$ and $\langle \epsilon^{max}_{ijk} \rangle_c$ hold the same form for the calculation but the restrictions change to be $R_{ij}, R_{jk} > R_{ik}$ and $|\epsilon_i| > |\epsilon_j| > |\epsilon_k|$, respectively. The coefficient $t^{(0)}_{\mathrm{max/min}}$ represents the maxmimum/minimum element in the matrix $t^{(0)}_{ij}$. In obtaining Eq. (15) we have approximated the configuration averaging of the exponential functions to be the configuration averaging of the exponents. This approximation loses the information of the power-law dependence of the AHC on the temperature, and it requires the dominant temperature dependence of the AHC should be in the exponential form. In the hopping conduction regime this condition is satisfied.

It is instructive to point out the underlying physics of the two limits. In the hopping regime, charge transport

may prefer a short and straight path in the forward direction with larger resistance than a long and meandrous path with somewhat smaller resistance[16,25]. This picture introduces an additional restriction complementary to the percolation theory for charge transport. What bonds in a triad play the major role for the current flowing through it is determined by the optimization of the resistance magnitudes and spatial configuration of the three bonds. A quantitative description can be obtained by phenomenologically introducing an additional probability factor to restrict the charge transport[16,25]. Here we only need to adopt this picture to present the two extreme situations corresponding to $\{\sigma_{xy}^{AH}\}_{min/max}$. To get the upper limit we assume that for each triad of the percolation cluster the two bonds with smaller direct conductance dominate the charge transport, i.e. the product of two smallest conductances minimize the denominator, and take the maximum value for the numerator of Eq. (14). For the opposite limit, the situation that the two bonds with larger conductances in each triad dominate the charge transport corresponds to the lower limit of the AHC.

We study first the lower limit of the AHC. For convenience we neglect the spin indices. According to Eq. (7), we know the configuration averaging $\langle R_{ij}+R_{jk}-R_{ik}\rangle_c|_{R_{ij},R_{jk}<R_{ik}}$ is a nine dimensional (9D) integral over the position $\int d^3\vec{R}_{ij}\int d^3\vec{R}_{jk}$ and the on-site energies $\int d\epsilon_i d\epsilon_j d\epsilon_k$. We shall first perform the integral over position. Denote by $\vec{R}_1=\vec{R}_{ij},\vec{R}_2=\vec{R}_{jk}$ for convenience, and then $R_3=R_{ik}=(R_1^2+R_2^2-2R_1R_2\cos\theta)^{1/2}$. We then study the integral $I=\frac{1}{\mathcal{N}_r}\int d^3\vec{R}_1\int d^3\vec{R}_2(R_1+R_2-R_3)$ with $\mathcal{N}_r=\int d^3\vec{R}_1\int d^3\vec{R}_2$. Note in the configuration integral we have the restrictions: $R_i\le R_{i,max}$ and $R_1,R_2\le R_3$, with $R_{i,max}$ determined through $2aR_{ij}^{max}+\frac{1}{2}\beta(|\epsilon_i|+|\epsilon_j|+|\epsilon_i-\epsilon_j|)=\beta\xi_c$ (from the condition $G_{ij}^{min}=G_c$ or $Z_{ij}^{max}=1/G_c$). For this we can show the integral satisfies

$$I \le \frac{1}{\mathcal{N}_r}8\pi^2\int_0^{R_{2max}}dR_2R_2^2\Big[\int_{\pi/2}^{\pi}d\theta\int_0^{R_{1max}}dR_1R_1^2\sin\theta(R_1+R_2-\sqrt{R_1^2+R_2^2-2R_1R_2\cos\theta}) + \int_{\pi/3}^{\pi/2}d\theta\int_{2R_2\cos\theta}^{R_{1max}}dR_1R_1^2\sin\theta(R_1+R_2-\sqrt{R_1^2+R_2^2-2R_1R_2\cos\theta})\Big], \quad (16)$$

where $\mathcal{N}_r=\frac{8}{3}\pi^2R_{2max}^3\int_{\pi/2}^{\pi}d\theta\int_0^{R_{1max}}dR_1R_1^2\sin\theta+8\pi^2\int_0^{R_{2max}}dR_2R_2^2\int_{\pi/3}^{\pi/2}d\theta\int_{2R_2\cos\theta}^{R_{1max}}dR_1R_1^2\sin\theta$. By a straightforward calculation we obtain further

$$I\simeq 0.424\pi^2R_{max}^7/\mathcal{N}_r, \quad (17)$$

with the normalization factor $\mathcal{N}_r=\frac{23}{18}\pi^2R_{max}^6$. Here $R_{max}=\max\{R_{1max},R_{2max}\}$. We should emphasize that to this step we cannot cancel the function R_{max}^7 in the numerator of the Eq. (17) by the normalization factor $\mathcal{N}_r$. This is because both of them are only part of the original configuration averaging $\langle R_{ij}+R_{jk}-R_{ik}\rangle_c$. The final result needs to further integrate over onsite energies, and gives that

$$\ln\langle R_{ijk}^{min}\rangle_c\simeq 0.156\beta\xi_c. \quad (18)$$

Now we evaluate the configuration averaging $\langle\epsilon_{ijk}^{min}\rangle_c$, which corresponds to a 3D integral over onsite energies. Similarly, the formula is given by

$$\ln\langle\epsilon_{ijk}^{min}\rangle_c=\frac{0.5\beta\int d\epsilon_i d\epsilon_j d\epsilon_k\rho(\epsilon_i)\sum_{l\ge3}P_l(\epsilon_i)\rho(\epsilon_i)\sum_{l\ge3}P_l(\epsilon_j)\rho(\epsilon_k)\sum_{l\ge3}P_l(\epsilon_k)(|\epsilon_i|+|\epsilon_j|+|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|)}{\int d\epsilon_i d\epsilon_j d\epsilon_k\rho(\epsilon_i)\sum_{l\ge3}P_l(\epsilon_i)\rho(\epsilon_i)\sum_{l\ge3}P_l(\epsilon_j)\rho(\epsilon_k)\sum_{l\ge3}P_l(\epsilon_k)}. \quad (19)$$

To simplify the above integral, we check $|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|$ with the restriction: $|\epsilon_i|<|\epsilon_j|<|\epsilon_k|$. For the case i) $\mathrm{sgn}(\epsilon_i)=\mathrm{sgn}(\epsilon_j)=\mathrm{sgn}(\epsilon_k)=\pm1$, we have $|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|=-|\epsilon_i-\epsilon_j|$; For ii) $\mathrm{sgn}(\epsilon_i)=\mathrm{sgn}(\epsilon_j)=-\mathrm{sgn}(\epsilon_k)=\pm1$, we have $|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|=-|\epsilon_i-\epsilon_j|$; For iii) $\mathrm{sgn}(\epsilon_i)=\mathrm{sgn}(\epsilon_k)=-\mathrm{sgn}(\epsilon_j)=\pm1$, we have $|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|=-|\epsilon_i-\epsilon_j|$; For iv) $\mathrm{sgn}(\epsilon_j)=\mathrm{sgn}(\epsilon_k)=-\mathrm{sgn}(\epsilon_i)=\pm1$, we have $|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|=|\epsilon_i-\epsilon_j|$. For this we obtain that $\langle|\epsilon_i|+|\epsilon_j|+|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|\rangle_c\simeq\langle|\epsilon_i|+|\epsilon_j|-\frac{1}{2}|\epsilon_i-\epsilon_j|\rangle_c$. Then by a straightforward calculation one can verify that

$$\ln\langle\epsilon_{ijk}^{min}\rangle_c=0.086\beta\xi_c. \quad (20)$$

Together with the results in Eq. (18) and Eq. (20) we

get

$$\langle R_{ijk}^{min}\rangle_c\langle\epsilon_{ijk}^{min}\rangle_c \simeq e^{0.242\beta\xi_c}, \tag{21}$$

The lower limit of the AHC is then obtained by

$$\{\sigma_{xy}^{AH}\}_{min} = 3L\sigma_{xx}^2\frac{k_BT}{e^2}\frac{1}{t_{max}^{(0)}}e^{0.242\beta\xi_c}. \tag{22}$$

Note the longitudinal conductivity σ_{xx} is calculated based on the 2-site function of G_{ij} which should be no less than G_c in a percolation path. The evaluation of σ_{xx} with percolation theory has been well studied in the published literatures[16,24,25]. It can be shown that the result of σ_{xx} equals G_c divided by the correlation length of the network and takes the form $\sigma_{xx} = \sigma_0(T)e^{-\beta\xi_c}$, where $\sigma_0(T)$ gives at most a power-law on T[24,25]. Comparing this form with the lower limit of the AHC obtained above, we reach that

$$\{\sigma_{xy}^{AH}\}_{min} = 3L\sigma_0^{0.242}\frac{k_BT}{e^2}\frac{1}{t_{max}^{(0)}}\sigma_{xx}^{1.758}. \tag{23}$$

The upper limit can be studied in the same way. Considering the different restrictions, we obtain $\langle R_{ij}+R_{jk}-R_{ik}\rangle_c|_{R_{ij},R_{jk}<R_{ik}} = 0.483\beta\xi_c/a$ and $\langle|\epsilon_i|+|\epsilon_j|+|\epsilon_j-\epsilon_k|-|\epsilon_i-\epsilon_k|\rangle_c|_{|\epsilon_i|>|\epsilon_j|>|\epsilon_k|} = 0.275\xi_c$. For this we obtain $\langle R_{ijk}^{max}\rangle_c \simeq e^{0.483\beta\xi_c}$, $\langle\epsilon_{ijk}^{max}\rangle_c \simeq e^{0.138\beta\xi_c}$, and the upper limit of the AHC by

$$\{\sigma_{xy}^{AH}\}_{max} = 3L\sigma_0^{0.621}\frac{k_BT}{e^2t_{min}^{(0)}}\sigma_{xx}^{1.379} \propto \sigma_{xx}^{\gamma}. \tag{24}$$

Comparing the above results with the AHC, we reach $\{\sigma_{xy}^{AH}\}_{min/max} \sim \sigma_0^{2-\gamma_{a/b}}\sigma_{xx}^{\gamma_{a/b}}$ with $\gamma_a = 1.76$ and $\gamma_b = 1.38$. This leads to the scaling relation between σ_{xy}^{AH} and σ_{xx} of the AHE in the Mott VRH regime:

$$\sigma_{xy}^{AH} \propto \sigma_{xx}^{\gamma}, \quad 1.38 < \gamma < 1.76. \tag{25}$$

The maximum (minimum) of the AHC corresponds to the smaller (larger) power index γ_b (γ_a). This scaling range can be confirmed with a numerical calculation of the Eq. (15). Furthermore, a direct numerical study for the configuration integral (14) gives the scaling exponent $\gamma \approx 1.62$, which is consistent with the analytical prediction of the lower and upper limits.

It is noteworthy that the configuration averaging over the position $\langle R_{ijk}\rangle_c$ undergoes a relatively large change in magnitude between the upper and lower limits. This result reflects an important property of the (variable range) hopping conduction regime presented below. In the VRH, the hopping process allows to go beyond between nearest neighbor impurity sites to minimize the resistivity. The optimization of the typical hopping length plays a major role in determining the scaling of the conductivities with respect to temperature[17]. The lower and upper limits correspond to the opposite extreme situations of the triad distribution which have distinct influences on the optimization of the hopping distances for the Hall transport and thus lead to very different results for the AHC after spatial averaging. We should emphasize that this remarkable difference between $\langle R_{ijk}^{max}\rangle_c$ and $\langle R_{ijk}^{min}\rangle_c$ is obtained in the case of a constant DOS around Fermi energy. One can expect this effect will be suppressed in the E-S hopping regime where the DOS is a parabolic function of the onsite energy and the difference between configuration integrals with respect to energies become more important (refer to the discussion in the next subsection).

We make a further remark here to compare our results with those obtained by Burkov et al[21]. In the final result of the AHC in Ref. 17, the configuration averaging was actually not performed but simply replaced by the maximum value of the integrand in their formalism. This procedure, not surprisingly, cannot predict the correct scaling relation. Here we have performed the exact calculation of the lower and upper limits of the AHC based on the correct configuration averaging formula. With our procedure, the key physics that the Hall currents are averaged over percolation cluster is completely considered and reflected in our evaluation.

So far in the calculation we have assumed a constant DOS. This approximation is applicable for the ferromagnetic system with strong exchange interaction between local magnetic moments and charge carriers (e.g. oxides, magnetites) and half metals in general. In this case we do not need to sum over spin-up and spin-down states which contribute oppositely to the AHE, and the previous results are valid.

However, when the Fermi energy crosses both spin-up and -down impurity states, a symmetric DOS with $\rho(\epsilon) = \rho(-\epsilon)$ leads to zero AHC. This is because under the transformation $\epsilon_{l,\sigma} \to -\epsilon_{l,-\sigma}$ ($l = i,j,k$), $\mathcal{G}_{ijk}$ changes sign, while G_{ij} is invariant. Thus the averaging for AHC over all spin states and on-site energies cancels[21]. We relax the previous simplifying restriction by expanding the DOS by $\rho(\epsilon) = \sum_n \frac{1}{n!}\frac{d^n\rho_0}{d\epsilon_F^n}\epsilon^n$, where we consider $\rho_0 = \rho(\epsilon_F)$ is finite and has only a relatively small variation in the range $|\epsilon| < \xi_c$. Substituting this expansion into Eq. (14) yields $\sigma_{xy}^{AH} = \sum_{n=0}^{\infty}\sigma_{xy}^{(n)}$, with the 1st and 2nd nonzero terms respectively proportional to $\frac{d\rho_0}{d\epsilon_F}$ and $\frac{d^3\rho_0}{d\epsilon_F^3}$. We can similarly evaluate the lower and upper limits of σ_{xy}^{AH} as before. The first two nonzero terms in the expansion are $\{\sigma_{xy}^{(1)}\}_{min/max} \sim M\frac{d\rho_0}{d\epsilon_F}\xi_c(T)\sigma_0^{2-\gamma_{a/b}}\sigma_{xx}^{\gamma_{a/b}}$ and $\{\sigma_{xy}^{(2)}\}_{min/max} \sim 0.002M\frac{d^3\rho_0}{d\epsilon_F^3}\xi_c^3(T)\sigma_0^{2-\gamma_{a/b}}\sigma_{xx}^{\gamma_{a/b}}$. The appearance of M is due to the summation over the spin-up and -down states. We have also employed the result $\langle|\epsilon|\rangle_c = 0.112\xi_c$. Note that $\sigma_{xy}^{(1)}$ and $\sigma_{xy}^{(2)}$ have different physical meanings. The term $\sigma_{xy}^{(1)}$ dominates when the DOS varies monotonically versus ϵ. Furthermore, when the DOS has a local minimum at the Fermi level, which may be obtained due to particle-particle interaction (coulomb interaction), we have $d\rho/d\epsilon_F = 0$. Then the term $\sigma_{xy}^{(1)}$ varnishes and $\sigma_{xy}^{(2)}$ dominates the AHE.

The result that the AHC σ_{xy}^{AH} proportional to $d\rho_0/d\epsilon_F$ (when the DOS varies monotonically with respect to energy around Fermi energy) or $d^3\rho_0/d\epsilon_F^3$ (when the DOS has a local minimum at Fermi level) indicates an interesting property that the AHC may change sign when the first or third order derivative of DOS with respect to energy changes sign. This result is consistent with the observation by Allen et al *et al*[7].

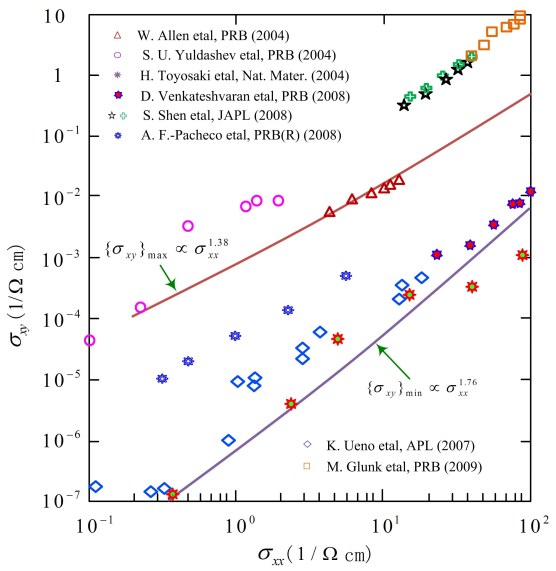

FIG. 5: (Color online) Scaling relation between the AHC and longitudinal conductivity. The theoretical results are compared with the experimental observations.

Fig. 5 shows our theoretical prediction is consistent with the experimental observations of the scaling relation in this regime, hence completing the understanding of the phase diagram of the AHE.

B. Efros-Shkolvskii regime

In the case with strong coulomb interaction, the DOS may be greatly reduced around the Fermi energy in disordered insulators[26,27]. In this case the assumption in previous subsection that the DOS has a small variation relative to ρ_0 is not valid. The limit situation is that both the DOS and the first derivative at Fermi level varnish (i.e. the E-S hopping regime), which corresponds to the appearance of a gap due to coulomb interaction. In this case $\rho_0 = 0$ and $d\rho/d\epsilon_F = 0$, and thus

$$\rho(\epsilon) \simeq \frac{1}{2}\frac{d^2\rho_0}{d\epsilon^2}\epsilon^2 + \frac{1}{6}\frac{d^3\rho_0}{d\epsilon^3}\epsilon^3. \tag{26}$$

The E-S hopping regime is different from the cases discussed in the previous section, since around the Fermi energy DOS is not dominated by a constant but by a parabolic function of on-site energy. This may lead to a quantitative variation of the probability function in the configuration averaging, and finally affect the quantitative but not qualitative result of the scaling relation. The formula of the connectivity $n(\epsilon_i, \xi_c)$ is now given by

$$\begin{aligned} n(\epsilon_i,\xi_c) &= \frac{1}{(2ak_BT)^3}\frac{2\pi}{3}\frac{d^2\rho_0}{d\epsilon^2}\big(\frac{1}{30}\xi_c^6 - \frac{1}{10}\xi_c^5|\epsilon_i| + \\ &+\frac{1}{4}\xi_c^4|\epsilon_i|^2 - \frac{1}{3}|\epsilon_i|^3\xi_c^3 + \frac{3}{10}|\epsilon_i|^5\xi_c - \\ &-\frac{3}{20}|\epsilon_i|^6\big). \end{aligned} \tag{27}$$

The configuration averaging of AHC can be calculated following a similar procedure. Specifically, for the lower limit we obtain

$$\langle R_{ijk}^{min}\rangle_c \simeq e^{0.092\beta\xi_c},\ \langle \epsilon_{ijk}^{min}\rangle_c \simeq e^{0.29\beta\xi_c}. \tag{28}$$

Comparing the above results with those obtained in the Mott VRH regime with a constant DOS, we can see the magnitude of energy averaging in the E-S hopping regime increases, while the magnitude of position averaging decreases. This is reasonable since the DOS varies as a function of ϵ^2, which increases the contribution to the Hall effect from the impurity states with energies far away from the Fermi energy and accordingly, decreases the contribution from hopping between impurity sites with large distances. The lower limit of the AHC is then obtained by

$$\{\sigma_{xy}^{AH}\}_{min} \simeq 0.059L\sigma_0^{0.38}(T)\frac{M_0}{e^2}\frac{1}{t_{max}^{(0)}}\frac{d^3\rho_0}{d\epsilon_F^3}\xi_c^3(T)\sigma_{xx}^{1.62}. \tag{29}$$

For the upper limit of AHC, we get $\langle R_{ijk}^{max}\rangle_c \simeq e^{0.29\beta\xi_c}$ and $\langle \epsilon_{ijk}^{max}\rangle_c \simeq e^{0.38\beta\xi_c}$. This leads to the scaling relation in the upper limit

$$\{\sigma_{xy}^{AH}\}_{max} \simeq 0.026L\sigma_0^{0.67}(T)\frac{M_0}{e^2}\frac{1}{t_{min}^{(0)}}\frac{d^3\rho_0}{d\epsilon_F^3}\xi_c^3(T)\sigma_{xx}^{1.33}. \tag{30}$$

Therefore in the E-S hopping regime the scaling relation between anomalous Hall and longitudinal conductivities becomes $\sigma_{xy}^{AH} \propto \sigma_{xx}^{\gamma}$ with $1.33 \leq \gamma \leq 1.62$, which has a small quantitative shift relative to the scaling obtained in the case with a constant DOS. This result is consistent with the observations in the experiments by Aronzon et al[4], and by Allen et al[7], who found the scaling relation as $1.4 \leq \gamma \leq 1.6$ for the E-S hopping conduction regime. Again, the AHC in the E-S regime σ_{xy}^{AH} is proportional to $d^3\rho_0/d\epsilon_F^3$, and thus may change sign when the third order derivative of DOS with respect to energy changes sign. Finally, it can be expected that the general situation with a reduced DOS (not necessarily zero) at Fermi level will be associated with a scaling falling in between the E-S hopping regime and the case with a constant DOS. This confirms that the scaling relation between σ_{xy}^{AH} and σ_{xx} is insensitive to what types of hopping conduction the material belongs to, and is therefore generic for the disordered insulating regime.

C. Activation E_3 hopping regime

Finally, we present a brief study on the AHE in the activation E_3 hopping regime, which dominates the charge transport in the disordered insulating system when the temperature $T > T_0$. Here T_0 is given by Eq. (9). In the activation E_3 hopping regime, the hopping between nearest neighbor impurity sites dominates the charge transport[27,31]. In this case the hopping configuration in the position space is not affected by temperature. Thus the configuration averaging over position space is independent of temperature. The temperature dependence of the conductivities is solely determined by the energy configuration integral. Again we consider that the impurity sites are homogeneously distributed in position space. Then connectivity $n(\epsilon_i)$ for a specific impurity site with on-site energy ϵ_i is given by

$$n(\epsilon_i) = \frac{4\pi R_c^3}{3} \int d\epsilon_j \rho(\epsilon_j) \Theta\big(E_3 - \frac{|\epsilon_i| + |\epsilon_j| + |\epsilon_i - \epsilon_j|}{2}\big), \tag{31}$$

where E_3 is the cut-off for on-site energy and R_c represents the typical distance between the neighbor impurity sites. For a constant DOS, one has

$$n(\epsilon_i) = \frac{4}{3}\pi R_c^3 \rho_0 (2E_3 - |\epsilon_i|), \tag{32}$$

with $|\epsilon| \leq E_3$. It can be seen that for the present regime, $n(E_3) = \frac{4}{3}\pi R_c^3 \rho_0 E_3 > 0$. This is different from the situation in the VRH regime considered in previous sections. Substituting the above formula into Eq. (8) one can calculate the relation between the cut-off E_3 and $\bar{n}$ straightforwardly, with which one can verify that E_3 is a constant independent of temperature and $E_3 \propto 1/(\rho_0 R_c^3)$[31]. The longitudinal conductivity is then given by

$$\sigma_{xx} = \sigma_0 e^{-E_3/k_B T}. \tag{33}$$

The AHC is still given by Eq. (14) with the function of $n(\epsilon_i)$ given by Eq. (31), but now the configuration integral over position is unrelated to that over on-site energies and thus does not affect the temperature dependence of σ_{xy}^{AH}. For this we obtain the upper and lower limits of the AHC that

$$\{\sigma_{xy}^{AH}\}_{max}^{min} \simeq 3L\sigma_{xx}^2 \frac{k_B T e^{aR_c}}{e^2 t_{max/min}^{(0)}} \langle \epsilon_{ijk}^{max} \rangle_c^{min}. \tag{34}$$

By a direct numerical evaluation we obtain that $\langle \epsilon_{ijk}^{\max} \rangle_c \approx e^{0.61\beta E_3}$, $\langle \epsilon_{ijk}^{min} \rangle_c \approx e^{0.34\beta E_3}$, with which we obtain the scaling relation $\sigma_{xy}^{AH} \propto \sigma_{xx}^{\gamma}$, where $1.39 \leq \gamma \leq 1.66$. With this result we conclude that the scaling in the activation E_3 hopping regime has only a quantitative small shift relative to the scaling in the VRH hopping regimes.

VI. CONCLUSIONS

We have developed a theory based on the phonon-assisted hopping mechanism and percolation theory to study the anomalous Hall effect (AHE) in the disordered insulating regime. A general formula for the anomalous Hall conductivity (AHC) has been derived for the hopping conduction regime, with the key physics that the Hall currents are averaged over percolation cluster being completely considered. We calculated the lower and upper limits of the AHC, and show it scales with the longitudinal conductivity as $\sigma_{xy}^{AH} \sim \sigma_{xx}^{\gamma}$ with γ predicted to be $1.33 \leq \gamma \leq 1.76$. The predicted scaling only slightly depends on the specific hopping types, and is quantitatively in agreement with the experimental observations.

From our theory the scaling relation in the insulating AHE is fully determined by the microscopic origin: phonon-assisted hopping conduction mechanism, and by the fact that the AHC is dominated by the percolation clusters. It is clear that these two aspects are generic for hopping conduction regime of the disordered insulators, and therefore the obtained scaling in this regime is qualitatively generic in the disordered insulating AHE. We have shown that this scaling remains similar regardless of whether the hopping process is Mott-variable-range-hopping, influenced by interactions, or activation E_3 hopping (nearest neighbor hopping) regime. Our results explain naturally how the scaling between the two quantities remain true even when the diagonal conductivity crosses regimes and why this type of scaling is so prevalent in the insulating regime. Our theory completes the understanding of the AHE phase diagram in the insulating regime.

This work is supported by NSF under Grant No. DMR-1105512, NSF-MRSEC DMR-0820414, NHARP, and by SWAN-NRI, and the Research Corporation for the Advancement of Science.

[1] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and P. Ong, Rev. Mod. Phys. **82**, 1539 (2010).

[2] R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954).

[3] J. S.-Y. Feng, R. D. Pashleyz and M.-A. Nicolet, J. Phys. C: Solid State Phys. **8**, 1010 (1975).

[4] B. A. Aronzon, D. Yu. Kovalev, A. N. Lagar'kov, E. Z. Meilikhov, V. V. Ryl'kov, M. A. Sedova, N. Negre, M. Goiran, and J. Leotin, JETP, **70** 90 (1999).

[5] A. V. Samoilov, G. Beach, C. C. Fu, N.-C. Yeh, and R. P. Vasquez, Phys. Rev. B **57**, 14032(R) (1998).

[6] H. Toyosaki, T. Fukumura, Y. Yamada, K. Nakajima, T. Chikyow, T. Hasegawa, H. Koinuma and M. Kawasaki, Nat. Mater. **3**, 221 (2004).

[7] W. Allen, E. G. Gwinn, T. C.Kreutz, and A. C. Gossard, Phys. Rev. B **70**, 125320 (2004).

[8] Sh. U. Yuldashev et al., Phys. Rev. B **70**, 193203 (2004).

[9] K. Ueno, T. Fukumura, H. Toyosaki, M. Nakano, and M. Kawasaki, Appl. Phys. Lett. **90**, 072103 (2007).

[10] S. Shen, et al., J. Appl. Phys. **103**, 07D134 (2008).

[11] A. Fernández-Pacheco, J. M. De Teresa, J. Orna, L. Morellon, P. A. Algarabel, J. A. Pardo, and M. R. Ibarra, Phys. Rev. B **77**, 100403(R) (2008).

[12] D. Venkateshvaran, W. Kaiser, A. Boger, M. Althammer, M. S. Ramachandra Rao, Sebastian T. B. Goennenwein, M. Opel, and R. Gross, Phys. Rev. B **78**, 092405 (2008).

[13] M. Glunk, J. Daeubler, W. Schoch, R. Sauer, and W. Limmer, Phys. Rev. B **80**, 125204 (2009).

[14] D. Chiba, A. Werpachowska, M. Endo, Y. Nishitani, F. Matsukura, T. Dietl, and H. Ohno, Phys. Rev. Lett. **104**, 106601 (2010).

[15] S. Onoda, N. Sugimoto, and N. Nagaosa, Phys. Rev. Lett. **97**, 126602 (2006).

[16] A. Miller and E. Abrahams, Phys. Rev. **120**, 745 (1960).

[17] N. F. Mott, Phil. Mag. **19**, 835 (1969).

[18] T. Holstein, Phys. Rev. **1**, 1329 (1961).

[19] S. H. Chun, M. B. Salamon, Y. Lyanda-Geller, P. M. Goldbart, and P. D. Han, Phys. Rev. Lett. **84**, 757 (2000).

[20] Y. Lyanda-Geller, S. Chun, M. Salamon, P. Goldbart, P. Han, Y. Tomioka, A. Asamitsu, and Y. Tokura, Phys. Rev. B **63**, 184426 (2001).

[21] A. A. Burkov and L. Balents, Phys. Rev. Lett. **91**, 057202 (2003).

[22] A metallic theory introducing strong disorder broadening showed an above unity scaling outside its range of validity ($k_F l \ll 1$), but predicts, expectedly, metallic conductivities at zero temperature and is therefore invalid in the insulating regime[1,15].

[23] G. A. Fiete, G. Zaránd, and K. Damle, Phys. Rev. Lett. **91**, 097202 (2003).

[24] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).

[25] M. Pollak, J. Non-Cryst. Solids **11**, 1-24 (1972).

[26] A. L. Efros and B. I. Shklovskii, J. Phys. C **8**, L49 (1975).

[27] B. I. Shklovskii and A. L. Efros, Electronic Properties of Doped Semiconductors, Springer-Verlag, Berlin (1984).

[28] G. E. Pike and C. H. Seager, Phys. Rev. B **10**, 1421 (1974).

[29] H. Overhof, Phys. Stat. Sol. (b) **67**, 709 (1975).

[30] H. Böttger and V. V. Bryksin., phys. stat. sol. (b) **81**, 433 (1977).

[31] Y. Imry, Introduction to Mesoscopic Physics, Oxford University Press, Newyork (1997).