# Time-reversal symmetric hierarchy of fractional incompressible liquids

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# Time-reversal symmetric hierarchy of fractional incompressible liquids

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We provide an effective description of fractional topological insulators that include the fractional quantum spin Hall effect by considering the time-reversal symmetric pendant to the topological quantum field theories that encode the Abelian fractional quantum Hall liquids. We explain the hierarchical construction of such a theory and establish for it a bulk-edge correspondence by deriving the equivalent edge theory for chiral bosonic fields. Further, we compute the Fermi-Bose correlation functions of the edge theory and provide representative ground state wave functions for systems described by the bulk theory.

## I. INTRODUCTION

Laughlin initiated the theoretical exploration of the fractional quantum Hall effect (FQHE) by proposing wave functions for the ground states of interacting electrons in the lowest Landau level at filling fractions $\nu = 1/(2m+1),\ m \in \mathbb{Z}$.[1] The experimental observation of a plethora of fractional Hall plateaus at other filling fractions lead to the construction of a hierarchy of wave functions out of Laughlin's wave function,[2–7] and the development of the composite fermion picture.[8] These approaches were later reconciled, and unified by the effective description of the FQHE in terms of multi-component Chern-Simons theories in $(2+1)$-dimensional space and time.[9–15] These topological effective theories for the hierarchy of FQHE deliver a correspondence between the physics in the two-dimensional bulk and the physics along one-dimensional boundaries at which the two-dimensional sample terminates.[16–21]

It is possible to double the Chern-Simons effective theory representing the universal properties of the FQHE at some filling fraction $\nu = 1/(2m+1),\ m \in \mathbb{Z}$ so as to obtain a time-reversal symmetric theory. This approach has been used to interpret a fully gaped superconductor as an example of a topological phase,[22–24] and – more generally – to explore the universal properties of interacting theories with an emergent local gauge $\mathbb{Z}_2$ symmetry (see Refs. 25–29) that signals the phenomenon of spin and charge separation.[30–34]

A more urgent impetus for the construction of effective time-reversal symmetric topological field theories in $(2+1)$-dimensional space and time arose with the theoretical prediction of time-reversal symmetric topological band insulators, shortly followed by their experimental discovery.[35–39] These band insulators realize the counterparts to the integer quantum Hall effect and their discovery suggests that a time-reversal symmetric counterpart to the FQHE might emerge from interacting itinerant electrons in a crystalline environment.

From the outset, this endeavor follows a different line of logic than the FQHE, as it is not based on pre-existing experimental evidence. Past experience with the FQHE has thus guided recent attempts to either construct time-reversal symmetric edge theories or to construct time-reversal symmetric bulk wave functions supporting local excitations carrying fractional quantum numbers.[37,40–45]

While numerical support for a time-reversal symmetric topological phase of matter was given by Neupert et al. in their study of a lattice model for interacting itinerant electrons,[44] a description in terms of an effective theory is desirable to reveal the universal properties of such a phase. In Ref. 44, the universal properties such as the topological degeneracies of the ground state manifold were explored with the help of a family of edge theories. In this paper, we are going to construct the corresponding bulk topological theory by generalizing the hierarchy of Abelian FQHEs to the hierarchy of Abelian fractional quantum spin Hall effects (FQSHEs) in Sec. II. We will show in Sec. III the correspondence between the bulk theory and the edge theory whose stability to the breaking of translation invariance and residual spin-1/2 U(1) symmetry was studied in Ref. 44. Finally, we shall generalize in Sec. IV the wave functions supporting the Abelian FQHE for a fractional filling of the lowest Landau level to wave functions supporting an Abelian FQSHE. These time-reversal symmetric wave functions are built from the holomorphic and antiholomorphic single-particle wave functions belonging to the lowest Landau level when the applied uniform magnetic field is pointing down or up, respectively. For the reader who wants to skip the derivations, we provide a detailed summary of our results in Sec. V.

## II. TIME-REVERSAL SYMMETRIC ABELIAN CHERN-SIMONS QUANTUM FIELD THEORY

Let us start by summarizing some of the results that we will derive in this section. We shall construct a class of incompressible liquids, each of which is the ground state of a time-reversal symmetric (2+1)-dimensional Chern-Simons quantum field theory that depends on $2N$ flavors of gauge fields $a_{i,\mu}(t,\boldsymbol{x})$, where $i = 1,\ldots,2N$ labels the flavors and $\mu = 0,1,2$ labels the space-time coordinates

$x^\mu \equiv (t,\boldsymbol{x})$, and with the action

$$\mathcal{S} := \int \mathrm{d}t\,\mathrm{d}^2\boldsymbol{x}\;\epsilon^{\mu\nu\rho}\left(-\frac{1}{4\pi}K_{ij}\;a_{i,\mu}\;\partial_\nu\,a_{j,\rho} + \frac{e}{2\pi}\,Q_i\;A_\mu\,\partial_\nu\,a_{i,\rho} + \frac{s}{2\pi}\,S_i\;B_\mu\,\partial_\nu\,a_{i,\rho}\right). \tag{2.1a}$$

Here, $K_{ij}$ are elements of the symmetric and invertible $2N\times 2N$ integer matrix $K$. The integer-valued component $Q_i$ of the $2N$-dimensional vector $Q$ represents the $i$-th electric charge in units of the electronic charge $e$, which couples to the electromagnetic gauge potential $A_\mu(t,\boldsymbol{x})$. Similarly, $S_i$ is an integer-valued component of the $2N$-dimensional vector $S$ that represents the $i$-th spin charge in units of $s$ associated to the up or down spin projection along a spin-1/2 quantization axis, which couples to the Abelian (spin) gauge potential $B_\mu(t,\boldsymbol{x})$. The operation of time reversal maps $a_{\mathsf{i},\mu}(t,\boldsymbol{x})$ into $-g^{\mu\nu}\,a_{\mathsf{i}+N,\nu}(-t,\boldsymbol{x})$ for $\mathsf{i}=1,\cdots,N$ and vice versa. Here, $g_{\mu\nu}:=\mathrm{diag}(+,-,-)\equiv g^{\mu\nu}$ is the Lorentz metric. In Eq. (2.1a) $\boldsymbol{x}\in\Omega$, where $\Omega\subset\mathbb{R}^2$ is a region of two-dimensional Euclidean space, which for the discussion of the bulk theory we consider to have no boundary, $\partial\,\Omega=\varnothing$. The domain of integration $\mathbb{R}$ is unbounded in time $t$. We will show that time-reversal symmetry imposes that the matrix $K$ and the vectors $Q$ and $S$ are of the block form

$$K=\begin{pmatrix}\kappa & \Delta\\ \Delta^{\mathsf{T}} & -\kappa\end{pmatrix},\quad Q=\begin{pmatrix}\varrho\\ \varrho\end{pmatrix},\quad S=\begin{pmatrix}\varrho\\ -\varrho\end{pmatrix}, \tag{2.1b}$$

with $\varrho$ an integer $N$-vector, while $\kappa=\kappa^{\mathsf{T}}$ and $\Delta=-\Delta^{\mathsf{T}}$ are symmetric and antisymmetric integer-valued $N\times N$ matrices, respectively.

The doubled structure of the theory is even more evident if we express it as a BF theory,[46,47] i.e., by defining

$$a^{(\pm)}_{\mathsf{i},\mu} := \frac{1}{2}\left(a_{\mathsf{i},\mu}\pm a_{\mathsf{i}+N,\mu}\right),\qquad \mathsf{i}=1,\ldots,N\;, \tag{2.2a}$$

for $\mu=0,1,2$. This basis allows to re-express the effective action (2.1a) as

$$\mathcal{S} :=\int \mathrm{d}t\,\mathrm{d}^2\boldsymbol{x}\;\epsilon^{\mu\nu\rho}\left(-\frac{1}{\pi}\varkappa_{\mathsf{ij}}\,a^{(+)}_{\mathsf{i},\mu}\;\partial_\nu\,a^{(-)}_{\mathsf{j},\rho} + \frac{e}{\pi}\;\rho_{\mathsf{i}}\;A_\mu\partial_\nu\,a^{(+)}_{\mathsf{i},\rho} + \frac{s}{\pi}\;\rho_{\mathsf{i}}\;B_\mu\partial_\nu\,a^{(-)}_{\mathsf{i},\rho}\right). \tag{2.2b}$$

In this representation, the indices in sans serif fonts $\mathsf{i},\mathsf{j}$ run from 1 to $N$. The coupling between the pair of gauge fields $a^{(+)}$ and $a^{(-)}$ is off-diagonal in the BF labels $\pm$. This is a consequence of time-reversal symmetry, which is implemented by

$$a^{(\pm)}_{\mu}(t,\boldsymbol{x}) \stackrel{\mathcal{T}}{\to} \mp g^{\mu\nu}\,a^{(\pm)}_{\nu}(-t,\boldsymbol{x})\;, \tag{2.2c}$$

that leaves the action (2.2b) invariant. In this representation, the electromagnetic gauge potential $A$ couples to the $+$-species only, while the spin gauge potential $B$ couples to the $-$-species only. The $N\times N$ integer-valued matrix $\varkappa$ in the BF representation is related to the block matrices $\kappa$ and $\Delta$ contained in $K$ from Eq. (2.1b) through

$$\varkappa=\kappa-\Delta. \tag{2.2d}$$

The degeneracy of the ground state is obtained for either description, i.e., the one in terms of the flavors $a_i$ with $i=1,\cdots,2N$ or the one in terms of the flavors $a^{(\pm)}_{\mathsf{i}}$ with $\mathsf{i}=1,\cdots,N$, from

$$\mathcal{N}_{\mathrm{GS}} \;=\; \left|\det\begin{pmatrix}0 & \varkappa\\ \varkappa^{\mathsf{T}} & 0\end{pmatrix}\right| = \left(\det\varkappa\right)^2. \tag{2.3}$$

If the underlying microscopic theory describes fermions with a residual spin-1/2 U(1) (easy plane $XY$) symmetry, it is then meaningful to define the quantized spin Hall resistance

$$\sigma_{\mathrm{sH}} := \frac{e}{2\pi}\times\nu_{\mathrm{s}}. \tag{2.4a}$$

The filling fraction $\nu_{\mathrm{s}}$ is here defined so that it is unity for the integer quantum spin Hall effect and therefore given by

$$\begin{aligned}\nu_{\mathrm{s}} &:= \frac{1}{2}Q^{\mathsf{T}}\,K^{-1}\,S\\ &= \varrho^{\mathsf{T}}\,\varkappa^{-1}\,\varrho.\end{aligned} \tag{2.4b}$$

We now turn to the hierarchical construction of the states described by thisS quantum field field theory. As a warm-up, we begin by reviewing how a one-component Chern-Simons quantum field theory in (2+1)-dimensional space and time is related to the quantum Hall effect. We then construct recursively the multi-component Chern-Simons quantum field theory in such a way that it respects time-reversal symmetry.

### A. Brief review of the one-component Chern-Simons theory

We start from the Lagrangian density

$$\mathcal{L}_{\mathrm{CS}} := -\frac{p}{4\pi}\,\epsilon^{\mu\nu\lambda}\,a_\mu\,\partial_\nu\,a_\lambda + \frac{e}{2\pi}\,\epsilon^{\mu\nu\lambda}\,A_\mu\,\partial_\nu\,a_\lambda \tag{2.5a}$$

in $(2+1)$-dimensional space and time with the action

$$\mathcal{S}_{\mathrm{CS}} := \int\limits_{\mathbb{R}}\mathrm{d}t\int\limits_{\Omega}\mathrm{d}^2\boldsymbol{x}\,\mathcal{L}_{\mathrm{CS}} \tag{2.5b}$$

and partition function

$$Z_{\mathrm{CS}}[A] := \int\mathcal{D}[a]\,e^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\mathrm{CS}}}. \tag{2.5c}$$

The dimensionless integer $p$ is positive. The electromagnetic coupling (electric charge) $e$ is dimensionfull.

It measures the strength of the interaction between an external electromagnetic gauge field $A$ with the components $A^{\mu} \equiv (A^{0}, \boldsymbol{A})$ and a dynamical gauge field $a$ with the components $a^{\mu} \equiv (a^{0}, \boldsymbol{a})$. The symbol $\mathcal{D}[a]$ represents the measure of all gauge orbits stemming from the Abelian group U(1).

The operation $\mathcal{T}$ for reversal of time is defined by

$$a_{\mu}(t,\boldsymbol{x}) \xrightarrow{\mathcal{T}} +g^{\mu\nu}\, a_{\nu}(-t,\boldsymbol{x}), \tag{2.6a}$$

$$A_{\mu}(t,\boldsymbol{x}) \xrightarrow{\mathcal{T}} +g^{\mu\nu}\, A_{\nu}(-t,\boldsymbol{x}), \tag{2.6b}$$

for $\mu = 0,1,2$. We also posit that $\mathcal{T}$ is an anti-unitary linear transformation. If so, one verifies that $\mathcal{L}_{\mathrm{CS}}$ is odd under reversal of time.

Define the electromagnetic current to be the 3-vector

$$\begin{aligned} J^{\mu}_{\mathrm{CS}} :=& \frac{1}{\hbar}\frac{\delta\mathcal{S}_{\mathrm{CS}}}{\delta A_{\mu}} \\ =& \frac{e}{2\pi\hbar}\,\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,a_{\lambda} \end{aligned} \tag{2.7a}$$

for $\mu = 0,1,2$. Because the Levi-Civita tensor with the component $\epsilon^{012} \equiv 1$ is fully antisymmetric, this current is conserved,

$$\partial_{\mu} J^{\mu}_{\mathrm{CS}} = 0. \tag{2.7b}$$

Now, the equations of motions

$$0 = \frac{\delta\mathcal{S}_{\mathrm{CS}}}{\delta a_{\mu}} = -\frac{p}{2\pi}\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,a_{\lambda} + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,A_{\lambda} \tag{2.8}$$

can be used in conjunction with Eq. (2.7) to yield the conserved electromagnetic current

$$J^{\mu}_{\mathrm{CS}} = \frac{1}{p}\,\frac{e^{2}}{h}\,\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,A_{\lambda} \tag{2.9}$$

which allows us to identify the filling fraction $\nu = p^{-1}$ in this simple example, so that the quantum Hall conductance is given by $\sigma_{\mathrm{H}} = \nu\,\frac{e^{2}}{h}$. From now on, we adopt units in which $\hbar = 1$.

### B. One-component BF theory

We start from the Lagrangian density in $(2+1)$-dimensional space and time

$$\begin{aligned} \mathcal{L}^{\mathrm{TRS}}_{\mathrm{BF}} :=& -\frac{p}{\pi}\,\epsilon^{\mu\nu\lambda}\,a^{(+)}_{\mu}\,\partial_{\nu}\,a^{(-)}_{\lambda} \\ &+\frac{e}{\pi}\,\epsilon^{\mu\nu\lambda}\,A_{\mu}\,\partial_{\nu}\,a^{(+)}_{\lambda} + \frac{s}{\pi}\,\epsilon^{\mu\nu\lambda}\,B_{\mu}\,\partial_{\nu}\,a^{(-)}_{\lambda} \end{aligned} \tag{2.10a}$$

with the action

$$\mathcal{S}^{\mathrm{TRS}}_{\mathrm{BF}} := \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^{2}\boldsymbol{x}\,\mathcal{L}^{\mathrm{TRS}}_{\mathrm{BF}} \tag{2.10b}$$

and partition function

$$Z^{\mathrm{TRS}}_{\mathrm{BF}}[A,B] := \int \mathcal{D}[a^{(+)},a^{(-)}]\,e^{\mathrm{i}\mathcal{S}^{\mathrm{TRS}}_{\mathrm{BF}}}. \tag{2.10c}$$

Equation (2.10) is a BF theory made of two copies of the Chern-Simons theory (2.5) with the specificity that the integer $p$ enters with opposite signs in the two copies. We have also introduced two external gauge fields $A$ and $B$ with the couplings $e$ and $s$, respectively. For the gauge field $A$, $e$ will be interpreted as a total U(1) charge. For the gauge field $B$, $s$ will be interpreted as a relative U(1) charge. If the underlying microscopic model is built from itinerant electrons, the gauge field $A$ is the U(1) electromagnetic gauge field that couples to the conserved electric charge whereas the gauge field $B$ is the U(1) gauge field that couples to the conserved projection along some quantization axis of the electronic spin, i.e., $s = 1/2$.

This theory is invariant under the operation of time reversal defined by the anti-linear extension of

$$a^{(\pm)}_{\mu}(t,\boldsymbol{x}) \xrightarrow{\mathcal{T}} \mp g^{\mu\nu}\,a^{(\pm)}_{\nu}(-t,\boldsymbol{x}) \equiv \mp a^{(\pm)\mu}(\tilde{t},\tilde{\boldsymbol{x}}), \tag{2.11a}$$

$$A_{\mu}(t,\boldsymbol{x}) \xrightarrow{\mathcal{T}} +g^{\mu\nu}\,A_{\nu}(-t,\boldsymbol{x}) \equiv +A^{\mu}(\tilde{t},\tilde{\boldsymbol{x}}), \tag{2.11b}$$

$$B_{\mu}(t,\boldsymbol{x}) \xrightarrow{\mathcal{T}} -g^{\mu\nu}\,B_{\nu}(-t,\boldsymbol{x}) \equiv -B^{\mu}(\tilde{t},\tilde{\boldsymbol{x}}), \tag{2.11c}$$

for $\mu = 0,1,2$. The component $A^{0}$ of the external electromagnetic gauge field $A$ is unchanged whereas its vector component $\boldsymbol{A}$ is reversed under reversal of time, just as the vector components of $a^{(-)}$. This behavior is reversed for the components of the external gauge field $B$ that couples to the conserved U(1) spin current and the gauge field $a^{(+)}$.

Since this theory is equivalent to two independent copies of the Chern-Simons theory (2.5), there are two independent conserved currents of the form (2.7),

$$J^{\mu}_{\pm} := \frac{e}{\pi}\epsilon^{\mu\nu\lambda}\partial_{\nu}a^{(\pm)}_{\lambda}, \tag{2.12}$$

for $\mu = 0,1,2$. Their transformation laws under reversal of time are

$$J^{\mu}_{\pm}(x) \xrightarrow{\mathcal{T}} \pm g_{\mu\nu} J^{\nu}_{\pm}(\tilde{x}), \tag{2.13}$$

for $\mu = 0,1,2$. If the microscopic model is made of itinerant electrons, we can thus interpret $J^{\mu}_{+}$ as the charge current and, if the model has a residual U(1) rotation symmetry of the electronic spin, $J^{\mu}_{-}$ represents the conserved spin current. The equations of motions

$$0 = \frac{\delta\mathcal{S}^{\mathrm{TRS}}_{\mathrm{BF}}}{\delta a^{(\pm)}_{\mu}} \tag{2.14a}$$

for the dynamical compact gauge fields $a^{(-)}$ and $a^{(+)}$, respectively, deliver the relations

$$\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,a^{(+)}_{\lambda} = \frac{s}{p}\,\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,B_{\lambda} \tag{2.14b}$$

and

$$\epsilon^{\mu\nu\lambda}\,\partial_\nu\, a^{(-)}_\lambda = \frac{e}{p}\,\epsilon^{\mu\nu\lambda}\,\partial_\nu\, A_\lambda, \tag{2.14c}$$

for $\mu = 0, 1, 2$, respectively. We conclude that, on the one hand, the charge current obeys the Hall response

$$J^\mu_+ = 2s \times \frac{e}{2\pi p}\epsilon^{\mu\nu\lambda}\,\partial_\nu\, B_\lambda, \tag{2.15a}$$

with $\mu = 0, 1, 2$ while, on the other hand, the spin current obeys the Hall response

$$J^\mu_- = 2e \times \frac{e}{2\pi p}\,\epsilon^{\mu\nu\lambda}\,\partial_\nu\, A_\lambda, \tag{2.15b}$$

with $\mu = 0, 1, 2$.

### C. Time-reversal symmetric hierarchy

The generic structure of the hierarchical construction is the following. Let $n > 0$ be any positive integer. Define at the level $n$ of the hierarchy the quantum field theory with the partition function

$$Z^{\mathrm{TRS}}_n[A, B] := \int \mathcal{D}\left[a^{(+)}_1, \cdots, a^{(+)}_n, a^{(-)}_1, \cdots, a^{(-)}_n\right] \times e^{\mathrm{i}\mathcal{S}^{\mathrm{TRS}}_n}, \tag{2.16a}$$

where the action is

$$\mathcal{S}^{\mathrm{TRS}}_n := \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^2\boldsymbol{x}\, \mathcal{L}^{\mathrm{TRS}}_n \tag{2.16b}$$

and the Lagrangian density is

$$\begin{aligned}\mathcal{L}^{\mathrm{TRS}}_n := &-\sum_{\mathsf{i},\mathsf{j}=1}^{n} \frac{1}{\pi}\, \varkappa^{(n)}_{\mathsf{ij}}\, \epsilon^{\mu\nu\lambda}\, a^{(+)}_{\mathsf{i},\mu}\,\partial_\nu\, a^{(-)}_{\mathsf{j},\lambda} \\ &+\sum_{\mathsf{i}=1}^{n} \frac{e}{\pi}\, \varrho^{(n)}_{\mathsf{i}}\, \epsilon^{\mu\nu\lambda}\, A_\mu\,\partial_\nu\, a^{(+)}_{\mathsf{i},\lambda} \\ &+\sum_{\mathsf{i}=1}^{n} \frac{s}{\pi}\, \varrho^{(n)}_{\mathsf{i}}\, \epsilon^{\mu\nu\lambda}\, B_\mu\,\partial_\nu\, a^{(-)}_{\mathsf{i},\lambda}.\end{aligned} \tag{2.16c}$$

Here, the dynamical gauge fields $a^{(\pm)}$ are the $n$-tuplet with the components

$$\left(a^{(\pm)}_{\mathsf{i}}\right) \equiv \left(a^{(\pm)}_1, \cdots, a^{(\pm)}_n\right)^{\mathsf{T}}. \tag{2.17a}$$

Moreover, the $n\times n$ matrix $\varkappa^{(n)}$ is invertible and has, by assumption, integer-valued matrix elements. The charge vector $\varrho^{(n)}$ has the integer-valued components

$$\varrho^{(n)} = (1, 0, \cdots, 0)^{\mathsf{T}} \in \mathbb{Z}^n. \tag{2.17b}$$

Finally, the compatibility condition

$$(-)^{\varkappa^{(n)}_{\mathsf{ii}}} = (-)^{\varrho^{(n)}_{\mathsf{i}}} \tag{2.17c}$$

for $\mathsf{i} = 1, \cdots, n$ is also assumed.

The operation of time reversal is the rule

$$x^\mu \xrightarrow{\mathcal{T}} \tilde{x}^\mu := -g_{\mu\nu}\, x^\nu \tag{2.18a}$$

together with the anti-linear extension of the rules

$$a^{(\pm)\mu}_{\mathsf{i}}(x) \xrightarrow{\mathcal{T}} \mp g_{\mu\nu}\, a^{(\pm)\nu}_{\mathsf{i}}(\tilde{x}), \tag{2.18b}$$

for $\mu = 0, 1, 2$ and $\mathsf{i} = 1, \cdots, n$ that leaves the Lagrangian density (2.16c) invariant.

The level $n+1$ of the hierarchical construction posits the existence of the pair of quasiparticle 3-currents $j_{\pm,n+1}$ that are conserved, i.e.,

$$\partial_\mu\, j^\mu_{\pm,n+1} = 0. \tag{2.19}$$

It also posits the existence of some *even* integer $p_{n+1}$ and $2n$ integers $l^{(+)}_{\mathsf{i}},\ l^{(-)}_{\mathsf{i}}$ with $\mathsf{i} = 1, \cdots, n$ such that the constraints

$$j^\mu_{\pm,n+1} = \frac{\epsilon^{\mu\nu\lambda}}{\pi p_{n+1}} \sum_{\mathsf{i}=1}^{n} l^{(\pm)}_{\mathsf{i}}\,\partial_\nu\, a^{(\pm)}_{\mathsf{i},\lambda} \tag{2.20}$$

for $\mu = 0, 1, 2$ hold. The constraint (2.20) means that any pair of flux quanta, arising when $a^{(+)}_{\mathsf{i}}$ and $a^{(-)}_{\mathsf{i}}$ each support a vortex, creates a quasi-particle with charge $2\, l^{(+)}_{\mathsf{i}}/p_{n+1}$ and spin $2\, l^{(-)}_{\mathsf{i}}/p_{n+1}$ for $\mathsf{i} = 1, \cdots, n$.

This construction can be achieved from the partition function

$$Z^{\mathrm{TRS}}_{n+1}[A, B] := \int \mathcal{D}\left[a^{(+)}_1, \cdots, a^{(+)}_{n+1}, a^{(-)}_1, \cdots, a^{(-)}_{n+1},\right] \times e^{\mathrm{i}\mathcal{S}^{\mathrm{TRS}}_{n+1}}, \tag{2.21a}$$

with the action

$$\mathcal{S}^{\mathrm{TRS}}_{n+1} := \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^2\boldsymbol{x}\, \mathcal{L}_{n+1} \tag{2.21b}$$

and Lagrangian density

$$\begin{aligned}\mathcal{L}^{\mathrm{TRS}}_{n+1} :=& \mathcal{L}^{\mathrm{TRS}}_n \\ &- \frac{p_{n+1}}{\pi}\, \epsilon^{\mu\nu\lambda}\, a^{(+)}_{n+1,\mu}\,\partial_\nu\, a^{(-)}_{n+1,\lambda} \\ &+ \frac{1}{\pi}\, \epsilon^{\mu\nu\lambda} \sum_{\mathsf{i}=1}^{n} l^{(+)}_{\mathsf{i}}\, a^{(+)}_{\mathsf{i},\mu}\,\partial_\nu\, a^{(-)}_{n+1,\lambda} \\ &+ \frac{1}{\pi}\, \epsilon^{\mu\nu\lambda} \sum_{\mathsf{i}=1}^{n} l^{(-)}_{\mathsf{i}}\, a^{(-)}_{\mathsf{i},\mu}\,\partial_\nu\, a^{(+)}_{n+1,\lambda}.\end{aligned} \tag{2.21c}$$

Indeed, we can then *define* the conserved quasiparticle currents of type $n$ to be

$$j^\mu_{\pm,n+1} := \frac{1}{\pi}\, \epsilon^{\mu\nu\lambda}\,\partial_\nu\, a^{(\pm)}_{n+1,\lambda} \tag{2.22}$$

for $\mu=0,1,2$ and use the equations of motion

$$
\begin{aligned}
&0=\frac{\delta\mathcal{S}^{\mathrm{TRS}}_{n+1}}{\delta a^{(\mp)}_{n+1,\mu}}\Longleftrightarrow\\
&\frac{p_{n+1}}{\pi}\epsilon^{\mu\nu\lambda}\partial_{\nu}a^{(\pm)}_{n+1,\lambda}=\frac{\epsilon^{\mu\nu\lambda}}{\pi}\sum_{\mathsf{i}=1}^{n}l^{(\pm)}_{\mathsf{i}}\,\partial_{\nu}\,a^{(\pm)}_{\mathsf{i},\lambda}
\end{aligned}
\tag{2.23}
$$

obeyed by the dynamical gauge fields $a^{(\pm)}_{n+1,\mu}$ to establish that they indeed obey the constraints imposed in Eq. (2.20).

Observe that if we introduce the two $(n+1)$-tuplets $a^{(\pm)}$ given by

$$
\left(a^{(\pm)}_{\mathsf{i}}\right)^{\mathsf{T}}\equiv\left(a^{(\pm)}_{1},\cdots,a^{(\pm)}_{n+1}\right)^{\mathsf{T}}
\tag{2.24a}
$$

of dynamical gauge fields, then the Lagrangian $\mathcal{L}^{\mathrm{TRS}}_{n+1}$ defined in Eq. (2.21c) takes the same form as $\mathcal{L}^{\mathrm{TRS}}_{n}$ defined in Eq. (2.16c) after the substitution $n\to n+1$. The $(n+1)\times(n+1)$ matrix $\varkappa^{(n+1)}$ is then given by

$$
\varkappa^{(n+1)}=\begin{pmatrix}\varkappa^{(n)}&-l^{(+)}\\-l^{(-)\mathsf{T}}&p_{n+1}\end{pmatrix}.
\tag{2.24b}
$$

The $(n+1)$-component charge vector $\varrho^{(n+1)}$ is given by

$$
\varrho^{(n+1)}=(1,0,\cdots,0)^{\mathsf{T}}\in\mathbb{Z}^{n+1},
\tag{2.24c}
$$

thus imposing a vanishing coupling of the external gauge fields $A$ and $B$ to $a^{(\pm)}_{n+1}$. The compatibility condition

$$
(-)^{\varkappa^{(n)}_{\mathsf{ii}}}=(-)^{\varrho^{(n)}_{\mathsf{i}}}
\tag{2.24d}
$$

for $\mathsf{i}=1,\cdots,n+1$ holds if and only if the integer $p_{n+1}$ is even.

The representation (2.24) is called the hierarchical representation.

The operation of time reversal obtained from Eq. (2.18) by allowing $\mathsf{i}$ to run from 1 up to $n+1$ leaves the Lagrangian of level $n+1$ invariant. Therefore, we have constructed a hierarchical time-reversal symmetric BF theory.

### D. Equivalent representations

We define an equivalence class on all the actions of the form (2.2b) when there exists a linear transformation $W$ with integer valued coefficients and unit determinant such that

$$
\varkappa=W^{\mathsf{T}}\,\varkappa'\,W
\tag{2.25a}
$$

and

$$
\varrho=W^{\mathsf{T}}\,\varrho',
\tag{2.25b}
$$

between any two given pairs $(\varkappa,\varrho)$ and $(\varkappa',\varrho')$ within an equivalence class.

*Example 1:* The lower-triangular transformation

$$
W^{\mathsf{T}}:=\begin{pmatrix}1&0&\cdots&0\\1&-1&\cdots&0\\\vdots&\vdots&\cdots&\vdots\\1&0&\cdots&-1\end{pmatrix}
\tag{2.26a}
$$

relates the hierarchical basis characterized by the charge vectors

$$
\varrho=(1,0,\cdots,0)^{\mathsf{T}}
\tag{2.26b}
$$

to the so-called symmetric basis characterized by the charge vector

$$
\varrho=(1,1,\cdots,1)^{\mathsf{T}}.
\tag{2.26c}
$$

*Example 2:* The block-diagonal transformation

$$
W^{\mathsf{T}}:=\begin{pmatrix}\mathbb{1}_{m-1}&&&&\\&0&&-1&\\&&\mathbb{1}_{n-1-m}&&\\&+1&&0&\\&&&&\mathbb{1}_{N-n}\end{pmatrix}
\tag{2.27}
$$

with $1\le m<n\le N$ that interchanges $\varkappa_{mm}$ with $\varkappa_{nn}$, $\varkappa_{mn}$ with $-\varkappa_{nm}$, while it substitutes $-\varrho_{n}$ for $\varrho_{m}$ and $+\varrho_{m}$ for $\varrho_{n}$.

## III. EDGE THEORY

In this Section, we study the quantum field theory for $2N$ Abelian Chern-Simons fields as defined in (2.1a) or, equivalently, (2.2b) in a system with a boundary by following a strategy pioneered in Refs. 48 and 49. However, before relaxing the condition $\partial\,\Omega=\varnothing$, we decompose the action (2.1a) of the bulk theory into

$$
\mathcal{S}:=\mathcal{S}_{K}+\mathcal{S}_{Q}+\mathcal{S}_{S},
\tag{3.1a}
$$

$$
\mathcal{S}_{K}:=-\frac{1}{4\pi}\int\limits_{\mathbb{R}}\mathrm{d}t\int\limits_{\Omega}\mathrm{d}^{2}\boldsymbol{x}\,K_{ij}\,\epsilon^{\mu\nu\rho}\,a_{i,\mu}\,\partial_{\nu}\,a_{j,\rho},
\tag{3.1b}
$$

$$
\mathcal{S}_{Q}:=+\int\limits_{\mathbb{R}}\mathrm{d}t\int\limits_{\Omega}\mathrm{d}^{2}\boldsymbol{x}\,\frac{e}{2\pi}\,Q_{i}\,\epsilon^{\mu\nu\rho}\,a_{i,\mu}\,\partial_{\nu}\,A_{\rho},
\tag{3.1c}
$$

$$
\mathcal{S}_{S}:=+\int\limits_{\mathbb{R}}\mathrm{d}t\int\limits_{\Omega}\mathrm{d}^{2}\boldsymbol{x}\,\frac{s}{2\pi}\,S_{i}\,\epsilon^{\mu\nu\rho}\,a_{i,\mu}\,\partial_{\nu}\,B_{\rho}.
\tag{3.1d}
$$

Notice that we have performed a partial integration in Eq. (3.1c) and Eq. (3.1d) as compared to Eq. (2.1a), so that the gauge fields $A$ and $B$ enter Eq. (3.1) in an explicitly gauge invariant form. In contrast, we are going to make a gauge choice for the fields $a_{i}$ with $i=1,\cdots,2N$

to derive the gauge-invariant effective theory of the edge, once we have relaxed the condition $\partial\,\Omega = \varnothing$.

Let us choose $\Omega$ to be the upper-half plane of $\mathbb{R}^2$, i.e.,

$$\Omega := \left\{ (x,y) \in \mathbb{R}^2 \middle| \, y \geq 0 \right\} \tag{3.2}$$

for notational simplicity but without loss of generality. Observe that under the $2N$ independent Abelian gauge transformations of the dynamical Chern-Simons fields

$$a_{i,\mu} \to a_{i,\mu} + \partial_\mu \chi_i \tag{3.3a}$$

for $\mu = 0,1,2$ where $\chi_i$ with $i = 1,\cdots,2N$ are real-valued and smooth, the action $\mathcal{S}$ defined in Eq. (3.1) obeys the transformation law

$$\mathcal{S} \to \mathcal{S} + \delta\mathcal{S} \tag{3.3b}$$

with

$$\delta\mathcal{S} = \int\limits_{-\infty}^{+\infty} \mathrm{d}\,t \int\limits_{-\infty}^{+\infty} \mathrm{d}\,x \, \left( \chi_i \, \mathcal{J}_i^2 \right)(t,x,0) \tag{3.3c}$$

and

$$\begin{aligned} \mathcal{J}_i^2(t,x,y) :=& -\frac{1}{4\pi} K_{ij}\, \epsilon^{2\nu\rho} \, \left( \partial_\nu \, a_{j,\rho} \right)(t,x,y) \\ &+ \frac{e}{2\pi}\, Q_i \, \epsilon^{2\nu\rho} \, \left( \partial_\nu \, A_\rho \right)(t,x,y) \\ &+ \frac{s}{2\pi}\, S_i \, \epsilon^{2\nu\rho} \, \left( \partial_\nu \, B_\rho \right)(t,x,y). \end{aligned} \tag{3.3d}$$

The equations of motion

$$K_{ij}\, \epsilon^{\mu\nu\rho}\, \partial_\nu \, a_{j,\rho} = e\; Q_i \, \epsilon^{\mu\nu\rho}\, \partial_\nu \, A_\rho + s\, S_i \, \epsilon^{\mu\nu\rho}\, \partial_\nu \, B_\rho \tag{3.4}$$

for the dynamical gauge field $a$ dictate here that

$$\mathcal{J}_i^\mu(t,x,y) = +\frac{1}{4\pi} K_{ij}\, \epsilon^{\mu\nu\rho}\, \partial_\nu \, a_{j,\rho} \tag{3.5}$$

for $i = 1,\cdots,2N$ and $\mu = 0,1,2$. Hence, the $2N$ components of the quasi-particle 3-current $\mathcal{J}_i$ obey the continuity equation $\partial_\mu \mathcal{J}_i^\mu = 0$ if $\left(\partial_\mu\, \partial_\nu - \partial_\nu\, \partial_\mu\right) a_{i,\rho} = 0$ holds for any $i = 1,\cdots,2N$ and $\rho = 0,1,2$.

We now assume that the $2N$-tuplet $\chi$ is constant along the boundary $\partial\Omega$ for all times,

$$\left(\partial_x \chi_i\right)(t,x,y=0) = \left(\partial_t\, \chi_i\right)(t,x,y=0) = 0 \tag{3.6}$$

for $i = 1,\cdots,2N$. In this case, each component $\chi_i$ can be pulled outside the integral in Eq. (3.3c) yielding

$$\delta\mathcal{S} = \chi_i \int\limits_{-\infty}^{+\infty} \mathrm{d}\,t \int\limits_{-\infty}^{+\infty} \mathrm{d}\,x \; \mathcal{J}_i^2(t,x,0). \tag{3.7}$$

Gauge invariance, i.e., $\delta\mathcal{S} = 0$, is then achieved if, in addition to the restriction (3.6), we demand that there is no net accumulation of quasi-particle charge along the boundary arising from the quasi-particle current normal to the boundary, i.e.,

$$0 = \int\limits_{-\infty}^{+\infty} \mathrm{d}\,t \int\limits_{-\infty}^{+\infty} \mathrm{d}\,x \; \mathcal{J}_i^2(t,x,0). \tag{3.8}$$

Observe that the stronger condition

$$\chi_i(t,x,y=0) = 0 \tag{3.9}$$

for $i = 1,\cdots,2N$ achieves gauge invariance, i.e., $\delta\mathcal{S} = 0$, without imposing condition (3.8).

Now that we understand under what conditions the quantum field theory with the action (3.1) is gauge invariant with the choice (3.2) for $\Omega$, we are ready to construct the bulk-edge correspondence. To this end, we are going to extract from the dynamical gauge field $a$ degrees of freedom that are localized on the edge $\partial\Omega$ and invariant under the gauge transformations induced by Eqs. (3.3a), (3.6), and (3.8) on the edge $\partial\Omega$.

## A. Bulk-edge correspondence

We start by fixing the gauge of the $2N$ Abelian Chern-Simons fields through the conditions

$$a_0 = K^{-1}\, V\, a_1. \tag{3.10a}$$

We demand here that $V$ is a symmetric, positive definite $2N \times 2N$ matrix that satisfies

$$V = \Sigma_1\, V\, \Sigma_1, \tag{3.10b}$$

where the $2N \times 2N$ matrices

$$\Sigma_\rho := \sigma_\rho \otimes \mathbb{1}_N, \qquad \rho = 1,2,3 \tag{3.10c}$$

are defined by taking the tensor product between any of the Pauli matrices $\sigma_1$, $\sigma_2$, and $\sigma_3$ and the unit $N \times N$ matrix $\mathbb{1}_N$. Condition (3.10b) guarantees that the gauge condition (3.10a) is consistent with reversal of time defined by

$$a_\mu(t,x,y) \xrightarrow{\mathcal{T}} -g^{\mu\nu}\Sigma_1\, a_\nu(-t,x,y). \tag{3.10d}$$

Indeed, the gauge condition (3.10a) then transforms under reversal of time into

$$-\,\Sigma_1\, a^0(-t,x,y) = K^{-1}\, V\, \Sigma_1\, a_1(-t,x,y), \tag{3.11}$$

which, upon using $K^{-1} = -\Sigma_1\, K^{-1}\, \Sigma_1$, coincides with Eq. (3.10a) if and only if we impose condition (3.10b).

Next, we use the gauge conditions (3.10a) to eliminate the time components $a_0$ of the dynamical gauge fields from the theory. For that, observe that their equations of motion

$$0 = \frac{\delta\mathcal{S}_K}{\delta a_0} \Longleftrightarrow \partial_1 a_2 - \partial_2 a_1 = 0, \tag{3.12a}$$

which require the vanishing of their field strengths, are automatically satisfied if

$$a_1 = \partial_1 \Phi, \qquad a_2 = \partial_2 \Phi, \tag{3.12b}$$

for

$$\left(\partial_1 \partial_2 - \partial_2 \partial_1\right) \Phi = 0 \tag{3.12c}$$

then follows if the $2N$ components $\Phi_i$ of the vector field $\Phi$ are smooth for $i = 1, \cdots, 2N$.

We rewrite the kinetic part (3.1b) of the action (3.1a) using the gauge conditions (3.10a) and the equations of motion (3.12a) and subsequently substitute the gauge fields $\Phi$ defined in Eq. (3.12b):

$$\begin{aligned}
\mathcal{S}_K =& -\frac{\epsilon^{0\nu\lambda}}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{0}^{+\infty} \mathrm{d}y \, \left(-a_\nu^{\mathsf{T}} \, K \, \partial_0 \, a_\lambda + a_\nu^{\mathsf{T}} \, V \, \partial_\lambda \, a_1\right) \\
=& -\frac{\epsilon^{0\nu\lambda}}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{0}^{+\infty} \mathrm{d}y \, \left(\partial_\nu \Phi\right)^{\mathsf{T}} \left(K \, \partial_0 \, \partial_\lambda \Phi - V \, \partial_\lambda \, \partial_1 \Phi\right) \\
=& -\frac{\epsilon^{0\nu\lambda}}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{0}^{+\infty} \mathrm{d}y \, \partial_\nu \left(\Phi^{\mathsf{T}} \, K \, \partial_0 \, \partial_\lambda \Phi - \Phi^{\mathsf{T}} \, V \, \partial_\lambda \, \partial_1 \Phi\right).
\end{aligned} \tag{3.13}$$

We shall demand that $\Phi(t, \boldsymbol{x})$ vanishes for $|\boldsymbol{x}| \to \infty$, in which case

$$\begin{aligned}
\mathcal{S}_K =& -\frac{1}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \, \left(\Phi^{\mathsf{T}} \, K \, \partial_0 \partial_1 \Phi - \Phi^{\mathsf{T}} \, V \, \partial_1 \, \partial_1 \Phi\right)(t, x, 0) \\
=& \frac{1}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \, \left[\left(\partial_1 \Phi\right)^{\mathsf{T}} \, K \, \partial_0 \Phi - \left(\partial_1 \Phi\right)^{\mathsf{T}} \, V \, \partial_1 \Phi\right](t, x, 0).
\end{aligned} \tag{3.14}$$

Under the gauge transformation (3.3a) subject to the constraints (3.6) and (3.8) the $2N$-tuplet $\Phi$ transforms as

$$\Phi(t, \boldsymbol{x}) \to \Phi(t, \boldsymbol{x}) + \chi. \tag{3.15}$$

The fact that $\chi$ is independent of time $t$ and space $x$ implies that (a) the edge theory (3.14) is unchanged under Eq. (3.15), as anticipated, and (b) $\left(\partial_1 \Phi\right)(t, x, 0)$ and $\left(\partial_0 \Phi\right)(t, x, 0)$ are unchanged under Eq. (3.15) and therefore are physical degrees of freedom at the edge. Their dynamics are controlled by the non-universal matrix $V$, which is fixed by microscopic details of the physical system near the edge.

So far, we have discussed only the kinetic part of the action. Let us now discuss the couplings to the external gauge potentials $A$ and $B$ given by the actions (3.1c) and (3.1d), respectively. We assume that the external gauge field $A$ is chosen so that (i) all its components are independent of $y$, i.e.,

$$A_\mu(t, x, y) = A_\mu(t, x), \tag{3.16a}$$

for $\mu = 0, 1, 2$ and (ii) they generate the Maxwell equations in a one-dimensional space defined by the boundary $y = 0$, i.e.,

$$A_2(t, x) = 0 \tag{3.16b}$$

for all times $t$ and for all positions $x$ along the one-dimensional boundary $y = 0$. Using (i) and (ii), we can recast $\mathcal{S}_Q$ as

$$\begin{aligned}
\mathcal{S}_Q =& + \frac{e}{2\pi} \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^2\boldsymbol{x} \, Q_i \, \epsilon^{2\nu\rho} \, a_{i,2} \, \partial_\nu \, A_\rho \\
=& + \frac{e}{2\pi} \int\limits_{\mathbb{R}} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^2\boldsymbol{x} \, Q_i \, \epsilon^{2\mu\nu} \, \partial_2 \, \left(\Phi_i \partial_\mu \, A_\nu\right) \\
=& - \frac{e}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}t \int\limits_{-\infty}^{\infty} \mathrm{d}x \, \left(\epsilon^{\mu\nu} \, A_\mu \, Q^{\mathsf{T}} \, \partial_\nu \, \Phi\right)(t, x, 0).
\end{aligned} \tag{3.17}$$

On the last line, the Levi-Civita tensor is defined for $(1+1)$ space and time.

Furthermore, the very same manipulations that lead to Eq. (3.17) can be carried out on $\mathcal{S}_S$ to deliver

$$\mathcal{S}_S = -\frac{s}{2\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}t\int\limits_{-\infty}^{+\infty}\mathrm{d}x\left(\epsilon^{\mu\nu}\,B_\mu\,S^{\mathsf{T}}\,\partial_\nu\,\Phi\right)(t,x,0). \tag{3.18}$$

Finally, the operation of time reversal stated in Eq. (3.10d) in the bulk reduces on the boundary to the transformation law

$$\begin{aligned} a_1(t,x) &= \left(\partial_x\Phi\right)(t,x) \\ &\stackrel{\mathcal{T}}{\to}\Sigma_1\,a_1(-t,x) = \left(\partial_x\,\Sigma_1\,\Phi\right)(-t,x). \end{aligned} \tag{3.19}$$

The transformation law of the $2N$-tuplet $\Phi$ under reversal of time is thus only fixed unambiguously up to an additive constant $2N$-tuplet. The choice

$$\Phi(t,x)\stackrel{\mathcal{T}}{\to}\Sigma_1\,\Phi(-t,x)+\pi K^{-1}\,\Sigma^{\downarrow}\,Q, \tag{3.20a}$$

with

$$\Sigma^{\uparrow} := \frac{1}{2}\left(\Sigma_0+\Sigma_3\right), \qquad \Sigma^{\downarrow} := \frac{1}{2}\left(\Sigma_0-\Sigma_3\right), \tag{3.20b}$$

guarantees that at least one Kramers doublet of fermions exists as local fields in the edge theory, as was shown in Ref. 44.

## B. Fermi-Bose edge correlation functions

Local excitations on the edge can be classified into two groups. There are quasiparticle excitations that carry rational charges and obey fractional statistics. There are Fermi-Bose excitations that carry integer charges and obey Fermi or Bose statistics. The former excitations are built from vertex operators of the form

$$V^{\mathrm{qp}}_i(t,x) := e^{-\mathrm{i}\Phi_i(t,x)} \tag{3.21a}$$

that are labeled by the flavor index $i=1,\cdots,2N$. The latter excitations are built from the vertex operators of the form

$$V^{\mathrm{fb}}_i(t,x) := e^{-\mathrm{i}K_{ij}\,\Phi_j(t,x)}, \tag{3.21b}$$

that are also labeled by the flavor index $i=1,\cdots,2N$. Establishing the statistics under exchange obeyed by these vertex operators can be achieved by computing their correlation functions, as we now show for the Fermi-Bose operators.

We shall choose for $\Omega$ a disk of unit radius centered at the origin of the complex plane with coordinate $z\in\mathbb{C}$. Thus, the boundary $\partial\,\Omega$ is the unit circle centered at the origin of $\mathbb{C}$. We are after the correlation function

$$\begin{aligned} &\Psi\left(\left\{z_{1,1},\bar z_{1,1},\cdots,z_{1,n_1},\bar z_{1,n_1}\right\};\cdots;\left\{z_{2N,1},\bar z_{2N,1},\cdots,z_{2N,n_{2N}},\bar z_{2N,n_{2N}}\right\}\right) := \\ &\qquad\left\langle e^{\mathcal{Q}}\,V^{\mathrm{fb}}_1(z_{1,1},\bar z_{1,1})\times\cdots\times V^{\mathrm{fb}}_1(z_{1,n_1},\bar z_{1,n_1})\times\cdots\times V^{\mathrm{fb}}_{2N}(z_{2N,1},\bar z_{2N,1})\times\cdots\times V^{\mathrm{fb}}_{2N}(z_{2N,n_{2N}},\bar z_{2N,n_{2N}})\right\rangle \end{aligned} \tag{3.22}$$

where the angular bracket denotes an expectation value using the quantum field theory with the action (3.14) and $\mathcal{Q}$ is a so-called background charge. This correlation function fixes the positions of $n_i$ particles of flavor $i=1,\cdots,2N$ at the locations $z_{i,1},\ z_{i,2},\ \cdots,\ z_{i,n_i}$ along the unit circle. We are omitting any reference to the time $t$ since all Fermi-Bose vertex operators are taken at equal time.

We shall use the rules that

$$\left\langle\tilde\Phi_I(z,\bar z)\tilde\Phi_J(w,\bar w)\right\rangle = \begin{cases}\log(z-w), & \text{if } I=J=1,\cdots,N,\\ \log(\bar z-\bar w), & \text{if } I=J=N+1,\cdots,2N,\\ 0 & \text{otherwise,}\end{cases} \tag{3.23}$$

where the capitalized index $I=1,\cdots,2N$ labels the basis of $\mathbb{R}^{2N}$ for which the $K$ matrix is represented by the diagonal matrix made of the signature of its eigenvalues

$$\Sigma_3 = \left(W^{-1}\right)^{\mathsf{T}}\,K\,\left(W^{-1}\right). \tag{3.24}$$

Observe that the linear transformation $W$ needs neither be integer-valued nor have unit determinant. It is a mere useful device to compute the correlation function (3.22). The relationship between the co-ordinates $\tilde\Phi_I$ and $\Phi_j$ is linear and given by

$$\tilde{\Phi}_I = W_{Ij}\,\Phi_j, \qquad I=1,\cdots,2N, \tag{3.25}$$

where the summation convention for the repeated small case indices is used.

In order to take advantage of Eq. (3.23) when evaluating Eq. (3.22), we use the decomposition

$$\begin{aligned} K_{ij}\Phi_j(z,\bar{z}) &= W_{Ii}\ (\Sigma_3)_{IJ}\ W_{Jj}\ \Phi_j(z,\bar{z}) \\ &= W_{Ii}\left(\Sigma^{\uparrow}_{IJ}-\Sigma^{\downarrow}_{IJ}\right)\tilde{\Phi}_J(z,\bar{z}) \\ &= \left(W_{Ii}\,\Sigma^{\uparrow}_{IJ}\,\tilde{\Phi}_J\right)(z)-\left(W_{Ii}\,\Sigma^{\downarrow}_{IJ}\tilde{\Phi}_J\right)(\bar{z}) \end{aligned} \tag{3.26a}$$

for $i=1,\cdots,2N$ where the matrices $\Sigma^{\uparrow}$ and $\Sigma^{\downarrow}$ were defined in Eq. (3.20b). Under the decomposition (3.26a), any Fermi-Bose vertex operator (3.21b) occurring in the correlation function (3.22) becomes

$$V^{\mathrm{fb}}_i(z,\bar{z}) = e^{-\mathrm{i}K_{ij}\,\Phi_j(z,\bar{z})} = \exp\left(-\mathrm{i}\left(W_{Ii}\,\Sigma^{\uparrow}_{IJ}\,\tilde{\Phi}_J\right)(z)\right)\times\exp\left(+\mathrm{i}\left(W_{Ii}\,\Sigma^{\downarrow}_{IJ}\,\tilde{\Phi}_J\right)(\bar{z})\right). \tag{3.26b}$$

We shall also decompose accordingly the background charge $\mathcal{Q}=\mathcal{Q}_{\uparrow}+\mathcal{Q}_{\downarrow}$. Now,

$$\begin{aligned} \Psi\left(\cdots;z_{i,1},\cdots,\bar{z}_{i,n_i};\cdots\right) &= \exp\left(\mathcal{Q}_{\uparrow}+\frac{1}{2}\left\langle\left(+\sum_{i=1}^{2N}\sum_{a_i=1}^{n_i}\left(W_{Ii}\,\Sigma^{\uparrow}_{IJ}\,\tilde{\Phi}_J\right)(z_{i,a_i})\right)^2\right\rangle\right) \\ &\quad\times\exp\left(\mathcal{Q}_{\downarrow}+\frac{1}{2}\left\langle\left(-\sum_{i=1}^{2N}\sum_{a_i=1}^{n_i}\left(W_{Ii}\,\Sigma^{\downarrow}_{IJ}\,\tilde{\Phi}_J\right)(\bar{z}_{i,a_i})\right)^2\right\rangle\right) \\ &= \left[\prod_{i=1}^{2N}\prod_{1\le a_i<b_i\le n_i}\left(z_{i,a_i}-z_{i,b_i}\right)^{W_{Ii}\Sigma^{\uparrow}_{IJ}W_{Ji}}\left(\bar{z}_{i,a_i}-\bar{z}_{i,b_i}\right)^{W_{Ii}\Sigma^{\downarrow}_{IJ}W_{Ji}}\right] \\ &\quad\times\left[\prod_{1\le i<j\le 2N}\prod_{a_i,b_j=1}^{n_i}\left(z_{i,a_i}-z_{j,b_j}\right)^{W_{Ii}\Sigma^{\uparrow}_{IJ}W_{Jj}}\left(\bar{z}_{i,a_i}-\bar{z}_{j,b_j}\right)^{W_{Ii}\Sigma^{\downarrow}_{IJ}W_{Jj}}\right]. \end{aligned} \tag{3.27}$$

The role of the background charge is to guarantee "charge neutrality". Observe that for any pair $i,j=1,\cdots,2N$, the exponents

$$\alpha_{ij} := W_{Ii}\,\Sigma^{\uparrow}_{IJ}\,W_{Jj} \tag{3.28}$$

and

$$\beta_{ij} := W_{Ii}\,\Sigma^{\downarrow}_{IJ}\,W_{Jj} \tag{3.29}$$

can be an irrational number! Nevertheless if $z_{i,a_i}\neq z_{j,b_j}$ the correlation function (3.27) is single-valued, for it is the product of functions of the form

$$\begin{aligned} f(z_{i,a_i},z_{j,b_j}) &:= \left(z_{i,a_i}-z_{j,b_j}\right)^{\alpha_{ij}}\left(\bar{z}_{i,a_i}-\bar{z}_{j,b_j}\right)^{\beta_{ij}} \\ &= \left(z_{i,a_i}-z_{j,b_j}\right)^{\alpha_{ij}-\beta_{ij}+\beta_{ij}}\left(\bar{z}_{i,a_i}-\bar{z}_{j,b_j}\right)^{\beta_{ij}} \\ &= \left(z_{i,a_i}-z_{j,b_j}\right)^{K_{ij}}\left|z_{i,a_i}-z_{j,b_j}\right|^{2\beta_{ij}} \end{aligned} \tag{3.30a}$$

where one verifies that

$$\alpha_{ij}-\beta_{ij}=K_{ij} \tag{3.30b}$$

is integer valued. This is consistent with the fact that the vertex operators (3.21b) describe either Fermions or Bosons.

## IV. WAVE FUNCTIONS

The family of topological quantum field theories defined by Eqs. (2.2b) and (2.25) encode the universal properties of a family of time-reversal symmetric fractional quantum liquids.

Any connection to a microscopic realization of a time-reversal symmetric fractional quantum liquid whose universal properties are captured by Eqs. (2.2b) and (2.25) must, however, be supplied.

For example, the very definition of an electron operator is ambiguous for any equivalence class of topological quantum field theories defined by Eqs. (2.2b) and (2.25). First, there is no unique definition of a local fermion in any topological quantum field theory of the form Eqs. (2.2b) and (2.25) that admits a hierarchical representation with one type of Kramers degenerate pairs of fermions, for this representation is equivalent to the symmetric representation that admits $N$ distinct types of Kramers degenerate pair of fermions [see Eq. (2.26)]. Second, for any representation of the universal data $(K, Q, S)$ from Eq. (2.1b) that admits fermions, a basis set of functions must be supplied to construct a representation of the microscopic electron operator. This basis set of functions is usually provided by some reference single-particle electron basis.

In the context of the FQHE observed in a GaAs accumulation layer, the basis set of functions is the single-particle basis of the Landau Hamiltonian describing an electron moving in a plane perpendicular to a uniform magnetic field. However, the Landau basis set and, in particular, the basis set for the lowest Landau level, is not appropriate for the recently discovered fractional quantum Hall phases in lattice models without external magnetic field.[50–53]

With this caveat in mind, we are going to construct some wave functions using the data $(K, Q, S)$ from Eq. (2.1b) as the universal input and using the Landau wave functions spanning the lowest Landau level as the microscopic input. We do this out of simplicity in view of the elegant analytic properties of these single-particle functions. Hence, we choose the symmetric gauge for which the Slater determinant in the lowest Landau level is

$$\Psi_{\nu_{\mathsf{i}}=1}\left(\{z_{\mathsf{i}}, \bar{z}_{\mathsf{i}}\}_{n_{\mathsf{i}}}\right) := \left[\prod_{1\le k<l\le n_{\mathsf{i}}} (z_{\mathsf{i},k} - z_{\mathsf{i},l})\right] \times \prod_{k=1}^{n_{\mathsf{i}}} \exp\left(-\frac{\bar{z}_{\mathsf{i},k} z_{\mathsf{i},k}}{4\ell^2}\right) \tag{4.1a}$$

for $n_{\mathsf{i}}$ electrons labeled by the flavor index $\mathsf{i} = 1, \cdots, N$ while it is

$$\Psi_{\nu_{\mathsf{i}}=1}\left(\{w_{\mathsf{i}}, \bar{w}_{\mathsf{i}}\}_{n_{\mathsf{i}}}\right) := \left[\prod_{1\le k<l\le n_{\mathsf{i}}} (\bar{w}_{\mathsf{i},k} - \bar{w}_{\mathsf{i},l})\right] \times \prod_{k=1}^{n_{\mathsf{i}}} \exp\left(-\frac{\bar{w}_{\mathsf{i},k} w_{\mathsf{i},k}}{4\ell^2}\right) \tag{4.1b}$$

for $n_{N+\mathsf{i}}$ electrons labeled by the flavor index $N+\mathsf{i} = N+1, \cdots, 2N$. Here,

$$\{z_{\mathsf{i}}, \bar{z}_{\mathsf{i}}\}_{n_{\mathsf{i}}} := \left\{z_{\mathsf{i},1}, \cdots, z_{\mathsf{i},n_{\mathsf{i}}}, \bar{z}_{\mathsf{i},1}, \cdots, \bar{z}_{\mathsf{i},n_{\mathsf{i}}}\right\} \tag{4.2}$$

denotes the complex coordinates of the particles of the first $N$ flavors, with $\bar{z}$ denoting their complex conjugates, and likewise $\{w_{\mathsf{i}}, \bar{w}_{\mathsf{i}}\}_{n_{\mathsf{i}}}$ denotes the complex coordinates for the last $N$ flavors; $\ell^2 := \phi_0/(2\pi|B|)$ is the square of the magnetic length $\ell$ in the presence of the uniform magnetic field of magnitude $|B|$, $\phi_0 := 2\pi/e$ is the quantum of magnetic flux, $|\Omega||B|$ is the magnitude of the flux threading the disk $\Omega$, and $\nu_{\mathsf{i}} := (n_{\mathsf{i}}\phi_0)/(|\Omega||B|)$ represents the filling fraction of the lowest Landau level.

It remains to decide on the number of electron flavors, a microscopic input. We shall assume that the hierarchical (symmetric) representation corresponds to a single pair ($N$ pairs) of microscopic flavors of electrons forming a Kramers doublet ($N$ Kramers doublets). We begin with the wave function for $N = 1$, in which case there is no distinction between the two representations. We then work out examples with $N = 2$ in the symmetric and the hierarchical representation.

### A. Wave function for $N = 1$

We choose the universal data to be

$$K = \begin{pmatrix} +m & 0 \\ 0 & -m \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}), \qquad Q = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^2, \tag{4.3}$$

for some given positive odd integer $m$. The spin filling fraction defined in Eq. (2.4) is

$$\nu_{\mathrm{s}} := \frac{1}{m}. \tag{4.4}$$

The putative ground state wave function that generalizes the $\nu = 1/m$ single-layer wave function from Laughlin (see Ref. 1) to the time-reversal symmetric case is

$$\Psi_{1/m}\left(\{z,\bar{z}\}_n \,|\, \{w,\bar{w}\}_n\right) = \left[\prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(z_i - z_j\right)^m \left(\bar{w}_i - \bar{w}_j\right)^m\right] \times \prod_{i=1}^{n} \exp\left(-\frac{\left|z_i\right|^2 + \left|\bar{w}_i\right|^2}{4\ell^2}\right). \tag{4.5}$$

By construction, it is invariant under the operation of time reversal represented by

$$z_i \xrightarrow{\mathcal{T}} \bar{w}_i\,, \qquad w_i \xrightarrow{\mathcal{T}} \bar{z}_i\,, \qquad i = 1, \cdots, n. \tag{4.6}$$

It thus realizes a time-reversal symmetric fractional incompressible state. Observe that this wave function factorizes into an holomorphic and an antiholomorphic sector. Time-reversal symmetry forbids a coupling between the holomorphic and antiholomorphic sector when $N = 1$.

### B. Wave functions in the symmetric representation

We choose the universal data to be

$$K = \begin{pmatrix} +\kappa & +\Delta \\ -\Delta & -\kappa \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}), \qquad Q = \begin{pmatrix} \varrho \\ \varrho \end{pmatrix} \in \mathbb{Z}^4. \tag{4.7a}$$

The $2\times 2$ matrix $\kappa$ is given by

$$\kappa = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}). \tag{4.7b}$$

We impose that the integers $m_1$ and $m_2$ are odd and positive while the integer $n$ is positive [$n \geq 0$ is not restrictive in view of Eq. (2.27)] whereby

$$m_1 m_2 - n^2 > 0, \tag{4.7c}$$

in order for $\kappa$ to be maximally chiral. In turn,

$$\Delta = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}\,, \tag{4.7d}$$

where the integer $d \geq 0$ is chosen to be non-negative. Finally, the charge vector

$$\varrho = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.7e}$$

enforces the presence of 4 fermions related pairwise by reversal of time. We assume that the fermions with the charge vector $Q$ in the topological quantum field theory represent $2N$ distinct flavors of electrons in a microscopic theory. For example, each flavor of electrons could be constrained to move with the dynamics dictated by the single-particle Landau Hamiltonian in its own two-dimensional layer in the presence of a uniform magnetic field pointing up for the first $N$ layers and down for the next $N$ layers. If the lowest Landau level of each layer is partially filled, interactions might select an incompressible ground state. The spin filling fraction defined in Eq. (2.4) is

$$\nu_{\mathrm{s}} := \frac{m_1 + m_2 - 2n}{m_1\, m_2 - n^2 + d^2}. \tag{4.8}$$

The putative ground state wave function that generalizes the $(m_1, m_2, n)$ bilayer wave function from Halperin (see Ref. 2) to the time-reversal symmetric case is

$$\begin{aligned} &\Psi^{\mathrm{symm}}_{m_1,m_2,n,d}\left(\{z_1,\bar{z}_1\}_{n_1};\{z_2,\bar{z}_2\}_{n_2} \,|\, \{w_1,\bar{w}_1\}_{n_1};\{w_2,\bar{w}_2\}_{n_2}\right) = \\ &\qquad \Psi_{1/m_1}\left(\{z_1,\bar{z}_1\}_{n_1} \,|\, \{w_1,\bar{w}_1\}_{n_1}\right) \times \Psi_{1/m_2}\left(\{z_2,\bar{z}_2\}_{n_2} \,|\, \{w_2,\bar{w}_2\}_{n_2}\right) \\ &\qquad \times \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(z_{1,i} - z_{2,j}\right)^n \left(\bar{w}_{1,i} - \bar{w}_{2,j}\right)^n \left(z_{1,i} - w_{2,j}\right)^d \left(\bar{w}_{1,i} - \bar{z}_{2,j}\right)^d. \end{aligned} \tag{4.9}$$

By construction, it is invariant under the operation of time reversal represented by

$$z_{\mathsf{i},i_{\mathsf{i}}} \xrightarrow{\mathcal{T}} \bar{w}_{\mathsf{i},i_{\mathsf{i}}}\,, \qquad w_{\mathsf{i},i_{\mathsf{i}}} \xrightarrow{\mathcal{T}} \bar{z}_{\mathsf{i},i_{\mathsf{i}}}\,, \qquad i_{\mathsf{i}} = 1, \cdots, n_{\mathsf{i}}\,, \tag{4.10}$$

for $\mathsf{i} = 1, 2$. It thus realizes a time-reversal symmetric fractional incompressible state.

### C. Wave functions in the hierarchical representation

We choose the universal data to be

$$K = \begin{pmatrix} +\kappa & +\Delta \\ -\Delta & -\kappa \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}), \qquad Q = \begin{pmatrix} \varrho \\ \varrho \end{pmatrix} \in \mathbb{Z}^4. \tag{4.11a}$$

The $2\times 2$ matrix $\kappa$ is given by

$$\kappa = \begin{pmatrix} +m & +1 \\ +1 & -p \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \tag{4.11b}$$

where $m$ is a positive odd integer and $p$ is an even integer larger than zero. The $2\times 2$ matrix $\Delta$ is given by

$$\Delta = \begin{pmatrix} 0 & +d \\ -d & 0 \end{pmatrix} \tag{4.11c}$$

with $d$ any positive integer. Finally, the charge vector

$$\varrho = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.11d}$$

enforces the presence of 2 fermions related pairwise by reversal of time. The spin filling fraction defined in Eq. (2.4) is

$$\nu_{\mathrm{s}} := \frac{p}{mp+1-d^2}. \tag{4.12}$$

The putative ground state wave function that generalizes the $\nu = \frac{p}{mp+1}$ single-layer wave function from Halperin (see Ref. 2) to the time-reversal symmetric case is

$$\begin{aligned} \Psi^{\mathrm{hier}}_{m,-p,1,d}\Big(\{z,\bar z\}_{pn}\,|\,\{w,\bar w\}_{pn}\Big) = {} & \left[\prod_{i=1}^{n}\int_{\Omega}\mathrm{d}^2\,\eta_i\int_{\Omega}\mathrm{d}^2\,\xi_i\right] \times \Psi_{1/m}\Big(\{z,\bar z\}_{pn}\,|\,\{w,\bar w\}_{pn}\Big) \times \Psi_{1/p}\Big(\{\xi,\bar\xi\}_n\,|\,\{\eta,\bar\eta\}_n\Big) \\ & \times \prod_{i=1}^{pn}\prod_{j=1}^{n}\left(z_i-\eta_j\right)\left(\bar w_i-\bar\xi_j\right)\left(z_i-\xi_j\right)^d\left(\bar w_i-\bar\eta_j\right)^d . \end{aligned} \tag{4.13}$$

By construction, it is invariant under the operation of time reversal represented by

$$z_i \stackrel{\mathcal{T}}{\to} \bar w_i, \qquad w_i \stackrel{\mathcal{T}}{\to} \bar z_i, \tag{4.14a}$$

for $i = 1,\cdots,pn$ and

$$\xi_i \stackrel{\mathcal{T}}{\to} \bar\eta_i, \qquad \eta_i \stackrel{\mathcal{T}}{\to} \bar\xi_i, \tag{4.14b}$$

for $i=1,\cdots,n$. It thus realizes a time-reversal symmetric fractional incompressible state.

## V. SUMMARY

In this paper, we first derived a hierarchy of FQSHEs, the universal properties of which are encoded by equivalence classes of BF theories of the form

$$\begin{aligned} \mathcal{L} = {} & -\frac{1}{\pi}\epsilon^{\mu\nu\lambda}\,a^{(+)\mathsf{T}}_{\mu}\,\varkappa\,\partial_\nu\,a^{(-)}_{\lambda} \\ & + \frac{e}{\pi}\epsilon^{\mu\nu\lambda}\,A_\mu\,\varrho^{\mathsf{T}}\,\partial_\nu\,a^{(+)}_{\lambda} + \frac{s}{\pi}\epsilon^{\mu\nu\lambda}\,B_\mu\,\varrho^{\mathsf{T}}\,\partial_\nu\,a^{(-)}_{\lambda}. \end{aligned} \tag{5.1}$$

The $N\times N$ invertible and integer-valued matrix $\varkappa$ couples the $N$ flavors of the dynamical gauge field $a^{(+)}$ to the $N$ flavors of the dynamical gauge field $a^{(-)}$. The $N$-tuplets $a^{(+)}$ and $a^{(-)}$ also couple linearly to the external gauge fields $A$ and $B$, respectively, through the vector $\varrho\in\mathbb{Z}^N$, where the integer $\varrho_{\mathsf{i}}$ shares the same parity as the integer $\varkappa_{\mathsf{ii}}$ for $\mathsf{i}=1,\cdots,N$. Correspondingly, there exists two independent conserved currents, a charge current associated to the gauge field $a^{(+)}$ and a spin current associated to the gauge field $a^{(-)}$.

Time-reversal symmetry implies the vanishing of the charge Hall conductivity

$$\sigma_{\mathrm{H}} = \frac{e^2}{2\pi}\times\nu = 0. \tag{5.2}$$

The non-vanishing spin filling fraction

$$\nu_{\mathrm{s}} := \varrho^{\mathsf{T}}\,\varkappa^{-1}\,\varrho \tag{5.3a}$$

can be interpreted as the spin Hall conductance

$$\sigma_{\mathrm{sH}} := \frac{e}{2\pi}\times\nu_{\mathrm{s}} \tag{5.3b}$$

if the U(1) conservation law associated to the current of $a^{(-)}$ arises microscopically from a residual spin-1/2 U(1) (easy plane $XY$) symmetry. The topological ground state degeneracy, if two-dimensional space $\Omega$ has a toroidal geometry,

$$(\det\varkappa)^2 \tag{5.4}$$

is always the square of an integer as a consequence of time-reversal symmetry. Equivalent pairs $(\varkappa, \varrho)$ and $(\varkappa', \varrho')$, as defined by Eq. (2.25), share the same spin Hall conductivity and topological degeneracy.

The theory (5.1) is topological when two-dimensional space $\Omega$ has no boundary, i.e., the Hamiltonian density associated to the Lagrangian density (5.1) vanishes. This is not true anymore if the boundary $\partial\Omega$ is a one-dimensional manifold. We have shown that imposing gauge invariance delivers a gapless theory with all excitations propagating along the boundary $\partial\Omega$. These excitations can all be constructed out of $N$ pairs of counter-propagating chiral bosons whose non-universal velocities along the boundary $\partial\Omega$ derive from a gauge-fixing condition. The stability of this edge theory to the (time-reversal symmetric) breaking of translation invariance along the boundary (including the breaking of the spin conservation law associated to the spin vector $S$) was studied in Ref. 44. The correlation functions for the Fermi-Bose excitations along the edge were computed and shown to be a product over the functions (3.30a).

Finally, we have proposed a time-reversal symmetric counterpart to the hierarchy of wave functions that have been proposed in the context of the FQHE by way of few examples, the $\nu_{\mathrm{s}} = 1/m$, $\nu_{\mathrm{s}} = p/(mp+1-d^2)$, and $\nu_{\mathrm{s}} = (m_1+m_2-2n)/(m_1m_1-n^2+d^2)$ sequences.

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