Z_{2} spin liquid and chiral antiferromagnetic phase in the Hubbard model on a honeycomb lattice

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Z_2 spin liquid and chiral antiferromagnetic phase in honeycomb-lattice Hubbard model

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In Schwinger-fermion representation we classify all 128 possible spin liquids that preserves $SU(2)$ spin rotational symmetry, honeycomb lattice group symmetry and time-reversal symmetry. Among them we identify a Z_2 spin liquid called the sublattice-pairing state (SPS) as the spin liquid phase discovered in recent numerical study on a honeycomb lattice[1]. Our method provides a systematic way to identify spin liquids close to Mott transition. We also show that SPS is identical to the 0-flux Z_2 spin liquid in Schwinger-boson representation[2] through an explicit duality transformation. SPS is connected to an *unusual* antiferromagnetic ordered phase, which we term as chiral-antiferromagnetic (CAF) phase, by an $O(4)$ critical point. CAF phase breaks the $SU(2)$ spin rotational symmetry completely and has three Goldstone modes. Our results indicate that there is likely a hidden phase transition between CAF phase and simple AF phase at large U/t. We also propose numerical measurements to reveal the CAF phase and the hidden phase transition.

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I. INTRODUCTION

A novel state of matter, quantum spin liquid (SL), has been recently discovered in organic spin-1/2 triangular lattice experimental systems[3–5]. This class of organic Mott insulators is in the vicinity of the Mott transition and can be driven into a Fermi liquid by applying pressure. A quantum SL is the ground state of a Mott insulator which does not break physical symmetries, and cannot be adiabatically connected to a band insulator. After Anderson's proposal of resonating valence bond (RVB) states[6], a lot of theoretical and experimental efforts have been made to show the existence of such novel phases of matter with fractionalized excitations[7]. For example, various exact or quasi-exact solvable models[8–11] hosting SL ground states have been constructed. The exciting experimental progress on the triangular lattice organics raises an interesting question: can SL be naturally realized in a Mott insulator close to the Mott transition?[12,13] If this is true, it can serve as a guideline in searching SLs in experimental systems. Physical intuition suggests this is likely to be the case because in the neighborhood of the Mott transition, quantum fluctuations of spins are strong which can suppress the classical spin ordering.

A recent remarkable numerical study[1] for the nearest neighbor Hubbard model on the honeycomb lattice

$$H = -t \sum_{<ij>\sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow} \qquad (1)$$

provides another evidence of this guideline, where an insulating phase respecting all physical symmetry is found in the neighborhood of the Mott transition. This phase has attracted quite some theoretical attention[2,14] because it cannot be explained as a band insulator due to the honeycomb lattice structure. It should be a SL with fractionalized excitations. There are many different SLs on the honeycomb lattice, characterized by different topological orders[15]. Which SL is realized in the simulated Hubbard model? And in a general context, is there a systematic way to identify the SLs in the neighborhood of a Mott transition? We provide our answers to these questions in this letter.

In Ref. 1, it is shown that this SL phase has a full energy gap, and is likely to be smoothly connected (*i.e.* through a continuous phase transition) to both the semi-metal phase for small U/t and an antiferromagnetic (AF) phase for large U/t. These three conditions strongly restrict the candidate SL phases.

There have been two popular approaches to describe SL phases: Schwinger-fermion[16–19] and Schwinger-boson[8,20], in which the low energy spin excitations are fermionic and bosonic spinons respectively. The fermionic approach is more natural to be used in the viicinity of a continuous Mott transition, whereas the bosonic model is more natural when close to a magnetic transition[50]. The possible underlying relation between the two seemingly very different approaches has been a long-standing puzzle.

In Schwinger-fermion representation, we classify all possible 128 different Z_2 spin liquids using projective symmetry group (PSG)[15]. In the vicinity of the Mott transition in simulated Hubbard model[1], we will show that there is only one natural SL among them, a Z_2 state coined the sublattice pairing state (SPS) which has a full energy gap and can be smoothly connected to the semi-metal phase. Moreover, Schwinger-boson method has been used to describe the magnetic phase transition[2] in the same system. It was found that only one SL phase, the 0-flux state, can be smoothly connected to an antiferromagnetically (AF) ordered phase. The 0-flux state is also a Z_2 state with a full energy gap. However, it is not clear whether this 0-flux state can be smoothly connected to the semi-metal phase. Can SPS be related to the 0-flux state? In this paper, we will demonstrate that remarkably, SPS and 0-flux state are identical by an explicit duality transformation in the low energy effective theory. This is the first identification of a state in both the Schwinger fermion and the Schwinger boson methods.

We will also show that the magnetically ordered phase connected to SPS is rather unusual and *not* the simple Neel phase, because it breaks the $SU(2)$ spin-rotation symmetry completely and has three Goldstone modes. We name this phase chiral-antiferromagnetic (CAF) phase. Aside from the usual AF spin order parameter $\vec{N}=(-1)^{i_s}\vec{S}_i$ where $i_s=0,1$ for A and B sublattices respectively, in CAF phase there is another vector-chirality spin order parameter $\vec{n}=\sum_{<<ij>>}\nu_{ij}\vec{S}_i\times\vec{S}_j$ whose expectation value satisfies $\langle\vec{n}\rangle\perp\langle\vec{N}\rangle$. $\nu_{ij}=+1(-1)$ if one goes from site i to j in a clockwise(counterclockwise) manner, as shown by the arrows in FIG. 1(b). Since the usual AF phase should exist in the large U/t limit[21], our results suggest a hidden phase transition, which might happen in the "Neel" phase of the previously mentioned numerical study[1] or at a larger U/t not studied before. We therefore propose the schematic phase diagram as shown in Fig.2(a).

This paper is organized in the following way. In section II we give a brief exposition of $SU(2)$ Schwinger-fermion approach to spin liquid states and projective symmetry group (PSG) classification of different spin liquids. Following the mathematical classification of all Z_2 spin liquids using PSG in Appendix B, we discuss the possible gapped Z_2 spin liquids continuously connected to a semimetal phase in section III. We find out there is only one natural candidate, the sublattice pairing state (SPS). A mean-field study of SPS in J_1-J_2 Heisenberg model is given in Appendix F. In section IV we discuss the continuous phase transition between SPS and a magnetically ordered phase and reveal a hidden order parameter of this chiral-antiferromagnetic (CAF) phase aside from Neel order parameter. In section V we identify our SPS state with the 0-flux state in Schwinger-boson mean-field approach[2] through an explicit duality transformation. Finally we summarize our results in section VI.

II. SCHWINGER-FERMION APPROACH AND PROJECTIVE SYMMETRY GROUP (PSG)

In the large U limit at half-filling, the charge flucutation in Hubbard model (1) is severely suppressed and the low-energy spin fluctuations are described by $S=1/2$ Heisenberg model[43]

$$H_{spin}=-\frac{4t^2}{U}\sum_{<ij>}\mathbf{S}_i\cdot\mathbf{S}_j+O(\frac{t^3}{U^2}) \tag{2}$$

In Schwinger-fermion approach, a spin-1/2 operator at site i is represented by:

$$\vec{S}_i=\frac{1}{2}f_{i\alpha}^\dagger\vec{\sigma}_{\alpha\beta}f_{i\beta}. \tag{3}$$

A Heisenberg spin Hamiltonian $H=\sum_{<ij>}J_{ij}\vec{S}_i\cdot\vec{S}_j$ is represented as $H=\sum_{<ij>}-\frac{1}{2}J_{ij}\big(f_{i\alpha}^\dagger f_{j\alpha}f_{j\beta}^\dagger f_{j\beta}+\frac{1}{2}f_{i\alpha}^\dagger f_{i\alpha}f_{j\beta}^\dagger f_{j\beta}\big)$. Because this representation enlarges the Hilbert space, states need to be constrained in the physical Hilbert space, i.e., one f-fermion per site:

$$f_{i\alpha}^\dagger f_{i\alpha}=1, \qquad f_{i\alpha}f_{i\beta}\epsilon_{\alpha\beta}=0. \tag{4}$$

Introducing mean-field parameters $\eta_{ij}\epsilon_{\alpha\beta}=-2\langle f_{i\alpha}f_{j\beta}\rangle$, $\chi_{ij}\delta_{\alpha\beta}=2\langle f_{i\alpha}^\dagger f_{j\beta}\rangle$, where $\epsilon_{\alpha\beta}$ is fully antisymmetric tensor, after Hubbard-Stratonovich transformation, the Lagrangian of the spin system can be written as[15]

$$\begin{aligned}L=\sum_i\psi_i^\dagger\partial_\tau\psi_i+\sum_{<ij>}\frac{3}{8}J_{ij}\big[\frac{1}{2}\mathrm{Tr}(U_{ij}^\dagger U_{ij})\\-(\psi_i^\dagger U_{ij}\psi_j+h.c.)\big]+\sum_i a_0^l(i)\psi_i^\dagger\tau^l\psi_i\end{aligned} \tag{5}$$

where two-component fermion notation $\psi_i=(f_{i,\uparrow},f_{i,\downarrow}^\dagger)^T$ is introduced for reasons that will be explained shortly. U_{ij} is a matrix of mean-field amplitudes:

$$U_{ij}=\begin{pmatrix}\chi_{ij}^\dagger & \eta_{ij}\\ \eta_{ij}^\dagger & -\chi_{ij}\end{pmatrix}. \tag{6}$$

$a_0^l(i)$ are the local Lagrangian multipliers that enforces the constraints Eq.(4).

In terms of ψ, Schwinger-fermion representation has an explicit $SU(2)$ gauge redundancy: a transformation $\psi_i\to W_i\psi_i$, $U_{ij}\to W_iU_{ij}W_j^\dagger$, $W_i\in SU(2)$ leaves the action invariant. This redundancy is originated from representation Eq.(3): this local $SU(2)$ transformation leaves the spin operators invariant[18] and thus does not change physical Hilbert space.

One can try to solve Eq.(5) by mean-field (or saddle-point) approximation. At mean-field level, U_{ij} and a_0^l are treated as complex numbers, and a_0^l must be chosen such that constraints Eq.(4) are satisfied at the mean field level: $\langle\psi_i^\dagger\tau^l\psi_i\rangle=0$. The mean-field ansatz can be written as:

$$H_{MF}=-\sum_{<ij>}\psi_i^\dagger u_{ij}\psi_j+\sum_i\psi_i^\dagger a_0^l\tau^l\psi_i. \tag{7}$$

where $u_{ij}=\frac{3}{8}J_{ij}U_{ij}$. A local $SU(2)$ gauge transformation modify $u_{ij}\to W_iu_{ij}W_j^\dagger$ but does not change the physical spin state described by the mean-field ansatz. By construction the mean-field amplitudes do not break spin rotation symmetry, and the mean field solutions describe spin liquid states if translational symmetry is preserved. Different $\{u_{ij}\}$ ansatz may be in different spin liquid phases. The mathematical language to classify different spin liquid phases is PSG[15].

PSG is the manifestation of topological order in the Schwinger-fermion representation: spin liquid states described by different PSG's are different phases. It is defined as the collection of all combinations of symmetry

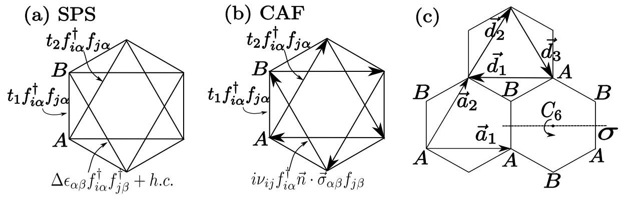

FIG. 1: Mean-field ansatz of (a) SPS phase and (b) CAF phase in terms of f-fermion. $\nu_{ij} = 1$ if $i \to j$ is along the arrow direction. (c)The honeycomb lattice and its Bravais lattice vector $\vec{a}_{1,2}$. $\vec{d}_{1,2,3}$ are the three vectors used in Eq.21. Two generators of symmetry group are also shown: 60° rotation C_6 the horizontal mirror σ.

group and $SU(2)$ gauge transformations that leave $\{u_{ij}\}$ invariant (as a_0^l are determined self-consistently by $\{u_{ij}\}$, these transformations also leave a_0^l invariant). The invariance of a mean-field ansatz $\{u_{ij}\}$ under an element of PSG $G_U U$ can be written as

$$\begin{aligned} G_U U(\{u_{ij}\}) &= \{u_{ij}\}, \\ U(\{u_{ij}\}) &\equiv \{\tilde{u}_{ij} = u_{U^{-1}(i),U^{-1}(j)}\}, \\ G_U(\{u_{ij}\}) &\equiv \{\tilde{u}_{ij} = G_U(i) u_{ij} G_U(j)^\dagger\}, \\ G_U(i) &\in SU(2). \end{aligned} \tag{8}$$

Here $U \in SG$ is an element of symmetry group (SG) of the spin liquid state. SG on a honeycomb lattice is generated by time reversal $\boldsymbol{T}$, reflection $\boldsymbol{\sigma}$, $\pi/3$ rotation C_6 and translations T_1, T_2 as illustrated in FIG. 1 (see also appendix A). G_U is the gauge transformation associated with U such that $G_U U$ leaves $\{u_{ij}\}$ invariant.

There is an important subgroup of PSG, Invariant Gauge Group (IGG), which is composed of all the pure gauge transformations in PSG: $IGG \equiv \{\{W_i\}|W_i u_{ij} W_j^\dagger = u_{ij}, W_i \in SU(2)\}$. One can always choose a gauge in which the elements in IGG is site-independent. In this gauge, IGG can be global Z_2 transformations: $\{W_i = \tau^0, W_i = -\tau^0\}$, the global $U(1)$ transformations: $\{W_i = e^{i\theta\tau^3}, \theta \in [0, 2\pi]\}$, or the global $SU(2)$ transformations: $\{W_i = e^{i\theta\hat{n}\cdot\vec{\tau}}, \theta \in [0, 2\pi], \hat{n} \in S^2\}$, and we dub them Z_2, $U(1)$ and $SU(2)$ state respectively.

The importance of IGG is that it controls the low-energy gauge fluctuations. Beyond mean-field level, fluctuations of U_{ij} and a_0^l need to be considered and the mean-field state may or may not be stable. The low-energy effective theory is described by fermionic spinon band structure coupled with a dynamical gauge field of IGG. For example, Z_2 state with gapped spinon dispersion can be a stable phase because the low-energy Z_2 dynamical gauge field can be in the deconfined phase[36,37]. But for a $U(1)$ state with gapped spinon dispersion, the $U(1)$ gauge fluctuations would generally drive the system into confinement due to monopole proliferation[38], and the mean-field state would be unstable. And an $SU(2)$ state with gapped spinon dispersion should also be in the confined phase because there is no known IR stable fixed point of pure $SU(2)$ gauge theory in 2+1 dimension. Because the purpose of this paper is to search for stable spin liquid phases that has a Schwinger fermion mean-field description, we will focus on Z_2 states.

If $G_U U \in PSG$ and $g \in IGG$, by definition we have $g G_U U \in PSG$. This means that the mapping $h : PSG \to SG : f(G_U U) = U$ is a many-to-one mapping. In fact it is easy to show that mapping h induces group homomorphism[15]:

$$PSG/IGG = SG. \tag{9}$$

Mathematically PSG is an extension of SG by IGG.

Our definition of PSG requires a mean-field ansatz $\{u_{ij}\}$. With Eq.(9), one can define algebraic-PSG which does not require ansatz $\{u_{ij}\}$. An algebraic-PSG is simply defined as a group satisfying Eq.(9). Obviously a PSG (realizable by an ansatz) must be an algebraic-PSG, but the reverse may not be true, because sometimes an algebraic-PSG cannot be realized by any mean-field ansatz.

To classifying all possible Z_2 Schwinger-fermion mean-field states, we need to find all possible PSG group extensions of the SG with a Z_2 IGG. Here SG is the direct product of the space group of honeycomb lattice and the time-reversal Z_2 group. In appendix A we show the general constraints that must be satisfied for such a group extension. In appendix B, using these constraints, we find there are in total 160 Z_2 algebraic-PSGs on honeycomb lattice. And at most 128 PSGs of them can be realized by an ansatz $\{u_{ij}\}$. These 128 PSGs completely classifies all Schwinger-fermion mean-field ansatz of Z_2 spin liquids on a honeycomb lattice.

III. Z_2 SPIN LIQUIDS ON A HONEYCOMB LATTICE AND THE SPS PHASE

Among the 128 states, can one further identify the candidate states for the spin liquid discovered in the numerical study[1]? The answer is yes. Numerically the spin liquid phase is found close to the Mott transition and it seems to be connected to the semimetal phase by a continuous phase transition. What are the Z_2 Schwinger-fermion states in the neighborhood of the semi-metal phase?

Are there Schwinger-fermion mean-field states that can be connected to the semi-metal phase via a continuous phase transition? Physically a continuous Mott transition is associated with the loss of charge coherence of the electronic quasi-particles. The spinons in the Schwinger-fermion approach exactly describe these quasiparticles whose charge coherence has been lost[22,23]. The natural resulting SL phase is nothing but the state with a spinon band structure identical to the electronic one on the metallic side. In the present case this SL is the uniform Resonating-Valence-Bond (u-RVB) state with a Dirac gapless spinon dispersion[23]. The nature of the Mott transtion between the SM phase and the u-RVB SL (referred to as algebraic spin liquid (ASL) in Ref. 23) was studied by Hermele[23]. However numerically it was shown that the SL phase is fully gapped. How to resolve this discrepancy?

This discrepancy is related to the stability issue of the ASL. The u-RVB (or ASL) ansatz can be simply expressed as a graphene-like nearest neighbor hopping of f-fermions:

$$H_{MF}^{uRVB} = \chi \sum_{<ij>} f_{i\alpha}^{\dagger} f_{j\alpha}, \tag{10}$$

where χ is real. The low energy effective theory of ASL is 2+1D $SU(2)$ QCD with $N_f = 2$ flavors of fermions[23], *i.e.* QCD$_3$. In the large-N_f limit QCD$_3$ has a stable IR fixed point with gapless excitations and can be a stable ASL phase[39]. However the $N_f = 2$ case remains unclear. When $N_f = 0$ the pure gauge QCD$_3$ is in a confined phase[40,41]. This indicates a critical N_c and when $N_f < N_c$ confinement occurs[39]. Although no controlled estimate of N_c is available, a self-consistent solution of the Schwinger-Dyson equations[39] suggests $N_c \approx \frac{64}{\pi^2}$, indicating $N_f = 2$ u-RVB (or ASL) state may not be a stable phase.

We find that the above-mentioned discrepancy can be resolved if we assume the ASL is not a stable phase but has one or more relavant couplings λ in the renormalization group sense. λ may be a four-fermion interaction. If λ is irrelevant at the Mott transition point (λ is dangerously irrelevant in this case), the Mott transition is still continuous and controlled by the fixed point studied in Ref. 23. We present the schematic RG flow in Fig.2(b), and propose this scenario for the Mott transition in the simulated Hubbard model.

If this scenario is correct, the mean-field ansatz of the Z_2 spin liquid should be connected to the u-RVB ansatz by a continuous Higgs condensation driven by the λ-flow, which breaks the $SU(2)$ IGG down to Z_2. During this transition, the u-RVB ansatz $\{u_{ij}^{uRVB}\} \to \{u_{ij}^{uRVB} + \delta u_{ij}\}$ and the δu_{ij} amplitudes play the role of the Higgs boson. We define a Z_2 state to be around (or in the neighborhood of) the u-RVB when the Z_2 state can be obtained by an infinitesimal change $\{u_{ij}^{uRVB}\} \to \{u_{ij}^{uRVB} + \delta u_{ij}\}$. Therefore this scenario dictates *the PSG of the Z_2 state to be a subgroup of the PSG of the ASL*. We propose this group theoretical observation as a systematic way of identifying the SLs close to a continuous Mott transition: *the PSGs of these SL are subgroups of the PSG of the "parent" SL whose spinon band structure is identical to the fermi liquid*.

In appendix C we classify all these possible PSG subgroups with the Z_2 IGG, which allows us to construct all possible Z_2 states around the u-RVB state. This technique was firstly developed by Wen[15]. We find that among the 128 Z_2 states, there are totally 24 gauge inequivalent Z_2 PSGs satisfying this condition, as listed in Table I in appendix C.

Can these 24 Z_2 SL states have a full energy gap? We find not all of them can have a gapped spinon spectrum.

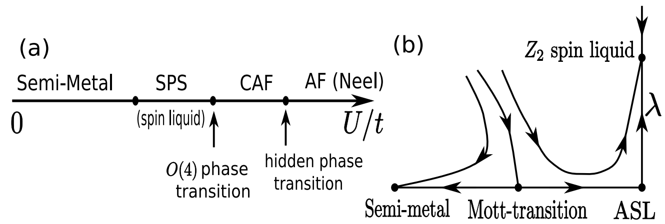

FIG. 2: (a)Proposed schematic phase diagram of the Hubbard model on the honeycomb lattice. (b)Schematic RG flow of the Mott transition.

This can be understood starting from a Dirac dispersion of the u-RVB state. To gap out the Dirac nodes, at least one mass term in the low-energy effective theory of a given Z_2 state must be allowed by symmetry. In appendix E we show that only 4 of the 24 Z_2 states allow mass term in the low energy theory. Thus only these 4 states are fully gapped Z_2 spin liquids around u-RVB state. The other 20 states have symmetry protected gapless spinon dispersions.

These four states are state #16,#17,#19, and #22 in Table I in appendix C. We can generate their mean-field ansatzs by these PSGs. We find that up to the 3rd neighbor mean-field amplitudes $u_{(\alpha,\beta,\gamma)}$ as shown in Fig.3, only one of these four states can be realized, which is state #19. As shown in appendix E 2, mean-field ansatzs up to the 3rd neighbor of the other three states actually have a $U(1)$ IGG. Only after introducing longer-range mean-field bonds can these three states have a Z_2 IGG. In particular, state #16 requires 5th neighbor, state #17 requires 4th neighbor and state #22 requires 9th neighbor amplitudes, while state #19 only requires 2nd neighbor amplitudes. Because the t/U expansion of the Hubbard model give a rather short-ranged spin interaction for the SL phase found in numerics[1] ($t/U \sim 1/4$), the other three states are unlikely to be realized in a Hubbard model on honeycomb lattice.

SPS is a fully-gapped Z_2 SL on the honeycomb lattice. Its mean-field fermionic spinon band structure, after choosing a proper gauge, is given as follows: (Fig.1 a)

$$H_{SPS}^{MF} = t_1 \sum_{<ij>} f_{i\alpha}^{\dagger} f_{j\alpha} + t_2 \sum_{<<ij>>} f_{i\alpha}^{\dagger} f_{j\alpha} - \mu \sum_{i} f_{i\alpha}^{\dagger} f_{i\alpha} + \Delta \sum_{<<ij>>} \epsilon_{\alpha\beta} f_{i\alpha}^{\dagger} f_{j\beta}^{\dagger} + h.c. \tag{11}$$

where $t_{1,2}$ are real numbers. In the Schwinger-fermion approach, f-spinons are coupled to an $SU(2)$ gauge field[15,18]. However due to non-zero t_2 and Δ, the $SU(2)$ gauge symmetry is reduced to Z_2 through Higgs mechanism. Thus at low energy f-spinons are coupled to a dynamical Z_2 gauge field and stay in the deconfined phase. A Schwinger-fermion mean-field study of J_1-J_2 Heisenberg model using SPS ansatz is presented in Appendix F.

IV. CONTINUOUS PHASE TRANSITION FROM SPS TO CAF PHASE

We start from discussing the continuous phase transition from SPS to CAF phase in the Schwinger-fermion approach. How to describe an AF order in this approach? In Ref. 24, it is shown that the easy-plane AF order on a honeycomb lattice is described by a quantum spin Hall (QSH) band structure of spinons (spin quantized along z-axis) coupled with a dynamical $U(1)$ gauge field. QSH effect binds the gauge fluctuation to S^z spin density fluctuation, and the Goldstone mode of the easy-plane Neel order is nothing but photon of the $U(1)$ gauge field. Monopole quantum number[24] of $U(1)$ gauge field shows the spin order is antiferromagnetic.

In the present spin rotation symmetric system, we consider the phase described by a fluctuating $O(3)$ QSH order parameter $\vec{n}$, coupled with a $U(1)$ gauge field a_μ. Its mean-field ansatz is: (Fig. 1b)

$$\begin{aligned} H_{CAF}^{MF} = & t_1 \sum_{<ij>} f_{i\alpha}^\dagger f_{j\alpha} + t_2 \sum_{<<ij>>} f_{i\alpha}^\dagger f_{j\alpha} \\ & - \mu \sum_i f_{i\alpha}^\dagger f_{i\alpha} + \vec{n} \cdot \sum_{<<ij>>} i\nu_{ij} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{j\beta}. \end{aligned} \quad (12)$$

There are three gapless modes in this phase: two $\hat{n}$ fluctuating modes and one photon mode. The photon mode is in-plane spin wave of AF order $\vec{N}$ ($\vec{N} \perp \hat{n}$)[24], and the spin $SU(2)$ symmetry is completely broken. Because $\hat{n}$ has the same symmetry as QSH order, Eq.(12) is the representation of CAF phase in Schwinger-fermion method. The operation $C_6 \cdot \boldsymbol{T}$ ($\boldsymbol{T}$: time-reversal, C_6: defined in Fig.1c) leaves both order parameters invariant, indicating that the magnetic order in CAF phase is still collinear.

Comparing Eq.(12) with Eq.(11), s-wave pairing Δ of spinons in SPS phase is replaced by the $O(3)$ QSH order $\vec{n}$ in the CAF phase. If we group these orders together into a 5-component vector $\vec{V} = (\mathrm{Re}\Delta, \mathrm{Im}\Delta, \vec{n})$, as pointed out in Ref. 25, fluctuations of $\vec{V}$ has a Wess-Zumino-Witten (WZW) Berry's phase[26]. Physically it means that a skyrmion (anti-skyrmion) of $\hat{n}$ in two spatial dimension actually carries fermion charge $2(-2)$. The hedgehog instanton of $\hat{n}$ in 2+1 dimension thus creates a charge-2 s-wave fermion pair. Therefore a continuous phase transition between a QSH insulator and an s-wave superconductor on the honeycomb lattice becomes possible[25].

To discuss the CAF-SPS phase transition, it is convenient to introduce the CP^1 representation of the $\hat{n}$ order parameter: $\hat{n} = w^\dagger \vec{\sigma} w$, where $w = (w_1, w_2)^T$ are two complex numbers satisfying $|w_1|^2 + |w_2|^2 = 1$. This representation has a $U(1)$ gauge redundancy and thus w-bosons couple to a $U(1)$ gauge field A_μ. After integrating out the f-spinons (see Appendix G for details), we obtain the effective Lagrangian[51]:

$$\begin{aligned} L = & |(\partial_\mu - \mathrm{i} A_\mu) w|^2 + r|w|^2 + s|w|^4 + \frac{1}{2g_a^2} f_{\mu\nu}^2 \\ & + \frac{1}{2g_A^2} F_{\mu\nu}^2 + \frac{\mathrm{i}}{\pi} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda, \end{aligned} \quad (13)$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are gauge field strengths. The last term, a mutual Chern-Simons (CS) term, is nothing but the WZW term in the gauge representation: it is well-known that a skyrmion of $\hat{n}$ is a 2π A_μ gauge flux, which also carries 2 units of a_μ gauge charge due to the WZW term.

What are the phases described by the effective Lagrangian Eq.(13)? When $r < 0$, w-boson condenses and $\hat{n}$ is ordered, corresponding to CAF phase. Here the mutual CS term does not qualitatively modify the low-energy dynamics due to the Higgs mechanism of A_μ. When $r > 0$, w-bosons are gapped and $\hat{n}$ is disordered, remarkably, Eq.(13) describes the Z_2 SPS phase. The identification of a $U(1)$ mutual CS theory and a Z_2 gauge theory has been studied before[27,28]. Here with the WZW term, we are able to further identify the PSG of the Z_2 theory.

This identification is easily shown by studying the the monopoles of a_μ and A_μ in Eq.(13). In $\hat{n}$ disordered phase, monopole events of both a_μ and A_μ are allowed. We denote their monopole creation operators as $V_a^\dagger$ and $V_A^\dagger$ respectively. The mutual CS term clearly dictates that an a_μ monopole creates 2 units of A_μ gauge charge, and vice versa. These events mean that $f_\alpha^\dagger f_\beta^\dagger$ and $w_\alpha^\dagger w_\beta^\dagger$ pairing terms exist, which break the $U(1)$ gauge group down to Z_2. The WZW term indicates that the f-spinon pairing is s-wave, and thus the system is in SPS phase.

Mutual CS term also dictates that f-spinon and w-boson satisfy mutual semion statistics. Namely they see each other as a π-flux and are dual degrees of freedom. We can focus on either set of dual variables, $f(V_A^\dagger)$ or $w(V_a^\dagger)$, to write down the effective theory. Because the phase transition from SPS to CAF phase is described by w-boson condensation in Eq.(13), we will use $w(V_a^\dagger)$ variables in the next section.

V. DUALITY BETWEEN SCHWINGER-FERMION AND SCHWINGER-BOSON REPRESENTATIONS

In this section we focus on the dual variables of f-spinons: the w-bosons. The SPS phase is then a Z_2 phase with w-bosons as Z_2 charges, but f-spinons as visons. In this formulation SPS-CAF phase transition is naturally presented as a Higgs condensation of w-bosons.

First we need to represent the order parameters of the CAF phase in terms of w. The QSH order is $\hat{n} = w^\dagger \vec{\sigma} w$, but what is the Neel order parameter? Neel order in CAF phase corresponds to the monopole of a_μ, namely a pairing of w-boson. There are two spin-1 bosonic pairing order parameters satisfying this requirement, *i.e.* the real and imaginary part of $(i\sigma_y w^*)^\dagger \vec{\sigma} w$:

$$\hat{n}_1 + i\hat{n}_2 = (i\sigma_y w^*)^\dagger_\alpha \vec{\sigma}_{\alpha\beta} w_\beta. \tag{14}$$

It is easy to verify that $\hat{n} = \hat{n}_1 \times \hat{n}_2$, so there are only two independent vectorial order parameters. The issue is, which one is the Neel order parameter $\vec{N}$: $\hat{n}_1$ or $\hat{n}_2$?

A $U(1)$ gauge transformation $w \to e^{i\theta} w$ generates a rotation in the $\hat{n}_1, \hat{n}_2$ plane. By fixing a proper gauge, we can always choose $\hat{n}_1$ as the Neel order. We will work within this gauge $\vec{N} = \hat{n}_1$ throughout the phase transition. Such a gauge fixing breaks the $U(1)$ gauge redundancy down to Z_2: $w \to \pm w$.

The physical symmetries of the QSH (or vector spin chirality) and the Neel order parameters completely determine the transformation rules of the w-boson up to a Z_2 gauge redundancy:

$$\begin{aligned}
T_1, T_2: &\quad \hat{n} \to \hat{n}, &\quad \vec{N} \to \vec{N}, &\quad w \to w,\\
\boldsymbol{T}: &\quad \hat{n} \to \hat{n}, &\quad \vec{N} \to -\vec{N}, &\quad w \to i w^*,\\
\sigma: &\quad \hat{n} \to -\hat{n}, &\quad \vec{N} \to -\vec{N}, &\quad w \to i\sigma_y w^*,\\
C_6: &\quad \hat{n} \to \hat{n}, &\quad \vec{N} \to -\vec{N} &\quad w \to i w.
\end{aligned} \tag{15}$$

where time-reversal transformation $\boldsymbol{T}$ is anti-unitary. The reason why there are no further arbitrariness on the transformation rules of w can be easily understood by the following construction. If we write w-boson as an SU(2) matrix:

$$U = \begin{pmatrix} w_1 & w_2^* \\ w_2 & -w_1^* \end{pmatrix}, \tag{16}$$

then the most general $O(4)$ transformation leaving $|w_1|^2 + |w_2|^2 = 1$ is $U \to V_L U V_R$, where V_L and V_R are both $SU(2)$ rotations (V_L is spin rotation), and $O(4) \sim SU(2)_L \times SU(2)_R$. In this representation, the vectors $\hat{n}_1, \hat{n}_2, \hat{n}$ are the 1st, 2nd and 3rd columns of a 3 by 3 rotation matrix R:[49]

$$R^{ab} = \frac{1}{2}\mathrm{Tr}(U^\dagger \sigma^a U \sigma^b) \tag{17}$$

Clearly, to leave R invariant, the transformations $V_{L,R}$ must be ± 1.

These symmetry transformation rules allow us to reveal the connection between the SPS state here and the 0-flux state in the Schwinger-boson representation obtained by Wang[2]. In Wang's work, the Neel order is represented by the z-boson as $\vec{N} = z^\dagger \vec{\sigma} z$ in the effective theory. From Eq.(17), we can easily construct the duality transformation between the w-boson and z-boson representations: $U_w = U_z V_R$, $V_R = e^{i\frac{\pi}{4}\sigma_y}$, namely:

$$w = \frac{1}{\sqrt{2}}(z - i\sigma^y z^*) \quad \text{or} \quad z = \frac{1}{\sqrt{2}}(w + i\sigma^y w^*) \tag{18}$$

Under duality transformation:

$$\begin{aligned}
\vec{N} &= \mathrm{Re}[(i\sigma_y w^*)^\dagger \vec{\sigma} w] = z^\dagger \vec{\sigma} z,\\
\hat{n} &= w^\dagger \vec{\sigma} w = -\mathrm{Re}[(i\sigma_y z^*)^\dagger \vec{\sigma} z].
\end{aligned} \tag{19}$$

From Eq.(15),(18), we can obtain transformation rules of z-bosons:

$$\begin{aligned}
T_1, T_2: &\quad z \to z,\\
\boldsymbol{T}: &\quad z \to \sigma_y z,\\
\sigma: &\quad z \to i\sigma_y z^*,\\
C_6: &\quad z \to \sigma_y z^*,
\end{aligned} \tag{20}$$

which are exactly the transformation rules found by Wang[2] for 0-flux state up to a Z_2 gauge arbitrariness. This explicitly confirms that the z-bosons constructed in Eq.(18) are the same z-bosons discussed by Wang, and the SPS phase here is identical to the 0-flux phase in Schwinger-boson description.

Following the discussion in Ref. 2, we can write down the general symmetry-allowed effective theory for the phase transition in terms of z-boson:

$$\begin{aligned}
L =& |\partial_\tau z|^2 + c^2|\nabla z|^2 + m^2|z|^2 + u(|z|^2)^2\\
&+ \lambda_H\ (i\sigma_y z^*)^\dagger \big[\sum_i (\vec{d}_i \cdot \nabla)^3\big] z + h.c.
\end{aligned} \tag{21}$$

Here λ_H is the Higgs coupling which reduces the gauge degrees of freedom in the z-boson formulation down to Z_2. $\vec{d}_1 = -\vec{a}_1, \vec{d}_2 = \vec{a}_2$, and $\vec{d}_3 = \vec{a}_1 - \vec{a}_2$ as shown in FIG. 1. For instance, the single time derivative term $z^\dagger \partial_\tau z$ is forbidden by σ, and $z^T(-i\sigma_y)\partial_\tau z$ is forbidden by C_6. The Higgs coupling can also be written as a pairing of w-bosons: $\lambda_H\ (i\sigma_y w^*)^\dagger \big[\sum_i (\vec{d}_i \cdot \nabla)^3\big] w + h.c.$ By naive power counting λ_H is irrelevant, therefore we have an $O(4)$ critical point between the CAF (z-condensed) phase and the SPS (z-gapped) phase. The critical behavior of this transition is well-studied[29–31].

VI. SUMMARY

In this study, our main prediction is the CAF phase. Unlike usual AF (Neel) phase, CAF phase has two order parameters: Neel order $\vec{N}$ and QSH order $\hat{n}$. As CAF phase is likely to be the magnetically ordered phase adjacent to the SL phase found in the numeric study[1], in the following we propose explicit numerical methods to detect the CAF phase.

One can directly measure QSH order by $\langle \vec{n}(x) \cdot \vec{n}(0) \rangle$ correlation function, or the spin vector-chirality correlation $\langle (\nu_{i+x,j+x} \vec{S}_{i+x} \times \vec{S}_{j+x}) \cdot (\nu_{ij} \vec{S}_i \times \vec{S}_j) \rangle$. Since QSH order is odd under $\sigma \cdot \boldsymbol{T}$ while Neel order is $\sigma \cdot \boldsymbol{T}$-even, one does not expect a long range correlation of QSH order in a usual Neel phase. Therefore, the long range QSH correlation is an intrinsic signature of CAF phase. In addition, one can check that the QSH vector is normal to the Neel vector. For example, one can pin the Neel order by an infinitesimal (in thermodynamic limit) staggered magnetic field along z-axis, and the measured QSH order parameters should only have x, y components. Experimentally such an exotic SL may be realized in candidate systems such as expanded graphene-like system in group IV elements[32,33] and fermions in optical lattices[34,35].

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Appendix A: General conditions on projective symmetry group of symmetric spin liquids on a honeycomb lattice

As mentioned in section II, SG on a honeycomb lattice is generated by time reversal transformation $\boldsymbol{T}$, translations along $\vec{a}_1, \vec{a}_2$: T_1, T_2, plaquette-centered 60° C_6 rotation, and a horizontal mirror reflection $\boldsymbol{\sigma}$ as shown in Fig.1. In the present problem, the symmetry group SG can be represented as

$$SG = \{U = \boldsymbol{T}^{\nu_{\boldsymbol{T}}} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot C_6^{\nu_{C_6}} \cdot \boldsymbol{\sigma}^{\nu_{\boldsymbol{\sigma}}}\}$$

where $\nu_{T_1}, \nu_{T_2} \in \boldsymbol{Z}$ and $\nu_{\boldsymbol{T}}, \nu_{\boldsymbol{\sigma}} \in \boldsymbol{Z}_2$, $\nu_{C_6} \in \boldsymbol{Z}_6$, since the generators satisfy

$$\boldsymbol{T}^2 = \boldsymbol{\sigma}^2 = (C_6)^6 = 1 \tag{A1}$$

Here 1 stands for the identity element of SG. To completely determine the multiplication rule of this group, we need to identify the multiplication rule of two different generators in an order different from $\boldsymbol{T}^{\nu_{\boldsymbol{T}}} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot C_6^{\nu_{C_6}} \cdot \boldsymbol{\sigma}^{\nu_{\boldsymbol{\sigma}}}$:

$$U\boldsymbol{T} = \boldsymbol{T}U \quad (U = T_1, T_2, C_6, \boldsymbol{\sigma}) \tag{A2}$$
$$T_1T_2 = T_2T_1 \tag{A3}$$
$$C_6T_1 = T_2C_6 \tag{A4}$$
$$C_6T_2 = T_1^{-1}T_2C_6 \tag{A5}$$
$$\boldsymbol{\sigma}T_1 = T_1\boldsymbol{\sigma} \tag{A6}$$
$$\boldsymbol{\sigma}T_2 = T_1T_2^{-1}\boldsymbol{\sigma} \tag{A7}$$
$$\boldsymbol{\sigma}C_6 = C_6^{-1}\boldsymbol{\sigma} \tag{A8}$$

The above relations can be written in an alternative way

$$\boldsymbol{T}^2 = \boldsymbol{\sigma}^2 = (C_6)^6 = 1 \tag{A9}$$
$$\boldsymbol{T}U\boldsymbol{T}^{-1}U^{-1} = 1 \quad (U = T_1, T_2, C_6, \boldsymbol{\sigma}) \tag{A10}$$
$$T_1T_2T_1^{-1}T_2^{-1} = 1 \tag{A11}$$
$$T_2^{-1}C_6T_1C_6^{-1} = 1 \tag{A12}$$
$$T_1^{-1}C_6T_1T_2^{-1}C_6^{-1} = 1 \tag{A13}$$
$$T_1^{-1}\boldsymbol{\sigma}T_1\boldsymbol{\sigma}^{-1} = 1 \tag{A14}$$
$$T_2^{-1}\boldsymbol{\sigma}T_1T_2^{-1}\boldsymbol{\sigma}^{-1} = 1 \tag{A15}$$
$$\boldsymbol{\sigma}C_6\boldsymbol{\sigma}C_6 = 1 \tag{A16}$$

which determines the inverse of all the group elements.

As introduced in section II, the mean-field ansatz $\{u_{ij}\}$ of a spin liquid is invariant under the action of any element G_UU of a projective symmetry group (PSG). The multiplication rule of the symmetry group would immediately enforce the following constraints on a PSG by its definition: if $U_1U_2 = U_3$ then

$$\begin{aligned} G_{U_1}U_1G_{U_2}U_2(\{u_{ij}\}) = G_{U_3}U_3(\{u_{ij}\}) \Longrightarrow \\ \big[G_{U_1}\big(U_1U_2(i)\big)G_{U_2}\big(U_2(i)\big)\big]u_{ij}\big[G_{U_1}\big(U_1U_2(i)\big)G_{U_2}\big(U_2(i)\big)\big]^\dagger \\ = \big[G_{U3}\big(U_3(i)\big)\big]u_{ij}\big[G_{U3}\big(U_3(i)\big)\big]^\dagger, \quad \forall\ i,j \end{aligned} \tag{A17}$$

On the other hand, we know those pure gauge transformations, under which the mean-field ansatz $\{u_{ij}\}$ is invariant, constitute a subgroup of PSG, coined the invariant gauge group (IGG):

$$IGG = \{W_i|W_iu_{ij}W_j^\dagger = u_{ij}, \quad W_i \in SU(2)\} \tag{A18}$$

Therefore from (A17) we have the following constraints on the elements of a PSG

$$\big[G_{U_1U_2}\big(U_1U_2(i)\big)\big]^\dagger G_{U_1}\big(U_1U_2(i)\big)G_{U_2}\big(U_2(i)\big) = \mathcal{G} \in IGG$$

The above condition holds for any two group elements U_1, U_2 of SG. Similar with SG, we can choose a set of generators in any given PSG: $\{G_{T_1}T_1, G_{T_2}T_2, G_{\boldsymbol{T}}\boldsymbol{T}, G_{C_6}C_6, G_{\boldsymbol{\sigma}}\boldsymbol{\sigma}\}$. Any given element in PSG can be written in the standard form:

$$\begin{aligned} G_UU =&(G_{\boldsymbol{T}}\boldsymbol{T})^{\nu_{\boldsymbol{T}}} \cdot (G_{T_1}T_1)^{\nu_{T_1}} \cdot (G_{T_2}T_2)^{\nu_{T_2}} \\ &\cdot (G_{C_6}C_6)^{\nu_{C_6}} \cdot (G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})^{\nu_{\boldsymbol{\sigma}}} \end{aligned} \tag{A19}$$

Since the multiplication rule of SG on a honeycomb lattice is completely determined by (A1)-(A8), or equivalently (A9)-(A16), the only independent constraints on the PSG generators are the following:

$$\begin{gathered} (G_{\boldsymbol{T}}\boldsymbol{T})^2 \in IGG \\ (G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})^2 \in IGG \\ (G_{C_6}C_6)^6 \in IGG \\ (G_{T_1}T_1)^{-1}(G_{T_2}T_2)^{-1}(G_{T_1}T_1)(G_{T_2}T_2) \in IGG \\ (G_{T_1}T_1)^{-1}(G_{\boldsymbol{T}}\boldsymbol{T})^{-1}(G_{T_1}T_1)(G_{\boldsymbol{T}}\boldsymbol{T}) \in IGG \\ (G_{T_2}T_2)^{-1}(G_{\boldsymbol{T}}\boldsymbol{T})^{-1}(G_{T_2}T_2)(G_{\boldsymbol{T}}\boldsymbol{T}) \in IGG \\ (G_{T_2}T_2)^{-1}(G_{C_6}C_6)(G_{T_1}T_1)(G_{C_6}C_6)^{-1} \in IGG \\ (G_{T_1}T_1)^{-1}(G_{C_6}C_6)(G_{T_1}T_1)(G_{T_2}T_2)^{-1}(G_{C_6}C_6)^{-1} \in IGG \\ (G_{\boldsymbol{T}}\boldsymbol{T})^{-1}(G_{C_6}C_6)^{-1}(G_{\boldsymbol{T}}\boldsymbol{T})(G_{C_6}C_6) \in IGG \\ (G_{T_1}T_1)^{-1}(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})(G_{T_1}T_1)(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})^{-1} \in IGG \\ (G_{T_2}T_2)^{-1}(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})(G_{T_1}T_1)(G_{T_2}T_2)^{-1}(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})^{-1} \in IGG \\ (G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})(G_{C_6}C_6)(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})(G_{C_6}C_6) \in IGG \\ (G_{\boldsymbol{T}}\boldsymbol{T})^{-1}(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma})^{-1}(G_{\boldsymbol{T}}\boldsymbol{T})(G_{\boldsymbol{\sigma}}\boldsymbol{\sigma}) \in IGG \end{gathered} \tag{A20}$$

or more specifically

$$\left[G_{\boldsymbol{T}}(i)\right]^2 \in IGG, \tag{A21}$$
$$G_{\boldsymbol{\sigma}}\big(\boldsymbol{\sigma}(i)\big)G_{\boldsymbol{\sigma}}(i) \in IGG,$$
$$G_{C_6}\big(C_6^{-1}(i)\big)G_{C_6}\big(C_6^{-2}(i)\big)G_{C_6}\big(C_6^{3}(i)\big)$$
$$\cdot G_{C_6}\big(C_6^{2}(i)\big)G_{C_6}\big(C_6(i)\big)G_{C_6}(i) \in IGG,$$
$$G_{T_1}^{-1}\big(T_2^{-1}T_1(i)\big)G_{T_2}^{-1}\big(T_1(i)\big)G_{T_1}\big(T_1(i)\big)G_{T_2}(i) \in IGG,$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{\boldsymbol{T}}^{-1}\big(T_1(i)\big)G_{T_1}\big(T_1(i)\big)G_{\boldsymbol{T}}(i) \in IGG,$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{\boldsymbol{T}}^{-1}\big(T_2(i)\big)G_{T_2}\big(T_2(i)\big)G_{\boldsymbol{T}}(i) \in IGG,$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{C_6}\big(T_2(i)\big)G_{T_1}\big(T_1C_6^{-1}(i)\big)G_{C_6}^{-1}(i) \in IGG,$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{C_6}\big(T_1(i)\big)G_{T_1}\big(C_6^{-1}T_1(i)\big)$$
$$\cdot G_{T_2}^{-1}\big(C_6^{-1}(i)\big)G_{C_6}^{-1}(i) \in IGG,$$
$$G_{\boldsymbol{T}}^{-1}\big(C_6^{-1}(i)\big)G_{C_6}^{-1}(i)G_{\boldsymbol{T}}(i)G_{C_6}(i) \in IGG,$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{\boldsymbol{\sigma}}\big(T_1(i)\big)G_{T_1}\big(T_1\boldsymbol{\sigma}^{-1}(i)\big)G_{\boldsymbol{\sigma}}^{-1}(i) \in IGG,$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{\boldsymbol{\sigma}}\big(T_2(i)\big)G_{T_1}\big(\boldsymbol{\sigma}T_2(i)\big)G_{T_2}^{-1}\big(\boldsymbol{\sigma}(i)\big)G_{\boldsymbol{\sigma}}^{-1}(i) \in IGG,$$
$$G_{\boldsymbol{\sigma}}(i)G_{C_6}\big(\boldsymbol{\sigma}(i)\big)G_{\boldsymbol{\sigma}}\big(\boldsymbol{\sigma}C_6(i)\big)G_{C_6}\big(C_6(i)\big) \in IGG,$$
$$G_{\boldsymbol{T}}^{-1}(\boldsymbol{\sigma}(i))G_{\boldsymbol{\sigma}}^{-1}(i)G_{\boldsymbol{T}}(i)G_{\boldsymbol{\sigma}}(i) \in IGG.$$

Above are all the general consistent conditions to be satisfied by the generators of a PSG on a honeycomb lattice.

We will use (x_1, x_2, s) to label a site i in a honeycomb lattice, where x_1, x_2 are the coordinates of the unit cell in basis $\vec{a}_1, \vec{a}_2$ and $s = 0, 1$ for A and B sublattice respectively. For convenience, we summarize the coordinate transformation of all the generators in the symmetry group on a honeycomb lattice as follows:

$$\boldsymbol{T}: \quad (x_1, x_2, s) \to (x_1, x_2, s), \tag{A22}$$
$$T_1: \quad (x_1, x_2, s) \to (x_1 + 1, x_2, s),$$
$$T_2: \quad (x_1, x_2, s) \to (x_1, x_2 + 1, s),$$
$$\boldsymbol{\sigma}: \quad (x_1, x_2, s) \to (x_1 + x_2, -x_2, 1 - s),$$
$$C_6: \quad (x_1, x_2, 0) \to (1 - x_2, x_1 + y_1 - 1, 1)$$
$$(x_1, x_2, 1) \to (-x_2, x_1 + y_1, 0)$$

Appendix B: Classification of Z_2 projective symmetry groups on a honeycomb lattice

As discussed in section II, the problem of classifying all possible Z_2 Schwinger-fermion mean-field states is mathematically reduced to finding all possible PSGs. Let us firstly find all algebraic PSGs.

1. General discussions

In the case of Z_2 spin liquids, the IGG of the corresponding PSG is a Z_2 group: $IGG = \{\pm\tau^0\}$. The constraints listed in appendix A now becomes

$$\left[G_{\boldsymbol{T}}(i)\right]^2 = \eta_{\boldsymbol{T}}\tau^0, \tag{B1}$$
$$G_{\boldsymbol{\sigma}}\big(\boldsymbol{\sigma}(i)\big)G_{\boldsymbol{\sigma}}(i) = \eta_{\boldsymbol{\sigma}}\tau^0, \tag{B2}$$
$$G_{C_6}\big(C_6^{-1}(i)\big)G_{C_6}\big(C_6^{-2}(i)\big)G_{C_6}\big(C_6^{3}(i)\big) \tag{B3}$$
$$\cdot G_{C_6}\big(C_6^{2}(i)\big)G_{C_6}\big(C_6(i)\big)G_{C_6}(i) = \eta_{C_6}\tau^0, \tag{B4}$$
$$G_{T_1}^{-1}\big(T_2^{-1}T_1(i)\big)G_{T_2}^{-1}\big(T_1(i)\big)$$
$$\cdot G_{T_1}\big(T_1(i)\big)G_{T_2}(i) = \eta_{12}\tau^0, \tag{B5}$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{\boldsymbol{T}}^{-1}\big(T_1(i)\big)G_{T_1}\big(T_1(i)\big)G_{\boldsymbol{T}}(i) = \eta_{1\boldsymbol{T}}\tau^0 \tag{B6}$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{\boldsymbol{T}}^{-1}\big(T_2(i)\big)G_{T_2}\big(T_2(i)\big)G_{\boldsymbol{T}}(i) = \eta_{2\boldsymbol{T}}\tau^0 \tag{B7}$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{C_6}\big(T_2(i)\big)$$
$$\cdot G_{T_1}\big(T_1C_6^{-1}(i)\big)G_{C_6}^{-1}(i) = \eta_{C_61}\tau^0, \tag{B8}$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{C_6}\big(T_1(i)\big)G_{T_1}\big(C_6^{-1}T_1(i)\big)$$
$$\cdot G_{T_2}^{-1}\big(C_6^{-1}(i)\big)G_{C_6}^{-1}(i) = \eta_{C_62}\tau^0, \tag{B9}$$
$$G_{\boldsymbol{T}}^{-1}\big(C_6^{-1}(i)\big)G_{C_6}^{-1}(i)G_{\boldsymbol{T}}(i)G_{C_6}(i) = \eta_{C_6\boldsymbol{T}}\tau^0, \tag{B10}$$
$$G_{T_1}^{-1}\big(T_1(i)\big)G_{\boldsymbol{\sigma}}\big(T_1(i)\big)$$
$$\cdot G_{T_1}\big(T_1\boldsymbol{\sigma}^{-1}(i)\big)G_{\boldsymbol{\sigma}}^{-1}(i) = \eta_{\boldsymbol{\sigma}1}\tau^0, \tag{B11}$$
$$G_{T_2}^{-1}\big(T_2(i)\big)G_{\boldsymbol{\sigma}}\big(T_2(i)\big)G_{T_1}\big(\boldsymbol{\sigma}T_2(i)\big)$$
$$\cdot G_{T_2}^{-1}\big(\boldsymbol{\sigma}(i)\big)G_{\boldsymbol{\sigma}}^{-1}(i) = \eta_{\boldsymbol{\sigma}2}\tau^0, \tag{B12}$$
$$G_{\boldsymbol{\sigma}}(i)G_{C_6}\big(\boldsymbol{\sigma}(i)\big)$$
$$\cdot G_{\boldsymbol{\sigma}}\big(\boldsymbol{\sigma}C_6(i)\big)G_{C_6}\big(C_6(i)\big) = \eta_{\boldsymbol{\sigma}C_6}\tau^0, \tag{B13}$$
$$G_{\boldsymbol{T}}^{-1}(\boldsymbol{\sigma}(i))G_{\boldsymbol{\sigma}}^{-1}(i)G_{\boldsymbol{T}}(i)G_{\boldsymbol{\sigma}}(i) = \eta_{\boldsymbol{\sigma}\boldsymbol{T}}\tau^0. \tag{B14}$$

where all the η's take value of ±1. Not all of these conditions are gauge independent. Because we can re-choose the gauge part of generators such as G_{T_1}, G_{T_2}... by multiplying them by $-\tau^0$ (an element of IGG), only those conditions in which the same generator shows up twice are gauge independent. We can use this gauge dependence to simplify these conditions. Because $G_{T_1}(G_{T_2})$ only show up once in the equation of $\eta_{C_61}(\eta_{C_62})$, we can always choose a gauge such that $\eta_{C_61} = \eta_{C_62} = 1$. All other η's are gauge independent.

In the following we will determine all the possible PSG's with different (gauge inequivalent) elements $\{G_U(i)\}$. These different PSG's characterize all the different type of Z_2 spin liquids on a honeycomb lattice, which might be constructed from mean-field ansatz $\{u_{ij}\}$.

First notice that under a local $SU(2)$ gauge transformation $u_{ij} \to W_i u_{ij} W_j^\dagger$, the PSG elements transform as $G_U(i) \to W_i G_U(i) W_{U^{-1}(i)}^\dagger$. Making use of such a degree of freedom, we can always choose proper gauge so that

$$G_{T_1}(x_1, x_2, s) = G_{T_2}(0, x_2, s) = \tau^0, \quad x_1, x_2 \in \mathbb{Z}.$$

Now taking (B5) into account, we have $G_{T_2}(\{x_1 + 1, x_2, s\}) = \eta_{12} G_{T_2}(\{x_1, x_2, s\})$ and therefore

$$G_{T_1}(x_1, x_2, s) = \tau^0 \tag{B15}$$
$$G_{T_2}(x_1, x_2, s) = \eta_{12}^{x_1}\tau^0$$

Meanwhile, from (B1), (B6) and (B7) we can immediately see that $\eta_{1\boldsymbol{T}} = \eta_{2\boldsymbol{T}} = 1$, and the gauge inequivalent choices of $G_{\boldsymbol{T}}(i)$ are the following

$$G_{\boldsymbol{T}}(x_1, x_2, s) = g_{\boldsymbol{T}}(s) = \begin{cases} \eta_t^s \tau^0, & \eta_{\boldsymbol{T}} = 1 \\ i\tau^3, & \eta_{\boldsymbol{T}} = -1 \end{cases} \tag{B16}$$

where $\eta_t = \pm 1$.

As discussed earlier, we can always choose a proper gauge so that $\eta_{C_6 1} = \eta_{C_6 2} = 1$. Then from conditions (B8) and (B9) we see that

$$G_{C_6}(x_1, x_2, s) = \eta_{12}^{x_1 x_2 + x_1(x_1-1)/2} g_{C_6}(s) \tag{B17}$$

similarly from conditions (B11) and (B12) we have

$$G_{\boldsymbol{\sigma}}(x_1, x_2, s) = \eta_{\boldsymbol{\sigma}1}^{x_1} \eta_{\boldsymbol{\sigma}2}^{x_2} \eta_{12}^{x_2(x_2-1)/2} g_{\boldsymbol{\sigma}}(s) \tag{B18}$$

where $g_{C_6}(s), g_{\boldsymbol{\sigma}}(s) \in SU(2)$. Note that (B2) and (B13) give further constraints to the above expression (B18):

$$\eta_{\boldsymbol{\sigma}1} = \eta_{\boldsymbol{\sigma}2} = \eta_{12} \tag{B19}$$

Now we see the elements of PSG can be expressed as

$$\begin{aligned} G_{T_1}(x_1, x_2, s) &= \tau^0 && \text{(B20)} \\ G_{T_2}(x_1, x_2, s) &= \eta_{12}^{x_1} \tau^0 \\ G_{\boldsymbol{T}}(x_1, x_2, s) &= g_{\boldsymbol{T}}(s) && \text{(B21)} \\ G_{C_6}(x_1, x_2, s) &= \eta_{12}^{x_1 x_2 + x_1(x_1-1)/2} g_{C_6}(s) \\ G_{\boldsymbol{\sigma}}(x_1, x_2, s) &= \eta_{12}^{x_1 + x_2(x_2+1)/2} g_{\boldsymbol{\sigma}}(s) \end{aligned}$$

Consistent conditions (B2), (B4), (B10), (B13) and (B14) correspond to the following constraints on $SU(2)$ matrices $g_{C_6}(s),\ g_{\boldsymbol{\sigma}}(s)$:

$$\begin{aligned} g_{\boldsymbol{\sigma}}(0) g_{\boldsymbol{\sigma}}(1) &= \eta_{\boldsymbol{\sigma}} \tau^0, \\ [g_{C_6}(s) g_{C_6}(1-s)]^3 &= \eta_{C_6} \eta_{12} \tau^0, \\ g_{\boldsymbol{T}}(s) g_{C_6}(s) &= g_{C_6}(s) g_{\boldsymbol{T}}(1-s) \eta_{C_6 \boldsymbol{T}} \\ g_{\boldsymbol{T}}(s) g_{\boldsymbol{\sigma}}(s) &= g_{\boldsymbol{\sigma}}(s) g_{\boldsymbol{T}}(1-s) \eta_{\boldsymbol{\sigma T}} \\ g_{\boldsymbol{\sigma}}(s) g_{C_6}(1-s) &= \begin{cases} \lambda_{C_6}^s \tau^0, & \eta_{\boldsymbol{\sigma} C_6} = 1 \\ \mathrm{i}\hat{n}_s \cdot \vec{\tau}, & \eta_{\boldsymbol{\sigma} C_6} = -1 \end{cases} \end{aligned} \tag{B22}$$

where $\lambda_{C_6} = \pm 1$ and $\hat{n}_s$ is a unit vector.

2. A summary of 160 different PSG's

Below we summarize all the 160 possible PSG's obtained through solving (B22). We use capital Roman numerals (I) and (II) to label $g_{\boldsymbol{T}} = \eta_t^s \tau^0$ and $g_{\boldsymbol{T}} = i\tau^3$ respectively. Roman numerals (i) and (ii) are used to label $\eta_{C_6 \boldsymbol{T}} = \pm 1$ respectively. (A) and (B) are used to label $\eta_{\boldsymbol{\sigma} C_6} = \pm 1$ respectively. Finally (α) and (β) are used to label $\eta_{\boldsymbol{\sigma T}}$ respectively.

(I) $g_{\boldsymbol{\sigma} T} = \eta_t^s \tau^0$:

It's easy to see that $\eta_{C_6 \boldsymbol{T}} = \eta_{\boldsymbol{\sigma T}} = \eta_t$ from (B22), so there is the only possibility among (i) and (ii).

(A) $g_{\boldsymbol{\sigma}}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$:
we have $\lambda_{C_6} = \eta_{\boldsymbol{\sigma}} \eta_{C_6} \eta_{12}$ and

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) = g_{\boldsymbol{\sigma}}(0) &= \eta_{C_6} \eta_{12} \tau^0, \\ g_{\boldsymbol{\sigma}}(1) &= \eta_{\boldsymbol{\sigma}} \eta_{C_6} \eta_{12} \tau^0. \end{aligned} \tag{B23}$$

This represents $2^4 = 16$ different PSG's in the class (I)(A) since $\eta_t, \eta_{C_6}, \eta_{\boldsymbol{\sigma}}, \eta_{12} = \pm 1$.

(B) $g_{\boldsymbol{\sigma}}(s) g_{C_6}(1-s) = \mathrm{i}\hat{n}_s \cdot \vec{\tau}$:
Choosing a proper gauge (so that $g_{C_6}(0) = \tau^0$) we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} e^{\mathrm{i}\psi_3 \tau^3}, \\ g_{\boldsymbol{\sigma}}(0) &= \mathrm{i}\tau^1 \eta_{C_6} \eta_{12} e^{-\mathrm{i}\psi_3 \tau^3}, \\ g_{\boldsymbol{\sigma}}(1) &= -\mathrm{i}\eta_{\boldsymbol{\sigma}} \eta_{C_6} \eta_{12} e^{\mathrm{i}\psi_3 \tau^3} \tau^1. \end{aligned} \tag{B24}$$

where $\psi_3 \equiv 0, \pm 2\pi/3$ stand for the multiples of $2\pi/3$ mod 2π. There are $2^4 \times 3 = 48$ different PSG's in this class (I)(B).

(II) $g_T(s) = \mathrm{i}\tau^3$:
(i) $\eta_{C_6 \boldsymbol{T}} = 1$:
(A) $g_{\boldsymbol{\sigma}}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$:
in this case $\lambda_{C6} = \eta_{\boldsymbol{\sigma}} \eta_{C_6} \eta_{12}$, so we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{\boldsymbol{\sigma}}(0) = g_{C_6}(1) &= \eta_{C_6} \eta_{12} \tau^0, \\ g_{\boldsymbol{\sigma}}(1) &= \eta_{\boldsymbol{\sigma}} \eta_{C_6} \eta_{12} \tau^0. \end{aligned} \tag{B25}$$

there are $2^3 = 8$ different PSG's in the class (II)(i)(A).

(B) $g_{\boldsymbol{\sigma}}(s) g_{C_6}(1-s) = \mathrm{i}\hat{n}_s \cdot \vec{\tau}$:
(α) $\eta_{\boldsymbol{\sigma T}} = 1$, *i.e.* $[g_{\boldsymbol{\sigma}}(s), \tau^3] = 0$:
here we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} \tau^0, \\ g_{\boldsymbol{\sigma}}(0) &= -\mathrm{i}\eta_{\boldsymbol{\sigma}} \tau^3, \\ g_{\boldsymbol{\sigma}}(1) &= \mathrm{i}\tau^3. \end{aligned} \tag{B26}$$

there are $2^3 = 8$ different PSG's in the class (II)(i)(B)(α).

(β) $\eta_{\boldsymbol{\sigma T}} = -1$, *i.e.* $\{g_{\boldsymbol{\sigma}}(s), \tau^3\} = 0$:
here we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} e^{\mathrm{i}\psi_3 \tau^3}, \\ g_{\boldsymbol{\sigma}}(0) &= -\mathrm{i}\eta_{\boldsymbol{\sigma}} \tau^1, \\ g_{\boldsymbol{\sigma}}(1) &= \mathrm{i}\tau^1. \end{aligned} \tag{B27}$$

there are $2^3 \times 3 = 24$ different PSG's in the class (II)(i)(B)(β) since $\psi_3 = 0, \pm 2\pi/3$.

(ii) $\eta_{C_6 \boldsymbol{T}} = -1$:
(A) $g_{\boldsymbol{\sigma}}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$:

here we must have $\eta_{\boldsymbol{\sigma T}}=-1$, $\lambda_{C_6}=\eta_{\boldsymbol{\sigma}}\eta_{C_6}\eta_{12}$ and

$$\begin{aligned} g_{C_6}(0) &= \mathrm{i}\tau^1, \\ g_{C_6}(1) &= -\mathrm{i}\eta_{C_6}\eta_{12}\tau^1, \\ g_{\boldsymbol{\sigma}}(0) &= \mathrm{i}\eta_{C_6}\eta_{12}\tau^1, \\ g_{\boldsymbol{\sigma}}(1) &= -\mathrm{i}\eta_{\boldsymbol{\sigma}}\eta_{C_6}\eta_{12}\tau^1. \end{aligned} \tag{B28}$$

there are $2^3=8$ different PSG's in the class (II)(ii)(A).
(B) $g_{\boldsymbol{\sigma}}(s)g_{C_6}(1-s)=\mathrm{i}\hat{n}_s\cdot\vec{\tau}$:
(α) $\eta_{\boldsymbol{\sigma T}}=1$, *i.e.* $[g_{\boldsymbol{\sigma}}(s),\tau^3]=0$:
here we have

$$\begin{aligned} g_{C_6}(0) &= \mathrm{i}\tau^1, \\ g_{C_6}(1) &= -\mathrm{i}\eta_{C_6}\eta_{12}\tau^1 e^{\mathrm{i}\psi_3\tau^3}, \\ g_{\boldsymbol{\sigma}}(0) &= \tau^0, \\ g_{\boldsymbol{\sigma}}(1) &= \eta_{\boldsymbol{\sigma}}\tau^0. \end{aligned} \tag{B29}$$

there are $2^3\times 3=24$ different PSG's in the class (II)(ii)(B)(α) since $\psi_3=0,\pm 2\pi/3$.
(β) $\eta_{\boldsymbol{\sigma T}}=-1$, *i.e.* $\{g_{\boldsymbol{\sigma}}(s),\tau^3\}=0$:
here we have

$$\begin{aligned} g_{C_6}(0) &= \mathrm{i}\tau^1, \\ g_{C_6}(1) &= -\mathrm{i}\eta_{C_6}\eta_{12}\tau^1 e^{\mathrm{i}\psi_3\tau^3}, \\ g_{\boldsymbol{\sigma}}(0) &= \mathrm{i}\tau^1, \\ g_{\boldsymbol{\sigma}}(1) &= -\mathrm{i}\eta_{\boldsymbol{\sigma}}\tau^1. \end{aligned} \tag{B30}$$

there are $2^3\times 3=24$ different PSG's in the class (II)(ii)(B)(β) since $\psi_3=0,\pm 2\pi/3$.

To summarize, above are the 160 different (algebraic) PSG's with $IGG=\{\pm\tau^0\}$ on a honeycomb lattice. They represent different Z_2 spin liquid states on a honeycomb lattice, which possess all the symmetries of the honeycomb lattice generated by $\{\boldsymbol{T},T_1,T_2,\boldsymbol{\sigma},C_6\}$. We also want to emphasize that any solution to the set of equation (B1)-(B14) may look different, but it will be gauge equivalent to one of these 160 PSG's.

On the other hand, such a (algebraic) PSG really corresponds to a spin liquid if and only if it can be realized by a mean-field ansatz $\{u_{ij}\}$ on a honeycomb lattice[15]. In fact, not all of these algebraic PSGs can be realized by an ansatz. After the time-reveral transformation, the mean field amplitude changes sign[15]: $\boldsymbol{T}(u_{ij})=-u_{ij}$. Gauge transformation $G_{\boldsymbol{T}}$ must change the sign again:

$$-u_{ij}=G_{\boldsymbol{T}}(i)u_{ij}G_{\boldsymbol{T}}(j)^\dagger \tag{B31}$$

If in an algebraic PSG, $G_{\boldsymbol{T}}(i)=\tau^0$ independent of site, u_{ij} must vanish.

Clearly at least 32 algebraic PSG's among the total 160 types cannot be realized by any mean-field ansatz $\{u_{ij}\}$. These are the PSG's with $G_{\boldsymbol{T}}(i)=g_{\boldsymbol{T}}(s)=\tau^0$ in the class (I)(I)(A)&(B). Since under time reversion $\boldsymbol{T}$ we require $-u_{ij}=G_{\boldsymbol{T}}(i)u_{ij}G^\dagger_{\boldsymbol{T}}(j)=u_{ij}$, this leads to the vanishing of all bonds $\{u_{ij}\equiv 0\}$ in the mean-field ansatz. Therefore, there are at the most 128 possible Z_2 spin liquids that can be realized by a mean-field ansatz on a honeycomb lattice.

Appendix C: Classification of Z_2 projective symmetry groups around u-RVB ansatz

In this section we focus on those Z_2 spin liquids near the u-RVB state, which is discussed in section III. These Z_2 spin liquids are plausibly connected to a semimetal through a continuous phase transition. The u-RVB state is realized by the following ansatz:

$$u_{ij}=(-1)^{s_i}\,\mathrm{i}\chi\tau^0 \tag{C1}$$

its mean-field bond is only nonzero between nearest neighbors $<ij>$, which have different sublattice indices $s_i=1-s_j$. By definition, its PSG has the following form:

$$\begin{aligned} G_{T_1}(x_1,x_2,s) &= g_1, \\ G_{T_2}(x_1,x_2,s) &= g_2, \\ G_{\boldsymbol{T}}(x_1,x_2,s) &= (-1)^s g_{\boldsymbol{T}}, \\ G_{C_6}(x_1,x_2,s) &= (-1)^s g_{C_6}, \\ G_{\boldsymbol{\sigma}}(x_1,x_2,s) &= (-1)^s g_{\boldsymbol{\sigma}}. \end{aligned} \tag{C2}$$

where $g_1,g_2,g_{\boldsymbol{T}},g_{C_6},g_{\boldsymbol{\sigma}}\in SU(2)$. To find out those Z_2 spin liquids around such a u-RVB state, we need to trace those PSG's with $IGG=\{\pm\tau^0\}$ that looks like (C2). Consistent conditions (B1)-(B14) now corresponds to constraints on the $SU(2)$ matrices $\{g_1,g_2,g_{\boldsymbol{T}},g_{C_6},g_{\boldsymbol{\sigma}}\}$:

$$\begin{aligned} &g_1^{-1}g_2^{-1}g_1g_2=\xi_{12}\tau^0, && g_{\boldsymbol{T}}^2=\xi_{\boldsymbol{T}}\tau^0, \\ &g_1^{-1}g_{\boldsymbol{T}}^{-1}g_1g_{\boldsymbol{T}}=\xi_{1\boldsymbol{T}}\tau^0, && g_2^{-1}g_{\boldsymbol{T}}^{-1}g_2g_{\boldsymbol{T}}=\xi_{2\boldsymbol{T}}\tau^0, \\ &g_2^{-1}g_{C_6}g_1g_{C_6}^{-1}=\xi_{C_61}\tau^0, && g_1^{-1}g_{C_6}g_1g_2^{-1}g_{C_6}^{-1}=\xi_{C_62}\tau^0, \\ &g_{\boldsymbol{T}}^{-1}g_{C_6}^{-1}g_{\boldsymbol{T}}g_{C_6}=\xi_{C_6\boldsymbol{T}}\tau^0, && g_{C_6}^6=\xi_{C_6}\tau^0, \\ &g_1^{-1}g_{\boldsymbol{\sigma}}g_1g_{\boldsymbol{\sigma}}^{-1}=\xi_{\boldsymbol{\sigma}1}\tau^0, && g_2^{-1}g_{\boldsymbol{\sigma}}g_1g_2^{-1}g_{\boldsymbol{\sigma}}^{-1}=\xi_{\boldsymbol{\sigma}2}\tau^0, \\ &g_{\boldsymbol{\sigma}}g_{C_6}g_{\boldsymbol{\sigma}}g_{C_6}=\xi_{\boldsymbol{\sigma}C_6}\tau^0, && g_{\boldsymbol{T}}^{-1}g_{\boldsymbol{\sigma}}^{-1}g_{\boldsymbol{T}}g_{\boldsymbol{\sigma}}=\xi_{\boldsymbol{\sigma T}}\tau^0, \\ &g_{\boldsymbol{\sigma}}^2=\xi_{\boldsymbol{\sigma}}\tau^0. \end{aligned} \tag{C3}$$

where all ξ's take value of ± 1. Again, as discussed in appendix B we can always make $\xi_{C_61}=\xi_{C_62}=1$ by choosing a proper gauge. After solving eqs. (C3), we find out there are 24 gauge inequivalent solutions in total, as summarized in TABLE I. In other words, there are 24 different Z_2 spin liquid around the u-RVB state, suggesting that they are promising candidates of the spin liquid connected to a semimetal on honeycomb lattice through a continuous phase transition.

Appendix D: Consistent conditions on the mean-field ansatz $\{u_{ij}\}$ on a honeycomb lattice

In this section we derive the consistent conditions on an arbitrary mean-field bond u_{ij}, which realizes a spin liquid with a certain PSG on a honeycomb lattice. The basic idea is to find all possible symmetry group elements that transform the two lattice sites $\{i,j\}$ into itself $\{i,j\}$ or into each other $\{j,i\}$, so that the corresponding PSG

#	$g_{\boldsymbol{T}}$	$g_{\boldsymbol{\sigma}}$	g_{C_6}	g_1	g_2
1	τ^0	τ^0	τ^0	τ^0	τ^0
2	τ^0	τ^0	$i\tau^3$	τ^0	τ^0
3	τ^0	τ^0	$i\tau^3$	$e^{\mathrm{i}\,2\pi/3\tau^1}$	$e^{-\mathrm{i}\,2\pi/3\tau^1}$
4	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0
5	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0
6	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	τ^0	τ^0
7	τ^0	$\mathrm{i}\tau^3$	$e^{\mathrm{i}\,\pi/6\tau^1}$	τ^0	τ^0
8	τ^0	$\mathrm{i}\tau^3$	$e^{\mathrm{i}\,\pi/3\tau^1}$	τ^0	τ^0
9	τ^0	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$e^{\mathrm{i}\,2\pi/3\tau^3}$	$e^{-\mathrm{i}\,2\pi/3\tau^3}$
10	τ^0	$\mathrm{i}\tau^3$	$e^{\mathrm{i}\,2\pi/3\tau^1}$	$\mathrm{i}\,(\frac{\tau^1}{\sqrt{3}}-\sqrt{\frac{2}{3}}\tau^2)$	$\mathrm{i}\,(\frac{\tau^3}{\sqrt{2}}-\frac{\tau^2}{\sqrt{6}}-\frac{\tau^1}{\sqrt{3}})$
11	$\mathrm{i}\tau^3$	τ^0	τ^0	τ^0	τ^0
12	$\mathrm{i}\tau^3$	τ^0	$\mathrm{i}\tau^3$	τ^0	τ^0
13	$\mathrm{i}\tau^3$	τ^0	$\mathrm{i}\tau^1$	τ^0	τ^0
14	$\mathrm{i}\tau^3$	τ^0	$\mathrm{i}\tau^1$	$e^{\mathrm{i}\,2\pi/3\tau^3}$	$e^{-\mathrm{i}\,2\pi/3\tau^3}$
15	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0	τ^0
16	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	τ^0	τ^0
17	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	τ^0	τ^0
18	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$e^{\mathrm{i}\,2\pi/3\tau^3}$	$e^{-\mathrm{i}\,2\pi/3\tau^3}$
19	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$\mathrm{i}\tau^1$	τ^0	τ^0
20	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$\mathrm{i}\tau^2$	τ^0	τ^0
21	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	τ^0	τ^0	τ^0
22	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$\mathrm{i}\tau^3$	τ^0	τ^0
23	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$e^{\mathrm{i}\,\pi/6\tau^3}$	τ^0	τ^0
24	$\mathrm{i}\tau^3$	$\mathrm{i}\tau^1$	$e^{\mathrm{i}\,\pi/3\tau^3}$	τ^0	τ^0

TABLE I: A summary of all 24 different PSG's with $IGG = \{\pm\tau^0\}$ around the u-RVB ansatz. They correspond to 24 different Z_2 spin liquids near the u-RVB state.

elements must transform mean-field bond u_{ij} into itself u_{ij} or its Hermitian conjugate $u_{ij}^\dagger = u_{ji}$.

As a special case, the identity element 1 always transform a bond into itself: correspondingly in PSG the IGG elements (*e.g.* τ^0 for a Z_2 ansatz) always transform any bond u_{ij} into itself. This is nothing but the definition of invariant gauge group (IGG).

Now we need to look at nontrivial symmetry group elements which transform two lattice sites (connected by the bond) into itself or into each other. Without loss of generality, we consider the following bond

$$\langle x_1, x_2, s\rangle \equiv u_{(x_1,x_2,s)(0,0,0)} \tag{D1}$$

1. Regarding time reversal $\boldsymbol{T}$

Any element of the symmetry group can be written as

$$U = \boldsymbol{T}^{\nu_{\boldsymbol{T}}} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot \boldsymbol{\sigma}^{\nu_{\boldsymbol{\sigma}}} \cdot C_6^{\nu_{C_6}} \tag{D2}$$

First we study the consistent conditions from time reversal transformation $\boldsymbol{T}$ and then turn to other group elements.

Notice that time reversal $\boldsymbol{T}$ doesn't change anything except the sign of bond:

$$G_{\boldsymbol{T}}(i) u_{ij} [G_{\boldsymbol{T}}(j)]^\dagger = -u_{ij} \tag{D3}$$

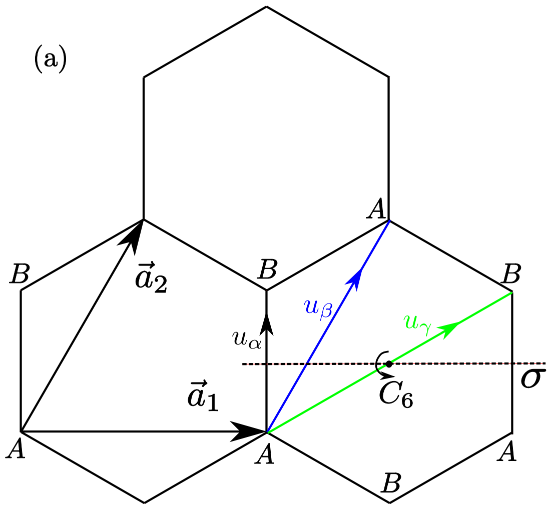

FIG. 3: (color online) Honeycomb lattice and generators of symmetry group. u_α, u_β and u_γ are representatives of 1st, 2nd and 3rd nearest neighbor (n.n.) mean-field amplitudes.

so this bond must satisfy the following constraint:

$$\begin{aligned} &G_{\boldsymbol{T}}(x_1, x_2, s)\langle x_1, x_2, s\rangle \\ &= -\langle x_1, x_2, s\rangle G_{\boldsymbol{T}}(0,0,0) \end{aligned} \tag{D4}$$

2. Conditions on a bond within the same sublattice: $s = 0$

First we study $s = 0$ case, *i.e.* a bond within the same sublattice. Since both mirror reflection $\boldsymbol{\sigma}$ and $\pi/3$ rotation C_6 will change the sublattice index s while the translations T_1, T_2 don't, we must have an even number of reflection and rotation, *i.e.* $\nu_{\boldsymbol{\sigma}} + \nu_{C_6} = 0 \mod 2$ to transform the bond to itself (or its Hermitian conjugate).

¿From (A22) it's easy to check the 5 nontrivial elements consisting of $\{\boldsymbol{\sigma},\ C_6\}$:

$$\begin{aligned} C_6^2(x_1,x_2,0) &= (1-x_1-x_2, x_1, 0), \\ C_6^{-2}(x_1,x_2,0) &= (x_2, 1-x_1-x_2, 0), \\ \boldsymbol{\sigma}C_6(x_1,x_2,0) &= (x_1, 1-x_1-y_1, 0), \\ \boldsymbol{\sigma}C_6^3(x_1,x_2,0) &= (1-x_1-x_2, x_2, 0), \\ \boldsymbol{\sigma}C_6^{-1}(x_1,x_2,0) &= (x_2, x_1, 0). \end{aligned} \tag{D5}$$

In order that the bond goes back after some translations, it's straightforward to find out all the consistent conditions on such a bond:

$$\begin{aligned} T_2^{-1}\boldsymbol{\sigma}C_6 :&\quad \langle -2x, x, 0\rangle \to \langle -2x, x, 0\rangle \\ T_2^{x-1}\boldsymbol{\sigma}C_6 :&\quad \langle 0, x, 0\rangle \to \langle 0, x, 0\rangle^\dagger \\ T_1^{-1}\boldsymbol{\sigma}C_6^3 :&\quad \langle x, -2x, 0\rangle \to \langle x, -2x, 0\rangle \\ T_1^{x-1}\boldsymbol{\sigma}C_6^3 :&\quad \langle x, 0, 0\rangle \to \langle x, 0, 0\rangle^\dagger \\ \boldsymbol{\sigma}C_6^{-1} :&\quad \langle x, x, 0\rangle \to \langle x, x, 0\rangle \\ T_1^x T_2^{-x}\boldsymbol{\sigma}C_6^{-1} :&\quad \langle x, -x, 0\rangle \to \langle x, -x, 0\rangle^\dagger \end{aligned} \tag{D6}$$

for $\forall\ \ x \in \mathbb{Z}$.

3. Conditions on a bond connecting different sublattices: $s=1$

In the $s=1$ case, such a bond connects different sublattices. So only an even number of reflection and rotation, *i.e.* $\nu_{\boldsymbol{\sigma}}+\nu_{C_6}=0 \mod 2$ might transform the bond to itself, while an odd number of reflection and rotation, *i.e.* $\nu_{\boldsymbol{\sigma}}+\nu_{C_6}=1 \mod 2$ can transform the bond $\langle x_1,x_2,1\rangle$ into its Hermitian conjugate $\langle x_1,x_2,1\rangle^\dagger$.

It's straightforward to obtain the following conditions on the bond $\langle x_1,x_2,1\rangle\equiv u_{(x_1,x_2,1)(0,0,0)}$:

$$\begin{aligned}
\boldsymbol{\sigma}:&\quad \langle -2x,x,1\rangle\to\langle -2x,x,1\rangle^\dagger \qquad\qquad (D7)\\
\boldsymbol{\sigma}C_6^{-1}:&\quad \langle x+1,x,1\rangle\to\langle x+1,x,1\rangle\\
T_1^{-2x-2}T_2^{x+1}\boldsymbol{\sigma}C_6^{-2}:&\quad \langle -2x-1,x,1\rangle\to\langle -2x-1,x,1\rangle^\dagger\\
T_1^{x_1-1}T_2^{x_2}C_6^3:&\quad \langle x_1,x_2,1\rangle\to\langle x_1,x_2,1\rangle^\dagger\\
T_1^{-1}\boldsymbol{\sigma}C_6^3:&\quad \langle x,-2x,1\rangle\to\langle x,-2x,1\rangle\\
T_1^{x-1}T_2^{x-1}\boldsymbol{\sigma}C_6^2:&\quad \langle x+1,x,1\rangle\to\langle x+1,x,1\rangle^\dagger\\
T_2^{-1}\boldsymbol{\sigma}C_6:&\quad \langle -2x-1,x,1\rangle\to\langle -2x-1,x,1\rangle
\end{aligned}$$

for $\forall\ \ x,x_1,x_2\in\mathbb{Z}$.

4. An example: mean-field ansatz $\{u_{ij}\}$ of Z_2 spin liquids near u-RVB state

To demonstrate the above consistent conditions, let's take a look at how they determine the mean-field ansatz $\{u_{ij}\}$ of any Z_2 spin liquid near u-RVB state, with PSG generators (C2).

Considering time reversion $\boldsymbol{T}$ we immediately have

$$g_{\boldsymbol{T}}\langle x_1,x_2,s\rangle=-(-1)^s\langle x_1,x_2,s\rangle g_{\boldsymbol{T}} \qquad (D8)$$

In other words, the bond connecting two sites belonging to the same (different) sublattice(s) anti-commutes(commutes) with $g_{\boldsymbol{T}}$.

For the nearest neighbor (n.n.) bond (see FIG. 3) $u_\alpha\equiv\langle 0,0,1\rangle$ we have $x_1=x_2=0,s=1$. Conditions (D7) and (D8) immediately lead to

$$\begin{aligned}
&[u_\alpha,g_{\boldsymbol{T}}]=0 \qquad (D9)\\
&g_{\boldsymbol{\sigma}}u_\alpha=-u_\alpha^\dagger g_{\boldsymbol{\sigma}}\\
&g_1^{-1}g_{C_6}^3u_\alpha=-u_\alpha^\dagger g_1^{-1}g_{C_6}^3
\end{aligned}$$

For 2nd n.n. bond $u_\beta\equiv\langle 0,1,0\rangle$ we have $x_1=0=s,x_2=1$ and (D6), (D8) lead to

$$\begin{aligned}
&\{u_\beta,g_{\boldsymbol{T}}\}=0 \qquad (D10)\\
&g_{\boldsymbol{\sigma}}g_{C_6}u_\beta=u_\beta^\dagger g_{\boldsymbol{\sigma}}g_{C_6}
\end{aligned}$$

For 3rd n.n. bond $u_\gamma\equiv\langle 1,0,1\rangle$ we have $x_2=0,x_1=s=1$. Conditions (D7) and (D8) lead to

$$\begin{aligned}
&[u_\gamma,g_{\boldsymbol{T}}]=0 \qquad (D11)\\
&g_{C_6}^3u_\gamma=-u_\gamma^\dagger g_{C_6}^3\\
&g_{\boldsymbol{\sigma}}g_{C_6}^{-1}u_\gamma=u_\gamma g_{\boldsymbol{\sigma}}g_{C_6}^{-1}
\end{aligned}$$

Constraints on further neighbors: *e.g.* 4th n.n. $\langle 0,1,1\rangle$, 5th n.n. $\langle 1,1,0\rangle$ and 6th n.n. $\langle 2,0,0\rangle$ can be similarly obtained.

Appendix E: A search of gapped spin liquids near the u-RVB state

In appendix C we showed that there are at most 24 Z_2 spin liquids around the u-RVB state, which are likely to connect with a semimetal through a continuous phase transition. In this section we search for those states with spectral gaps among the 24 spin liquid ansatz. In the end we find out most of the 24 states are gapless. More specifically, they cannot open up a mass gap through any perturbation around the u-RVB state, which has two graphenelike Dirac cones in the 1st Brillouin zone. It turns out that only 4 of them, *i.e.* #16, #17, #19 and #22 in TABLE I, are gapped spin liquids near the u-RVB state.

1. Symmetry-allowed masses in a graphenelike u-RVB state

We start from the low-energy effective Hamiltonian of the u-RVB state, which is described by a massless 8-component Dirac equation. These 8 components contain 2 spin indices (labeled by Pauli matrices $\{\tau^i\}$), 2 sublattice indices (labeled by Pauli matrices $\{\mu^i\}$) and 2 valley indices (labeled by Pauli matrices $\{\nu^i\}$). Just like graphene, the two valleys are located at $\mathbf{K}$ and $\mathbf{K}'$, *i.e.* the vertices in the honeycomb-shaped 1st Brillouin zone. Following the convention shown in FIG. 1, the momentum of these two cones are $\mathbf{K}=\frac{4\pi}{3}\vec{b}_1+\frac{2\pi}{3}\vec{b}_2$ and $\mathbf{K}'=\frac{2\pi}{3}\vec{b}_1+\frac{4\pi}{3}\vec{b}_2$ respectively, where $\{\vec{b}_1=(\sqrt{3},-1)/\sqrt{3}a,\ \vec{b}_2=(0,2)/\sqrt{3}a\}$ are the reciprocal lattice vectors corresponding to lattice vectors $\{\vec{a}_1=(a,0),\ \vec{a}_2=(1,\sqrt{3})a/2\}$.

Expanding the mean-field Hamiltonian of a u-RVB state with $u_\alpha=\mathrm{i}\tau^0$ (here $\mathbf{k}=\frac{2}{\sqrt{3}a}(k_x,k_y)=k_1\vec{b}_1+k_2\vec{b}_2$)

$$\begin{aligned}
H_{uRVB}=&\ \mathrm{i}(\psi^\dagger_{\mathbf{k},A},\psi^\dagger_{\mathbf{k},B})\cdot\\
&\begin{bmatrix}0 & -\tau^0(1+e^{-\mathrm{i}k_2}+e^{\mathrm{i}(k_1-k_2)})\\ \tau^0(1+e^{\mathrm{i}k_2}+e^{\mathrm{i}(k_2-k_1)}) & 0\end{bmatrix}\\
&\cdot\begin{pmatrix}\psi_{\mathbf{k},A}\\ \psi_{\mathbf{k},B}\end{pmatrix}
\end{aligned}$$

around $\mathbf{K}$ and $\mathbf{K}'$ we immediately obtain the Dirac equations

$$H_{\mathbf{K}}=(\psi^\dagger_{\mathbf{k},A},\psi^\dagger_{\mathbf{k},B})\begin{bmatrix}0 & \tau^0(k_y+\mathrm{i}k_x)\\ \tau^0(k_y-\mathrm{i}k_x) & 0\end{bmatrix}\begin{pmatrix}\psi_{\mathbf{k},A}\\ \psi_{\mathbf{k},B}\end{pmatrix}$$

$$H_{\mathbf{K}'}=(\psi^\dagger_{\mathbf{k}',A},\psi^\dagger_{\mathbf{k}',B})\begin{bmatrix}0 & \tau^0(k'_y-\mathrm{i}k'_x)\\ \tau^0(k'_y+\mathrm{i}k'_x) & 0\end{bmatrix}\begin{pmatrix}\psi_{\mathbf{k}',A}\\ \psi_{\mathbf{k}',B}\end{pmatrix}$$

Defining the following 8-component spinor:

$$\Psi^T\equiv(\psi^T_{\mathbf{k},A},\psi^T_{\mathbf{k},B},\psi^T_{\mathbf{k}',B},\psi^T_{\mathbf{k}',A}) \qquad (E1)$$

we can write the above effective Hamiltonian of u-RVB state as

$$H = \Psi^\dagger \mu^3(\mu^2\partial_x + \mu^1\partial_y)\otimes\tau^0\otimes\nu^0\Psi \tag{E2}$$

Therefore only those mass terms $M = \mu^3\otimes\tau^a\otimes\nu^b,\ \ a,b=0,1,2,3$ satisfy that $\{H,\Psi^\dagger M\Psi\}=0$ so that a mass gap can be opened in the Dirac spectrum. In the following we study how the mass term changes under the action of symmetry transformation such as spin rotations, time reversal $\boldsymbol{T}$ and translations T_1,T_2 *etc.* The physical symmetry of a spin liquid state realized by mean-field ansatz only allow those masses that are invariant under the corresponding PSG. If a PSG already rules out all possible mass terms $M=\mu^3\otimes\tau^a\otimes\nu^b,\ \ a,b=0,1,2,3$, we conclude the corresponding spin liquid realized by mean-field ansatz is gapless.

First we work out the transformation rules of Dirac spinor Ψ and M under a PSG. We focus on the 24 PSG's near the u-RVB state with the form (C2) as summarized in TABLE I.

a. Spin rotations

It's straightforward to show that a spin rotation along $\hat{z}$-axis by 2θ angle is realized by

$$\Psi\to e^{\mathrm{i}\,\theta}\Psi \tag{E3}$$

while a spin rotation along $\hat{y}$-axis by π angle is realized by

$$\Psi\to \mathrm{i}\tau^2\mu^1\nu^1\Psi^* \tag{E4}$$

Apparently S_z rotations leave the mass term invariant, while under S_y rotations by π the mass term transforms in the following way

$$M\to -\mu^1\otimes\nu^1\otimes\tau^2 M^T\tau^2\otimes\mu^1\otimes\nu^1 \tag{E5}$$

Since the mass term is invariant under spin rotations, its allowed form as seen from the above constraint can only be

$$M_A^{(a)} = \mu^3\otimes\nu^3\otimes\tau^a,\quad a=1,2,3 \tag{E6}$$

or

$$M_B^{(b)} = \mu^3\otimes\nu^b\otimes\tau^0,\quad b=0,1,2 \tag{E7}$$

b. Time reversal $\boldsymbol{T}$

Since a mean-field bond u_{ij} becomes $-(-1)^{s_i}g_T u_{ij} g_T^\dagger(-1)^{s_j}$ under the time reversal transformation in a PSG (C2), clearly $\boldsymbol{T}$ is realized by

$$\begin{aligned}\Psi &\to g_T^\dagger\otimes\mu^3\otimes\nu^3\Psi\\ M&\to -M\end{aligned} \tag{E8}$$

so the mass term is invariant under time reversal $\boldsymbol{T}$ if

$$M = -g_T\otimes\mu^3\otimes\nu^3 M g_T^\dagger\otimes\mu^3\otimes\nu^3 \tag{E9}$$

10 spin liquids near the u-RVB state, *i.e.* #1-#10 in TABLE I has $g_T=\tau^0$. In these cases, mass terms $M_A^{(a)},\ a=1,2,3$ will violate transformation rule (E9), and the only allowed masses are $M_B^{(1)},\ M_B^{(2)}$.

The other 14 spin liquids around u-RVB state (#11-#24 in TABLE I) are characterized by $g_T=\mathrm{i}\tau^3$. In this case the allowed masses are $M_B^{(1)},\ M_B^{(2)}$ and $M_A^{(1)},\ M_A^{(2)}$.

c. Translations $T_1,\ T_2$

Under translations T_1,T_2 in a PSG (C2) the 8-component spinor changes as

$$\begin{aligned}T_1:&\qquad \Psi\to e^{-\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_1^\dagger\Psi,\\ T_2:&\qquad \Psi\to e^{\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_2^\dagger\Psi.\end{aligned} \tag{E10}$$

since $\mathbf{K}\cdot\vec{a}_{1,2}=\mp\frac{2\pi}{3}$ and $\mathbf{K}'\cdot\vec{a}_{1,2}=\pm\frac{2\pi}{3}$. In order for the mass term to be invariant

$$\begin{aligned}M &= e^{\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_1 M e^{-\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_1^\dagger\\ &= e^{-\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_2 M e^{\mathrm{i}\,\frac{2\pi}{3}\nu^3}\otimes g_2^\dagger\end{aligned} \tag{E11}$$

the symmetry-allowed masses can only be:
$M_B^{(0)}$ and $M_A^{(a)},\ a=1,2,3$ if $g_1=g_2=\tau^0$;
$M_B^{(0)}$ and $M_A^{(3)}$ if $g_1=g_2^{-1}=e^{\mathrm{i}\,2\pi/3\tau^3}$;
$M_B^{(0)}$ for the special case #10 in TABLE I.

Combining conditions (E9) and (E11) we can see that $\{M_B^{(b)},\ b=0,1,2\}$ are not allowed by symmetry in any of the 24 spin liquids near u-RVB state. In the following study we will focus on masses $M_A^{(a)},\ a=1,2,3$.

d. Reflection $\boldsymbol{\sigma}$

Similar to time reversal $\boldsymbol{T}$, under reflection along $\hat{x}$-axis the spinor transforms as

$$\Psi\to\mu^1\cdot g_{\boldsymbol{\sigma}}^\dagger\otimes\mu^3\otimes\nu^3\Psi = -\mathrm{i}g_{\boldsymbol{\sigma}}^\dagger\otimes\mu^2\otimes\nu^3\Psi \tag{E12}$$

The mass term is invariant under reflection $\boldsymbol{\sigma}$ if

$$M = g_{\boldsymbol{\sigma}}\otimes\mu^2\otimes\nu^3 M g_{\boldsymbol{\sigma}}^\dagger\otimes\mu^2\otimes\nu^3 \tag{E13}$$

The symmetry-allowed masses are:
none if $g_{\boldsymbol{\sigma}}=\tau^0$;
$M_A^{(a)},\ a\neq b$ if $g_{\boldsymbol{\sigma}}=\mathrm{i}\tau^b$.

e. $\pi/3$ *rotation* C_6

Under C_6, *i.e.* a rotation by $\pi/3$ the spinor transforms as

$$\Psi \to g_{C_6}^\dagger \otimes e^{\mathrm{i}\frac{5\pi}{6}\mu^3} \otimes (\frac{\sqrt{3}}{2}\nu^1 - \frac{1}{2}\nu^2)\Psi \tag{E14}$$

The mass term is invariant under reflection $\boldsymbol{\sigma}$ if

$$\begin{aligned} M = g_{C_6} \otimes e^{-\mathrm{i}\frac{5\pi}{6}\mu^3} \otimes (\tfrac{\sqrt{3}}{2}\nu^1 - \tfrac{1}{2}\nu^2) \cdot M \\ \cdot g_{C_6}^\dagger \otimes e^{\mathrm{i}\frac{5\pi}{6}\mu^3} \otimes (\tfrac{\sqrt{3}}{2}\nu^1 - \tfrac{1}{2}\nu^2) \end{aligned} \tag{E15}$$

The symmetry-allowed masses are:
none if $g_{C_6} = \tau^0,\ e^{\mathrm{i}\theta\tau^{1,3}}$ with $\theta \neq 0 \mod \pi/2$;
$M_A^{(a)},\ a \neq b$ if $g_{C_6} = \mathrm{i}\tau^b$.

2. Realizing the 4 gapped Z_2 spin liquids near the u-RVB state

Among all 24 spin liquids near the u-RVB states, it turns out that there are no symmetry-allowed masses for 20 of them. In other words, these 20 spin liquids cannot open up a mass gap through a perturbation around the u-RVB state. Only the following 4 spin liquids near the u-RVB state can obtain an energy gap in the spectrum through adding a symmetry-allowed mass term:

#16 with two symmetry-allowed masses $M_A^{(1,2)} = \mu^3 \otimes \nu^3 \otimes \tau^{1,2}$;

#17 with one symmetry-allowed mass $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$;

#19 with one symmetry-allowed mass $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$;

#22 with one symmetry-allowed mass $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$.

In fact, as summarized in TABLE II, all these 4 gapped spin liquids can be realized by mean-field ansatz $\{u_{ij}\}$, which satisfies consistent conditions from the corresponding PSG as discussed in appendix D. In the following we describe the mean-field ansatz for these 4 gapped Z_2 spin liquids. In the end only one gapped Z_2 spin liquid, *i.e.* #19 can be realized by a mean-field ansatz up to 3rd n.n. bonds.

a. Z_2 *spin liquid #16: up to 5th n.n. bonds needed*

The mean-field ansatz $\{u_{ij}\}$ for Z_2 spin liquid #16 is summarized in TABLE II, up to 5th n.n. bonds. The corresponding spin liquid has a Z_2 gauge structure, if and only if $[u_\beta, u_\varepsilon] \neq 0$, so that the IGG of this mean-field ansatz is a Z_2 group $\{\pm\tau^0\}$.

It's straightforward to check that 2nd n.n. bond $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$ open up a mass gap $M \sim \mu^3 \otimes \nu^3 \otimes (\beta_1\tau^1 + \beta_2\tau^2) = \beta_1 M_A^{(1)} + \beta_2 M_A^{(2)}$.

#	u_α	u_β	u_γ	u_δ	u_ε	9th n.n. $\langle 1,2,0\rangle$
16	$\mathrm{i}\tau^0$	$\{\tau^1,\tau^2\}$	$\mathrm{i}\tau^0$	$\mathrm{i}\tau^0$	$\{\tau^1,\tau^2\}$	$\cdots$
17	$\mathrm{i}\tau^0$	τ^2	$\mathrm{i}\tau^0$	$\{\mathrm{i}\tau^0,\tau^3\}$	$\cdots$	$\cdots$
19	$\{\mathrm{i}\tau^0,\tau^3\}$	$\{\tau^1,\tau^2\}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
22	$\mathrm{i}\tau^0$	τ^2	$\mathrm{i}\tau^0$	$\mathrm{i}\tau^0$	τ^2	$\{\tau^1,\tau^2\}$

TABLE II: Symmetry-allowed mean-field ansatz of the 4 possible gapped spin liquids near the u-RVB state. We follow the notations for mean-field bonds in appendix D. We only summarize the mean-field bonds that are necessary to realize a gapped Z_2 spin liquid. Ellipsis represents those longer-range mean-field bonds unnecessary for a Z_2 spin liquid, which are not listed in this table. Up to 3rd n.n. mean-field bonds $\{u_\alpha, u_\beta, u_\gamma\}$, only one Z_2 spin liquid, *i.e.* #19 can be realized on a honeycomb lattice.

b. Z_2 *spin liquid #17: up to 4th n.n. bonds needed*

The mean-field ansatz $\{u_{ij}\}$ for Z_2 spin liquid #17 is summarized in TABLE II, up to 4th n.n. bonds. The corresponding spin liquid has a Z_2 gauge structure, if and only if $[u_\beta, u_\delta] \neq 0$, so that the IGG of this mean-field ansatz is a Z_2 group $\{\pm\tau^0\}$.

It's straightforward to check that 2nd n.n. bond $u_\beta = \beta\tau^2$ open up a mass gap $M \sim \beta\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta M_A^{(2)}$.

c. Z_2 *spin liquid #19: up to 2nd n.n. bonds needed*

The mean-field ansatz $\{u_{ij}\}$ for Z_2 spin liquid #17 is summarized in TABLE II, up to 2nd n.n. bonds. The corresponding spin liquid has a Z_2 gauge structure, if and only if $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$ with $\beta_1, \beta_2 \neq 0$, so that the IGG of this mean-field ansatz is a Z_2 group $\{\pm\tau^0\}$. This is the only gapped Z_2 spin liquid near the u-RVB state, that can be realized in a mean-field ansatz up to 3rd n.n. bonds.

It's straightforward to check that 2nd n.n. bond $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$ open up a mass gap $M \sim \beta_2\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta_2 M_A^{(2)}$.

d. Z_2 *spin liquid #22: up to 9th n.n. bonds needed*

The mean-field ansatz $\{u_{ij}\}$ for Z_2 spin liquid #17 is summarized in TABLE II, up to 9th n.n. bonds. The corresponding spin liquid has a Z_2 gauge structure, if and only if $[u_\beta, u_9] \neq 0$, so that the IGG of this mean-field ansatz is a Z_2 group $\{\pm\tau^0\}$. $u_9 \equiv \langle 1,2,0\rangle$ is the 9th n.n. mean-field bond. In this Z_2 spin liquid, the symmetry-allowed consistent mean-field bonds for 6th, 7th and 8th n.n. are:

$$\begin{aligned} u_6 &\equiv \langle 2,0,0\rangle \propto \tau^2, \\ u_7 &\equiv \langle 2,0,1\rangle \propto \mathrm{i}\tau^0, \\ u_8 &\equiv \langle 0,2,1\rangle \propto \mathrm{i}\tau^0. \end{aligned}$$

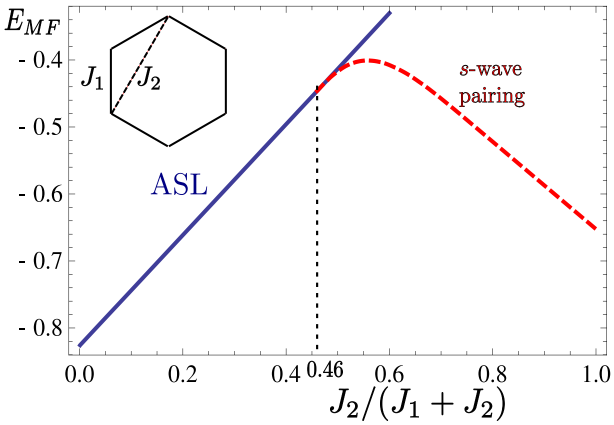

FIG. 4: Mean-field phase diagram of $J_1 - J_2$ model by Schwinger-fermion approach.

It's straightforward to check that 2nd n.n. bond $u_\beta = \beta\tau^2$ open up a mass gap $M \sim \beta\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta M_A^{(2)}$.

Appendix F: Schwinger-fermion mean-field study of the $J_1 - J_2$ model on honeycomb lattice

Can SPS be realized in the Hubbard model when $t/U \sim 1/4$, where numerics shows a gapped SL phase? In particular, by the Mott transition theory of Hermele[23], the u-RVB (or ASL) state is in the neighborhood of the Mott transition. Can SPS be more favorable than the ASL state? To address this question, we use t/U expansion of the Hubbard model[43] to obtain an effective $J_1 - J_2$ spin model on honeycomb lattice:

$$H = J_1 \sum_{<ij>} \vec{S}_i \cdot S_j + J_2 \sum_{<<ij>>} \vec{S}_i \cdot S_j \tag{F1}$$

where J_1 and J_2 are the 1st neighbor and 2nd neighbor antiferromagnetic coupling. Following Ref. 43, we find up to t^4/U^3 order, the effective J_1 and J_2 are:

$$J_1 = 4t^2/U - 16t^4/U^3, \qquad J_2 = 4t^4/U^3. \tag{F2}$$

Naively plugging in $t/U \sim 1/4$ gives $J_2/J_1 \sim 1/12$.

We use the following variationally mean-field ansatz:

$$\begin{aligned} H_{MF} =& \chi \sum_{<ij>} f_{i\alpha}^\dagger f_{j\alpha} + \Delta e^{i\theta} \sum_{<<ij>>\in A} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger \\ &+ \Delta e^{-i\theta} \sum_{<<ij>>\in B} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger + \text{h.c.} \end{aligned} \tag{F3}$$

It is equivalent to the SPS ansatz (11) by a $SU(2)$ gauge transformation. Note that this mean-field study is biased towards spin disordered ground state. For example, we do not include Neel order which is known to be the ground state at $J_2 = 0$, and we also do not include the spiral spin order which is found by semiclassical study of $J_1 - J_2$ model[44,45]. The purpose of the current mean-field study is to understand whether a gapped spin liquid can be more favorable compared to the gapless ASL state when J_2 is tuned up and frustration becomes important.

By minimizing the mean-field energy in Eq.(5), the phase diagram of $J_1 - J_2$ model is obtained and shown in Fig.4, where we fix $J_1 + J_2 = 1$ and E_{MF} is scaled from Eq.(5) by 8/3. We find that when $J_2/J_1 < 0.85$ (or $J_2/(J_1+J_2) < 0.46$), the ground state is u-RVB(or ASL) state: $\chi \neq 0$ and $\Delta = 0$. When $J_2/J_1 > 0.85$, the ground state is an s-wave pairing state: $\chi, \Delta \neq 0$ and $\theta = 0$. The s-wave pairing state opens an energy gap for spinons but has remaining $U(1)$ gapless gauge fluctuation. Due to monopole proliferation[38] the s-wave pairing state is not a stable phase. In this mean-field study, the gauge fluctuations are not considered and this is the reason why we find s-wave pairing state as a ground state. Taking gauge fluctuations into account, the likely fate of the s-wave pairing state is that θ becomes nonzero and the Z_2 SPS state is realized.

We propose to study the $J_1 - J_2$ model by Gutzwiller projected wavefunction variational approach[46] because it can be viewed as a method to include the gauge fluctuation. We leave this projected wavefunction study as a direction of future research, which may realize SPS as the ground state. Projected wavefunctions are also classified by PSG, so the present work also provide guideline for the search of ground states in the projected wavefunction space.

Appendix G: Derivation of the mutual Chern-Simons term

We start from the following low energy effective Lagrangian of spinon fields ψ in imaginary time (*i.e.* Euclidean space-time):

$$\begin{aligned} \mathcal{L} =& \psi^\dagger \gamma_0 (\partial_\mu - i a_\mu) \gamma_\mu \psi + m\, \hat{n} \cdot \psi^\dagger \vec{M} \psi \\ &+ \frac{1}{4g_a^2} f_{\mu\nu}^2 + \frac{1}{2u} (\partial_\mu \hat{n})^2. \end{aligned} \tag{G1}$$

We're aiming for an effective action of gauge fields a_μ obtained by integrating out the spinon fields ψ in

$$\mathcal{L}_{eff} = \bar{\psi} \big[\mathrm{i}\gamma_\mu (\partial_\mu - \mathrm{i} a_\mu) + \mathrm{i} m\, \hat{n} \cdot \vec{\sigma}\ \big] \psi \tag{G2}$$

where we define $\bar{\psi} \equiv \psi^\dagger \gamma^0$. For simplicity let's denote $-\mathrm{i}\mathcal{G}^{-1} = \gamma_\mu(\partial_\mu - \mathrm{i} a_\mu) + m\hat{n} \cdot \vec{\sigma}$, then integrating out spinon fields ψ yield the effective action $\mathcal{S} = -\ln\det(\mathcal{G}^{-1}) = -\mathrm{Tr}\ln(\mathcal{G}^{-1})$. Following the spirit of Abanov and Wiegmann[26,47], we use large-m expansion to obtain the low energy effective theory in the longwavelength limit $\omega \ll m$. By defining $\mathcal{G}_0^{-1} = \mathrm{i}(\gamma_\mu \partial_\mu + m\hat{n} \cdot \vec{\sigma})$ we have $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + a_\mu \gamma_\mu$. Let's denote $\not{\partial} \equiv \gamma_\mu \partial_\mu$ and similarly $\not{a} \equiv \gamma_\mu a_\mu$ and we have

$$\begin{aligned} \mathcal{S} &= -\mathrm{Tr}\ln(\mathcal{G}_0^{-1} + \not{a}) = -\mathrm{Tr}\ln(\mathcal{G}_0^{-1}) - \mathrm{Tr}\ln(1 + \mathcal{G}_0 \not{a}) \\ &= \mathcal{S}_0 + \textstyle\sum_{l=1}^{\infty} (-1)^l \mathrm{Tr}(\mathcal{G}_0 \not{a})^l \end{aligned}$$

Here $\mathcal{S}_0$ gives the nonlinear-sigma-model dynamics $\sim (\partial_\mu \hat{n})^2$ of vector $\hat{n}$, while the coupling between vector $\hat{n}$

and gauge field a_μ is given by the 2nd term. In the large-m expansion we consider only the leading-order term:

$$\mathcal{S}_1 = -\mathrm{Tr}(\mathcal{G}_0 \not{a}) = -\mathrm{Tr}\Big\{\mathcal{G}_0^{-1}\big[\mathcal{G}_0^{-1}(\mathcal{G}_0^{-1})^\dagger\big]^{-1}\not{a}\Big\}$$

It's straightforward to check that $\mathcal{G}_0^{-1}(\mathcal{G}_0^{-1})^\dagger = -\partial^2 + m^2 - m\vec{\sigma}\cdot\not{\partial}\hat{n}$, therefore large-$m$ expansion leads to

$$\big[\mathcal{G}_0^{-1}(\mathcal{G}_0^{-1})^\dagger\big]^{-1} = (-\partial^2 + m^2)^{-1}\sum_{l=0}^{\infty}\big(\frac{m\vec{\sigma}\cdot\not{\partial}\hat{n}}{-\partial^2 + m^2}\big)^l$$

and consequently

$$\mathcal{S}_1 = -\sum_{l=0}^{\infty}\mathrm{Tr}\Big\{\frac{\mathrm{i}(\not{\partial} + m\hat{n}\cdot\vec{\sigma})}{-\partial^2 + m^2}\big(\frac{m\vec{\sigma}\cdot\not{\partial}\hat{n}}{-\partial^2 + m^2}\big)^l\not{a}\Big\}$$

It turns out that $l = 0, 1$ terms both vanish and the leading-order correction to the low energy effective action is the following topological term:

$$\begin{aligned}\mathcal{S}_{topo} &= -a_\mu \mathrm{Tr}\Big[\gamma_\mu \tfrac{\mathrm{i}m\hat{n}\cdot\vec{\sigma}}{-\partial^2+m^2}\big(\tfrac{m\gamma_\nu(\partial_\nu\hat{n})\cdot\vec{\sigma}}{-\partial^2+m^2}\big)^2\Big] \\ &= \tfrac{\mathrm{i}}{4\pi}\epsilon_{\mu\nu\lambda}\ a_\mu\ \hat{n}\cdot(\partial_\nu\hat{n}\times\partial_\lambda\hat{n})\end{aligned} \qquad \text{(G3)}$$

Notice that in the CP^1 parametrization of order parameter $\hat{n} = w^\dagger\sigma w$, spinor w is the eigenvector of $\hat{n}\cdot\vec{\sigma}$ whose spin orientation is along unit vector $\hat{n}$. Therefore the Skyrmion current of $\hat{n}$

$$J_\mu^{Sk} = \frac{1}{2}\cdot\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\hat{n}\cdot(\partial_\nu\hat{n}\times\partial_\lambda\hat{n}) \qquad \text{(G4)}$$

which equals half the winding number of $\hat{n}$ wrapping around S^2, is nothing but the Berry's phase[48] for spinor w. Since spinor w obtains π phase (*i.e.* a minus sign) as $\hat{n}$ wraps around S^2 once (*i.e.* $\hat{n}$ covers 4π solid angle), this gives a direct correspondence between the Skyrmion current density and the $U(1)$ gauge field strength $F_{\mu\nu}$ coupled to w:

$$\frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\hat{n}\cdot(\partial_\nu\hat{n}\times\partial_\lambda\hat{n}) = \frac{1}{\pi}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \qquad \text{(G5)}$$

Therefore the topological term (G3) is exactly the mutual Chern-Simons term mentioned earlier

$$\mathcal{S}_{topo} = \frac{\mathrm{i}}{\pi}\epsilon_{\mu\nu\lambda}\ a_\mu\ \partial_\nu A_\lambda$$

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[50] This is because close to a Mott transition, physically the low energy spinons should be the electrons which lose their charge coherence. And bosonic spinon condensation can easily explain the transition into a magnetic ordered phase.
[51] The constraint $|w_1|^2 + |w_2|^2 = 1$ can be enforced by a Lagrangian multiplier λ. (13) can be obtained by the saddle point expansion of λ.