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Z_2 spin liquids in $S = 1/2$ Heisenberg model on kagome lattice: projective symmetry group study of Schwinger-fermion mean-field states

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Due to strong geometric frustration and quantum fluctuation, $S = 1/2$ quantum Heisenberg antiferromagnets on the kagome lattice has long been considered as an ideal platform to realize spin liquid (SL), a novel phase exhibiting fractionalized excitations without any symmetry breaking. A recent numerical study[1] of Heisenberg $S = 1/2$ kagome lattice model (HKLM) shows that in contrast to earlier results, the ground state is a singlet-gapped SL with signatures of Z_2 topological order. Motivated by this numerical discovery, we use projective symmetry group to classify all 20 possible Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice. Among them we found only one gapped Z_2 SL (which we call $Z_2[0, \pi]\beta$ state) in the neighborhood of $U(1)$ -Dirac SL state. Since its parent state, *i.e.* $U(1)$ -Dirac SL is found[2] to be the lowest among many other candidate $U(1)$ SLs including the uniform resonating-valence-bond states, we propose this $Z_2[0, \pi]\beta$ state to be the numerically discovered SL ground state of HKLM.

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I. INTRODUCTION

At zero temperature all degrees of freedom tend to freeze and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. However, in a quantum system with a large zero-point energy, one may expect a liquid-like ground state to exist even at $T = 0$. In a system consisting of localized quantum magnets, we call such a quantum-fluctuation-driven disordered ground state a quantum spin liquid (SL)³. It is an exotic phase with novel “fractionalized” excitations carrying only a fraction of the electron quantum number, *e.g.* spinons which carry spin but no charge. The internal structures of these SLs are so rich that they are beyond the description of Landau’s symmetry breaking theory⁴ of conventional ordered phases. Instead they are characterized by long-range quantum entanglement^{5,6} encoded in the ground state, which is coined “topological order”^{7,8} in contrast to the conventional symmetry-breaking order.

Geometric frustration in a system of quantum magnets would lead to a huge degeneracy of classical ground state configurations. The quantum tunneling among these classical ground states provides a mechanism to realize quantum SLs. The quest for quantum SLs in frustrated magnets (for a recent review see Ref. 9) has been pursued for decades. Among them the Heisenberg $S = 1/2$ kagome lattice model (HKLM)

$$H_{HKLM} = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \quad (1)$$

has long been thought as a promising candidate. Here $\langle i, j \rangle$ denotes i, j being a nearest neighbor pair. Experimental evidence of SL^{10–13} has been observed in $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$ (called herbertsmithite), a spin-half antiferromagnet on the kagome lattice. Theoretically, in

lack of an exact solution of the two-dimensional (2D) quantum Hamiltonian (1) in the thermodynamic limit, in previous studies either a honeycomb valence bond crystal^{14–18} (HVBC) with an enlarged 6×6 -site unit cell, or a gapless SL¹⁹ were proposed as the ground state of HKLM. However, recently an extensive density-matrix-renormalization-group (DMRG) study¹ on HKLM reveals the ground state of HKLM as a gapped SL, which substantially lowers the energy compared to HVBC. Besides, they also observe numerical signatures of Z_2 topological order in the SL state.

Motivated by this important numerical discovery, we try to find out the nature of this gapped Z_2 SL. Different Z_2 SLs on the kagome lattice have been previously studied using Schwinger-boson representation^{20,21}. Here we propose the candidate states of symmetric Z_2 SLs on kagome lattice by Schwinger-fermion mean field approach^{22–28}. Following is the summary of our results. First we use projective symmetry group⁸ (PSG) to classify all 20 possible Schwinger-fermion mean-field ansatz of Z_2 SLs which preserve all the symmetry of HKLM, as shown in TABLE I. We analyze these 20 states and rule out some obviously unfavorable states: *e.g.* gapless states, and those states whose 1st nearest neighbor (n.n.) mean-field amplitudes must vanish due to symmetry. Then we focus on those Z_2 SLs in the neighborhood of the $U(1)$ -Dirac SL². In Ref. 2 it is shown that $U(1)$ -Dirac SL has a significantly lower energy compared with other candidate $U(1)$ SL states, such as the uniform resonating-valence-bond (RVB) state (or the $U(1)$ SL-[0, 0] state in notation of Ref. 2). We find out that there is only one gapped Z_2 SL, which we label as $Z_2[0, \pi]\beta$, in the neighborhood of (or continuously connected to) $U(1)$ -Dirac SL. Therefore we propose this $Z_2[0, \pi]\beta$ state as a promising candidate state for the ground state of HKLM. The mean-field ansatz of $Z_2[0, \pi]\beta$ state is shown in FIG. 1(b). Our work also provides guideline for choosing variational

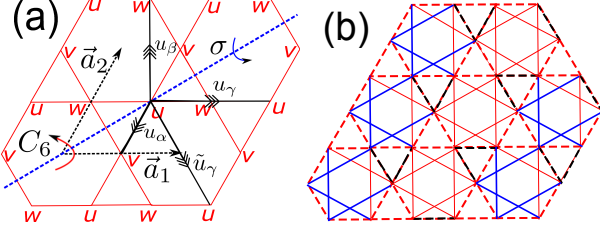


FIG. 1: (color online) (a) kagome lattice and the elements of its symmetry group. $\vec{a}_{1,2}$ are the translation unit vectors, C_6 denotes $\pi/3$ rotation around honeycomb center and σ represents mirror reflection along the dashed blue line. Here u_α and u_β denote 1st and 2nd nearest neighbor (n.n.) mean-field bonds while u_γ and \tilde{u}_γ represent two kinds of independent 3rd n.n. mean-field bonds. (b) Mean-field ansatz of $Z_2[0, \pi]\beta$ state up to 2nd nearest neighbor. Colors in general denote the sign structure of mean-field bonds. Dashed lines denote 1st n.n. real hopping terms $\chi_1 \sum_{\langle ij \rangle} (\nu_{ij} f_{i\alpha}^\dagger f_{j\alpha} + h.c.)$: red ones have $\nu_{ij} = 1$ and black ones have $\nu_{ij} = -1$. Solid lines stand for 2nd n.n. hopping $\chi_2 \sum_{\langle\langle ij \rangle\rangle} \nu_{ij} (f_{i\alpha}^\dagger f_{j\alpha} + h.c.)$ and singlet pairing $\sum_{\langle\langle ij \rangle\rangle} \epsilon_{\alpha\beta} \nu_{ij} (\Delta_2 f_{i\alpha}^\dagger f_{j\beta}^\dagger + h.c.)$: again red ones have $\nu_{ij} = 1$ and blue ones have $\nu_{ij} = -1$. Here $\chi_{1,2}$ and Δ_2 are real parameters after choosing a proper gauge.

states in future numeric studies of SL ground state on kagome lattice.

II. SCHWINGER-FERMION CONSTRUCTION OF SPIN LIQUIDS AND PROJECTIVE SYMMETRY GROUP (PSG)

A. Schwinger-fermion construction of symmetric spin liquids

In the Schwinger-fermion construction^{22–27}, we represent a spin-1/2 operator at site i by fermionic spinons $\{f_{i\alpha}, \alpha = \uparrow, \downarrow\}$:

$$\vec{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta}. \quad (2)$$

Heisenberg hamiltonian $H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$ is represented as $H = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} (f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} + \frac{1}{2} f_{i\alpha}^\dagger f_{i\alpha} f_{j\beta}^\dagger f_{j\beta})$. This construction enlarges the Hilbert space of the original spin system. To obtain the physical spin state from a mean-field state of f -spinons, we need to enforce the following one- f -spinon-per-site constraint:

$$f_{i\alpha}^\dagger f_{i\alpha} = 1, \quad f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta} = 0. \quad (3)$$

Mean-field parameters of symmetric SLs are $\Delta_{ij} \epsilon_{\alpha\beta} = -2\langle f_{i\alpha} f_{j\beta} \rangle$, $\chi_{ij} \delta_{\alpha\beta} = 2\langle f_{i\alpha}^\dagger f_{j\beta} \rangle$, where $\epsilon_{\alpha\beta}$ is the completely antisymmetric tensor. Both terms are invariant

under global $SU(2)$ spin rotations. After a Hubbard-Stratonovich transformation, the lagrangian of the spin system can be written as

$$L = \sum_i \psi_i^\dagger \partial_\tau \psi_i + \sum_{\langle ij \rangle} \frac{3}{8} J_{ij} \left[\frac{1}{2} \text{Tr}(U_{ij}^\dagger U_{ij}) - (\psi_i^\dagger U_{ij} \psi_j + h.c.) \right] + \sum_i a_0^l(i) \psi_i^\dagger \tau^l \psi_i \quad (4)$$

where two-component fermion notation $\psi_i \equiv (f_{i\uparrow}, f_{i\downarrow})$ is introduced for reasons that will be explained shortly. We use τ^0 to denote the 2×2 identity matrix and $\tau^{1,2,3}$ are the three Pauli matrices. U_{ij} is a matrix of mean-field amplitudes:

$$U_{ij} = \begin{pmatrix} \chi_{ij}^\dagger & \Delta_{ij} \\ \Delta_{ij}^\dagger & -\chi_{ij} \end{pmatrix}. \quad (5)$$

$a_0^l(i)$ are the local lagrangian multipliers that enforce the constraints Eq.(3).

In terms of ψ , Schwinger-fermion representation has an explicit $SU(2)$ gauge redundancy: a transformation $\psi_i \rightarrow W_i \psi_i$, $U_{ij} \rightarrow W_i U_{ij} W_j^\dagger$, $W_i \in SU(2)$ leaves the action invariant. This redundancy is originated from representation Eq.(2): this local $SU(2)$ transformation leaves the spin operators invariant and does not change physical Hilbert space. One can try to solve Eq.(4) by mean-field (or saddle-point) approximation. At mean-field level, U_{ij} and a_0^l are treated as complex numbers, and a_0^l must be chosen such that constraints (3) are satisfied at the mean field level: $\langle \psi_i^\dagger \tau^l \psi_i \rangle = 0$. The mean-field ansatz can be written as:

$$H_{MF} = - \sum_{\langle ij \rangle} \psi_i^\dagger \langle i|j \rangle \psi_j + \sum_i \psi_i^\dagger a_0^l \tau^l \psi_i. \quad (6)$$

where we defined $\langle i|j \rangle \equiv \frac{3}{8} J_{ij} U_{ij}$. Under a local $SU(2)$ gauge transformation $\langle i|j \rangle \rightarrow W_i \langle i|j \rangle W_j^\dagger$, but the physical spin state described by the mean-field ansatz $\{\langle i|j \rangle\}$ remains the same. By construction the mean-field ansatz does not break spin rotation symmetry, and the mean field solutions describe SL states if lattice symmetry is preserved. Different $\{\langle i|j \rangle\}$ ansatz may be in different SL phases. The mathematical language to classify different SL phases is projective symmetry group (PSG)⁸.

B. Projective symmetry group (PSG) classification of topological orders in spin liquids

PSG characterizes the topological order in Schwinger-fermion representation: SLs described by different PSGs are different phases. It is defined as the collection of all combinations of symmetry group and $SU(2)$ gauge transformations that leave mean-field ansatz $\{\langle i|j \rangle\}$ invariant (as a_0^l are determined self-consistently by $\{\langle i|j \rangle\}$, these transformations also leave a_0^l invariant). The invariance

of a mean-field ansatz $\{\langle i|j\rangle\}$ under an element of PSG $G_U U$ can be written as

$$\begin{aligned} G_U U(\{\langle i|j\rangle\}) &= \{\langle i|j\rangle\}, \\ U(\{\langle i|j\rangle\}) &\equiv \{\langle i|\tilde{j}\rangle = \langle U^{-1}(i)|U^{-1}(j)\rangle\}, \\ G_U(\{\langle i|j\rangle\}) &\equiv \{\langle i|\tilde{j}\rangle = G_U(i)\langle i|j\rangle G_U(j)^\dagger\}, \\ G_U(i) &\in SU(2). \end{aligned} \quad (7)$$

Here $U \in SG$ is an element of symmetry group (SG) of the corresponding SL. In our case of symmetric SLs on the kagome lattice, we use (x, y, s) to label a site with sublattice index $s = u, v, w$ and $x, y \in \mathbb{Z}$. Bravais unit vector are chosen as $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$ as shown in FIG. 1(a). The symmetry group is generated by time reversal operation \mathbf{T} , lattice translations $T_{1,2}$ along $\vec{a}_{1,2}$ directions, $\pi/3$ rotation C_6 around honeycomb plaquette center and the mirror reflection σ (for details see Appendix A). For example, if $U = T_1$ is the translation along \vec{a}_1 -direction in Fig.1(a), $T_1(\{x, y, s\}) = \{x+1, y, s\}$. G_U is the gauge transformation associated with U such that $G_U U$ leave $\{\langle i|j\rangle\}$ invariant. Notice this condition (7) allows us to generate all symmetry-related mean-field bonds from one by the following relation:

$$\langle i|j\rangle = G_U(i)\langle U^{-1}(i)|U^{-1}(j)\rangle G_U^\dagger(j) \quad (8)$$

There is an important subgroup of PSG, the invariant gauge group (IGG), which is composed of all the pure gauge transformations in PSG: $IGG \equiv \{\{W_i\}|W_i\langle i|j\rangle W_j^\dagger = \langle i|j\rangle, W_i \in SU(2)\}$. In other words, $W_i = G_e(i)$ is the pure gauge transformation associated with identity element $e \in SG$ of the symmetry group. One can always choose a gauge in which the elements in IGG is site-independent. In this gauge, IGG can be the global Z_2 transformations: $\{G_e(i) \equiv G_e = \pm\tau^0\}$, the global $U(1)$ transformations: $\{G_e(i) \equiv e^{i\theta\tau^3}, \theta \in [0, 2\pi]\}$, or the global $SU(2)$ transformations: $\{G_e(i) \equiv e^{i\theta\hat{n}\cdot\vec{\tau}}, \theta \in (0, 2\pi], \hat{n} \in S^2\}$, and we term them as Z_2 , $U(1)$ and $SU(2)$ state respectively.

The importance of IGG is that it controls the low energy gauge fluctuations of the corresponding SL states. Beyond mean-field level, fluctuations of $\langle i|j\rangle$ and a_0^l need to be considered and the mean-field state may or may not be stable. The low energy effective theory is described by fermionic spinon band structure coupled with a dynamical gauge field of IGG. For example, Z_2 state with gapped spinon dispersion can be a stable phase because the low energy Z_2 dynamical gauge field can be in the deconfined phase^{29,30}.

Notice that the condition $\{G_e(i) \equiv G_e = \pm\tau^0\}$ for a Z_2 SL leads to a series of consistent conditions for the gauge transformations $\{G_U(i)|U \in SG\}$, as shown in Appendix A. Gauge inequivalent solutions of these conditions (A4)-(A11) lead to different Z_2 SLs. Soon we will show that there are 20 Z_2 SLs on the kagome lattice that can be realized by a Schwinger-fermion mean-field ansatz $\{\langle i|j\rangle\}$.

III. Z_2 SPIN LIQUIDS ON THE KAGOME LATTICE AND $Z_2[0, \pi]\beta$ STATE

Following previous discussions, we use PSG to classify all possible 20 Z_2 SL states on kagome lattice in this section. As will be shown later, among them there is one gapped Z_2 SL labeled as $Z_2[0, \pi]\beta$ state in the neighborhood of $U(1)$ -Dirac SL. This $Z_2[0, \pi]\beta$ SL state is the most promising candidate for the SL ground state of HKLM.

A. PSG classification of Z_2 spin liquids on kagome lattice

Applying the condition $G_e(i) \equiv G_e = \pm\tau^0$ to kagome lattice with symmetry group described in Appendix A, we obtain a series of consistent conditions for the gauge transformation $G_U(i)$, *i.e.* conditions (A4)-(A11). Solving these conditions we classify all the 20 different Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice, as summarized in TABLE I. These 20 mean-field states correspond to different Z_2 SL phases, which cannot be continuously tuned into each other without a phase transition.

As discussed in Appendix B2, from PSG elements $G_U(i)$ one can obtain all other symmetry-related mean-field bonds from one using symmetry condition (8). Therefore we use $u_\alpha \equiv \langle 0, 0, v|0, 0, u\rangle$ to represent 1st nearest neighbor (n.n.) mean-field bonds. $u_\beta \equiv \langle 0, 1, w|0, 0, u\rangle$ is the representative of 2nd n.n. mean-field bonds. There are two kinds of symmetry-unrelated 3rd n.n. mean-field bonds, represented by $u_\gamma = \langle 1, 0, u|0, 0, u\rangle$ and $\tilde{u}_\gamma = \langle 1, -1, u|0, 0, u\rangle$. The symmetry conditions for these mean-field bonds are summarized in (B13)-(B16). Besides, the on-site chemical potential terms $\Lambda(i)$ (which guarantee the physical constraint (3) on the mean-field level) also satisfy symmetry conditions (B12). We can show that $\Lambda(x, y, s) \equiv \Lambda_s$ for these 20 Z_2 SL states. The symmetry-allowed mean-field amplitudes/bonds are also summarized in TABLE I.

From TABLE I we can see there are 6 states, *i.e.* #7 – #12 that don't allow nonzero 1st n.n. mean-field amplitudes due to symmetry. Moreover, they cannot realize Z_2 SLs with up to 3rd n.n. mean-field amplitudes. Therefore they are unlikely to be the HKLM ground state. Ruling out these six Z_2 SLs, we can see the other 14 Z_2 SL states fall into 4 classes. To be specific, they are continuously connected to different parent $U(1)$ gapless SL states on kagome lattice. These parent $U(1)$ SL states in general have the following mean-field ansatz

$$H_{U(1)SL} = \chi_1 \sum_{\langle ij \rangle} \nu_{ij} (f_{i\alpha}^\dagger f_{j\alpha} + h.c.) \quad (9)$$

where $\nu_{ij} = \pm 1$ characterizes the sign structure of hopping terms with $\chi_1 \in \mathbb{R}$. Different parent $U(1)$ SL

| # | η_{12} | Λ_s | u_α | u_β | u_γ | \tilde{u}_γ | Label | Gapped? |
|----|-------------|------------------|------------------|------------------|------------------|--------------------|--------------------|---------|
| 1 | +1 | τ^2, τ^3 | τ^2, τ^3 | τ^2, τ^3 | τ^2, τ^3 | τ^2, τ^3 | $Z_2[0,0]A$ | Yes |
| 2 | -1 | τ^2, τ^3 | τ^2, τ^3 | τ^2, τ^3 | τ^2, τ^3 | 0 | $Z_2[0,\pi]\beta$ | Yes |
| 3 | +1 | 0 | τ^2, τ^3 | 0 | 0 | 0 | $Z_2[\pi,\pi]A$ | No |
| 4 | -1 | 0 | τ^2, τ^3 | 0 | 0 | τ^2, τ^3 | $Z_2[\pi,0]A$ | No |
| 5 | +1 | τ^3 | τ^2, τ^3 | τ^3 | τ^3 | τ^3 | $Z_2[0,0]B$ | Yes |
| 6 | -1 | τ^3 | τ^2, τ^3 | τ^3 | τ^3 | τ^2 | $Z_2[0,\pi]\alpha$ | No |
| 7 | +1 | 0 | 0 | τ^2, τ^3 | 0 | 0 | - | - |
| 8 | -1 | 0 | 0 | τ^2, τ^3 | 0 | 0 | - | - |
| 9 | +1 | 0 | 0 | 0 | τ^2, τ^3 | 0 | - | - |
| 10 | -1 | 0 | 0 | 0 | τ^2, τ^3 | 0 | - | - |
| 11 | +1 | 0 | 0 | τ^2 | τ^2 | 0 | - | - |
| 12 | -1 | 0 | 0 | τ^2 | τ^2 | 0 | - | - |
| 13 | +1 | τ^3 | τ^3 | τ^2, τ^3 | τ^3 | τ^3 | $Z_2[0,0]D$ | Yes |
| 14 | -1 | τ^3 | τ^3 | τ^2, τ^3 | τ^3 | 0 | $Z_2[0,\pi]\gamma$ | No |
| 15 | +1 | τ^3 | τ^3 | τ^3 | τ^2, τ^3 | τ^3 | $Z_2[0,0]C$ | Yes |
| 16 | -1 | τ^3 | τ^3 | τ^3 | τ^2, τ^3 | 0 | $Z_2[0,\pi]\delta$ | No |
| 17 | +1 | 0 | τ^2 | τ^3 | 0 | 0 | $Z_2[\pi,\pi]B$ | No |
| 18 | -1 | 0 | τ^2 | τ^3 | 0 | τ^3 | $Z_2[\pi,0]B$ | No |
| 19 | +1 | 0 | τ^2 | 0 | τ^2 | 0 | $Z_2[\pi,\pi]C$ | No |
| 20 | -1 | 0 | τ^2 | 0 | τ^2 | τ^3 | $Z_2[\pi,0]C$ | No |

TABLE I: Mean-field ansatz of 20 possible Z_2 SLs on a kagome lattice. In our notation of mean-field amplitudes $\langle x, y, s | 0, 0, u \rangle \equiv [x, y, s]$, this table summarizes all symmetry-allowed mean-field bonds up to 3rd n.n., *i.e.* 1st n.n. bond $u_\alpha = [0, 0, v]$, 2nd n.n. bond $u_\beta = [0, 1, w]$, 3rd n.n. bonds $u_\gamma = [1, 0, u]$ and $\tilde{u}_\gamma = [1, -1, u]$ as shown in FIG. 1(a). Λ_s denote the on-site chemical potential terms which enforce the constraint (11). τ^0 is 2×2 identity matrix while $\tau^{1,2,3}$ are three Pauli matrices. $\tau^{0,3}$ denote hopping while $\tau^{1,2}$ denote pairing terms. 0 means the corresponding mean-field amplitudes must vanish due to symmetry. Red color denotes the shortest mean-field bonds necessary to realize a Z_2 SL. In other words, the mean-field amplitudes with red color break the $U(1)$ gauge redundancy down to Z_2 through Higgs mechanism. So in #3, #19 and #7–#12 states a Z_2 SL cannot be realized with up to 3rd n.n. mean-field amplitudes. Note that #15 state needs only 3rd n.n. bond u_γ to realize a Z_2 SL (\tilde{u}_γ not necessary), while state #20 needs only \tilde{u}_γ to realize a Z_2 SL (u_γ not necessary). Notice that when $\eta_{12} = -1$ the mean-field ansatz (instead of the SL itself) will break translational symmetry and double the unit cell. There are six Z_2 SLs, *i.e.* #7–#12 that don't allow any 1st n.n. mean-field bonds. Among the other 14 Z_2 SLs with nonvanishing 1st n.n. mean-field bonds, only five Z_2 SL states, *i.e.* #1, #2, #5, #13, #15 have gapped spinon spectra. #2 or $Z_2[0, \pi]\beta$ state in neighborhood of $U(1)$ -Dirac SL is the most promising candidate of Z_2 SL for the HKLM ground state.

states are featured by the flux of f -spinon hopping phases around basic plaquette: honeycombs and triangles on the kagome lattice.

The simplest example is the so-called uniform RVB state with $\nu_{ij} \equiv +1$ for all 1st n.n. mean-field bonds. The hopping phase around any plaquette is $1 = \exp[i0]$, and the corresponding flux is $[0, 0]$ for [triangle, honeycomb] motifs. The 4 possible Z_2 spin liquids in the neighborhood³¹ of uniform RVB states (*i.e.* $U(1)$ SL- $[0, 0]$ state in Ref. 2) are classified in Appendix D. They

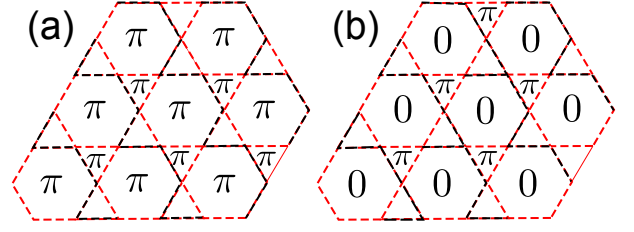


FIG. 2: (color online) Mean-field ansatz of (a) $U(1)$ SL- $[\pi, \pi]$ state and (b) $U(1)$ SL- $[\pi, 0]$ state, with 1st n.n. real hopping terms $H_{MF} = \chi_1 \sum_{\langle ij \rangle} (\nu_{ij} f_{i\alpha}^\dagger f_{j\alpha} + h.c.)$. Colors again denote the sign structure of mean-field bonds: red dashed lines have $\nu_{ij} = +1$ and black dashed lines have $\nu_{ij} = -1$.

are #1, #5, #15, #13 in TABLE I and TABLE II. We label them as $Z_2[0, 0]A$, $Z_2[0, 0]B$, $Z_2[0, 0]C$ and $Z_2[0, 0]D$ states. They all have gapped spectra of spinons.

The ansatz of two other parent $U(1)$ SLs are shown in FIG. 2. They both have π -flux piercing through a triangle basic plaquette. Following the above notations of hopping phase in [triangle, honeycomb] motifs, with either π -flux or 0-flux through the honeycomb plaquette, they are called $U(1)$ SL- $[\pi, \pi]$ state and $U(1)$ SL- $[\pi, 0]$ state. There are three Z_2 SLs in the neighborhood of both $U(1)$ SL states, *i.e.* #3, #17, #19 around $U(1)$ SL- $[\pi, \pi]$ state and #4, #18, #20 around $U(1)$ SL- $[\pi, 0]$ state. All these six Z_2 SLs have gapless spinon spectra, inherited from the two parent gapless $U(1)$ SLs. To be precise, the spinon band structure of these six Z_2 SL states are featured by a doubly-degenerate flat band and a Dirac cone at Brillouin-zone center. This is in contrast to the numerically observed gap in two-spinon spectrum¹, thus we can also rule out these 6 Z_2 SLs for the HKLM ground state.

Another $U(1)$ SL state is the so called $U(1)$ -Dirac SL or $U(1)$ SL- $[0, \pi]$ state. Its mean-field ansatz is shown by the 1st n.n. bonds in FIG. 1(b). Clearly π -flux pierces through certain triangle plaquette with no flux through the honeycomb plaquette. According to variational Monte Carlo studies^{2,32}, this $U(1)$ -Dirac SL have substantially lower energy compared to many other competing phases, including the uniform RVB state. Therefore we shall focus on those Z_2 SLs in the neighborhood of the $U(1)$ -Dirac SL in our search of the HKLM ground state. We need to mention that although unlikely, the four Z_2 SLs in the neighborhood of uniform RVB state, or $U(1)$ SL- $[0, 0]$ state are potentially possible to be the HKLM ground state.

In a previous study using PSG in Schwinger-boson representation²¹, it was shown that there are 8 different Schwinger-boson mean-field ansatz of Z_2 SLs on the kagome lattice which preserve all lattice symmetry. However, these 8 Z_2 SLs may or may not preserve time-reversal symmetry. One can show that requiring all lattice symmetry and time-reversal symmetry, there are 16 different Schwinger-boson Z_2 SLs on the kagome lattice. The relation between the 20 Z_2 SLs in Schwinger-

fermion representation (see TABLE I) and the 16 Z_2 SLs in Schwinger-boson representation are not clear. To clarify the relation between SL states in these two different representations, one can compare the neighboring (ordered) phases of the SLs, *e.g.* by computing the vison quantum numbers³⁴ of SL states.

B. $Z_2[0, \pi]\beta$ state as a promising candidate for the HKLM ground state

How to find those Z_2 SLs in the neighborhood of (or continuously connected to) the $U(1)$ -Dirac SL? Naively, we expect the mean-field ansatz of these Z_2 SLs can be obtained from that of $U(1)$ -Dirac SL by adding an infinitesimal perturbation. To be specific, we require an infinitesimal spinon pairing term on top of the $U(1)$ -Dirac SL mean-field ansatz (9) or (C1) to break the IGG from $U(1)$ to Z_2 through Higgs mechanism. Mathematically, we need to find those Z_2 SL states whose PSG is a subgroup of the $U(1)$ -Dirac SL's PSG³¹. Such Z_2 SL states are defined to be in the neighborhood of $U(1)$ -Dirac SL. Similar criterion applies to the neighboring Z_2 SL states of any parent $U(1)$ or $SU(2)$ SL state.

We find out all four Z_2 SLs in the neighborhood of $U(1)$ -Dirac SLs in Appendix C. They are states #6, #2, #14, #16 in TABLE I, labeled as $Z_2[0, \pi]\alpha$, $Z_2[0, \pi]\beta$, $Z_2[0, \pi]\gamma$ and $Z_2[0, \pi]\delta$ states respectively. Since the effective theory of a $U(1)$ -Dirac SL is an 8-component Dirac fermion coupled with dynamical $U(1)$ gauge field^{2,33}, we can find out all symmetry-allowed mass terms that can open up a gap in the Dirac-like spinon spectrum. Following detailed calculations in Appendix C, we can see that among the four Z_2 SLs around the $U(1)$ -Dirac SL, only one state, *i.e.* $Z_2[0, \pi]\beta$ (state #2 in TABLE I and II) can generate a mass gap in the spinon spectrum. In other 3 states the Dirac cone in spinon spectrum is protected by symmetry. The mean-field ansatz of $Z_2[0, \pi]\beta$ SL state up to 2nd n.n. is shown in FIG. 1(b):

$$H_{MF} = \sum_i (\lambda_3 \sum_\alpha f_{i\alpha}^\dagger f_{i\alpha} + \lambda_1 f_{i\uparrow}^\dagger f_{i\downarrow}^\dagger + h.c.) \quad (10) \\ + \chi_1 \sum_{\langle ij \rangle \alpha} \nu_{ij} (f_{i\alpha}^\dagger f_{j\alpha} + h.c.) + \\ \sum_{\langle\langle ij \rangle\rangle} \nu_{ij} (\chi_2 \sum_\alpha f_{i\alpha}^\dagger f_{j\alpha} + \Delta_2 \sum_{\alpha\beta} \epsilon^{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta} + h.c.)$$

where $\epsilon^{\alpha\beta}$ is the completely anti-symmetric tensor. We only list up to 2nd n.n. mean-field amplitudes because as shown in TABLE I (see also Appendix C), this $Z_2[0, \pi]\beta$ state only needs 2nd n.n. pairing terms to realize a Z_2 SL. We can always choose a proper gauge so that mean-field parameters $\chi_{1,2}$ and Δ_2 are all real. The sign structure of $\nu_{ij} = \pm 1$ are shown in FIG. 1(b), with red denoting $\nu_{ij} = +1$ and other colors representing $\nu_{ij} = -1$. As discussed in Appendix C, the 2nd n.n. singlet-pairing term $\Delta_2 \neq 0$ not only break the $U(1)$ gauge symmetry down to Z_2 , but also opens up a mass gap in the spinon spectrum. The on-site chemical potential $\lambda_{1,3}$ are self-

consistently determined by the following constraint:

$$\sum_i \langle f_{i\uparrow}^\dagger f_{i\downarrow}^\dagger \rangle = \sum_i \langle f_{i\uparrow} f_{i\downarrow} \rangle = 0, \\ \sum_i (\sum_{\alpha=\uparrow, \downarrow} f_{i\alpha}^\dagger f_{i\alpha} - 1) = 0. \quad (11)$$

For further n.n. mean-field ansatz see discussions in Appendix C.

IV. CONCLUSION

To summarize, motivated by the strong evidence of a Z_2 SL as the HKLM ground state in recent DMRG study¹, we classify all possible Z_2 SL states in Schwinger-fermion mean-field approach using PSG. We found 20 different Schwinger-fermion mean-field states of Z_2 SLs on kagome lattice, among which 6 states are unlikely due to vanishing 1st n.n. mean-field amplitude. In other 14 Z_2 SLs only 5 possess a gapped spinon spectrum, which is observed in the DMRG result¹. These five symmetric Z_2 SL states are all in the neighborhood of certain parent $U(1)$ gapless SLs. To be precise, four are in the neighborhood of gapless uniform RVB (or $U(1)$ SL-[0, 0]) state, while the other one, *i.e.* $Z_2[0, \pi]\beta$ is in the neighborhood of gapless $U(1)$ -Dirac SL (or $U(1)$ SL-[0, π]) state. Previous variational Monte Carlo study² showed that gapless $U(1)$ -Dirac SL has a substantially lower energy in comparison to the uniform RVB state. This suggests Z_2 SLs in the neighborhood of $U(1)$ -Dirac SL should have lower energy compared to those in the neighborhood of uniform RVB state. Therefore we propose this $Z_2[0, \pi]\beta$ state with mean-field ansatz (10) shown in FIG. 1(b) as the HKLM ground state numerically detected in Ref. 1. Our work provides important insight for future numeric study, *e.g.* variational Monte Carlo study of Gutzwiller projected wavefunctions.

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Appendix A: Symmetry group of kagome lattice and algebra conditions for Z_2 spin liquids

As shown in FIG. 1(a), we label the three lattice sites in each unit cell with sublattice index $\{s = u, v, w\}$. Choosing Bravais unit vector as $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$, the positions of the three atoms in a unit cell labeled by indices $i = (x, y, s)$ are

$$\vec{r}(x, y, u) = (x + \frac{1}{2})\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2, \quad (A1) \\ \vec{r}(x, y, v) = (x + \frac{1}{2})\vec{a}_1 + y\vec{a}_2, \\ \vec{r}(x, y, w) = x\vec{a}_1 + (y + \frac{1}{2})\vec{a}_2.$$

The symmetry group of such a two-dimensional kagome lattice is generated by the following operations

$$\begin{aligned}
T_1 : & (x, y, s) \rightarrow (x+1, y, s); \\
T_2 : & (x, y, s) \rightarrow (x, y+1, s); \\
\sigma : & (x, y, u) \rightarrow (y, x, u), \\
& (x, y, v) \rightarrow (y, x, w), \\
& (x, y, w) \rightarrow (y, x, v); \\
C_6 : & (x, y, u) \rightarrow (-y-1, x+y+1, v), \\
& (x, y, v) \rightarrow (-y, x+y, w), \\
& (x, y, w) \rightarrow (-y-1, x+y, u).
\end{aligned} \tag{A2}$$

together with time reversal \mathbf{T} .

The symmetry group of a kagome lattice is defined by the following algebraic relations between its generators:

$$\begin{aligned}
\mathbf{T}^2 = \sigma^2 = (C_6)^6 = \mathbf{e}, \\
g^{-1}\mathbf{T}^{-1}g\mathbf{T} = \mathbf{e}, \quad \forall g = T_{1,2}, \sigma, C_6, \\
T_2^{-1}T_1^{-1}T_2T_1 = \mathbf{e}, \\
\sigma^{-1}T_1^{-1}\sigma T_2 = \mathbf{e}, \\
\sigma^{-1}T_2^{-1}\sigma T_1 = \mathbf{e}, \\
C_6^{-1}T_2^{-1}C_6T_1 = \mathbf{e}, \\
C_6^{-1}T_2^{-1}T_1C_6T_2 = \mathbf{e}, \\
\sigma^{-1}C_6\sigma C_6 = \mathbf{e}.
\end{aligned} \tag{A3}$$

where \mathbf{e} stands for the identity element in the symmetry group. Therefore the consistent conditions for a generic Z_2 PSGs on a kagome lattice is written as

$$[G_{\mathbf{T}}(i)]^2 = \eta_{\mathbf{T}}\tau^0, \tag{A4}$$

$$G_{\sigma}(\sigma(i))G_{\sigma}(i) = \eta_{\sigma}\tau^0, \tag{A5}$$

$$G_{T_1}^{\dagger}(i)G_{\mathbf{T}}^{\dagger}(i)G_{T_1}(i)G_{\mathbf{T}}(T_1^{-1}(i)) = \eta_{T_1}\tau^0, \tag{A6}$$

$$G_{T_2}^{\dagger}(i)G_{\mathbf{T}}^{\dagger}(i)G_{T_2}(i)G_{\mathbf{T}}(T_2^{-1}(i)) = \eta_{T_2}\tau^0, \tag{A7}$$

$$G_{\sigma}^{\dagger}(i)G_{\mathbf{T}}^{\dagger}(i)G_{\sigma}(i)G_{\mathbf{T}}(\sigma^{-1}(i)) = \eta_{\sigma}\tau^0, \tag{A8}$$

$$G_{C_6}^{\dagger}(i)G_{\mathbf{T}}^{\dagger}(i)G_{C_6}(i)G_{\mathbf{T}}(C_6^{-1}(i)) = \eta_{C_6}\tau^0, \tag{A9}$$

$$G_{T_2}^{\dagger}(T_1^{-1}(i))G_{T_1}^{\dagger}(i)G_{T_2}(i)G_{T_1}(T_2^{-1}(i)) = \eta_{12}\tau^0 \tag{A10}$$

$$G_{C_6}(C_6^{-1}(i))G_{C_6}(C_6^{-2}(i))G_{C_6}(C_6^{-3}(i))G_{C_6}(C_6^{-4}(i)) \cdot$$

$$G_{C_6}(C_6^{-5}(i))G_{C_6}(C_6(i))G_{C_6}(i) = \eta_{C_6}\tau^0, \tag{A11}$$

$$G_{\sigma}^{\dagger}(T_2^{-1}(i))G_{T_2}^{\dagger}(i)G_{\sigma}(i)G_{T_1}(\sigma(i)) = \eta_{\sigma T_1}\tau^0, \tag{A12}$$

$$G_{\sigma}^{\dagger}(T_1^{-1}(i))G_{T_1}^{\dagger}(i)G_{\sigma}(i)G_{T_2}(\sigma(i)) = \eta_{\sigma T_2}\tau^0, \tag{A13}$$

$$G_{\sigma}^{\dagger}(C_6(i))G_{C_6}(C_6(i))G_{\sigma}(i)G_{C_6}(\sigma(i)) = \eta_{\sigma C_6}\tau^0 \tag{A14}$$

$$G_{C_6}^{\dagger}(T_2^{-1}(i))G_{T_2}^{\dagger}(i)G_{C_6}(i)G_{T_1}(C_6^{-1}(i)) = \eta_{C_6 T_1}\tau^0 \tag{A15}$$

$$G_{C_6}^{\dagger}(T_2^{-1}T_1(i))G_{T_2}^{\dagger}(T_1(i))G_{T_1}(T_1(i)) \cdot$$

$$G_{C_6}(i)G_{T_2}(C_6^{-1}(i)) = \eta_{C_6 T_2}\tau^0. \tag{A16}$$

for any lattice site $i = (x, y, s)$. Here all η s are Z_2 integers characterizing different SLs: different (gauge inequivalent) choices of these Z_2 integers (different Z_2 PSGs) correspond to different Z_2 SLs. Notice that under a local

gauge transformation $W(i) \in SU(2)$ the PSG element $G_U(i)$ transforms as

$$G_U(i) \rightarrow W(i)G_U(i)W^{\dagger}(U^{-1}(i)) \tag{A17}$$

Appendix B: Classification of all Z_2 spin liquids on kagome lattice

1. Classification of Z_2 algebraic PSGs on kagome lattice

In this section we classify all possible Z_2 spin liquids on a kagome lattice. Mathematically we need to find out all gauge-inequivalent solutions of algebraic conditions (A4)-(A15) for Z_2 PSGs.

First from condition (A10) we can always choose a proper gauge so that

$$G_{T_1}(x, y, s) = \eta_{12}^y \tau^0, \quad G_{T_2}(x, y, s) \equiv \tau^0. \tag{B1}$$

From (A12) and (A13) we can see $G_{\sigma}(x, y, s) = \eta_{\sigma T_1}^y \eta_{\sigma T_2}^x \eta_{12}^{xy} g_{\sigma}(s)$. Condition (A5) further determine $\eta_{\sigma T_1} = \eta_{\sigma T_2}$ and therefore we have

$$G_{\sigma}(x, y, s) = \eta_{\sigma T_1}^{x+y} \eta_{12}^{xy} g_{\sigma}(s)$$

where $SU(2)$ matrices $g_{\sigma}(s)$ satisfy

$$g_{\sigma}(w)g_{\sigma}(v) = [g_{\sigma}(u)]^2 = \eta_{\sigma}\tau^0 \tag{B2}$$

Notice that we can always choose a proper global Z_2 gauge on $G_{T_1}(x, y, s)$ (which doesn't change the mean-field ansatz) so that $\eta_{C_6 T_2} = 1$ in (A16). From (A15) and (A16) it's straightforward to show that $G_{C_6}(x, y, u/v) = \eta_{C_6 T_1}^{x+y} \eta_{12}^{xy+x(x+1)/2} g_{C_6}(u/v)$ and $G_{C_6}(x, y, w) = \eta_{C_6 T_1}^{x+y} \eta_{12}^{xy+xy+x(x+1)/2} g_{C_6}(w)$. It's condition (A14) that determines $\eta_{C_6 T_1} = \eta_{\sigma T_1} \eta_{12}$ and finally we have

$$G_{C_6}(x, y, u/v) = \eta_{\sigma T_1}^{x+y} \eta_{12}^{xy+\frac{x(x+1)}{2}} g_{C_6}(u/v),$$

$$G_{C_6}(x, y, w) = (\eta_{12} \eta_{\sigma T_1})^{x+y} \eta_{12}^{xy+\frac{x(x+1)}{2}} g_{C_6}(w).$$

where $SU(2)$ matrices $g_{C_6}(s)$ satisfy

$$[g_{C_6}(w)g_{C_6}(v)g_{C_6}(u)]^2 = \eta_{12}\eta_{C_6}\tau^0, \tag{B3}$$

$$[g_{\sigma}(v)g_{C_6}(w)]^2 = g_{\sigma}(w)g_{C_6}(v)g_{\sigma}(u)g_{C_6}(u) = \eta_{\sigma}\eta_{\sigma C_6}\tau^0. \tag{B4}$$

according to (A11) and (A14).

Now through a gauge transformation $W(x, y, s) = \eta_{\sigma T_1}^y$ we can fix $\eta_{\sigma T_1,2} = 1$ and the PSG elements become

$$G_{\sigma}(x, y, s) = \eta_{12}^{xy} g_{\sigma}(s); \tag{B5}$$

$$G_{C_6}(x, y, u/v) = \eta_{12}^{xy+\frac{x(x+1)}{2}} g_{C_6}(u/v),$$

$$G_{C_6}(x, y, w) = \eta_{12}^{xy+xy+\frac{x(x+1)}{2}} g_{C_6}(w). \tag{B6}$$

| # | η_{12} | $g_{\sigma}(u)$ | $g_{\sigma}(v)$ | $g_{\sigma}(w)$ | $g_{C_6}(u)$ | $g_{C_6}(v)$ | $g_{C_6}(w)$ | Label |
|----|-------------|-----------------|-----------------|-----------------|--------------|--------------|--------------|--------------------|
| 1 | +1 | τ^0 | τ^0 | τ^0 | τ^0 | τ^0 | τ^0 | $Z_2[0,0]A$ |
| 2 | -1 | τ^0 | τ^0 | τ^0 | τ^0 | τ^0 | τ^0 | $Z_2[0,\pi]\beta$ |
| 3 | +1 | τ^0 | τ^0 | τ^0 | τ^0 | $-\tau^0$ | $i\tau^1$ | $Z_2[\pi,\pi]A$ |
| 4 | -1 | τ^0 | τ^0 | τ^0 | τ^0 | $-\tau^0$ | $i\tau^1$ | $Z_2[\pi,0]A$ |
| 5 | +1 | τ^0 | τ^0 | τ^0 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $Z_2[0,0]B$ |
| 6 | -1 | τ^0 | τ^0 | τ^0 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $Z_2[0,\pi]\alpha$ |
| 7 | +1 | $i\tau^1$ | τ^0 | $-\tau^0$ | τ^0 | $i\tau^1$ | τ^0 | - |
| 8 | -1 | $i\tau^1$ | τ^0 | $-\tau^0$ | τ^0 | $i\tau^1$ | τ^0 | - |
| 9 | +1 | $i\tau^1$ | τ^0 | $-\tau^0$ | τ^0 | $-i\tau^1$ | $i\tau^1$ | - |
| 10 | -1 | $i\tau^1$ | τ^0 | $-\tau^0$ | τ^0 | $-i\tau^1$ | $i\tau^1$ | - |
| 11 | +1 | $i\tau^1$ | τ^0 | $-\tau^0$ | $i\tau^3$ | $-i\tau^2$ | $i\tau^3$ | - |
| 12 | -1 | $i\tau^1$ | τ^0 | $-\tau^0$ | $i\tau^3$ | $-i\tau^2$ | $i\tau^3$ | - |
| 13 | +1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $Z_2[0,0]D$ |
| 14 | -1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $Z_2[0,\pi]\gamma$ |
| 15 | +1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | τ^0 | τ^0 | τ^0 | $Z_2[0,0]C$ |
| 16 | -1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | τ^0 | τ^0 | τ^0 | $Z_2[0,\pi]\delta$ |
| 17 | +1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | τ^0 | τ^0 | $i\tau^1$ | $Z_2[\pi,\pi]B$ |
| 18 | -1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | τ^0 | τ^0 | $i\tau^1$ | $Z_2[\pi,0]B$ |
| 19 | +1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $-i\tau^3$ | $i\tau^2$ | $Z_2[\pi,\pi]C$ |
| 20 | -1 | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $i\tau^3$ | $-i\tau^3$ | $i\tau^2$ | $Z_2[\pi,0]C$ |

TABLE II: A summary of all 20 gauge-inequivalent PSG's with $G_{\mathbf{T}}(x, y, s) = i\tau^1$ on the kagome lattice. Notice that there is a free Z_2 integer $\eta_{12} = \pm 1$ in other PSG elements (B1), (B5) and (B6). They correspond to 20 different Z_2 spin liquids on the kagome lattice.

According to (A4), (A6) and (A7) we can see that $G_{\mathbf{T}}(x, y, s) = \eta_{T_1\mathbf{T}}^x \eta_{T_2\mathbf{T}}^y g_{\mathbf{T}}(s)$. (A9) and (A8) further determines $\eta_{T_1\mathbf{T}} = \eta_{T_2\mathbf{T}} = 1$ and by choosing a proper gauge we have

$$G_{\mathbf{T}}(x, y, s) = g_{\mathbf{T}}(s) \equiv \begin{cases} \tau^0, & \eta_{\mathbf{T}} = 1. \\ i\tau^1, & \eta_{\mathbf{T}} = -1. \end{cases} \quad (\text{B7})$$

which satisfy

$$g_{\sigma}(u)g_{\mathbf{T}}(u) = \eta_{\sigma\mathbf{T}}g_{\mathbf{T}}(u)g_{\sigma}(u), \quad (\text{B8})$$

$$g_{\sigma}(v)g_{\mathbf{T}}(w) = \eta_{\sigma\mathbf{T}}g_{\mathbf{T}}(v)g_{\sigma}(v),$$

$$g_{\sigma}(w)g_{\mathbf{T}}(v) = \eta_{\sigma\mathbf{T}}g_{\mathbf{T}}(w)g_{\sigma}(w);$$

$$g_{C_6}(u)g_{\mathbf{T}}(w) = \eta_{C_6\mathbf{T}}g_{\mathbf{T}}(u)g_{C_6}(u), \quad (\text{B9})$$

$$g_{C_6}(v)g_{\mathbf{T}}(u) = \eta_{C_6\mathbf{T}}g_{\mathbf{T}}(v)g_{C_6}(v),$$

$$g_{C_6}(w)g_{\mathbf{T}}(v) = \eta_{C_6\mathbf{T}}g_{\mathbf{T}}(w)g_{C_6}(w).$$

according to (A9) and (A8).

In the following we find out all the gauge-inequivalent solutions of $SU(2)$ matrices $g_{\mathbf{T},\sigma,C_6}(s)$ satisfying the above conditions. They are summarized in TABLE .

(I) $g_{\mathbf{T}}(s) = \tau^0$ and therefore $\eta_{\mathbf{T}} = \eta_{\sigma\mathbf{T}} = \eta_{C_6\mathbf{T}} = 1$:

Conditions (B8) and (B9) are automatically satisfied.

(i) $\eta_{\sigma} = 1$:

Notice that under a global gauge transformation $W(x, y, s) \equiv W_s \in SU(2)$ the PSG elements transform

as

$$\begin{aligned} g_{\sigma}(u) &\rightarrow W_u g_{\sigma}(u) W_u^{\dagger}, \\ g_{\sigma}(v) &\rightarrow W_v g_{\sigma}(v) W_v^{\dagger}, \\ g_{\sigma}(w) &\rightarrow W_w g_{\sigma}(w) W_w^{\dagger}; \\ g_{C_6}(u) &\rightarrow W_u g_{C_6}(u) W_u^{\dagger}, \\ g_{C_6}(v) &\rightarrow W_v g_{C_6}(v) W_v^{\dagger}, \\ g_{C_6}(w) &\rightarrow W_w g_{C_6}(w) W_w^{\dagger}. \end{aligned}$$

Thus from (B2) and (B4) we can always have $g_{\sigma}(s) = \tau^0$ and $g_{C_6}(u) = \tau^0$, $g_{C_6}(v) = \eta_{\sigma C_6} \tau^0$ by choosing a proper gauge.

(A) $\eta_{\sigma C_6} = \eta_{12} \eta_{C_6} = 1$:

from (B3) we have $g_{C_6}(w) = \tau^0$.

(B) $\eta_{\sigma C_6} = \eta_{12} \eta_{C_6} = -1$:

from (B3) we have $g_{C_6}(w) = i\tau^3$ by gauge fixing.

(ii) $\eta_{\sigma} = -1$:

from (B2) we have $g_{\sigma}(v) = -g_{\sigma}(w) = \tau^0$ and $g_{\sigma}(u) = i\tau^3$ by gauge fixing. Also from (B4) we can choose a gauge so that $g_{C_6}(u) = \tau^0$ and $g_{C_6}(v) = -i\eta_{\sigma C_6} \tau^3$.

(A) $\eta_{\sigma C_6} = -1$:

In this case (B4) requires $g_{C_6}(w) = \tau^0$ and thus $\eta_{12} \eta_{C_6} = -1$ according to (B3).

(B) $\eta_{\sigma C_6} = 1$:

(a) $\eta_{12} \eta_{C_6} = -1$:

Now from (B4) and (B3) we have $g_{C_6}(w) = i\tau^1$ by gauge fixing.

(b) $\eta_{12} \eta_{C_6} = 1$:

by (B4) and (B3) we must have $g_{C_6}(w) = i\tau^3$.

To summarize there are $2 \times (2 + 3) = 10$ different algebraic PSGs with $\eta_{\mathbf{T}} = 1$ and $g_{\mathbf{T}}(s) = \tau^0$.

(II) $g_{\mathbf{T}}(s) = i\tau^1$ and $\eta_{\mathbf{T}} = -1$:

(i) $\eta_{\sigma} = 1$:

According to (B2) and (B8), by choosing a proper gauge we can have $g_{\sigma}(s) = \tau^0$ and $\eta_{\sigma\mathbf{T}} = 1$. From (B3) and (B4) we also have $[g_{C_6}(w)]^2 = g_{C_6}(v)g_{C_6}(u) = \eta_{\sigma C_6} \tau^0 = \eta_{12} \eta_{C_6} \tau^0$.

(A) $\eta_{12} \eta_{C_6} = \eta_{\sigma C_6} = 1$:

From (B9), (B3) and (B4), by choosing gauge we have $g_{C_6}(s) = \tau^0$ and $\eta_{C_6\mathbf{T}} = 1$.

(B) $\eta_{12} \eta_{C_6} = \eta_{\sigma C_6} = -1$:

(a) $\eta_{C_6\mathbf{T}} = 1$:

In this case we have $g_{C_6}(u) = -g_{C_6}(v) = \tau^0$ and $g_{C_6}(w) = i\tau^1$ by choosing a proper gauge.

(b) $\eta_{C_6\mathbf{T}} = -1$:

In this case we can have $g_{C_6}(s) = i\tau^3$ by choosing a proper gauge.

(ii) $\eta_{\sigma} = -1$:

(A) $\eta_{\sigma\mathbf{T}} = 1$:

From (B8) and (B2) we have $g_{\sigma}(u) = i\tau^1$ and $g_{\sigma}(v) = -g_{\sigma}(w) = \tau^0$ by proper gauge fixing. Also from (B4) we know $[g_{C_6}(w)]^2 = -\eta_{\sigma C_6} \tau^0$ and $g_{C_6}(u)g_{C_6}(v) = -i\eta_{\sigma C_6} \tau^1$.

(a) $\eta_{\sigma C_6} = -1$:

from (B9) and (B4), (B3) it's clear that $\eta_{C_6}\mathbf{T} = 1$, $g_{C_6}(u) = g_{C_6}(w) = \tau^0$ and $g_{C_6}(v) = i\tau^1$ through gauge fixing. Also we have $\eta_{12}\eta_{C_6} = -1$.

(b) $\eta_{\sigma C_6} = 1$:

(b1) $\eta_{C_6}\mathbf{T} = 1$:

In this case $\eta_{12}\eta_{C_6} = 1$, and we can always choose a proper gauge so that $g_{C_6}(u) = \tau^0$, $g_{C_6}(w) = -g_{C_6}(v) = i\tau^1$.

(b2) $\eta_{C_6}\mathbf{T} = -1$:

In this case $\eta_{12}\eta_{C_6} = -1$, and we can always choose a proper gauge so that $g_{C_6}(v) = -i\tau^2$, $g_{C_6}(u) = g_{C_6}(w) = i\tau^3$.

(B) $\eta_{\sigma}\mathbf{T} = -1$:

Conditions (B8) and (B2) assert that $g_{\sigma}(s) = i\tau^3$ by proper gauge choosing.

(a) $\eta_{\sigma C_6} = -1$:

In this case from (B4) we know $g_{C_6}(w) = i\tau^3$, hence $\eta_{C_6}\mathbf{T} = -1$. Then we can always choose a gauge so that $g_{C_6}(u) = g_{C_6}(v) = i\tau^3$ and so $\eta_{12}\eta_{C_6} = -1$ from (B3).

(b) $\eta_{\sigma C_6} = 1$:

(b1) $\eta_{C_6}\mathbf{T} = 1$:

In this case from (B8),(B4) we have $g_{C_6}(u) = g_{C_6}(v) = \tau^0$ by a proper gauge choice. Meanwhile conditions (B3) and (B4) become $[g_{C_6}(w)]^2 = \eta_{12}\eta_{C_6}\tau^0$ and $[i\tau^3 g_{C_6}(w)]^2 = -\tau^0$.

(b.1.1) $\eta_{12}\eta_{C_6} = 1$:

here we have $g_{C_6}(w) = \tau^0$.

(b.1.2) $\eta_{12}\eta_{C_6} = -1$:

here we have $g_{C_6}(w) = i\tau^1$.

(b2) $\eta_{C_6}\mathbf{T} = -1$:

In this case from (B8) and (B4) we can always choose a proper gauge so that $g_{C_6}(u) = -g_{C_6}(v) = i\tau^3$. We also have $g_{C_6}(w) = i\tau^2$ and $\eta_{12}\eta_{C_6} = -1$ from (B3).

To summarize there are $2 \times (3 + 7) = 20$ different algebraic PSGs with $\eta_{\mathbf{T}} = -1$ and $g_{\mathbf{T}}(s) = i\tau^1$.

So in summary we have $10 + 20 = 30$ different Z_2 algebraic PSGs satisfying conditions (A4)-(A16). Among them there are at most 20 solutions that can be realized by a mean-field ansatz, since those PSGs with $g_{\mathbf{T}}(s) = \tau^0$ would require all mean-field bonds to vanish due to (B11). As a result there are 20 different Z_2 spin liquids on a kagome lattice.

2. Symmetry conditions on mean-field ansatz

Let's denote the mean-field bonds connecting sites $(0, 0, u)$ and (x, y, s) as $[x, y, s] \equiv \langle x, y, s | 0, 0, u \rangle$. Using (8) we can generate any other mean-field bonds through symmetry operations (such as translations $G_{T_{1,2}}T_{1,2}$ and mirror reflection $G_{\sigma}\sigma$) from $[x, y, s]$. However these mean-field bonds cannot be chosen arbitrarily since they possess symmetry relation (8):

$$\langle i | j \rangle = G_U(i) \langle U^{-1}(i) | U^{-1}(j) \rangle G_U^\dagger(j) \quad (\text{B10})$$

where U is any element in the symmetry group. Notice that for time reversal \mathbf{T} we have

$$G_{\mathbf{T}}(i) \langle i | j \rangle G_{\mathbf{T}}^\dagger(j) = -\langle i | j \rangle \quad (\text{B11})$$

We summarize these symmetry conditions on the mean-field bonds here:

(i) For $s = u$

$$\begin{aligned} \mathbf{T} : \quad g_{\mathbf{T}}[x, y, u] g_{\mathbf{T}}^\dagger &= -[x, y, u], \\ T_1^x T_2^{-x} \sigma : \quad [x, -x, u] &\rightarrow [x, -x, u]^\dagger, \\ T_1^{x+1} T_2^{y+1} C_6^3 : \quad [x, y, u] &\rightarrow [x, y, u]^\dagger, \\ \sigma : \quad [x, x, u] &\rightarrow [x, x, u]. \end{aligned}$$

(ii) For $s = v$

$$\begin{aligned} \mathbf{T} : \quad g_{\mathbf{T}}[x, y, v] g_{\mathbf{T}}^\dagger &= [x, y, v], \\ T_2^{y+1} \sigma C_6^2 : \quad [0, y, v] &\rightarrow [0, y, v]^\dagger, \\ T_1^{2-2y} T_2^{y-1} \sigma C_6^{-1} : \quad [1-2y, y, v] &\rightarrow [1-2y, y, v]^\dagger. \end{aligned}$$

(iii) For $s = w$

$$\begin{aligned} \mathbf{T} : \quad g_{\mathbf{T}}[x, y, w] g_{\mathbf{T}}^\dagger &= [x, y, w], \\ T_1^{x-1} T_2^{2-2x} \sigma C_6 : \quad [x, 1-2x, w] &\rightarrow [x, 1-2x, w]^\dagger, \\ T_1^{x+1} \sigma C_6^{-2} : \quad [x, 0, w] &\rightarrow [x, 0, w]^\dagger. \end{aligned}$$

Now let's consider several simplest examples. At first, on-site chemical potential terms $\Lambda(x, y, s) = \Lambda_s$ satisfy the following consistent conditions:

$$\begin{aligned} \tau^1 \Lambda_s \tau^1 &= -\Lambda_s; \\ g_{\sigma}(u) \Lambda_u g_{\sigma}^\dagger(u) &= \Lambda_u, \\ g_{\sigma}(v) \Lambda_v g_{\sigma}^\dagger(v) &= \Lambda_v, \\ g_{\sigma}(w) \Lambda_w g_{\sigma}^\dagger(w) &= \Lambda_w; \\ g_{C_6}(u) \Lambda_u g_{C_6}^\dagger(u) &= \Lambda_u, \\ g_{C_6}(v) \Lambda_v g_{C_6}^\dagger(v) &= \Lambda_v, \\ g_{C_6}(w) \Lambda_w g_{C_6}^\dagger(w) &= \Lambda_w. \end{aligned} \quad (\text{B12})$$

In fact in all 20 Z_2 spin on a kagome lattice we all have $\Lambda_u = \Lambda_v = \Lambda_w \equiv \Lambda_s$ with a proper gauge choice.

All the 1st n.n. mean-field bonds can be generated from $u_\alpha \equiv [0, 0, v]$. For a generic Z_2 spin liquid with PSG elements $G_{\mathbf{T}}(x, y, s) = i\tau^1$ and (B1)(B5)(B6), the bond $u_\alpha = [0, 0, v]$ satisfies the following consistent conditions:

$$\begin{aligned} \tau^1 u_\alpha \tau^1 &= -u_\alpha, \\ g_{\sigma}(u) g_{C_6}(u) g_{C_6}(w) u_\alpha g_{C_6}^\dagger(v) g_{C_6}^\dagger(w) g_{\sigma}^\dagger(v) &= u_\alpha^\dagger. \end{aligned} \quad (\text{B13})$$

It follows immediately that for six Z_2 spin liquids, *i.e.* #7–12 in TABLE II all n.n. mean-field bonds must vanish since $u_\alpha = 0$ as required by (B13). Therefore it's unlikely that the Z_2 spin liquid realized in kagome Hubbard model would be one of these 6 states. In the following we study the rest 14 Z_2 spin liquids on the kagome lattice.

All 2nd n.n. mean-field bonds can be generated from $u_\beta \equiv [0, 1, w]$ which satisfies the following symmetry conditions

$$\begin{aligned}\tau^1 u_\beta \tau^1 &= -u_\beta, \\ g_\sigma(u) g_{C_6}(u) u_\beta g_{C_6}^\dagger(v) g_\sigma^\dagger(w) &= u_\beta^\dagger.\end{aligned}\quad (\text{B14})$$

There are two kinds of 3rd n.n. mean-field bonds: the first kind can all be generated by $u_\gamma \equiv [1, 0, u]$ which satisfies

$$\begin{aligned}\tau^1 u_\gamma \tau^1 &= -u_\gamma, \\ g_{C_6}(u) g_{C_6}(v) g_{C_6}(w) u_\gamma [g_{C_6}(u) g_{C_6}(v) g_{C_6}(w)]^\dagger &= u_\gamma^\dagger.\end{aligned}\quad (\text{B15})$$

the second kind can all be generated by $\tilde{u}_\gamma \equiv [1, -1, u]$ which satisfies

$$\begin{aligned}\tau^1 \tilde{u}_\gamma \tau^1 &= -\tilde{u}_\gamma, \\ g_\sigma(u) \tilde{u}_\gamma g_\sigma^\dagger(u) &= \tilde{u}_\gamma^\dagger, \\ g_{C_6}(u) g_{C_6}(w) g_{C_6}(v) \tilde{u}_\gamma [g_{C_6}(u) g_{C_6}(w) g_{C_6}(v)]^\dagger &= \eta_{12} \tilde{u}_\gamma^\dagger.\end{aligned}\quad (\text{B16})$$

Appendix C: Z_2 spin liquids in the neighborhood of $U(1)$ SL-[0, π] state

1. Mean-field ansatz of $U(1)$ SL-[0, π] state

Following $SU(2)$ Schwinger fermion formulation with $\psi_i \equiv (f_{i\uparrow}, f_{i\downarrow})^T$, we focus on those Z_2 spin liquids (SLs) in the neighborhood of $U(1)$ SL-[0, π] state with the following mean-field ansatz:

$$\begin{aligned}\langle x, y, u | x, y, v \rangle &= -\langle x, y, u | x, y, w \rangle = (-1)^x \chi \tau^3, \\ \langle x+1, y, w | x, y, u \rangle &= \langle x, y+1, v | x, y, u \rangle = -\langle x, y, v | x, y, w \rangle \\ &= \langle x+1, y-1, w | x, y, v \rangle = \chi \tau^3.\end{aligned}\quad (\text{C1})$$

where χ is a real hopping parameter. We define mean-field bonds $\langle x, y, s | x', y', s' \rangle$ in the following way

$$H_{MF} = \sum_{i,j} \psi_i^\dagger \langle i | j \rangle \psi_j + h.c. \quad (\text{C2})$$

For convenience of later calculation we implement the following gauge transformation

$$\psi_{x,y,u} \rightarrow i\tau^3 \psi_{x,y,u} \quad (\text{C3})$$

and the original mean-field ansatz (C1) transforms to be

$$\begin{aligned}\langle x, y, u | x, y, v \rangle &= -\langle x, y, u | x, y, w \rangle = i(-1)^x \chi \tau^0, \\ \langle x+1, y, w | x, y, u \rangle &= \langle x, y+1, v | x, y, u \rangle = -i\chi \tau^0, \\ -\langle x, y, v | x, y, w \rangle &= \langle x+1, y-1, w | x, y, v \rangle = \chi \tau^3.\end{aligned}\quad (\text{C4})$$

The projected symmetry group (PSG) corresponds to

the above mean-field ansatz (C4) is

$$\begin{aligned}G_{\mathbf{T}}(x, y, v) &= G_{\mathbf{T}}(x, y, w) = -G_{\mathbf{T}}(x, y, u) = g_{\mathbf{T}}, \\ g_{\mathbf{T}} \tau^3 g_{\mathbf{T}}^\dagger &= -\tau^3; \\ G_{T_2}(x, y, s) &= g_{T_2}, \quad g_{T_2} \tau^3 g_{T_2}^\dagger = \tau_3; \\ G_{T_1}(x, y, v) &= G_{T_1}(x, y, w) = -G_{T_1}(x, y, u) \\ &= (-1)^{x+y} g_{T_1}, \quad g_{T_1} \tau^3 g_{T_1}^\dagger = \tau_3; \\ G_{\sigma}(x, y, v) &= G_{\sigma}(x, y, w) = (-1)^{x+y+1} G_{\sigma}(x, y, u) \\ &= (-1)^{(x+y)(x+y+1)/2} g_{\sigma}, \quad g_{\sigma} \tau^3 g_{\sigma}^\dagger = \tau^3; \\ G_{C_6}(x, y, u) &= (-1)^{\frac{x(x+1)+y(y-1)}{2}} g_{C_6}, \\ G_{C_6}(x, y, v) &= -(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6}, \\ G_{C_6}(x, y, w) &= i(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6} \tau^3, \\ g_{C_6} \tau^3 g_{C_6}^\dagger &= \tau^3.\end{aligned}\quad (\text{C5})$$

so that the mean-field ansatz satisfy (8).

2. Classification of Z_2 spin liquids around $U(1)$ SL-[0, π] state

Plugging (C5) into algebraic consistent conditions (A4)-(A15) yields four algebraic solutions of Z_2 PSGs around the $U(1)$ SL-[0, π] state. Choosing a proper gauge they all satisfy

$$\begin{aligned}g_{\mathbf{T}} &= i\tau_1, \quad g_{T_1} = g_{T_2} = \tau^0, \\ \eta_{\mathbf{T}} &= \eta_{12} = \eta_{C_6 T_1} = -1, \\ \eta_{T_{1,2} \mathbf{T}} &= \eta_{\sigma T_{1,2}} = \eta_{C_6 T_2} = 1.\end{aligned}\quad (\text{C6})$$

The four Z_2 PSGs near the $U(1)$ SL-[0, π] state are featured by

$$\begin{aligned}(\#6) \ Z_2[0, \pi] \alpha : \quad &g_{\sigma} = g_{C_6} = \tau^0, \\ &\eta_{\sigma} = \eta_{\sigma \mathbf{T}} = 1, \\ &\eta_{\sigma C_6} = \eta_{C_6 \mathbf{T}} = -\eta_{C_6} = -1;\end{aligned}\quad (\text{C7})$$

$$\begin{aligned}(\#2) \ Z_2[0, \pi] \beta : \quad &g_{\sigma} = \tau^0, \quad g_{C_6} = i\tau^3, \\ &\eta_{\sigma} = \eta_{\sigma \mathbf{T}} = 1, \\ &\eta_{\sigma C_6} = \eta_{C_6 \mathbf{T}} = -\eta_{C_6} = 1;\end{aligned}\quad (\text{C8})$$

$$\begin{aligned}(\#14) \ Z_2[0, \pi] \gamma : \quad &g_{\sigma} = i\tau^3, \quad g_{C_6} = \tau^0, \\ &\eta_{\sigma} = \eta_{\sigma \mathbf{T}} = -1, \\ &\eta_{\sigma C_6} = \eta_{C_6 \mathbf{T}} = -\eta_{C_6} = -1;\end{aligned}\quad (\text{C9})$$

$$\begin{aligned}(\#16) \ Z_2[0, \pi] \delta : \quad &g_{\sigma} = g_{C_6} = i\tau^3 \\ &\eta_{\sigma} = \eta_{\sigma \mathbf{T}} = -1, \\ &\eta_{\sigma C_6} = \eta_{C_6 \mathbf{T}} = -\eta_{C_6} = 1.\end{aligned}\quad (\text{C10})$$

Of course they belong to the 20 Z_2 spin liquids summarized in TABLE II.

3. Four possible Z_2 spin liquids around $U(1)$ SL-[0, π] state: mean-field ansatz

a. Consistent conditions on mean-field bonds

Implementing the generic conditions mentioned earlier on several near neighbor mean-field bonds with PSG (C6)-(C10), we obtain the following consistent conditions:

(0) For on-site chemical potential terms $\Lambda_s(x, y, s) = \vec{\lambda}(x, y, s) \cdot \vec{\tau}$, translations operations $G_{T_{1,2}} T_{1,2}$ in PSG guarantee that $\Lambda_s(x, y, s) = \lambda_s(0, 0, s) \equiv \Lambda_s$, $s = u, v, w$. They satisfy

$$\begin{aligned} g_{\mathbf{T}} \Lambda_s g_{\mathbf{T}}^\dagger &= -\Lambda_s; \\ g_{\sigma} \Lambda_u g_{\sigma}^\dagger &= \Lambda_u, \quad g_{\sigma} \Lambda_v g_{\sigma}^\dagger = \Lambda_w, \quad g_{\sigma} \Lambda_w g_{\sigma}^\dagger = \Lambda_v; \\ g_{C_6} \Lambda_u g_{C_6}^\dagger &= \Lambda_v, \quad (g_{C_6} \tau^3) \Lambda_v (g_{C_6} \tau^3)^\dagger = \Lambda_w, \\ g_{C_6} \Lambda_w g_{C_6}^\dagger &= \Lambda_u. \end{aligned} \quad (\text{C11})$$

(I) For 1st neighbor mean-field bond $u_a \equiv [0, 0, v]^\dagger$ (there is only one independent mean-field bond, meaning all other 1st neighbor bonds can be generated from $[0, 0, v]$ through symmetry operations)

$$\begin{aligned} g_{\mathbf{T}} u_a^\dagger g_{\mathbf{T}}^\dagger &= u_a^\dagger, \\ (g_{\sigma} g_{C_6}^2 \tau^3) u_a^\dagger (g_{\sigma} g_{C_6}^2 \tau^3)^\dagger &= -u_a^\dagger. \end{aligned} \quad (\text{C12})$$

(II) For 2nd neighbor mean-field bond $u_b \equiv [0, 1, w]$ we have

$$\begin{aligned} g_{\mathbf{T}} u_b g_{\mathbf{T}}^\dagger &= u_b, \\ (g_{\sigma} g_{C_6}) u_b (g_{\sigma} g_{C_6})^\dagger &= -u_b^\dagger. \end{aligned} \quad (\text{C13})$$

(II) For 3rd neighbor mean-field bonds $u_{c1} \equiv [1, 0, u]$ and $u_{c2} \equiv [1, -1, u]$ we have

$$\begin{aligned} g_{\mathbf{T}} u_{c1} g_{\mathbf{T}}^\dagger &= -u_{c1}, \\ (g_{C_6}^3 \tau^3) u_{c1} (g_{C_6}^3 \tau^3)^\dagger &= u_{c1}^\dagger. \end{aligned} \quad (\text{C14})$$

and

$$\begin{aligned} g_{\mathbf{T}} u_{c2} g_{\mathbf{T}}^\dagger &= -u_{c2}, \\ g_{\sigma} u_{c2} g_{\sigma}^\dagger &= u_{c2}^\dagger, \\ (g_{C_6}^3 \tau^3) u_{c2} (g_{C_6}^3 \tau^3)^\dagger &= -u_{c2}^\dagger. \end{aligned} \quad (\text{C15})$$

b. Mean-field ansatz of the four Z_2 spin liquids near $U(1)$ SL-[0, π] state

For $Z_2[0, \pi]\alpha$ state with $g_{\sigma} = g_{C_6} = \tau^0$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a &= i a_0 \tau^0 + a_1 \tau^1, \quad u_b = i b_0 \tau^0, \\ u_{c1} &= c_3 \tau^3, \quad u_{c2} = c_2 \tau^2, \\ \Lambda_s &= \lambda_3 \tau^3, \quad s = u, v, w. \end{aligned} \quad (\text{C16})$$

Since we are considering a phase perturbed from the $U(1)$ SL-[0, π] state, we shall always assume $a_0 \neq 0$ (1st neighbor hopping terms) in the following discussion. A $Z_2[0, \pi]\alpha$ spin liquid can be realized by 1st neighbor mean-field singlet pairing terms with $a_1 \neq 0$.

For $Z_2[0, \pi]\beta$ state with $g_{\sigma} = \tau^0$, $g_{C_6} = i\tau^3$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a &= i a_0 \tau^0 + a_1 \tau^1, \quad u_b = i b_0 \tau^0 + b_1 \tau^1, \\ u_{c1} &= c_2 \tau^2 + c_3 \tau^3, \quad u_{c2} = 0, \\ \Lambda_u &= \lambda_2 \tau^2 + \lambda_3 \tau^3, \quad \Lambda_{v,w} = -\lambda_2 \tau^2 + \lambda_3 \tau^3. \end{aligned} \quad (\text{C17})$$

A $Z_2[0, \pi]\beta$ spin liquid can be realized by 2nd neighbor pairing terms with $a_0 b_1 - a_1 b_0 \neq 0$.

For $Z_2[0, \pi]\gamma$ state with $g_{\sigma} = i\tau^3$, $g_{C_6} = \tau^0$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a &= i a_0 \tau^0, \quad u_b = i b_0 \tau^0 + b_1 \tau^1, \\ u_{c1} &= c_3 \tau^3, \quad u_{c2} = 0, \\ \Lambda_s &= \lambda_3 \tau^3, \quad s = u, v, w. \end{aligned} \quad (\text{C18})$$

A $Z_2[0, \pi]\gamma$ spin liquid can be realized by 2nd neighbor pairing terms with $b_1 \neq 0$.

For $Z_2[0, \pi]\delta$ state with $g_{\sigma} = g_{C_6} = i\tau^3$ the mean-field ansatz are (up to 3rd neighbor mean-field bonds)

$$\begin{aligned} u_a &= i a_0 \tau^0, \quad u_b = i b_0 \tau^0, \\ u_{c1} &= c_2 \tau^2 + c_3 \tau^3, \quad u_{c2} = 0, \\ \Lambda_s &= \lambda_3 \tau^3, \quad s = u, v, w. \end{aligned} \quad (\text{C19})$$

A $Z_2[0, \pi]\delta$ spin liquid can be realized by 3rd neighbor pairing terms with $c_2 \neq 0$.

4. Low-energy effective theory

The reciprocal unit vectors (corresponding to unit vectors $\vec{a}_{1,2}$) on a kagome lattice are $\vec{b}_1 = \frac{1}{a}(\hat{x} - \frac{1}{\sqrt{3}}\hat{y})$ and $\vec{b}_2 = \frac{1}{a}\frac{2}{\sqrt{3}}\hat{y}$, satisfying $\vec{a}_i \cdot \vec{b}_j = \delta_{i,j}$. In the mean-field ansatz (C4) of $U(1)$ SL-[0, π] the unit cell is doubled whose translation unit vectors are $\vec{A}_1 = 2\vec{a}_1$ and $\vec{A}_2 = \vec{a}_2$. Accordingly the 1st BZ for such a mean-field ansatz is only half of the original 1st BZ with new reciprocal unit vectors being $\vec{B}_1 = \vec{b}_1/2$ and $\vec{B}_2 = \vec{b}_2$. Denoting the momentum as $\mathbf{k} \equiv (k_x, k_y)/a = k_1 \vec{B}_1 + k_2 \vec{B}_2$ with $|k_{1,2}| \leq \pi$, we have

$$k_1 = 2k_x, \quad k_2 = (k_x + \sqrt{3}k_y)/2. \quad (\text{C20})$$

The two Dirac cones in the spectra of $U(1)$ SL-[0, π] state (C4) are located at $\pm \mathbf{Q}$ with

$$\mathbf{Q} = (0, \frac{\pi}{\sqrt{3}}) = \frac{\pi}{2} \vec{B}_2 \quad (\text{C21})$$

with the proper chemical potential $\Lambda(i) = \langle i|i \rangle = \chi(\sqrt{3} - 1)\tau^3$ added to mean-field ansatz (C4).

For convenience we choose the following basis for Dirac-like Hamiltonian obtained from expansion around $\pm\mathbf{Q}$:

$$\begin{aligned}\phi_{+,\uparrow,A} &= \frac{1}{\sqrt{6}}e^{-i\frac{1}{24}\pi}, \\ (e^{-i\frac{11}{12}\pi}, 0, e^{i\frac{11}{12}\pi}, 0, 0, 0, e^{-i\frac{11}{12}\pi}, 0, e^{i\frac{5}{12}\pi}, 0, \sqrt{2}, 0)^T, \\ \phi_{+,\uparrow,B} &= \frac{1}{\sqrt{6}}e^{-i\frac{1}{24}\pi}, \\ (1, 0, e^{-i\frac{4}{3}\pi}, 0, \sqrt{2}e^{-i\frac{11}{12}\pi}, 0, -1, 0, e^{-i\frac{5}{6}\pi}, 0, 0, 0)^T, \\ \phi_{-,\uparrow,b} &= R_{T_1}(k_1 = 0, k_2 = -\frac{\pi}{2})\phi_{+,\uparrow,b}, \\ \phi_{\pm,\downarrow,b} &= R_T\phi_{\pm,\uparrow,b}.\end{aligned}\quad (\text{C22})$$

where \pm are valley index for two Dirac cones at $\pm\mathbf{Q}$ with Pauli matrices $\boldsymbol{\mu}$ and $b = A, B$ are band indices with Pauli matrices $\boldsymbol{\nu}$. Spin indices $\Sigma = \uparrow, \downarrow$ are as usual, with Pauli matrices $\boldsymbol{\sigma}$. The corresponding creation operators for these modes are $\Psi_{\pm,\Sigma,b}^\dagger = \psi_{\pm\mathbf{Q}}^\dagger \phi_{\pm,\Sigma,b}$ in the order of $(0, 0, u), (0, 0, v), (0, 0, w), (1, 0, u), (1, 0, v), (1, 0, w)$ for the six sites per doubled new unit cell.

$$\begin{aligned}\text{Here } R_T &\equiv I_{2 \times 2} \otimes \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes g_T, \quad R_{T_2}(\mathbf{k}) = \\ e^{-ik_2} I_{6 \times 6} &\otimes g_{T_2} \quad \text{and} \quad R_{T_1}(\mathbf{k}) = \begin{bmatrix} 0 & -e^{-ik_1} \\ 1 & 0 \end{bmatrix} \otimes \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} &\otimes g_{T_1} \text{ are transformation matrices on 12-}\end{aligned}$$

component eigenvectors for time reversal \mathbf{T} and translation $T_{1,2}$ operations. By definition of PSG the eigenvectors $\phi_{\mathbf{k}}$ with momentum $\mathbf{k} = k_1\vec{B}_1 + k_2\vec{B}_2 \equiv (k_1, k_2)$ and energy E have the following symmetric properties:

$$\begin{aligned}\mathbf{T}: \quad \tilde{\phi}_{(k_1, k_2)}(E) &= R_T \phi_{(k_1, k_2)}(-E), \\ T_1: \quad \tilde{\phi}_{(k_1, k_2)}(E) &= R_{T_1}(k_1, k_2) \phi_{(k_1, k_2 + \pi)}(E), \\ T_2: \quad \tilde{\phi}_{(k_1, k_2)}(E) &= R_{T_2}(k_1, k_2) \phi_{(k_1, k_2)}(E).\end{aligned}$$

$\tilde{\phi}$ and ϕ are the basis after and before the symmetry operations.

In such a set of basis the Dirac Hamiltonian obtained by expanding the $U(1)$ SL-[0, π] mean-field ansatz (C4) around the two cones at $\pm\mathbf{Q}$ is

$$H_{\text{Dirac}} = \sum_{\mathbf{k}} \frac{\chi}{\sqrt{2}} \Psi_{\mathbf{k}}^\dagger \mu^0 \sigma^3 (-k_x \nu^1 + k_y \nu^2) \Psi_{\mathbf{k}} \quad (\text{C23})$$

\mathbf{k} should be understood as small momenta measured from $\pm\mathbf{Q}$. Possible mass terms are $\mu^{0,1,2,3} \sigma^{1,2} \nu^0$ and $\mu^{0,1,2,3} \sigma^0 \nu^3$. However not all of them are allowed by symmetry. Here we numerate all symmetry operations and associated operator transformations:

Spin rotation along \hat{z} -axis by angle θ :

$$\Psi_{\mathbf{k}}^\dagger \rightarrow \Psi_{\mathbf{k}}^\dagger e^{i\frac{\theta}{2}}$$

Spin rotation along \hat{y} -axis by π :

$$\Psi_{\mathbf{k}}^\dagger \rightarrow \Psi_{-\mathbf{k}}^T \mu^2 \sigma^2 \nu^2$$

Time reversal \mathbf{T} :

$$\Psi_{\mathbf{k}}^\dagger \rightarrow \Psi_{\mathbf{k}}^\dagger (-i\sigma^2)$$

Translation T_1 :

$$\Psi_{\mathbf{k}}^\dagger \rightarrow \Psi_{\mathbf{k}}^\dagger (-\mu^3)$$

Translation T_2 :

$$\Psi_{\mathbf{k}}^\dagger \rightarrow \Psi_{\mathbf{k}}^\dagger (-i\mu^3)$$

Considering the above conditions, the only symmetry-allowed mass terms are $\sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger m_{1,2} \Psi_{\mathbf{k}}$ with $m_1 = \mu^0 \sigma^1 \nu^0$ and $m_2 = \mu^3 \sigma^3 \nu^3$.

The transformation rules for mirror reflection σ and $\pi/3$ rotation C_6 depend on the choice of g_σ, g_{C_6} in the PSG. In general we have

$$\begin{aligned}\sigma: \quad \Psi_{\mathbf{k}}^\dagger &\rightarrow \Psi_{\sigma\mathbf{k}}^\dagger M_\sigma(g_\sigma), \\ C_6: \quad \Psi_{\mathbf{k}}^\dagger &\rightarrow \Psi_{C_6\mathbf{k}}^\dagger M_{C_6}(g_{C_6}).\end{aligned}$$

Using the basis (C22) the 8×8 matrices M_{σ, C_6} can be expressed in terms of Pauli matrices $\boldsymbol{\mu} \otimes \boldsymbol{\sigma} \otimes \boldsymbol{\nu}$. For the four Z_2 spin liquid we have

$$\begin{aligned}M_\sigma(g_\sigma = \tau^0) &= \mu^3 \otimes \sigma^0 \otimes \begin{pmatrix} 0 & e^{-i\frac{1}{12}\pi} \\ e^{-i\frac{5}{12}\pi} & 0 \end{pmatrix}, \\ M_\sigma(g_\sigma = i\tau^3) &= \mu^3 \otimes \sigma^3 \otimes \begin{pmatrix} 0 & e^{i\frac{5}{12}\pi} \\ e^{i\frac{1}{12}\pi} & 0 \end{pmatrix}; \\ M_{C_6}(g_{C_6} = \tau^0) &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \sigma^0 \otimes e^{i\frac{7}{6}\pi\nu^3}, \\ M_{C_6}(g_{C_6} = i\tau^3) &= \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \otimes \sigma^0 \otimes e^{i\frac{1}{6}\pi\nu^3}.\end{aligned}$$

It turns out in $Z_2[0, \pi]\beta$ state, only the 1st mass term $m_1 = \mu^0 \sigma^1 \nu^0$ is invariant under σ and C_6 operations. In other 3 states neither mass terms $m_{1,2}$ are symmetry-allowed. As a result we only have one gapped Z_2 spin liquid, *i.e.* $Z_2[0, \pi]\beta$ state in the neighborhood of $U(1)$ Dirac SL-[0, π] state.

Let's consider mean-field bonds up to 2nd neighbor for ansatz $Z_2[0, \pi]\beta$. Perturbations to the two Dirac cones of $U(1)$ SL-[0, π] with $\lambda_3 = (\sqrt{3} - 1)a_0$ in general has the following form

$$\begin{aligned}\delta H_0 &= [\lambda_3 - (\sqrt{3} - 1)a_0 - (\sqrt{3} + 1)b_0] \mu^0 \sigma^3 \nu^0 \\ &+ [(\sqrt{3} + 1)b_1 - \lambda_2 - (\sqrt{3} - 1)a_1] \mu^0 \sigma^1 \nu^0\end{aligned}\quad (\text{C24})$$

This means we need either 1st neighbor (a_1) or 2nd neighbor (b_1) pairing term to open up a gap in the spectrum. Meanwhile these pairing terms break the original $U(1)$ symmetry down to Z_2 symmetry.

Appendix D: Z_2 spin liquids in the neighborhood of uniform RVB state

The mean-field ansatz of the uniform RVB state is simple:

$$H_{MF} = \chi \sum_{\langle ij \rangle, \sigma} f_{i,\sigma}^\dagger f_{j,\sigma} \quad (D1)$$

where χ is a real parameter and $\langle ij \rangle$ represents sites i, j being nearest neighbor (n.n.) of each other. It's straightforward to show the PSG elements of such a mean-field ansatz are

$$G_U(x, y, s) = g_U, \quad U = T_{1,2}, \mathbf{T}, \boldsymbol{\sigma}, C_6. \quad (D2)$$

and $SU(2)$ matrices g_U satisfy

$$\begin{aligned} g_{\mathbf{T}} \tau^3 g_{\mathbf{T}}^\dagger &= -\tau^3, \\ g_U \tau^3 g_U^\dagger &= \tau^3, \quad U = T_{1,2}, \boldsymbol{\sigma}, C_6. \end{aligned} \quad (D3)$$

It turns out there are only 4 gauge-inequivalent Z_2 PSGs as solutions to (A4)-(A16) with the form (D2). In other words, there are only 4 different Z_2 in the neighborhood of a uniform RVB states. Choosing a proper gauge they all satisfy $g_{\mathbf{T}} = i\tau^1$, $g_{T_{1,2}} = \tau^0$ and $\eta_{T_{1,2}\mathbf{T}} = \eta_{12} = \eta_{C_6 T_{1,2}} = \eta_{\boldsymbol{\sigma} T_{1,2}} = 1$, $\eta_{\mathbf{T}} = -1$. These four states are characterized by:

$$\begin{aligned} (\#1) \ Z_2[0,0]A: \quad & g_{\boldsymbol{\sigma}} = g_{C_6} = \tau^0, \\ & \eta_{\boldsymbol{\sigma}\mathbf{T}} = \eta_{C_6\mathbf{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = 1. \end{aligned} \quad (D4)$$

$$\begin{aligned} (\#5) \ Z_2[0,0]B: \quad & g_{\boldsymbol{\sigma}} = \tau^0, \ g_{C_6} = i\tau^3, \\ & \eta_{\boldsymbol{\sigma}\mathbf{T}} = \eta_{\boldsymbol{\sigma}} = 1, \ \eta_{C_6\mathbf{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = -1. \end{aligned} \quad (D5)$$

$$\begin{aligned} (\#15) \ Z_2[0,0]C: \quad & g_{\boldsymbol{\sigma}} = i\tau^3, \ g_{C_6} = \tau^0, \\ & \eta_{\boldsymbol{\sigma}\mathbf{T}} = \eta_{\boldsymbol{\sigma}} = -1, \ \eta_{C_6\mathbf{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = 1. \end{aligned} \quad (D6)$$

$$\begin{aligned} (\#13) \ Z_2[0,0]D: \quad & g_{\boldsymbol{\sigma}} = g_{C_6} = i\tau^3, \\ & \eta_{\boldsymbol{\sigma}\mathbf{T}} = \eta_{C_6\mathbf{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma}C_6} = -1. \end{aligned} \quad (D7)$$

It turns out these four Z_2 SLs around uniform RVB state are all gapped as shown in TABLE II.

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