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Solvable Limit For SU(N) Kondo Model

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We study a single channel one dimensional Kondo Model where the impurity spin is replaced by an su(n) spin. Using Abelian bosonization and canonical transformation we explicitly show that this system has an exactly solvable point. The calculation also shows that there are n collective excitation modes in the system, one charged and $n-1$ neutral spin excitation modes.

I. INTRODUCTION

The Kondo problem[1] and its subsequent multichannel generalization[2] is a classic problem of condensed matter physics. Over the years different approaches have been used to address both the single and the generalized multi-channel Kondo problem. This classic problem is now considered to be one of the class of condensed matter physics problems where a local degree of freedom interacts with a gap-less continuum. Some of the more powerful methods applied to understand properties of Kondo systems includes the renormalization group (RG) theory[2–4], boundary conformal field theory (BCFT)[5], an exact solution by Bethe Ansatz[6–8], exact solutions using bosonization and canonical transformations[9–11], and numerical methods[12].

With the advancement of new methods in micro-fabrication and other experimental techniques enabled physicists to design and fabricate artificial atoms in nano-structures. These developments renewed the interest in Kondo physics in novel heterostructures, where the effect can be observed when an artificial magnetic impurity sits on an artificial metal (a two dimensional electron gas). Such experiments have been conducted using semiconductor quantum dots (SCQD), such as GaAs/AlGaAs and carbon nanotube quantum dots (CNQD)[13,14]. In these experiments, a tuneable magnetic impurity is formed by controlling the tunneling of electrons between the artificial atom and the 2D electron gas.

The conventional Kondo problem has a spin rotation or su(2) symmetry. However, in nano-structures other higher symmetries are also possible, either due to additional internal degrees of freedom, or because of the way these hetero-structures are built in. In particular, there is growing interest in su(4) symmetry both in SCQD[15] and CNQD[16], the case relevant to carbon nanotubes.

In this paper we study a single channel Kondo system that has an su(n) symmetry. It was discovered by Toulouse[9] that the conventional su(2) Kondo model has a simple solvable limit in the parameter space of the coupling constants. The su(2) Toulouse solution was subsequently extended to provide useful insights into the multichannel and Kondo lattice problems[10,17]. Here we demonstrate that the single channel Kondo model, with a generalized su(n) symmetry, has an exactly solvable limit. We begin by solving the su(4) model before turning our attention to the generalized su(n) model.

II. THE SU(N) KONDO MODEL

We consider a single channel wire where electrons in the lead are assumed to be non-interacting. The magnetic impurity is placed at the center of the wire so that it interacts with the free electron gas in the metal via exchange coupling. As we are interested in higher symmetries we assume electrons to have n internal degrees of freedom. The case $n = 2$ corresponds to an electron with spin. Higher n values result if the electronic states are labeled by a sub-band index, as in the case of nanotubes where the orbital degeneracy is denoted by + and −, or by a valley index, as in the case of silicon. We denote the Hamiltonian of the Fermi sea by H_0 and the exchange interaction of the impurity and the free electron gas by H_{Kondo}. Since the critical behavior of the Kondo system depends mainly on the interaction of the impurity and the s angular momentum state of the Fermi sea, the radial equation can be used to describe the system. Following Schotte and Schotte[18] we write the linearized Hamiltonian in terms of chiral left moving fermions $\psi_\alpha(x)$ as[24]

$$H = H_0 + H_{Kondo}, \tag{1}$$

where the kinetic energy is given by

$$H_0 = \sum_{\alpha=1}^{n} \int_{-\infty}^{\infty} \psi_\alpha^\dagger(x)(-i\partial_x)\psi_\alpha(x)dx \tag{2}$$

and the exchange term has the form

$$H_{Kondo} = \sum_{\nu=1}^{n^2-1} J_\nu S^\nu \tau^\nu. \tag{3}$$

Here we are working in units of $\hbar$=v_F=1, where v_F is the Fermi velocity. $\vec{\tau}$ is the su(n) impurity "spin" and

$$\vec{S} = \sum_{\alpha,\beta=1}^{n} \psi_\alpha^\dagger(0)\vec{\Sigma}_{\alpha\beta}\psi_\beta(0) \tag{4}$$

is the su(n) "spin" density of the conduction electrons at the origin. J_ν is the exchange coupling, which we assume to be independent of energy and the $\vec{\Sigma}$'s are the $n \times n$ traceless Hermitian matrices that represent the su(n) "spin" operators. These are a set of $n^2 - 1$ matrices that constitute the basis for the set of $n \times n$ traceless hermitian matrices. Evidently $n-1$ of them are diagonal. They satisfy the "orthogonality" condition

$$Tr(\Sigma_\alpha \Sigma_\beta) = 2\delta_{\alpha\beta}. \tag{5}$$

The Σ matrices are called the *Pauli* matrices in su(2) case, the *Gell-mann* matrices for su(3), etc.

III. TOULOUSE LIMIT FOR SU(4) MODEL

We now focus on the su(4) case to find its solvable limit. Later on we use similar formalism to generalize the result to the su(n) case. In su(4) symmetry the Hilbert space of the 4×4 spin space can be spanned by the fifteen traceless Σ matrices. We choose the three diagonal Σ matrices which satisfy eq (5) as

$$D_1 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \tfrac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$D_3 = \tfrac{1}{\sqrt{12}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The twelve off-diagonal matrices are selected from the matrices $O(\alpha,\beta)$ and $\tilde{O}(\alpha,\beta)$ where

$$\begin{aligned} O(\alpha,\beta)_{ij} &= \delta_{\alpha i}\delta_{\beta j} + \delta_{\alpha j}\delta_{\beta i} \\ \tilde{O}(\alpha,\beta)_{ij} &= -i(\delta_{\alpha i}\delta_{\beta j} - \delta_{\alpha j}\delta_{\beta i}). \end{aligned} \tag{7}$$

These matrices are the generalizations of the *Pauli* matrices σ_x and σ_y and we denote them by O_i where $i = 1, ...12$ and

$$\begin{aligned} O_1 = O(1,2), \quad O_2 = O(1,3), \quad O_3 = O(1,4) \\ O_4 = O(2,3), \quad O_5 = O(2,4), \quad O_6 = O(3,4) \\ O_7 = \tilde{O}(1,2), \quad O_8 = \tilde{O}(1,3), \quad O_9 = \tilde{O}(1,4) \\ O_{10} = \tilde{O}(2,3), \; O_{11} = \tilde{O}(2,4), \; O_{12} = \tilde{O}(3,4). \end{aligned} \tag{8}$$

The d's and the O's, together, constitute the su(4) Lie Algebra. If the exchange coupling J_ν in eq (3) is independent of ν the Kondo model has a full su(n) symmetry. Here we consider an anisotropic case for the exchange coupling where J_ν takes either $J_\nu = J_\parallel$ or $J_\nu = J_\perp$. This reduces the interaction part of the Hamiltonian into parallel and perpendicular components,

$$H_{Kondo} = H_{Kondo}^{\parallel} + H_{Kondo}^{\perp} \tag{9}$$

where

$$H^{\parallel}_{Kondo} = J_{\parallel} \sum_{\alpha,\beta=1}^{4} \sum_{\nu=1}^{3} \tau^{\nu}_{\parallel} \psi^{\dagger}_{\alpha}(0)(D_{\nu})_{\alpha\beta}\psi_{\beta}(0) \tag{10}$$

and

$$H^{\perp}_{Kondo} = J_{\perp} \sum_{\alpha,\beta=1}^{4} \sum_{\nu=1}^{12} \tau^{\nu}_{\perp} \psi^{\dagger}_{\alpha}(O_{\nu})_{\alpha\beta}\psi_{\beta}. \tag{11}$$

IV. BOSONIZATION AND UNITARY TRANSFORMATION

The Hamiltonian of the system can take the form of a free Hamiltonian by Bosonizing the fermionic operators and then making a canonical transformation. Since the spin dynamics of the system depend only on the algebra that the spin operators satisfy, we prefer to work on the canonically transformed operators. The bosonization procedure can be done using the Mandelstam formula[19–21] where we can write chiral fermionic fields ψ_{α}'s in terms of the bosonic fields ϕ_{α}'s as

$$\psi_{\alpha\sigma}(x) = \frac{1}{\sqrt{2\pi\epsilon}} F_{\alpha\sigma} e^{-i\phi^{-}_{\alpha\sigma}(x)}, \tag{12}$$

where

$$\phi^{-}_{\alpha\sigma}(x) = \sqrt{\pi}\left[\int_{-\infty}^{x} dy \Pi_{\alpha\sigma}(y) + \phi_{\alpha\sigma}(x)\right]. \tag{13}$$

Here ϵ is the cutoff, which goes to zero in the continuum limit. $\Pi_{\alpha\sigma}(x)$ is the conjugate momentum of $\phi_{\alpha\sigma}(x)$ which satisfies the commutation relations

$$[\phi_{\alpha\sigma}(x), \Pi_{\beta\sigma'}(y)] = i\delta_{\alpha\beta}\delta_{\sigma\sigma'}\delta(x-y). \tag{14}$$

The $F_{\alpha\sigma}$ are the Klein factors that keeps check the correct anti-commutation relations. They commute with the $\phi^{-}_{\alpha\sigma}$'s and satisfy the following algebra:

$$\begin{array}{ll} F^{\dagger}_{\alpha\sigma}F_{\alpha\sigma} = F_{\alpha\sigma}F^{\dagger}_{\alpha\sigma} = 1, & \\ F^{\dagger}_{\alpha\sigma}F_{\alpha'\sigma'} = -F_{\alpha'\sigma'}F^{\dagger}_{\alpha\sigma} & for \quad (\alpha\sigma) \neq (\alpha'\sigma'), \\ F_{\alpha\sigma}F_{\alpha'\sigma'} = -F_{\alpha'\sigma'}F_{\alpha\sigma} & for \quad (\alpha\sigma) \neq (\alpha'\sigma'). \end{array} \tag{15}$$

Since the single fermion operators comes in to the Hamiltonian in pair, the Klein factors will completely vanishes from the bosonised versions of H_0. More over, one can also show[22,23] that the remaining Klein factors can be absorbed into the impurity part of the Hamiltonian. Hence, from here on the Klein factors are not retained explicitly for further calculations.

For convenience we define the following excitations, which we call spin(s), flavor(f), spin-flavor(fs) and charge(c) excitations as

$$\begin{array}{rcl} \phi^{-}_{s} &=& \frac{1}{\sqrt{2}}(\phi^{-}_{1} - \phi^{-}_{2}) \\ \phi^{-}_{f} &=& \frac{1}{\sqrt{6}}(\phi^{-}_{1} + \phi^{-}_{2} - 2\phi^{-}_{3}) \\ \phi^{-}_{sf} &=& \frac{1}{\sqrt{12}}(\phi^{-}_{1} + \phi^{-}_{2} + \phi^{-}_{3} - 3\phi^{-}_{4}) \\ \phi^{-}_{c} &=& \frac{1}{2}(\phi^{-}_{1} + \phi^{-}_{2} + \phi^{-}_{3} + \phi^{-}_{4}). \end{array} \tag{16}$$

Applying the bosonizing procedure in the free part of the Hamiltonian we have

$$H_0 = \frac{1}{2} \sum_{\alpha=c,s,f,sf} \int_{-\infty}^{\infty} dx \left[\left(\partial_x \phi^{-}_{\alpha}(x)\right)^2 + \Pi^{-2}_{\alpha}(x)\right]. \tag{17}$$

Similarly bosonization of the parallel part of the interaction Hamiltonian gives

$$H^{\parallel}_{Kondo} = \frac{J_{\parallel}}{\sqrt{\pi}} \left(\tau^{\parallel}_{1}(0)\frac{\partial\phi^{-}_{s}}{\partial x} + \tau^{\parallel}_{2}(0)\frac{\partial\phi^{-}_{f}}{\partial x} + \tau^{\parallel}_{3}(0)\frac{\partial\phi^{-}_{sf}}{\partial x}\right)\Bigg|_{x=0}. \tag{18}$$

Bosonization of the perpendicular term, $H^{\perp}_{Kondo}$, leads to a more complicated expression where spin-flip terms get coupled in pair wise fashion. However, these $\tau^{\perp}$'s are coupled only through an effective rotations of $(\phi^-_i - \phi^-_j)$, for $i \neq j$. Thus unitary transformation in the space of τ will remove the coupling. For a generic operator, U, its rotation is given by

$$U(t) = e^{iFt}U(0)e^{-iFt} \tag{19}$$

where t is a parameter and F is the generator of the unitary transformation. We choose this generator to be

$$F = \left(\tau^{\|}_1(0)\phi^-_s + \tau^{\|}_2(0)\phi^-_f + \tau^{\|}_3(0)\phi^-_{sf}\right)\Big|_{x=0}. \tag{20}$$

Application of the canonical transformation on the bosonized $H^{\perp}_{Kondo}$ completely decouples the $\tau^{\perp}$'s at $t = \sqrt{4\pi}$; *i. e.*

$$\begin{aligned} H^{\perp}_{Kondo}(\sqrt{4\pi}) &= e^{iFt}\; H^{\perp}_{Kondo}\; e^{-iFt}\,\big|_{t=\sqrt{4\pi}} \\ &= \frac{J_\perp}{\pi\epsilon}\sum_{\imath=1}^{6}\tau^{\perp}_{2i-1}(0) \end{aligned} \tag{21}$$

The same canonical transformation on $H^{\|}_{Kondo}$ will give no additional terms. However, H_0 will be transformed in such a way that the transformation of H after bosonization can be written in the form[25]

$$\begin{aligned} H &= e^{iFt}He^{-iFt}\,\big|_{t=\sqrt{4\pi}} \\ &= \frac{1}{2}\sum_{k=c,s,f,sf}\int_{-\infty}^{\infty}dx\left[\left(\partial_x\phi^-_k(x)\right)^2 + \Pi^{-\,2}_k(x)\right] \\ &\quad + \left(\frac{J_\|}{\sqrt{\pi}} - t\right)\left(\tau^{\|}_1(0)\frac{\partial\phi^-_s}{\partial x} + \tau^{\|}_2(0)\frac{\partial\phi^-_f}{\partial x} + \tau^{\|}_3(0)\frac{\partial\phi^-_{sf}}{\partial x}\right)\Bigg|_{x=0} \\ &\quad + \frac{J_\perp}{\pi\epsilon}\sum_{i=1}^{6}\tau^{\perp}_{2i-1}(0). \end{aligned} \tag{22}$$

We clearly see that for $J_\| = 2\pi$ the terms in the middle line of eq (22), which couples the free electron gas with the localized impurity spin, vanishes. Hence for $J_\| = 2\pi$ the su(4) Kondo problem is exactly solvable.

V. TOULOUSE LIMIT FOR SU(N) MODEL

A direct generalization of the same procedure reveals that the su(n) single channel Kondo model has the same solvable limit as that of the su(4) model, *i.e.* $J_\| = 2\pi$. The su(n) generalization can be studied by bosonizing the Hamiltonian in eq (1) and extending eq (20) to get the generalized form of the generator of the rotation in the nxn dimensional matrix spin space. The appropriate choice for the generator is

$$\mathcal{F} = \sum_{k=1}^{n-1}\tau^{\|}_k(0)\varphi^-_k \tag{23}$$

where the $\tau^{\|}_k$'s are the diagonal spin operators in their representations and φ^-_k are the n-1 different collective spin excitation modes, which are the generalizations of eq (16). Here we span the spin space with n^2-1 hermitian matrices. As in the case of su(4) symmetry, a convenient choice of the $n-1$ diagonal matrices will be

$$[D_k]_{ij} = \frac{d^j_k}{\sqrt{\sum_{j=1}^{n}(d^j_k)^2}}\delta_{ij} \tag{24}$$

where

$$d^j_k = \begin{cases} 1 & \text{if } j < k+1 \\ -k & \text{if } j = k+1 \\ 0 & \text{if } j > k+1 \end{cases}. \tag{25}$$

The d_k^j's are the j^{th} elements of the k^{th} diagonal matrix. The off-diagonal matrices are given by extending eq (7) for the nxn case. The collective spin excitation modes, φ_k^-, can be written in terms of the left moving Bose fields as

$$\varphi_k^- = \sum_{i=1}^{n} [D_k]_{ii} \, \phi_i^- \tag{26}$$

and the charge mode is also given by

$$\varphi_c^- = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \phi_k^- . \tag{27}$$

The canonical transformation of the off-diagonal spin matrices $\tau^\perp$ is obtained from the evolution equation

$$-i\frac{\partial}{\partial t}\tau_j^\perp(t) = e^{i\mathcal{F}t}[\mathcal{F}, \tau_j^\perp(0)]e^{-i\mathcal{F}t}. \tag{28}$$

Again here the spin operators and the n^2-1 hermitian matrices that span the Hilbert space satisfy the same Lie Algebra, the commutator of $\tau_j^\perp$ and $\tau_k^\parallel$ can be obtained from the commutator of D's and $O's(\tilde{O}'s)$, which is given by

$$[O(j,k), D_l] = -i\left(d_l^j - d_l^k\right)\tilde{O}(j,k) \tag{29}$$

$$\left[\tilde{O}(j,k), D_l\right] = i\left(d_l^j - d_l^k\right)O(j,k) \tag{30}$$

where $O(j,k)$ and $\tilde{O}(j,k)$ are given by eq (7).

Bosonization and canonical transformation of the su(n) Hamiltonian, eq (1), gives us

$$\begin{aligned} H = {} & \frac{1}{2}\sum_{k=1}^{n}\int_{-\infty}^{\infty} dx \left[\left(\partial_x \varphi_k^-(x)\right)^2 + \Pi_k^{-\,2}(x)\right] \\ & + \left(\frac{J_\parallel}{\sqrt{\pi}} - t\right)\sum_{k=1}^{n-1} \tau_k^\parallel \frac{\partial \varphi_k^-}{\partial x}\bigg|_{x=0} \\ & + \frac{J_\perp}{\pi\epsilon}\sum_{i=1}^{\frac{1}{2}(n^2-n)} \tau_{2i-1}^\perp\Big|_{t=0} , \end{aligned} \tag{31}$$

where again here we considered the energy independent anisotropic case of the exchange coupling, namely that J_ν takes is either $J_\nu = J_\parallel$ or $J_\nu = J_\perp$. Clearly eq (31) shows that for the model we considered the solvable point is the same as in the su(n) model.

VI. SUMMARY AND CONCLUSION

In this work we have studied su(n) Kondo spin in a one dimensional single channel wire with electrons in the lead assumed to be non-interacting. Using Abelian Bosonization of chiral fermions and canonical transformation we have found a solvable point for the problem, which is the su(n) generalization of the Toulouse limit[9]. This result may be used to test the large n approximation for the Kondo problem and a straightforward extension of this analysis can be applied to the multi-channel su(n) single impurity Kondo model and Kondo lattice problem. Finally the exact solution obtained here may be used to compute the transport properties of nanostructures, a task to which we will return in future work.

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Appendix A

Linearization of Kondo Hamiltonian

Here we show the linearization of the Hamiltonian of a single Kondo impurity in a formalism similar to that of Schotte and Schotte[18]. Consider a field operator $\psi(x)$ which we write it in terms of its Fourier components as

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{ikx}c_k \\ &= \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{-k_F} e^{ikx}c_k + \frac{1}{\sqrt{L}}\sum_{k=-k_F}^{-k_F} e^{ikx}c_k + \frac{1}{\sqrt{L}}\sum_{k=k_F}^{\infty} e^{ikx}c_k \end{aligned} \tag{32}$$

where $c_k^\dagger$ is electron creation operator and k_F is the one dimensional Fermi operator. Suppose we are interested only in the low lying excitations near the Fermi surface. Then the field operator can be approximated as

$$\psi(x) \approx \frac{1}{\sqrt{L}}\sum_{p=-\Lambda}^{\Lambda} e^{i(k_F+p)x}c_{-(k_F+p)} + \frac{1}{\sqrt{L}}\sum_{p=-\Lambda}^{\Lambda} e^{i(k_F+p)x}c_{k_F+p}. \tag{33}$$

If we rename $c^\dagger_{k_F+p} = \alpha_p$ and $c^\dagger_{-(k_F+p)} = \beta_p$, which creates electrons near the Fermi surface at $k = k_F$ and $k = k_{-F}$ respectively, then the approximated field operator in eq (33) can be written as

$$\psi(x) = \frac{1}{\sqrt{L}}e^{ik_Fx}\psi_+(x) + \frac{1}{\sqrt{L}}e^{-ik_Fx}\psi_-(x). \tag{34}$$

where

$$\psi_+(x) = \frac{1}{\sqrt{L}}\sum_{p=-\Lambda}^{\Lambda} e^{ipx}\alpha_p \ \text{ and } \ \psi_-(x) = \frac{1}{\sqrt{L}}\sum_{p=-\Lambda}^{\Lambda} e^{-ipx}\beta_p. \tag{35}$$

Here L is the length of the 1D wire. The operators $\psi_+^\dagger$ and $\psi_-^\dagger$ are called right and left moving chiral fermionic operators respectively, for a reason that will become clear, from the Hamiltonian form, shortly. The Hamiltonian of a free electron gas can be written in terms of the left and right moving chiral fermions as

$$\begin{aligned} H_0 &= \int dx\ \psi^\dagger(x)\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi(x) \\ &= k_F\int dx\left[\psi_+^\dagger(x)(-i\partial_x)\psi_+(x) + \psi_-^\dagger(x)(i\partial_x)\psi_-(x)\right] + \\ &\quad \frac{k_F^2}{2} + \text{highly oscillating terms.}\end{aligned} \tag{36}$$

If we rescaled the energy with respect to the Fermi level and neglect highly oscillating terms we obtain a Hamiltonian whose form is similar to that of left- and right-handed massless fermions. In terms of α_p and β_p the Hamiltonian is given by

$$H_0 = \frac{k_F^2}{2} + k_F\sum_p\left[\alpha_p^\dagger\alpha_p - \beta_p^\dagger\beta_p\right]. \tag{37}$$

Appendix B

Details of Canonical Transformation

In this section we show how we derived eq (21) and (22). We begin first with the construction of the su(4) spin representations using Schwinger's method of oscillators. We assume that $d_j^\dagger$ creates an electron on the impurity site with spin state $|j\rangle$. Then the su(4) spin space can be generated using the sixteen number conserving bilinear, $S_{ij} = d_i^\dagger d_j$. For a base state $|\mu\rangle$ the bilinear acts according to $S_{ij}|\mu\rangle = \delta_{j\mu}|i\rangle$. The commutator of these bilinear is given by

$$[S_{ij}, S_{kl}] = S_{il}\delta_{jk} - S_{jk}\delta_{il}. \tag{38}$$

Using these bilinear we write the number and spin operators as

$$\begin{aligned} N &= \sum_j d_j^\dagger d_j \\ \tau_\mu^{\|} &= \sum_{ij} d_i^\dagger [D_\mu]_{ij} d_j \quad \text{for } \nu = 1,2,3 \\ \tau_\nu^{\perp} &= \sum_{ij} d_i^\dagger [O_\nu]_{ij} d_j \quad \text{for } \nu = 1, \dots 12 \end{aligned} \tag{39}$$

where O_ν is given by eq (8). Using the su(4) algebra, eq (38), one can get the commutation $[\tau_\mu^{\|}, \tau_\nu^{\perp}]$ for any μ and ν. In fact, these spin operators will satisfy the same algebra as the fifteen matrices that we used to span the spin space in eqns (6) and (8). We now consider the derivation of eq (21). Bosonization of $H_{Kondo}^{\perp}$ from eq (11) results in

$$\begin{aligned} \frac{\pi\epsilon}{J_\perp} H_{Kondo}^{\perp} \;=\; & \tau_1^\perp(0)\cos[\sqrt{4\pi}(\phi_1^- - \phi_2^-)] + \tau_2^\perp(0)\sin[\sqrt{4\pi}(\phi_1^- - \phi_2^-)] + \\ & \tau_3^\perp(0)\cos[\sqrt{4\pi}(\phi_1^- - \phi_3^-)] + \tau_4^\perp(0)\sin[\sqrt{4\pi}(\phi_1^- - \phi_3^-)] + \\ & \tau_5^\perp(0)\cos[\sqrt{4\pi}(\phi_1^- - \phi_4^-)] + \tau_6^\perp(0)\sin[\sqrt{4\pi}(\phi_1^- - \phi_4^-)] + \\ & \tau_7^\perp(0)\cos[\sqrt{4\pi}(\phi_2^- - \phi_3^-)] - \tau_8^\perp(0)\sin[\sqrt{4\pi}(\phi_2^- - \phi_3^-)] + \\ & \tau_9^\perp(0)\cos[\sqrt{4\pi}(\phi_2^- - \phi_4^-)] - \tau_{10}^\perp(0)\sin[\sqrt{4\pi}(\phi_2^- - \phi_4^-)] + \\ & \tau_{11}^\perp(0)\cos[\sqrt{4\pi}(\phi_3^- - \phi_4^-)] - \tau_{12}^\perp(0)\sin[\sqrt{4\pi}(\phi_3^- - \phi_4^-)]. \end{aligned} \tag{40}$$

A straight forward calculation of the commutation $[\tau_\mu^{\|}, \tau_\nu^{\perp}]$ determines the evolution of the spin operators through eq (28); i.e.,

$$-i\frac{\partial}{\partial t}\tau_j^\perp(t) = e^{iFt}[F, \tau_j^\perp(0)]e^{-iFt}$$

where F is the generator of the rotation which was defined in eq (20). These differential equations are coupled in a pair wise fashion and their solution are given as follows:

$$\begin{aligned} \tau_1^\perp(t) &= \tau_1^\perp(0)\cos[(\phi_1^- - \phi_2^-)t] - \tau_2^\perp(0)\sin[(\phi_1^- - \phi_2^-)t] \\ \tau_2^\perp(t) &= \tau_2^\perp(0)\cos[(\phi_1^- - \phi_2^-)t] + \tau_1^\perp(0)\sin[(\phi_1^- - \phi_2^-)t] \\ \tau_3^\perp(t) &= \tau_3^\perp(0)\cos[(\phi_1^- - \phi_3^-)t] - \tau_4^\perp(0)\sin[(\phi_1^- - \phi_3^-)t] \\ \tau_4^\perp(t) &= \tau_4^\perp(0)\cos[(\phi_1^- - \phi_3^-)t] + \tau_3^\perp(0)\sin[(\phi_1^- - \phi_3^-)t] \\ \tau_5^\perp(t) &= \tau_5^\perp(0)\cos[(\phi_1^- - \phi_4^-)t] - \tau_6^\perp(0)\sin[(\phi_1^- - \phi_4^-)t] \\ \tau_6^\perp(t) &= \tau_6^\perp(0)\cos[(\phi_1^- - \phi_4^-)t] + \tau_5^\perp(0)\sin[(\phi_1^- - \phi_4^-)t] \\ \tau_7^\perp(t) &= \tau_7^\perp(0)\cos[(\phi_2^- - \phi_3^-)t] + \tau_8^\perp(0)\sin[(\phi_2^- - \phi_3^-)t] \\ \tau_8^\perp(t) &= \tau_8^\perp(0)\cos[(\phi_2^- - \phi_3^-)t] - \tau_7^\perp(0)\sin[(\phi_2^- - \phi_3^-)t] \\ \tau_9^\perp(t) &= \tau_9^\perp(0)\cos[(\phi_2^- - \phi_4^-)t] + \tau_{10}^\perp(0)\sin[(\phi_2^- - \phi_4^-)t] \\ \tau_{10}^\perp(t) &= \tau_{10}^\perp(0)\cos[(\phi_2^- - \phi_4^-)t] - \tau_9^\perp(0)\sin[(\phi_2^- - \phi_4^-)t] \\ \tau_{11}^\perp(t) &= \tau_{11}^\perp(0)\cos[(\phi_3^- - \phi_4^-)t] + \tau_{12}^\perp(0)\sin[(\phi_3^- - \phi_4^-)t] \\ \tau_{12}^\perp(t) &= \tau_{11}^\perp(0)\cos[(\phi_3^- - \phi_4^-)t] - \tau_{12}^\perp(0)\sin[(\phi_3^- - \phi_4^-)t] \end{aligned} \tag{41}$$

Application of the transformation $e^{iFt}\ H_{Kondo}^{\perp}\ e^{-iFt}$, which utilizes eq (41), completely decouples the diagonal and off-diagonal spin operators at $t = \sqrt{4\pi}$, giving the final result shown in eq (21).

To make a canonical transformation on H_0 and $H_{Kondo}^{\|}$ we first write both of these terms in terms of the spin excitation fields. The time evolution of these field operators (φ_c^-, φ_s^-, φ_f^- and φ_{sf}^-) and their conjugate fields (Π_c^-, Π_s^-, Π_f^- and Π_{sf}^-) are determined by

$$\begin{aligned} -i\tfrac{\partial}{\partial t}\varphi_j^-(x,t) &= e^{iFt}[F, \varphi_j^-(x)]e^{-iFt} \\ -i\tfrac{\partial}{\partial t}\Pi_j^-(x,t) &= e^{iFt}[F, \Pi_j^-(x)]e^{-iFt}. \end{aligned} \tag{42}$$

However, the commutation relation between the generator F and the fields are give by

$$[F, \varphi_j^-(x)] = -i\tau_k^{\|}(0)\Theta(-x)\delta_{jk} \tag{43}$$

and

$$[F, \Pi_j^-(x)] = i\tau_k^{\|}(0)\delta(x)\delta_{jk} \tag{44}$$

where $\Theta(-x)$ is the Heaviside step function. Utilizing the solutions of eq (42), one can show that the parallel component of the Kondo Hamiltonian is canonically transformed in to

$$\begin{aligned} H_{Kondo}^{\|} &= e^{iFt} H_{Kondo}^{\|} e^{-iFt} i \\ &= \frac{J_{\|}}{\sqrt{\pi}} \left(\tau_1^{\|}(0) \frac{\partial \phi_s^-}{\partial x} + \tau_2^{\|}(0) \frac{\partial \phi_f^-}{\partial x} + \tau_3^{\|}(0) \frac{\partial \phi_{sf}^-}{\partial x} \right)\Bigg|_{x=0} \\ &\quad + \text{diverging constant} . \end{aligned} \tag{45}$$

Similarly the kinetic energy part is also transformed as

$$\begin{aligned} H_0 &= e^{iFt} H_0 e^{-iFt} \\ &= \frac{1}{2} \sum_{k=c,s,f,sf} \int_{-\infty}^{\infty} dx \left[\left(\partial_x \phi_k^-(x) \right)^2 + \Pi_k^{-2}(x) \right] \\ &\quad t \left(\tau_1^{\|}(0) \frac{\partial \phi_s^-}{\partial x} + \tau_2^{\|}(0) \frac{\partial \phi_f^-}{\partial x} + \tau_3^{\|}(0) \frac{\partial \phi_{sf}^-}{\partial x} \right)\Bigg|_{x=0} \\ &\quad + \text{diverging constant} \end{aligned} \tag{46}$$

We throw away the the diverging constant as it is a term that can be renormalized, and hence arrive at eq (22). A straight forward, and similar, procedure is applied to get the solvable point of the su(n) Kondo Model, where the same algebra of eq (30) is used to get the commutator of the su(n) spin operators.

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[24] See Appendix A for the derivation.
[25] For detailed derivations of eqns 21 and 22 look at Appendix B.