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Counting Majorana zero modes in superconductors

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A counting formula for computing the number of (Majorana) zero modes bound to topological point defects is evaluated in a gradient expansion for systems with charge-conjugation symmetry. This semi-classical counting of zero modes is applied to some examples that include graphene and a chiral p-wave superconductor in two-dimensional space. In all cases, we explicitly relate the counting of zero modes to Chern numbers.

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I. INTRODUCTION

The counting of zero modes, eigenstates annihilated by a single-particle Hamiltonian $\mathcal{H}$, has a long history in physics. Charge-conjugation symmetry, the existence of a norm-preserving linear (antilinear) transformation $\mathcal{C}$ that anticommutes with $\mathcal{H}$, protects the parity of the number of zero modes. When the parity is odd, at least one zero mode must be robust to any perturbation that preserves the charge-conjugation symmetry. This paper aims at calculating the parity of zero modes of a single-particle Hamiltonian $\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x})\big)$ when (1) it obeys charge-conjugation symmetry, (2) it describes fermionic quasiparticles, and (3) it depends on a position dependent vector-valued order parameter $\boldsymbol{\varphi}(\boldsymbol{x})$. These three assumptions are often met in mean-field treatments of electrons interacting with each other or with collective excitations such as phonons or magnons in condensed matter physics.

Zero mode solutions can be found by direct means, in practice solving a differential equation. This requires a non-universal definition of the model since both microscopic and macroscopic data must be supplied, say the boundary conditions to be obeyed at the origin and at infinity in space.

Is there an alternative approach to calculating the parity in the number of zero modes that is more universal? The celebrated index theorem for elliptic differential operators gives a positive answer to this question for those problems in physics for which this theorem applies (Dirac Hamiltonians for example).[1] The index theorem achieves this by relating some (not all!) zero modes to a topological number (a global property of the Hamiltonian that can only take integer values). However, the index theorem cannot be applied to most Hamiltonians of relevance to condensed matter physics.

In this paper, we start from an exact integral representation of the total number of *unoccupied* zero modes (up to exponential accuracy) N for fermionic single-particle Hamiltonians with charge-conjugation symmetry

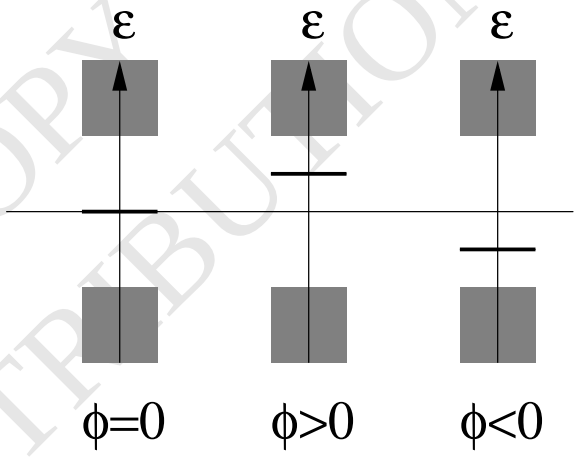

FIG. 1: The charge-conjugation-symmetry-breaking parameter ϕ moves in energy a mid-gap state upward or downward depending on its sign. The continuum parts of the energy-eigenvalue spectrum are denoted by the shaded boxes. The thin horizontal line denotes the band center about which the spectrum is symmetric when $\phi = 0$.

in terms of a conserved quasiparticle charge Q,

$$N = -2Q. \tag{1.1}$$

This counting formula appears implicitly in Ref. 2 and explicitly in Ref. 3, both in the context of polyacetylene.[4] For polyacetylene, it relates the number of unoccupied zero modes to the *conserved electric charge* $Q_{\rm dw}$ induced by domain walls in the spontaneous bond ordering triggered by the coupling of electrons to phonons. Remarkably, the electric charge $Q_{\rm dw} = \pm 1/2$ induced by a single domain wall is fractional and counts a single zero mode, with the sign ambiguity resolved by whether the midgap state is filled or empty.[5–7] Alternatively, this sign ambiguity can be removed by the application of a small charge-conjugation-symmetry-breaking perturbation that shifts the energy of the zero mode up or down (see Fig. 1). Hence, the counting formula (1.1) becomes

$$N = 2Q \bmod 2 \tag{1.2}$$

if no prescription is given as to whether the filled Fermi sea includes or not a zero mode. The same assignment of quantum numbers also relates a single zero mode and the conserved electric charge $Q_{\rm Kekule}$ induced by a vortex with unit vorticity in the Kekulé dimerization pattern of graphene.[8–12]

For polyacetylene and graphene, the charge-conjugation symmetry is approximate, for it originates from a sublattice symmetry that is broken as soon as next-nearest-neighbor hopping is included in the tight-binding model. To the extend that the Bardeen-Cooper-Schrieffer (BCS) mean-field approximation to superconductivity is empirically observed to be excellent, single-particle Bogoliubov-de-Gennes (BdG) Hamiltonians realize a much more robust charge-conjugation symmetry.[13] For BdG Hamiltonians, zero modes are associated to Majorana fermions. Majorana fermions do not carry a well-defined electric charge since global electro-magnetic gauge invariance is broken in any mean-field treatment of superconductivity. The counting formula (1.1) nevertheless applies to any BdG Hamiltonian $\mathcal{H}_{\rm BdG}$ with the important caveat that the conserved quasiparticle charge $Q_{\rm BdG}$ is unrelated to the electric charge. Rather, the conservation of $Q_{\rm BdG}$ encodes, for any local BdG Hamiltonian, a local continuity equation obeyed by the Bogoliubov quasiparticles that is responsible for the conservation of the thermal flow in the mean-field treatment of superconductivity.[14–16]

In this paper, we represent the counting formula (1.1) in terms of single-particle Green functions. The advantage of this choice is that it easily lends itself to a perturbative (gradient) expansion of the conserved quasiparticle charge Q that forgoes the non-universal short-distance data.[17–19] To leading order, this expansion is akin to an adiabatic approximation. We thus propose the adiabatic approximation to the conserved quasiparticle charge Q as an efficient mean to compute the parity in the number N of unoccupied zero modes of charge-conjugation-symmetric single-particle Hamiltonians.

We then apply this formula to charge-conjugation-symmetric Dirac insulators with time-reversal symmetry in arbitrary dimensions and chiral p-wave BdG superconductor in two-dimensional space that all support a point defect. One of the main results of this paper is an integral representation for the number (zero or one) of unoccupied zero mode induced by a unit vortex in a two-dimensional chiral p-wave BdG superconductor. A by-product of this integral representation is that it is closely related to *the second Chern number*. We also relate *the d-th Chern number* to the number of unoccupied zero modes for Dirac fermions in d-dimensional space that interact with a d-tuplet of Higgs field supporting point defects.

The gradient expansion is presented in Sec. II and applied to point defects in Sec. III. We conclude in Sec. IV.

II. GRADIENT EXPANSION OF THE COUNTING FORMULA

Let the charge-conjugation-symmetric single-particle Hamiltonian $\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}(\boldsymbol{x})\big)$ be such that its dependence on the space coordinate

$$\boldsymbol{x}\in\mathbb{R}^d \tag{2.1}$$

is implicit through that of a static vector-valued order parameter

$$\boldsymbol{\varphi}(\boldsymbol{x})\in\mathbb{R}^D, \tag{2.2}$$

while its dependence on the canonical momentum

$$\hat{\boldsymbol{p}}\equiv-\mathrm{i}\boldsymbol{\partial}\equiv-\mathrm{i}\frac{\partial}{\partial\boldsymbol{x}} \tag{2.3}$$

is explicit. (We have chosen natural units, i.e., $\hbar=1$.)

The Hermiticity of the single-particle Hamiltonian $\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}(\boldsymbol{x})\big)$ implies the existence of a conserved quasiparticle charge, which we construct explicitly with the help of the second quantized Hamiltonian[20]

$$\widehat{H}\equiv\int\limits_{\boldsymbol{x}}\widehat{\Psi}^\dagger(\boldsymbol{x})\,\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}(\boldsymbol{x})\big)\,\widehat{\Psi}(\boldsymbol{x}). \tag{2.4a}$$

The creation operators $\widehat{\Psi}^\dagger(\boldsymbol{x})$ and the annihilation operators $\widehat{\Psi}(\boldsymbol{x})$ obey here the fermion algebra

$$\begin{aligned}&\Big\{\widehat{\Psi}_r(\boldsymbol{x}),\widehat{\Psi}^\dagger_{r'}(\boldsymbol{x}')\Big\}=\delta_{r,r'}\delta(\boldsymbol{x}-\boldsymbol{x}'),\\&\Big\{\widehat{\Psi}_r(\boldsymbol{x}),\widehat{\Psi}_{r'}(\boldsymbol{x}')\Big\}=0,\qquad r,r'=1,\cdots,R,\end{aligned} \tag{2.4b}$$

in some R-dimensional representation of the charge-conjugation-symmetric single-particle Hamiltonian

$$\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}(\boldsymbol{x})\big)=-\,\mathcal{C}^{-1}\,\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}(\boldsymbol{x})\big)\,\mathcal{C}. \tag{2.4c}$$

The norm-preserving operation for charge conjugation in the R-dimensional single-particle representation is denoted by $\mathcal{C}$. The short-hand notation $\int\limits_{\boldsymbol{x}}$ stands for the space integration $\int\mathrm{d}^d\boldsymbol{x}$. The definition of the conserved quasiparticle charge requires filling of the Fermi sea for the quasiparticles created by the field $\widehat{\Psi}^\dagger_r(\boldsymbol{x})$.

We begin with an expression for a quasiparticle charge density in the vicinity of a point $\boldsymbol{x}$. To do so, we shall assume that the static vector-valued order parameter can be decomposed additively, i.e.,[21]

$$\boldsymbol{\varphi}(\boldsymbol{x})=\boldsymbol{\varphi}_0+\delta\boldsymbol{\varphi}(\boldsymbol{x}), \tag{2.5}$$

in such a way that: (i) the single-particle Hamiltonian

$$\mathcal{H}_0(\hat{\boldsymbol{p}})\equiv\mathcal{H}\big(\hat{\boldsymbol{p}},\boldsymbol{\varphi}_0\big) \tag{2.6}$$

is translation invariant, (ii) the single-particle eigenvalue spectrum of $\mathcal{H}_0(\hat{\boldsymbol{p}})$ is fully gaped with the gap $2\Delta_0>0$,

(iii) the changes between $\delta\boldsymbol{\varphi}(\boldsymbol{x})$ and $\delta\boldsymbol{\varphi}(\boldsymbol{y})$ when $\boldsymbol{x}$ and $\boldsymbol{y}$ are a distance of order $1/\Delta_0$ apart are small. Condition (iii) implies that the gradient of $\delta\boldsymbol{\varphi}(\boldsymbol{x})$ can be viewed as a smooth and small perturbation that can be treated perturbatively in the gradient expansion that will follow shortly.

Condition (i) implies that

$$\begin{aligned}\widehat{H}_0 &\equiv \int_{\boldsymbol{x}} \widehat{\Psi}^\dagger(\boldsymbol{x})\, \mathcal{H}_0(\hat{\boldsymbol{p}})\, \widehat{\Psi}(\boldsymbol{x}) \\ &= \int_{\boldsymbol{p}} \widehat{\Psi}^\dagger(\boldsymbol{p})\, \mathcal{H}_0(\boldsymbol{p})\, \widehat{\Psi}(\boldsymbol{p})\end{aligned} \tag{2.7a}$$

with the symmetric Fourier conventions

$$\begin{aligned}\widehat{\Psi}^\dagger(\boldsymbol{x}) &:= (2\pi)^{+d/2} \int_{\boldsymbol{p}} e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, \widehat{\Psi}^\dagger(\boldsymbol{p}), \\ \widehat{\Psi}^\dagger(\boldsymbol{p}) &:= (2\pi)^{-d/2} \int_{\boldsymbol{x}} e^{+\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, \widehat{\Psi}^\dagger(\boldsymbol{x}),\end{aligned} \tag{2.7b}$$

for the annihilation (creation) operators and the asymmetric convention

$$\begin{aligned}\mathcal{K}(\boldsymbol{x}) &:= \int_{\boldsymbol{p}} e^{+\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \mathcal{K}(\boldsymbol{p}), \\ \mathcal{K}(\boldsymbol{p}) &:= \int_{\boldsymbol{x}} e^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \mathcal{K}(\boldsymbol{x}),\end{aligned} \tag{2.7c}$$

for any kernel $\mathcal{K}$ (such as the Hamiltonian $\mathcal{H}$). Here, $\int_{\boldsymbol{p}}$ is a short-hand notation for the momentum-space integration $\int \mathrm{d}^d\boldsymbol{p}/(2\pi)^d$.

Condition (ii) implies the existence of the characteristic length scale $1/\Delta_0$.

Define the quasiparticle charge density

$$\rho_\gamma(\boldsymbol{x}) := \int_\gamma \frac{\mathrm{d}\omega}{2\pi} \left\langle \boldsymbol{x} \left| \mathrm{tr}_R \left[\mathcal{G}(\omega) - \mathcal{G}_0(\omega)\right] \right| \boldsymbol{x} \right\rangle . \tag{2.8a}$$

Here, tr_R denotes the trace over the R-dimensional degrees of freedom and we have introduced the Euclidean single-particle Green functions

$$\mathcal{G}(\omega) := \frac{1}{\mathrm{i}\omega - \mathcal{H}}, \qquad \mathcal{G}_0(\omega) := \frac{1}{\mathrm{i}\omega - \mathcal{H}_0}. \tag{2.8b}$$

The quasiparticle charge density $\rho_\gamma(\boldsymbol{x})$ depends on the contour of integration γ. The latter is chosen as in Appendix A so that the integration over the ω-complex plane picks up *only the first-order poles from the non-vanishing and negative energy eigenvalues* of $\mathcal{H}$ and of $\mathcal{H}_0$ [see Eq. (A15)].

It is shown in Appendix A [see Eq. (A16c)] that Eq. (2.4c) and condition (ii) imply that the *total number N of unoccupied zero modes* of $\mathcal{H}$ is related to the conserved quasiparticle charge Q_γ by

$$\frac{N}{2} = -\int \mathrm{d}^d\boldsymbol{x}\, \rho_\gamma(\boldsymbol{x}) \equiv -Q_\gamma. \tag{2.9}$$

According to the counting formula (2.9), computing N reduces to computing the conserved quasiparticle charge Q_γ induced by the smooth variation of $\delta\boldsymbol{\varphi}(\boldsymbol{x})$ through space defined in Eq. (2.1). The Hermiticity (2.4c) of the single-particle Hamiltonian guarantees the existence of this conserved quasiparticle charge. A local law for the conservation of quasiparticle current follows when the single-particle Hamiltonian is local. For single-particle Hamiltonians with the global U(1) symmetry delivering the conservation of the total electric charge, the conserved quasiparticle charge Q_γ is nothing but the electric charge in units in which the electron charge is unity. For single-particle Hamiltonians describing Bogoliubov-de-Gennes (BdG) quasiparticles, this global U(1) symmetry is spontaneously broken. The conserved quasiparticle charge Q_γ is then related to the conserved thermal current of BdG quasiparticles. The counting formula (2.9) appears implicitly in Ref. 2 and explicitly in Ref. 3.[4]

There is an alternative to specifying γ in the counting formula (2.9). We can regulate the first-order pole of the Green function $\mathcal{G}(\omega)$ at $\omega = 0$ by adding a perturbation that moves all zero modes to strictly positive energies. This perturbation must be small if all these positive energy increments are to remain much smaller than the threshold Δ_0 to the continuum. We can then safely replace the contour of integration γ in the counting formula (2.9) by $\mathbb{R}$ since the subtraction of $\mathcal{G}_0$ from $\mathcal{G}$ insures the convergence of the ω integration for large ω.

For example, we imagine that it is possible to augment the vector-valued order parameter (2.5) by the conjugation-symmetry-breaking real-valued field ϕ without loosing conditions (i)-(iii). More precisely, we define the $(D+1)$-tuplet

$$\boldsymbol{\phi}(\boldsymbol{x}) \equiv \begin{pmatrix} \phi_1(\boldsymbol{x}) \\ \vdots \\ \phi_D(\boldsymbol{x}) \\ \phi_{D+1} \end{pmatrix} \equiv \begin{pmatrix} \varphi_1(\boldsymbol{x}) \\ \vdots \\ \varphi_D(\boldsymbol{x}) \\ \phi \end{pmatrix} \tag{2.10a}$$

and we assume that Eq. (2.4c) becomes

$$\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x}), \phi\big) = -\,\mathcal{C}^{-1}\, \mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x}), -\phi\big)\, \mathcal{C}. \tag{2.10b}$$

It then follows that

$$Q(\phi) := \int \mathrm{d}^d\boldsymbol{x}\, \rho(\boldsymbol{x}, \phi) = -Q(-\phi) \tag{2.11}$$

where the quasiparticle charge density $\rho(\boldsymbol{x}, \phi)$ is obtained from Eq. (2.8) with $\mathbb{R}$ substituting for γ, Hamiltonian (2.10b) substituting for $\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x})\big)$, and $\mathcal{H}_0 \equiv \mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}_0, \phi\big)$ substituting for $\mathcal{H}_0 \equiv \mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}_0\big)$. The smoking gun for the *unoccupied zero modes* is now a discontinuity at the conjugation-symmetric point $\phi = 0$ of the odd function $Q(\phi)$ of ϕ, i.e., the counting formula (2.9) has become

$$\frac{N}{2} = -\lim_{\phi\to 0} \int \mathrm{d}^d\boldsymbol{x}\, \rho(\boldsymbol{x}, \phi) \equiv -\lim_{\phi\to 0} Q(\phi) \tag{2.12}$$

where the sign of ϕ is to be chosen so as to move the zero mode in energy to positive energies. Furthermore, we can relax the condition that the symmetry breaking ϕ is constant everywhere in space ($\boldsymbol{x}$) provided that the condition

$$\boldsymbol{\phi}^2(\boldsymbol{x}) \approx \boldsymbol{\varphi}_0^2 \equiv \Delta_0^2 \tag{2.13}$$

holds everywhere in $\mathbb{R}^d$. Condition (iii) then means that $\delta\boldsymbol{\phi}(\boldsymbol{x})$ varies slowly on the characteristic length scale $1/\Delta_0$.

Rather than computing $\rho(\boldsymbol{x})$ exactly, say with the help of numerical tools, we are after the leading contribution to the gradient expansion of the quasiparticle charge density $\rho(\boldsymbol{x})$, which we shall denote as $\rho_{\text{adia}}(\boldsymbol{x})$ where the subscript "adia"a refers to the adiabatic approximation contained in condition (iii).

The order parameter (2.10a) enters linearly in all the single Hamiltonians that we shall consider explicitly in this paper. Hence, there follows the additive law

$$\begin{aligned}\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\phi}(\boldsymbol{x})\big) &= \mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\phi}_0\big) + \mathcal{V}\big(\hat{\boldsymbol{p}}, \delta\boldsymbol{\phi}(\boldsymbol{x})\big) \\ &\equiv \mathcal{H}_0 + \mathcal{V}\big(\hat{\boldsymbol{p}}, \delta\boldsymbol{\phi}(\boldsymbol{x})\big)\end{aligned} \tag{2.14}$$

upon insertion of

$$\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\phi}_0 + \delta\boldsymbol{\phi}(\boldsymbol{x}). \tag{2.15}$$

Upon second-quantization, this implies that

$$\widehat{H} = \int\limits_{\boldsymbol{p}} \int\limits_{\boldsymbol{q}} \widehat{\Psi}^\dagger(\boldsymbol{p}) \left[\mathcal{H}_0(\boldsymbol{p})\, \delta(\boldsymbol{p}-\boldsymbol{q}) + \mathcal{V}(\boldsymbol{p};\boldsymbol{q})\right] \widehat{\Psi}(\boldsymbol{q}) \tag{2.16a}$$

holds with

$$\mathcal{V}(\boldsymbol{p};\boldsymbol{q}) := \left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\phi}}\right)_0 \left(\frac{\boldsymbol{p}+\boldsymbol{q}}{2}\right) \cdot \delta\boldsymbol{\phi}(\boldsymbol{p}-\boldsymbol{q}). \tag{2.16b}$$

The subscript 0 means setting $\delta\boldsymbol{\phi}$ to zero so that the gradient

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\phi}}\right)_0 = -\left(\frac{\partial \mathcal{G}^{-1}}{\partial \boldsymbol{\phi}}\right)_0 \tag{2.16c}$$

depends only on the single-particle canonical momentum operator $\hat{\boldsymbol{p}} \equiv -\mathrm{i}\boldsymbol{\partial}$. We have also adopted the convention that matrix elements of $\hat{\boldsymbol{p}}$ are to be symmetrized, i.e.,

$$\left(f^* \hat{\boldsymbol{p}} g\right)(\boldsymbol{x}) \equiv -\frac{\mathrm{i}}{2}\left[f^*\left(\frac{\partial g}{\partial \boldsymbol{x}}\right) - \left(\frac{\partial f^*}{\partial \boldsymbol{x}}\right) g\right](\boldsymbol{x}) \tag{2.16d}$$

for any differentiable functions f and g.

We now expand the quasiparticle charge density up to the first non-trivial order in an expansion in powers of $\mathcal{V}$ with the help of the geometrical series

$$\mathcal{G}(\omega) - \mathcal{G}_0(\omega) \approx \sum_{n=1}^{\infty} \left(\mathcal{G}_0(\omega)\mathcal{V}\right)^n \mathcal{G}_0(\omega). \tag{2.17}$$

This gives the expansion

$$\rho(\boldsymbol{x}) \approx \sum_{n=1}^{\infty} \rho_n(\boldsymbol{x}) \tag{2.18a}$$

with

$$\rho_n(\boldsymbol{x}) := \int\limits_{\omega} \int\limits_{\boldsymbol{p}} \int\limits_{\boldsymbol{q}_1} \cdots \int\limits_{\boldsymbol{q}_n} e^{\mathrm{i}(\boldsymbol{q}_1+\cdots+\boldsymbol{q}_n)\cdot\boldsymbol{x}} \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_n=1}^{D+1} I_{\mathsf{a}_n,\cdots,\mathsf{a}_1}(\omega,\boldsymbol{p},\boldsymbol{q}_1,\cdots,\boldsymbol{q}_n)\, \delta\phi_{\mathsf{a}_n}(\boldsymbol{q}_n)\cdots\delta\phi_{\mathsf{a}_1}(\boldsymbol{q}_1) \tag{2.18b}$$

where $\int\limits_{\omega} \equiv \int \frac{\mathrm{d}\omega}{2\pi}$ and the integrand

$$
\begin{aligned}
I_{\mathsf{a}_n,\cdots,\mathsf{a}_1}(\omega,\boldsymbol{p},\boldsymbol{q}_1,\cdots,\boldsymbol{q}_n) := \mathrm{tr}_R\Bigg[& \\
& \mathcal{G}_0\left(\omega,\boldsymbol{p}+\boldsymbol{q}_1+\boldsymbol{q}_2+\cdots+\boldsymbol{q}_n\right)\left(\frac{\partial\mathcal{H}}{\partial\phi_{\mathsf{a}_n}}\right)_0\left(\omega,\boldsymbol{p}+\boldsymbol{q}_1+\boldsymbol{q}_2+\cdots+\boldsymbol{q}_{n-1}+\frac{\boldsymbol{q}_n}{2}\right)\times \\
& \mathcal{G}_0\left(\omega,\boldsymbol{p}+\boldsymbol{q}_1+\boldsymbol{q}_2+\cdots+\boldsymbol{q}_{n-1}\right)\left(\frac{\partial\mathcal{H}}{\partial\phi_{\mathsf{a}_{n-1}}}\right)_0\left(\omega,\boldsymbol{p}+\boldsymbol{q}_1+\boldsymbol{q}_2+\cdots+\boldsymbol{q}_{n-2}+\frac{\boldsymbol{q}_{n-1}}{2}\right)\times \\
& \qquad\vdots \\
& \mathcal{G}_0\left(\omega,\boldsymbol{p}+\boldsymbol{q}_1\right)\left(\frac{\partial\mathcal{H}}{\partial\phi_{\mathsf{a}_1}}\right)_0\left(\omega,\boldsymbol{p}+\frac{\boldsymbol{q}_1}{2}\right)\times \\
& \mathcal{G}_0\left(\omega,\boldsymbol{p}\right)\Bigg].
\end{aligned}
\tag{2.18c}
$$

Finally, we expand the integrand (2.18c) in powers of the coordinates q_{1i_1}, $\cdots$, q_{ni_n} of the momenta $\boldsymbol{q}_1$, $\cdots$, $\boldsymbol{q}_n$, which are then to be integrated over. Condition (iii) suggests that we keep only first-order derivatives in the slowly varying fluctuations $\delta\boldsymbol{\phi}(\boldsymbol{x})$ of the order parameter (2.5). This approximation yields the leading contribution $\rho_{\mathrm{adia}}(\boldsymbol{x})$ in the gradient expansion of the quasiparticle charge density (2.18a) and becomes exact in the limit when the ratio between the characteristic length scale $1/\Delta_0$ and the characteristic length scale over which the change in $\delta\boldsymbol{\phi}(\boldsymbol{x})$ becomes significant vanishes.

However, to each order n in this expansion there are terms for which not all $\delta\boldsymbol{\phi}(\boldsymbol{x})$ are differentiated. These terms do not have to be small. Hence, they should be treated non-perturbatively. This is achieved by replacing the Euclidean single-particle Green function

$$
\mathcal{G}_0(\omega,\boldsymbol{p}) := \frac{1}{\mathrm{i}\omega-\mathcal{H}\big(\boldsymbol{p},\boldsymbol{\phi}_0\big)} \tag{2.19}
$$

with the Euclidean semi-classical Green function

$$
\mathcal{G}_{\text{s-c}}(\omega,\boldsymbol{p},\boldsymbol{x}) := \frac{1}{\mathrm{i}\omega-\mathcal{H}\big(\boldsymbol{p},\boldsymbol{\phi}(\boldsymbol{x})\big)} \tag{2.20}
$$

to any given order n in the expansion (2.18). This result is rooted in the fact that the choice of $\boldsymbol{\phi}_0$ in the additive decomposition (2.15) is arbitrary from the point of view of the expansion (2.18). Such arbitrariness should not appear in the final result, i.e., the final result should only depend on $\boldsymbol{\phi}(\boldsymbol{x})$ or $\partial_i\phi_{\mathsf{a}}(\boldsymbol{x}) \equiv \frac{\partial\phi_{\mathsf{a}}}{\partial x^i}(\boldsymbol{x})$ with $\mathsf{a}=1,\cdots,d+1$ and $i=1,\cdots,d$.[22]

Thus, by collecting the appropriate derivatives of the spatially varying order parameter $\delta\boldsymbol{\phi}(\boldsymbol{x})$, we arrive at expressions for the induced quasiparticle charge density. We now analyze this expression according to which term is the first non-vanishing contribution to the expansion.

When the adiabatic approximation for the quasiparticle charge density is the non-vanishing $n=1$ term in the gradient expansion, it is given by

$$
\rho_{\mathrm{adia}}(\boldsymbol{x}) = \mathcal{I}_{i_1\mathsf{a}_1}(-\mathrm{i})\left(\partial_{i_1}\phi_{\mathsf{a}_1}\right)(\boldsymbol{x}). \tag{2.21a}
$$

Here, the summation convention is assumed over repeated indices and the coefficients are given by

$$
\mathcal{I}_{i_1\mathsf{a}_1} := \int\limits_{\omega}\int\limits_{\boldsymbol{p}} \frac{1}{2}\mathrm{tr}_R\left(\left[\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_1}},\frac{\partial\mathcal{G}}{\partial p_{i_1}}\right]\mathcal{G}\right)_0(\omega,\boldsymbol{p}). \tag{2.21b}
$$

The subscript 0 refers to the semi-classical Green function (2.20). This case is the relevant one for the study of point defects in one-dimensional space.

When the first non-vanishing contribution to the adiabatic expansion occurs at $n=2$, then the quasiparticle charge density contains two gradients and is given by

$$
\rho_{\mathrm{adia}}(\boldsymbol{x}) = \mathcal{I}_{i_2\mathsf{a}_2 i_1\mathsf{a}_1}(-\mathrm{i})^2\left(\partial_{i_2}\phi_{\mathsf{a}_2}\right)\left(\partial_{i_1}\phi_{\mathsf{a}_1}\right)(\boldsymbol{x}). \tag{2.22a}
$$

By assumption, $\mathcal{I}_{i_1\mathsf{a}_1}$ defined by Eq. (2.21b) vanishes, but

$$
\begin{aligned}
\mathcal{I}_{i_2\mathsf{a}_2 i_1\mathsf{a}_1} := -\int\limits_{\omega}\int\limits_{\boldsymbol{p}} \frac{1}{2}\mathrm{tr}_R\Bigg(& 2\frac{\partial\mathcal{G}}{\partial p_{i_2}}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_2}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_1}}\frac{\partial\mathcal{G}}{\partial p_{i_1}} \\
& + \mathcal{G}\frac{\partial^2\mathcal{G}^{-1}}{\partial p_{i_2}\partial\phi_{\mathsf{a}_2}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_1}}\frac{\partial\mathcal{G}}{\partial p_{i_1}} \\
& + \frac{\partial\mathcal{G}}{\partial p_{i_2}}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_2}}\mathcal{G}\frac{\partial^2\mathcal{G}^{-1}}{\partial p_{i_1}\partial\phi_{\mathsf{a}_1}}\mathcal{G}\Bigg)_0(\omega,\boldsymbol{p})
\end{aligned}
\tag{2.22b}
$$

does not. Again, the summation convention is assumed over repeated indices and the subscript 0 refers to the

semi-classical Green function (2.20). We have chosen to represent Eq. (2.22) so as to make the reality of $\rho_{\text{adia}}(\boldsymbol{x})$ manifest. This case is the relevant one for the study of point defects in two-dimensional space.

Observe that, whenever $n > 1$, we must allow for the possibility that

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial \phi_{\mathsf{a}}} \equiv -\frac{\partial^2 \mathcal{G}^{-1}}{\partial p_i \partial \phi_{\mathsf{a}}} \tag{2.23}$$

is non-vanishing for some $i = 1, \cdots, d$ and some $\mathsf{a} = 1, \cdots, D+1$. These terms occur when dealing with a p-wave superconductor in $d = 2$ dimensions as we do in Sec. III B; however, we find by explicit calculation that these terms vanish upon integration over ω and $\boldsymbol{p}$.

When the order parameter is independent of momentum,

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial \phi_{\mathsf{a}}} \equiv -\frac{\partial^2 \mathcal{G}^{-1}}{\partial p_i \partial \phi_{\mathsf{a}}} = 0 \tag{2.24a}$$

for any $i = 1, \cdots, d$ and any $\mathsf{a} = 1, \cdots, D+1$. For example this is the case with $D = d$ for Dirac fermions in d-dimensional space interacting with $(d+1)$ real-valued Higgs fields, in which case it is the coefficient[23]

$$\begin{aligned}\mathcal{I}_{i_n \mathsf{a}_n \cdots i_1 \mathsf{a}_1} = -\,\mathrm{i} \int\limits_{\omega} \int\limits_{\boldsymbol{p}} \mathrm{tr}_R \Bigg[\left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial p_{i_n}} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \phi_{\mathsf{a}_n}} \right) \cdots \\ \cdots \left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial p_{i_1}} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \phi_{\mathsf{a}_1}} \right) \left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \omega} \right) \Bigg]_0 (\omega, \boldsymbol{p})\end{aligned} \tag{2.24b}$$

that controls the adiabatic approximation to n-th order through

$$\rho_{\text{adia}}(\boldsymbol{x}) = (-\mathrm{i})^d \mathcal{I}_{i_n \mathsf{a}_n \cdots i_1 \mathsf{a}_1} \left(\partial_{i_n} \phi_{\mathsf{a}_n} \right) \cdots \left(\partial_{i_1} \phi_{\mathsf{a}_1} \right) (\boldsymbol{x}) \tag{2.24c}$$

as we shall show in Sec. III C. The subscript 0 refers to the semi-classical Green function (2.20).

Finally, with the expression for the *local* quasiparticle charge density $\rho_{\text{adia}}(\boldsymbol{x})$ within the adiabatic approximation in hand, we can compute the *total* quasiparticle charge Q_{adia}. Naturally, one goes from the local density to the total charge by integrating the former over all space. We conclude that

$$\begin{aligned} Q_{\text{adia}} &= \int d^d x \, \rho_{\text{adia}}(\boldsymbol{x}) \\ &= \int \mathrm{d}^d x \, (-\mathrm{i})^d \, \mathcal{I}_{i_n \mathsf{a}_n \cdots i_1 \mathsf{a}_1}(\boldsymbol{x}) \\ &\quad \times \left(\partial_{i_n} \phi_{\mathsf{a}_n} \right) \cdots \left(\partial_{i_1} \phi_{\mathsf{a}_1} \right) (\boldsymbol{x}), \end{aligned} \tag{2.25a}$$

where it is the Euclidean semi-classical Green function

$$\mathcal{G}_{\text{s-c}}(\omega, \boldsymbol{p}, \boldsymbol{x}) := \frac{1}{\mathrm{i}\omega - \mathcal{H}\big(\boldsymbol{p}, \boldsymbol{\phi}(\boldsymbol{x})\big)} \tag{2.25b}$$

that enters the kernel $\mathcal{I}$.

III. ZERO MODES INDUCED BY POINT DEFECTS

We are going to apply the adiabatic expansion from Sec. II to the case of point defects supported by the static order parameter

$$\boldsymbol{\phi}(\boldsymbol{x}) \equiv \begin{pmatrix} \phi_1(\boldsymbol{x}) \\ \vdots \\ \phi_D(\boldsymbol{x}) \\ \phi_{D+1}(\boldsymbol{x}) \end{pmatrix} \equiv \begin{pmatrix} \varphi_1(\boldsymbol{x}) \\ \vdots \\ \varphi_D(\boldsymbol{x}) \\ \phi(\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^{d+1} \tag{3.1a}$$

and

$$\boldsymbol{\phi}^2(\boldsymbol{x}) \approx \boldsymbol{\varphi}_0^2 \equiv \Delta_0^2 \tag{3.1b}$$

when space is d-dimensional, i.e., when

$$\boldsymbol{x} \in \mathbb{R}^d. \tag{3.1c}$$

The component $\phi_{D+1}(\boldsymbol{x}) \equiv \phi(\boldsymbol{x})$ breaks locally the charge conjugation symmetry. This component determines if a zero mode bound to a defect at $\boldsymbol{x}$ is occupied or unoccupied. All remaining components of the order parameter (3.1a) are compatible with the charge-conjugation symmetry of the single-particle Hamiltonian. The condition (3.1b) suggests that the homotopy group

$$\Pi_d(S^d) = \mathbb{Z} \tag{3.2}$$

of smooth maps from the compactification of $\mathbb{R}^d$ into the d-sphere S^d to the d-sphere S^d in order-parameter space $\mathbb{R}^{d+1}$ might play an important role.[24]

We begin in one dimensional space (x) with a generic single-particle Hamiltonian. We show that the conserved quasiparticle charge Q_{adia} that enters the counting formula (2.9) is *the first Chern number* if we relax the condition that the charge-symmetry-breaking component ϕ of the order parameter is infinitesimally small and if we compactify space (3.1c) and the order-parameter space (3.1a).

We continue with the chiral p-wave superconductor in two-dimensions when the superconducting order parameter supports a vortex. We show that the number of unoccupied zero modes bound to a vortex with unit vorticity computed from the adiabatic approximation agrees with the direct construction of zero modes once all microscopic data have been supplied. Moreover, we show that the conserved quasiparticle charge Q_{adia} that enters the counting formula (2.9) is also related to *the second Chern number* after compactification of both space (3.1c) and order-parameter space (3.1a). This is apriori surprising since the superconducting order parameter couples to the momentum contrary to the simpler case of Dirac fermions.

We also study Dirac single-particle Hamiltonians in d-dimensional space (3.1c) that are represented by Dirac matrices of dimension $R = 2^d$. We show how the conserved quasiparticle charge Q_{adia} that enters the counting

formula (2.9) is related to *the d-th Chern number* when the Dirac spinor couples to a $(d+1)$-tuplet of Higgs field.

We close by discussing how to interpret the adiabatic approximation.

A. Generic single-particle $\mathcal{H}$ when $d = 1$

We compactify space and the order-parameter space,

$$x \in S^1, \qquad \boldsymbol{\phi}(\theta) \in S^1 \subset \mathbb{R}^2, \tag{3.3}$$

respectively, such that

$$\boldsymbol{\phi}(x) := \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = m \begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \end{pmatrix}, \tag{3.4}$$

and denote with $\mathcal{G} = (\mathrm{i}\omega - \mathcal{H})^{-1}$ the single-particle Green function in Euclidean space for any suitable $R\times R$ matrix-valued single-particle Hamiltonian $\mathcal{H}\big(\hat{p}, \boldsymbol{\phi}(x)\big)$. Suitability means that the dependence on the momentum operator $\hat{p} \equiv -\mathrm{i}\partial_x$ in $\mathcal{H}\big(\hat{p}, \boldsymbol{\phi}(x)\big)$ is only restricted by locality while that on the two-component order parameter $\boldsymbol{\phi}(x)$ is linear. Furthermore, $\mathcal{H}\big(\hat{p}, \boldsymbol{\phi}(x)\big)$ obeys Eq. (2.10) for $D = 1$ and supports the spectral gap $2m$ for a uniform $\boldsymbol{\phi}_0$. Under these conditions, for any p the semi-classical $R\times R$ matrix $\mathcal{H}\big(p, \boldsymbol{\phi}(x)\big)$ is traceless and its square is proportional to the unit matrix with the smallest nonvanishing eigenvalue no smaller than m^2. We have introduced the spherical coordinate θ of the one-sphere $S^1 \subset \mathbb{R}^2$ in order-parameter space.

According to Eq. (2.21), the conserved quasiparticle charge Q_{adia} becomes

$$\begin{aligned} Q_{\mathrm{adia}} = \int_{\omega} \int_{p\in S^1} \int_{x\in S^1} \frac{1}{2} \mathrm{tr}_R \Big(& \mathcal{G}\partial_p \mathcal{G}^{-1} \mathcal{G} \partial_\theta \mathcal{G}^{-1} \mathcal{G} \\ & - \mathcal{G}\partial_\theta \mathcal{G}^{-1} \mathcal{G} \partial_p \mathcal{G}^{-1} \mathcal{G} \Big)_0 (-\mathrm{i}) \partial_x \theta. \end{aligned} \tag{3.5}$$

The subscript 0 refers to the semi-classical Green function (2.20). We use the identity

$$\partial_\omega \mathcal{G}^{-1} = \mathrm{i}, \tag{3.6}$$

introduce the family of indices labeled by ν,

$$\nu_1, \nu_2, \nu_3 = 0, 1, 2, \tag{3.7}$$

and the Euclidean 3-momentum

$$K_\nu := (\omega, p, \theta). \tag{3.8}$$

With the help of the manipulations made between Eqs. (C34) and (C45), it is possible to re-write Eq. (3.5) as

$$\begin{aligned} Q_{\mathrm{adia}} = & -\frac{1}{24\pi^2} \int \mathrm{d}\omega \int_0^{2\pi} \mathrm{d}p \int_0^{2\pi} \mathrm{d}\theta \; \epsilon_{\nu_1\nu_2\nu_3} \\ & \times \mathrm{tr}_R \Big(\mathcal{G}\partial_{\nu_1} \mathcal{G}^{-1} \mathcal{G} \partial_{\nu_2} \mathcal{G}^{-1} \mathcal{G} \partial_{\nu_3} \mathcal{G}^{-1} \Big)_0 . \end{aligned} \tag{3.9}$$

Equation (3.9) is *the first Chern number*.[25] Thus, the conserved quasiparticle charge Q_{adia} obeying Eq. (3.9) takes integer values. We defer to Sec. III D and Appendix C 1 for a more detailed discussion of the connection between *the first Chern number* and the number of unoccupied zero modes.

B. Chiral p-wave superconductor when $d = 2$

1. Definition

One of the main results of this paper consists in applying the counting formula (2.12) to the two-dimensional chiral p-wave superconductor with the single-particle BdG Hamiltonian

$$\begin{aligned} \mathcal{H}^{\mathrm{BdG}}_{p_x+\mathrm{i}p_y} &:= \begin{pmatrix} \varepsilon(\hat{\boldsymbol{p}}) & \frac{1}{2}\{\hat{p}_1 - \mathrm{i}\hat{p}_2, \Delta(\boldsymbol{x})\} \\ \frac{1}{2}\{\hat{p}_1 + \mathrm{i}\hat{p}_2, \Delta^*(\boldsymbol{x})\} & -\varepsilon(\hat{\boldsymbol{p}}) \end{pmatrix} \\ &= \mathcal{H}^{\mathrm{BdG}\,\dagger}_{p_x+\mathrm{i}p_y}, \\ \varepsilon(\hat{\boldsymbol{p}}) &:= \frac{\hat{\boldsymbol{p}}^2}{2m} - \mu, \end{aligned} \tag{3.10}$$

for the case when the superconducting order parameter supports the vorticity $n_{\mathrm{v}} = \pm 1$ at the origin of two-dimensional Euclidean space, i.e., when

$$\Delta(\boldsymbol{x}) := \Delta_0(r)\, e^{\mathrm{i}\theta\, n_{\mathrm{v}}} \tag{3.11}$$

where r and θ are the polar coordinates of $\boldsymbol{x} \in \mathbb{R}^2$.

The applicability of the counting formula (2.12) follows from the antiunitary conjugation symmetry

$$\mathcal{C}_{\mathrm{ph}}^{-1}\, \mathcal{H}^{\mathrm{BdG}}_{p_x+\mathrm{i}p_y}\, \mathcal{C}_{\mathrm{ph}} = -\mathcal{H}_{p_x+\mathrm{i}p_y}, \tag{3.12a}$$

where

$$\mathcal{C}_{\mathrm{ph}} := \rho_1\, \mathsf{K}. \tag{3.12b}$$

The Pauli matrices ρ_1, ρ_2, and ρ_3 encode the particle and hole block structure of the single-particle BdG Hamiltonian (3.10). Complex conjugation is denoted by K.

For a unit vorticity, Read and Green have shown in Ref. 26 the existence of a single zero mode bound to the vortex by solving the eigenvalue problem at zero energy. A zero mode in a single-particle BdG Hamiltonian realizes a Majorana fermion, for it cannot be distinguished from its particle-hole conjugate. For an arbitrary vorticity $n_{\mathrm{v}} \in \mathbb{Z}$, it is shown in Refs. 27 and 28 by solving the eigenvalue problem at zero energy that the number of Majorana fermions is one if n_{v} is odd and zero otherwise.

We do not expect the adiabatic approximation to the counting formula (2.12) to capture this subtle parity effect since it is only sensitive to the net vorticity n_{v} trapped in region of space much larger than the characteristic length scale $\ell \gg 1/\Delta_0$ over which $|\delta\boldsymbol{\varphi}|$ changes by the amount of order Δ_0. The adiabatic approximation fails to distinguish the case of a single vortex with

vorticity $n_{\rm v}$ and $n_{\rm v}$ vortices with unit vorticity that are separated by a distance of order ℓ. This parity effect is a non-perturbative effect from the point of view of the gradient expansion that we now present.

2. Counting zero modes

To count the unoccupied zero modes induced by a vortex in the superconducting order parameter with the gradient expansion of Sec. II, we need to simultaneously move in energy the zero mode and properly regulate the singularity at the core of the vortex. Achieving both goals is impossible with the 2×2 BdG Hamiltonian for the chiral p-wave single-particle Hamiltonian. On the other hand, both conditions are met after doubling the BdG single-particle Hamiltonian so as to introduce an auxiliary chiral-symmetry-breaking perturbation in two ways (denoted by the subscripts $\mp$),

$$\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big) := \begin{pmatrix} \mathcal{H}^{\rm BdG}_{p_x+{\rm i}p_y} & \frac{1}{2}\rho_0\,\{\hat{p}_{\mp},\phi_3(\boldsymbol{x})\} \\ \frac{1}{2}\rho_0\,\{\hat{p}_{\pm},\phi_3(\boldsymbol{x})\} & -\mathcal{H}^{\rm BdG}_{p_x+{\rm i}p_y} \end{pmatrix}. \tag{3.13a}$$

The short-hand notations $\hat{p}_{\pm} := \hat{p}_x \pm {\rm i}\hat{p}_y$ was introduced and ρ_0 is the 2×2 unit matrix. The triplet

$$\boldsymbol{\phi}(\boldsymbol{x}) \equiv \begin{pmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ \phi_3(\boldsymbol{x}) \end{pmatrix} := \begin{pmatrix} \Delta_1(\boldsymbol{x}) \\ \Delta_2(\boldsymbol{x}) \\ \phi_3(\boldsymbol{x}) \end{pmatrix} \tag{3.13b}$$

is real-valued and is made of the static superconducting order parameter $\Delta(\boldsymbol{x}) \equiv \Delta_1(\boldsymbol{x}) + {\rm i}\Delta_2(\boldsymbol{x})$ and of the static auxiliary charge-conjugation-symmetry-breaking field $\phi_3(\boldsymbol{x})$.

The spectrum of the pair of auxiliary single-particle Hamiltonians (3.13a) given a uniform order parameter $\boldsymbol{\phi}_0$ in Eq. (3.13b) is

$$\varepsilon_0^2(\boldsymbol{p}) = \left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)^2 + \boldsymbol{p}^2\boldsymbol{\phi}_0^2. \tag{3.14}$$

Conditions (i) and (ii) from Sec. II are thus fulfilled since the Fermi surface $\boldsymbol{p}^2 = 2m\mu$ is fully gaped.

When $\phi_3 = 0$, the auxiliary single-particle Hamiltonian (3.13) represents two independent copies of the original chiral p-wave BdG Hamiltonian (3.10). When $\phi_3 = 0$, the spectrum of $\mathcal{H}^{\rm aux}_{\mp}$ is identical to the spectrum of $\mathcal{H}^{\rm BdG}_{p_x+{\rm i}p_y}$ up to a two-fold degeneracy arising from the doubling. Furthermore, $\phi_3 = 0$ implies that, in addition to the antiunitary charge conjugation symmetry

$$\mathcal{C}_{\rm ph}^{-1}\,\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big)\,\mathcal{C}_{\rm ph} = -\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big) \tag{3.15a}$$

with

$$\mathcal{C}_{\rm ph} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_1 \end{pmatrix}\mathsf{K}, \tag{3.15b}$$

where K denotes as before the operation of complex conjugation, that originates from Eq. (3.12), there exists an auxiliary unitary charge-conjugation symmetry

$$\mathcal{C}_{\rm ch}^{-1}\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big)\,\mathcal{C}_{\rm ch} = -\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big) \tag{3.16a}$$

with the generator of the auxiliary chiral transformation

$$\mathcal{C}_{\rm ch} := \begin{pmatrix} 0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix}. \tag{3.16b}$$

As we shall see in Sec. III C, Dirac Hamiltonians can also share unitary and antiunitary charge-conjugation symmetries. There are differences with Sec. III C however.

A first difference with Sec. III C is that

$$\mathcal{C}_{\rm ch}^{-1}\mathcal{H}^{\rm aux}_{\mp}\big(\hat{\boldsymbol{p}},\boldsymbol{\phi}(\boldsymbol{x})\big)\,\mathcal{C}_{\rm ch} = -\mathcal{H}^{\rm aux}_{\pm}\big(\hat{\boldsymbol{p}},C_{\rm ch}\boldsymbol{\phi}(\boldsymbol{x})\big) \tag{3.17a}$$

with

$$C_{\rm ch}\begin{pmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ \phi_3(\boldsymbol{x}) \end{pmatrix} \equiv \begin{pmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \\ -\phi_3(\boldsymbol{x}) \end{pmatrix}. \tag{3.17b}$$

A second difference with Sec. III C is the absence of relativistic invariance. A third difference is that Eq. (2.23) is non-vanishing.

We start from the expansion (2.18) of the quasiparticle charge density induced by the order parameter, the static triplet $\boldsymbol{\phi}$. We compute first the contribution from $n = 1$ for the pair of auxiliary Hamiltonians. It vanishes. The adiabatic approximation to the quasiparticle charge density for any one of the pair of auxiliary Hamiltonians is given by Eq. (2.22)

$$\begin{aligned} \rho^{\rm aux}_{\mp{\rm adia}}(\boldsymbol{x}) &= \pm\int\frac{{\rm d}\omega{\rm d}^2\boldsymbol{p}}{(2\pi)^3}\frac{8\boldsymbol{p}^2\left[\varepsilon(\boldsymbol{p})+2\mu\right]}{\left[\omega^2+\varepsilon^2(\boldsymbol{p})+\boldsymbol{p}^2\boldsymbol{\phi}^2(\boldsymbol{x})\right]^3} \\ &\quad\times\epsilon_{\mathsf{abc}}\Big(\left(\partial_1\phi_{\mathsf{a}}\right)\left(\partial_2\phi_{\mathsf{b}}\right)\phi_{\mathsf{c}}\phi_3\Big)(\boldsymbol{x}) \\ &= \pm\int\frac{{\rm d}^2\boldsymbol{p}}{(2\pi)^2}\frac{3\boldsymbol{p}^2\left[\varepsilon(\boldsymbol{p})+2\mu\right]}{2\left[\varepsilon^2(\boldsymbol{p})+\boldsymbol{p}^2\boldsymbol{\phi}^2(\boldsymbol{x})\right]^{5/2}} \\ &\quad\times\epsilon_{\mathsf{abc}}\Big(\left(\partial_1\phi_{\mathsf{a}}\right)\left(\partial_2\phi_{\mathsf{b}}\right)\phi_{\mathsf{c}}\phi_3\Big)(\boldsymbol{x}) \\ &= \pm\frac{1+{\rm sgn}(\mu)}{2\pi\left|\boldsymbol{\phi}(\boldsymbol{x})\right|^4}\epsilon_{\mathsf{abc}}\Big(\left(\partial_1\phi_{\mathsf{a}}\right)\left(\partial_2\phi_{\mathsf{b}}\right)\phi_{\mathsf{c}}\phi_3\Big)(\boldsymbol{x}). \end{aligned} \tag{3.18}$$

When $\mu < 0$, Eq. (3.18) gives $\rho^{\rm aux}_{\mp}(\boldsymbol{x}) = 0$. This is consistent with the absence of a normalizable zero energy solution for negative values of the chemical potential.[26–28]

When $\mu > 0$ and ϕ_3 is independent of space ($\boldsymbol{x}$), the adiabatic approximation to the conserved quasiparticle charge of the auxiliary Hamiltonian is

$$Q^{\rm aux}_{\mp{\rm adia}}(\phi_3) = \pm\frac{1}{\pi}\int{\rm d}\Theta\int_0^{\Delta_0}{\rm d}\rho\,\rho\,\frac{\phi_3^2}{\left(\rho^2+\phi_3^2\right)^2}, \tag{3.19a}$$

where the parametrization

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho(r)\cos\Theta(\theta) \\ \rho(r)\sin\Theta(\theta) \end{pmatrix} \tag{3.19b}$$

is assumed for the superconducting order parameter with r and θ denoting the polar coordinates of $\boldsymbol{x} \in \mathbb{R}^2$. In the limit in which ϕ_3 tends to zero, we get for the induced charge of the auxiliary Hamiltonian

$$\begin{aligned} Q^{\rm aux}_{\mp{\rm adia}} &= \pm \frac{1}{2\pi} \int {\rm d}\Theta \\ &= \pm \text{ winding number in } (\phi_1, \phi_2). \end{aligned} \tag{3.20}$$

To compute the number $N^{\rm aux}$ of unoccupied zero modes with the counting formula (2.12) induced by a vortex with vorticity unity, we choose the charge-conjugation-symmetry-breaking perturbation such that it shifts the zero mode in energy above the chemical potential. In the limit $\phi_3 \to 0$, we find that

$$N^{\rm aux} = 2. \tag{3.21}$$

Equation (3.21) implies that the adiabatic approximation for the number N of unoccupied zero modes of the original BdG Hamiltonian (3.10) induced by a vortex with unit vorticity is

$$N = 1. \tag{3.22}$$

3. Chern number

Next, we compactify space (3.1c) and the order-parameter space (3.1a),

$$\boldsymbol{x} \in S^2, \qquad \boldsymbol{\phi}(\boldsymbol{\theta}) \in S^2 \subset \mathbb{R}^3 \tag{3.23}$$

where we have introduced the spherical coordinates $\boldsymbol{\theta} = (\theta_1, \theta_2)$ of the two-sphere $S^2 \subset \mathbb{R}^3$. Motivated by Eq. (2.22), we separate the adiabatic approximation to the conserved quasiparticle charge of the auxiliary Hamiltonian into two contributions,

$$Q^{\rm aux}_{\rm adia} = Q^{\rm aux\prime}_{\rm adia} + Q^{\rm aux\prime\prime}_{\rm adia} \tag{3.24a}$$

where

$$\begin{aligned} Q^{\rm aux\prime}_{\rm adia} := -\int_\omega \int_{\boldsymbol{p}\in S^2} \int_{\boldsymbol{x}\in S^2} {\rm tr}_4 \Big(& \mathcal{G}\partial_{i_2}\mathcal{G}^{-1}\mathcal{G}\partial_{\mathsf{a}_2}\mathcal{G}^{-1} \\ & \times \mathcal{G}\partial_{\mathsf{a}_1}\mathcal{G}^{-1}\mathcal{G}\partial_{\mathsf{i}_1}\mathcal{G}^{-1}\mathcal{G}\Big)_0 \partial_{i_1}\phi_{\mathsf{a}_1}\partial_{i_2}\phi_{\mathsf{a}_2} \end{aligned} \tag{3.24b}$$

while

$$\begin{aligned} Q^{\rm aux\prime\prime}_{\rm adia} := \frac{1}{2}\int_\omega \int_{\boldsymbol{p}\in S^2} \int_{\boldsymbol{x}\in S^2} {\rm tr}_4 \Big(& \mathcal{G}\partial^2_{i_2\mathsf{a}_2}\mathcal{G}^{-1}\mathcal{G}\partial_{\mathsf{a}_1}\mathcal{G}^{-1}\partial_{i_1}\mathcal{G} \\ & + \partial_{i_2}\mathcal{G}\partial_{\mathsf{a}_2}\mathcal{G}^{-1}\mathcal{G}\partial^2_{i_1\mathsf{a}_1}\mathcal{G}^{-1}\mathcal{G}\Big)_0 \partial_{i_1}\phi_{\mathsf{a}_1}\partial_{i_2}\phi_{\mathsf{a}_2} \end{aligned} \tag{3.24c}$$

and $\mathcal{G} := ({\rm i}\omega - \mathcal{H}^{\rm aux}_{\mp})^{-1}$. The subscript 0 refers to the semi-classical Green function (2.20). The first contribution comes about when the order parameter decouples from the momentum. The second contribution arises when the order parameter and the momentum couple.

By explicit computation of the trace in Eq. (3.24b), one verifies that this trace is fully antisymmetric in all the indices $i_2, \mathsf{a}_2, i_1, \mathsf{a}_1$ where the family i and the family a of indices are distinct with

$$i_1, i_2 = 1, 2, \qquad \mathsf{a}_1, \mathsf{a}_2 = 1, 2, 3. \tag{3.25}$$

Furthermore, the trace in Eq. (3.24b) gives an integrand with even contributions in ω so that it does not vanish upon integration over ω. By explicit computation of the trace in Eq. (3.24c), one verifies that this trace yields an integrand that is odd in ω and thus vanishes upon integration over ω,

$$Q^{\rm aux\prime\prime}_{\rm adia} = 0. \tag{3.26}$$

With the help of the manipulations made between Eqs. (C34) and (C45), it is possible to write

$$\begin{aligned} Q^{\rm aux}_{\rm adia} &= Q^{\rm aux\prime}_{\rm adia} \\ &= \frac{-{\rm i}(2\pi)^2}{60} \int_K \epsilon_{\nu_1\cdots\nu_5} {\rm tr}_4 \left(\mathcal{G}\partial_{\nu_1}\mathcal{G}^{-1}\cdots\mathcal{G}\partial_{\nu_5}\mathcal{G}^{-1}\right)_0 \end{aligned} \tag{3.27a}$$

where we have introduced the family of indices labeled by ν

$$\nu_1, \cdots, \nu_5 = 1, \cdots, 5, \tag{3.27b}$$

the five-momentum

$$K_\nu = (\omega, p_1, p_2, \theta_1, \theta_2), \tag{3.27c}$$

and the domain of integration

$$\int_K \equiv \int \frac{{\rm d}\omega}{2\pi} \int_{\boldsymbol{p}\in S^2} \frac{{\rm d}\Omega_2(\boldsymbol{p})}{(2\pi)^2} \int_{\boldsymbol{\theta}\in S^2} \frac{{\rm d}\Omega_2(\boldsymbol{\theta})}{(2\pi)^2}. \tag{3.27d}$$

The "surface" element of the sphere S^2 is here denoted by ${\rm d}\Omega_2$. The subscript 0 refers to the semi-classical Green function (2.20). Equation (3.27) is *the second Chern number*.[29] It takes integer values only. It should be compared with Eq. (C28). We defer to Secs. III C, III D, and C 2 for a more detailed discussion of the connection between *this second Chern number* and the number of unoccupied zero modes.

We have verified that the *second Chern number* (3.27) vanishes for a s-wave superconducting order parameter that supports an isolated vortex with unit vorticity.

C. Dirac single-particle $\mathcal{H}^{\rm Dirac}_d$ for any d

Our purpose here is to apply the counting formula (2.12) to a Dirac single-particle Hamiltonian defined in the d-dimensional Euclidean space with the coordinates (2.1). We choose the dimensionality of the Dirac matrices to be

$$R = 2^d. \tag{3.28a}$$

The integer R is the smallest dimensionality compatible with an irreducible representation of the Clifford algebra generated by the unit $R \times R$ matrix $\mathbb{1}$ and the traceless and Hermitian matrices

$$\Gamma_\mu = \Gamma_\mu^\dagger, \quad \mathrm{tr}_R\, \Gamma_\mu = 0, \quad \{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu,\nu}\mathbb{1}, \quad (3.28b)$$

where $\mu, \nu = 1, \cdots, 2d+1$.[30] We choose the single-particle Dirac Hamiltonian such that a R-dimensional spinor is coupled to a $(d+1)$-tuplet of real-valued static Higgs fields

$$\boldsymbol{\phi}(\boldsymbol{x}) \in \mathbb{R}^{d+1}, \qquad \boldsymbol{x} \in \mathbb{R}^d. \quad (3.28c)$$

Accordingly,

$$\mathcal{H}_d^{\mathrm{Dirac}}\big(\hat{\boldsymbol{p}}, \boldsymbol{\phi}(\boldsymbol{x})\big) := \sum_{i=1}^{d} \Gamma_i \hat{p}_i + \sum_{\mathsf{a}=1}^{d+1} \Gamma_{d+\mathsf{a}} \phi_{\mathsf{a}}(\boldsymbol{x}). \quad (3.28d)$$

The decomposition (2.5) is generalized to

$$\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{\phi}_0 + \delta\boldsymbol{\phi}(\boldsymbol{x}) \quad (3.29)$$

and is assumed to hold together with conditions (iii) from Sec. II, once we have established the existence of charge-conjugation symmetry. Conditions (i) and (ii) follow from the spectrum

$$\varepsilon_0^2(\boldsymbol{p}) = \boldsymbol{p}^2 + \boldsymbol{\phi}_0^2 \quad (3.30)$$

of $\mathcal{H}_d^{\mathrm{Dirac}}\big(\hat{\boldsymbol{p}}, \boldsymbol{\phi}_0\big)$.

The Dirac Hamiltonian (3.28d) has the charge-conjugation symmetry

$$\mathcal{H}_d^{\mathrm{Dirac}} = -\Gamma_{d+\mathsf{a}} \mathcal{H}_d^{\mathrm{Dirac}} \Gamma_{d+\mathsf{a}} \quad (3.31a)$$

for any $\mathsf{a} = 1, \cdots, d+1$ as soon as $\phi_{\mathsf{a}} = 0$ everywhere in space. Without loss of generality, we choose the generator of the so-called chiral symmetry (3.31a) to be Γ_{2d+1}, i.e., we identify the charge-conjugation symmetry (2.4c) with the unitary operation represented by

$$\mathcal{C}_{\mathrm{ch}} := \Gamma_{2d+1} = (-\mathrm{i})^d \Gamma_1 \cdots \Gamma_{2d}. \quad (3.31b)$$

Equation (3.31) follows from $2d+1$ being an odd integer.[30] With this choice, the vector-valued order parameter (2.2) is

$$\varphi_a := \phi_a, \qquad a = 1, \cdots, d, \quad (3.32)$$

i.e.,

$$D = d. \quad (3.33)$$

It is shown in Appendix B that there exists a symmetric matrix $\mathcal{T}_d$ in the Clifford algebra (3.28b) that implements the time-reversal symmetry

$$\mathcal{H}_d^{\mathrm{Dirac}} = \mathcal{T}_d^{-1}\, \mathcal{H}_d^{\mathrm{Dirac}\,*}\, \mathcal{T}_d. \quad (3.34)$$

When

$$\phi_{d+1} = 0, \quad (3.35)$$

the symmetries (3.31a) and (3.34) of the Dirac Hamiltonian (3.28d) can be combined into the antiunitary operation

$$\mathcal{C}_{\mathrm{ph}} := \mathcal{T}_d\, \mathsf{K}\, \mathcal{C}_{\mathrm{ch}} \quad (3.36)$$

that implements a particle-hole transformation and under which the Dirac Hamiltonian (3.28d) is invariant,

$$\mathcal{H}_d^{\mathrm{Dirac}} = -\left(\mathcal{T}_d \Gamma_{2d+1}\right)^{-1} \mathcal{H}_d^{\mathrm{Dirac}\,*} \left(\mathcal{T}_d \Gamma_{2d+1}\right). \quad (3.37)$$

We present our main result for Dirac fermions in three steps. The reader is referred to Appendix C for their detailed derivations when $d = 1$ and $d = 2$, and a sketch of the derivation when $d > 2$.

First, the adiabatic approximation to the counting formula (2.12) is determined by the conserved quasiparticle adiabatic charge Q_{adia}. However, Q_{adia} can be computed without assuming charge-conjugation symmetry. It is only when using the counting formula (2.12) that Q_{adia} must be restricted to a charge-conjugation-symmetric configuration of the Higgs order parameter (3.1a). After relaxing the condition of charge-conjugation symmetry, we find

$$\begin{aligned} Q_{\mathrm{adia}} &= \mathrm{Chern}_d \\ &\equiv \frac{(-1)^{d(d-1)/2}(-\mathrm{i})^{d+1} d!}{(2d+1)!} (2\pi)^d \int_K \epsilon_{\nu_1 \cdots \nu_{2d+1}} \\ &\quad \times \mathrm{tr}_R \left(\mathcal{G} \partial_{\nu_1} \mathcal{G}^{-1} \cdots \mathcal{G} \partial_{\nu_{2d+1}} \mathcal{G}^{-1} \right)_0 \end{aligned} \quad (3.38a)$$

if we compactify momentum space and the target manifold, i.e., we have introduced the Euclidean momentum

$$K_\nu := (\omega, p_1, \cdots, p_d, \theta_1, \cdots, \theta_d) \quad (3.38b)$$

with $(p_1, \cdots, p_d) \in S^d$ and the spherical coordinates $\boldsymbol{\theta} \equiv (\theta_1, \cdots, \theta_d)$ with $\boldsymbol{\phi}(\boldsymbol{\theta}) \in S^d \subset \mathbb{R}^{d+1}$. The last inclusion serves to emphasize that the coordinates on the d-sphere in order-parameter space can involve a sizable breaking of the conjugation symmetry through $\phi_{d+1}(\boldsymbol{\theta}) \equiv \phi(\boldsymbol{\theta})$. The short-hand notation $\int_K$ stands for the integral over $\mathbb{R} \times S^d \times S^d$ defined in Eq. (C45a). The Euclidean single-particle Green function $\mathcal{G} = \big(\mathrm{i}\omega - \mathcal{H}_d^{\mathrm{Dirac}}\big)^{-1}$ is the one for the Dirac Hamiltonian. The subscript 0 refers to the semi-classical Green function (2.20). The second equality in Eq. (3.38a) *defines the d-th Chern number* in term of these Dirac Euclidean single-particle Green functions. It takes integer values and, conversely, to each integer N_{hedgehog} there corresponds a static $\boldsymbol{\phi}_{N_{\mathrm{hedgehog}}}(\boldsymbol{x}) \in S^d$ for which $Q_{\mathrm{adia}} = N_{\mathrm{hedgehog}}$. We call $\boldsymbol{\phi}_{N_{\mathrm{hedgehog}}}(\boldsymbol{x}) \in S^d$ a hedgehog with *the d-th Chern number* N_{hedgehog}.

Second, if we relax the condition that $\boldsymbol{\phi}(\boldsymbol{\theta}) \in S^d$ and replace it with Eq. (3.1b) instead, then the local quasiparticle current induced by the adiabatic variations of

the Higgs fields in space ($\boldsymbol{x}$) and time (x_0) is

$$j^\nu := \frac{\Omega_d^{-1}}{|\boldsymbol{\phi}|^{d+1}} \frac{\epsilon^{\nu\nu_1\cdots\nu_d}}{d!} \epsilon_{\mathtt{a}\mathtt{a}_1\cdots\mathtt{a}_d} \phi_{\mathtt{a}} \partial_{\nu_1} \phi_{\mathtt{a}_1} \cdots \partial_{\nu_d} \phi_{\mathtt{a}_d}. \quad (3.39a)$$

It obeys the continuity equation

$$\partial_\nu j^\nu = 0. \quad (3.39b)$$

Here,

$$|\boldsymbol{\phi}| := \sqrt{\phi_1^2 + \cdots + \phi_{d+1}^2}, \quad (3.39c)$$

the $(d+1)$ indices $\nu, \nu_1, \cdots, \nu_d$ run over space ($\boldsymbol{x}$) and time (x_0), and the $(d+1)$ indices $\mathtt{a}_1, \cdots, \mathtt{a}_{d+1}$ run over the $(d+1)$ components of the vector-valued Higgs field (3.28c). The Minkowski metric is used in space ($\boldsymbol{x}$) and time (x_0). Finally, Ω_d denotes the area of the d-sphere S^d. Equation (3.39a) was obtained by Goldstone an Wilczek in Ref. 31 when $d = 1$ and $d = 3$ and by Jaroszewicz in Ref. 32 when $d = 2$. The conserved quasiparticle charge induced by a hedgehog configuration is the integer

$$\begin{aligned} Q_{\text{adia}} &= \int \mathrm{d}^d \boldsymbol{x}\, j^0_{\text{adia}}(\boldsymbol{x}) \\ &= \frac{1}{\Omega_d} \int_{S^d} \mathrm{d}\Omega_d \\ &= N_{\text{hedgehog}} \in \mathbb{Z}. \end{aligned} \quad (3.40)$$

The "surface" element ("area") of the sphere S^d is here denoted by $\mathrm{d}\Omega_d$ (Ω_d).

Third, we need to define a point defect that is compatible with the charge-conjugation symmetry and consistent with the adiabatic approximation (see Sec. III D for a discussion of the latter caveat). A charge-conjugation symmetric point defect is a half hedgehog, i.e., a static Higgs $(d+1)$-tuplet (3.1a) that satisfies Eq. (3.1b) and wraps the d-sphere S^d the half integer $\pm 1/2$ number of times (a domain wall in $d = 1$, a vortex with unit vorticity in $d = 2$, etc). The counting formula (2.12) predicts that there is one unoccupied zero modes for a half-hedgehog provided we choose the sign of the infinitesimal conjugation-symmetry-breaking ϕ in Eq. (3.1a) to be opposite to the sign of $\pm 1/2$.

The *d-th Chern number* (3.38) is non-vanishing for any single-particle Hamiltonian topologically equivalent to the Dirac Hamiltonian (3.28).

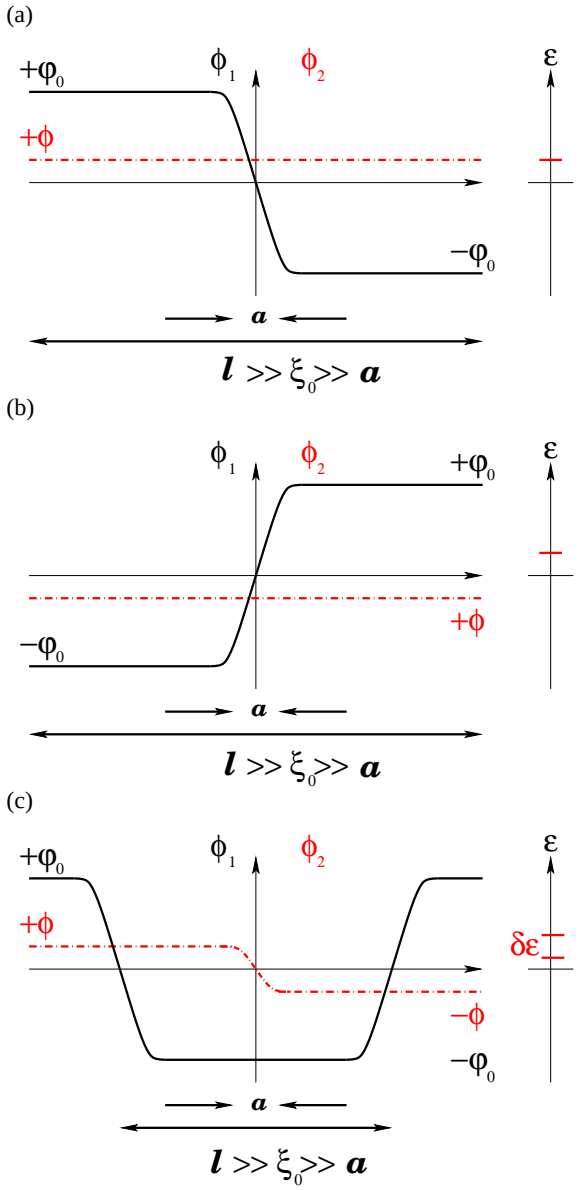

FIG. 2: (Color online) The dependence on space (x) of the Higgs doublet in Eq. (C1) is depicted in panels (a), (b), and (c) together with the number and position of the zero modes along the single-particle energy eigenvalue axis ε. The magnitude of the asymptotic value of the order parameter ϕ_1 is $\varphi_0 > 0$. The magnitude of the asymptotic value of the charge-conjugation symmetry ϕ_2 is $\phi > 0$. The size of the point defect, a domain wall, is $\mathfrak{a}$. The exponential decay of the envelope of the bound state is controlled by the length scale $\xi_0 = 1/|\boldsymbol{\phi}_0|$. The characteristic length ℓ is defined in Eq. (3.41).

D. Physical interpretation of the adiabatic approximation

The relevant length scales in d-dimensional space are: (1) the characteristic linear size $\mathfrak{a}$ of a point defect, (2) the characteristic linear size $\xi_0 = 1/\Delta_0$ of the support in space of the zero modes bound to the point defect, (3) the characteristic length ℓ over which the variation of the order parameter $\boldsymbol{\phi}$ is of order of the gap $2\Delta_0$,

$$|\boldsymbol{\nabla}\boldsymbol{\phi}|\, \ell = \Delta_0, \quad (3.41)$$

(4) and the linear size R of the region of space in which the conserved quasiparticle charge Q and the total num-

ber N of unoccupied zero modes is to be computed (measured). The gradient approximation relies on the hierarchy

$$\mathfrak{a} < \xi_0 \ll \ell \ll R. \tag{3.42}$$

In the gradient approximation, all microscopic data for length scales smaller than ℓ are dispensed with. This fact dictates how to properly interpret the results from the gradient approximation when point defects can be assigned an additive label $q_{\rm def}$. Here, the subscript stands for point defect.

When $d = 1$, $q_{\rm def} \equiv q_{\rm dw}$, with $q_{\rm dw} = +1$ representing a domain wall and $q_{\rm def} = -1$ representing an antidomain wall. Observe that we can exchange the terminology domain and antidomain wall, for there is no absolute notion of a positive or negative $q_{\rm dw}$. When $d = 2$, $q_{\rm def} \equiv q_{\rm vor} \in \mathbb{Z}$ encodes the vorticity of the vortex. In either case, these charges can be thought of as "classical Coulomb charges".

Any two point defects with the labels $q_{\rm def}^{(1)}$ and $q_{\rm def}^{(2)}$ within a distance of order $\mathfrak{a}$ of each other fuse into a point defect with the label

$$q_{\rm def}^{(1+2)} = q_{\rm def}^{(1)} + q_{\rm def}^{(2)}. \tag{3.43}$$

When $d = 1$, domain walls are necessarily ordered. Consecutive domain walls necessarily carry the opposite label. Fusion of two domain walls in $d = 1$ necessarily results in their annihilation. When $d = 2$, vortices can fuse to yield larger vorticities.

When $d = 1$, we have shown in Sec. C 1 that the number of unoccupied zero modes N bound to a single domain wall in a large region of size R is $N = 1$ in the adiabatic approximation. This result agrees with solving the differential equation for the zero mode once all microscopic data have been supplied.[5] However, we have also shown in Sec. III A and Sec. C 1 that the number of unoccupied zero modes N in a large region of size R can be made to be an arbitrary integer number in the adiabatic approximation, in apparent contradiction with the fact that the net number of domain walls is either 0 or 1 along any finite segment of the line. This is explained with the help of Fig. 2 as follows.

Occupying or leaving empty any single-particle level is a physical operation forcing a fermionic quasiparticle in or out of this single-particle level. This physical process is, for zero modes, achieved by the local sign of the charge-conjugation-symmetry-breaking ϕ_2. In Fig. 2 (a), we plot as a function of position x along the line the dependence of the order parameter ϕ_1 and of the charge-conjugation-symmetry-breaking ϕ_2. Given the shape of the domain wall and its asymptotic value φ_0, we choose the sign $\phi_2(x) = +\phi$ so that the zero mode bound to the domain wall is shifted to a positive energy. If the chemical potential is chosen to be at 0, then the zero mode is unoccupied. The zero mode bound to an antidomain wall remains unoccupied if we choose the sign $\phi_2(x) = -\phi$ as depicted in Fig. 2(b). In Fig. 2(c) a domain wall is followed by an antidomain wall a distance ℓ apart. The closing of the gap $|\boldsymbol{\phi}|$ at the center of each domain walls when $\phi_1 = 0$ is avoided by the charge-conjugation-symmetry-breaking field saturating to its asymptotic magnitude ϕ. A phase twist in ϕ by π half way between the two consecutive domain walls insures that the zero-modes are shifted to positive energies and thus remain unoccupied. Gap closing at this phase twist is again avoided because ϕ_1 has healed to its asymptotic value. Everywhere along the line, $|\boldsymbol{\nabla\phi}|$ can thus be made as small as needed. The energy splitting $\delta\varepsilon$, which is of order $\Delta_0 \exp(-2\ell/\xi_0)$, is exponentially small in ℓ/ξ_0. In the gradient approximation, the order of limit is $\ell/\xi_0 \to \infty$ first followed by $\phi \to 0$. Evidently, this order of limit does not commute with $\phi \to 0$ first followed by $\ell/\xi_0 \to \infty$. In the latter order of limits, the unoccupied number of zero modes always vanishes.

We close the discussion of the results in one-dimensional space from Sec. III A by observing that since the number of unoccupied zero mode is an integer, all higher contributions to the gradient expansion beyond the adiabatic (leading) order must vanish identically. We have verified this expectation explicitly for the first subleading order.

When $d = 2$, the adiabatic approximation of Sec. III B applied to a chiral p-superconductor predicts that the total number of unoccupied zero modes in a large region of size R equals in magnitude the net vorticity in this region. On the other hand, it is known from solving the differential equation for a single vortex of vorticity $q_{\rm vort}$ in a p-wave superconductor that the number of zero modes is one if $q_{\rm vort}$ is odd or zero otherwise.[26–28] This is not a paradox if the adiabatic approximation in this paper is limited to point defects each of which bind at most a single zero mode. We now argue that this interpretation of the adiabatic approximation is a necessary one.

The adiabatic approximation is not sensitive to the microscopic data on length scales smaller than ℓ. However, it is precisely those microscopic data that determines if more than one zero modes can be bound to a point defect. Consider the case of the relativistic Dirac Hamiltonian in two-dimensional space from Sec. C 2. It respects two charge-conjugation symmetries, the chiral symmetry and the particle-hole symmetry. Correspondingly, the order parameter $\boldsymbol{\varphi}$ can either be interpreted as a bond-order (Kekulé order),[8] in which case the electron charge is a good quantum number, or as an s-wave superconductor,[33] in which case the electron charge is not anymore a good quantum number (the thermal quasiparticle charge is). In either interpretations, an index theorem guarantees that the number of zero modes equals in magnitude the vorticity of the order parameter.[34,35] For example, a vorticity of two implies that there are two zero modes. Of course, they must be orthogonal. Perturb now the Dirac Hamiltonian with a perturbation, whose characteristic energy scale is η, that breaks the chiral symmetry but preserves the particle-hole symmetry.[12] We assume that η is not sufficiently large to close the gap. The delicate balance that allowed the two zero modes to be orthogonal is destroyed by the perturbation

η. The two zero modes split pairwise, one migrating to positive energy, the other migrating to negative energy. This level repulsion is encoded by the microscopic data in a region centered about the point defect of linear size ξ_0, a window of length scales inaccessible to the adiabatic approximation. If we split the single vortex with vorticity two into two vortices with vorticity one separated by a distance ℓ, we can use the adiabatic approximation. The energy shift induced by the chiral-symmetry-breaking ϕ is of order $\phi \exp(-2\ell/\xi_0)$. The level splitting induced by the particle-hole-symmetric η is of order $\eta \exp(-2\ell/\xi_0)$. They vanish in the adiabatic limit $\ell/\xi_0 \to \infty$ first, $\phi \to 0$ second.

The adiabatic approximation saturates the number of unoccupied zero modes N. Consistency demands that higher-order corrections vanish identically. This suggests that the adiabatic approximation cannot capture the parity effect by which a charge-conjugation-symmetric perturbation splits pairwise the degeneracy of zero modes. Such a parity effect is an essential singularity for the adiabatic expansion presented in this paper.

IV. CONCLUSION

In this paper, we have derived a procedure to count the zero-energy eigenvalues of a single-particle Hamiltonian $\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x})\big)$ that possesses a charge-conjugation symmetry, when the order parameter $\boldsymbol{\varphi}(\boldsymbol{x})$ supports point defects. This procedure is based on counting the conserved quasiparticle charge Q induced by a point defect of $\boldsymbol{\varphi}(\boldsymbol{x})$.

The conserved quasiparticle charge Q exists because of the Hermiticity of $\mathcal{H}\big(\hat{\boldsymbol{p}}, \boldsymbol{\varphi}(\boldsymbol{x})\big)$. For instance, in problems for which the electric charge is a good quantum number, the number of zero modes can be determined from the induced electric charge near point defects. In superconductors for which the mean-field approximation is accurate, the conserved charge is associated to the thermal currents carried by the Bogoliubov quasiparticles.

Irrespective of the origin of the conserved quasiparticle charge (electric or thermal), we showed that one can use counting arguments similar to those appearing implicitly in Ref. 2 and explicitly in Ref. 3 to compute the conserved fractional quasiparticle charge Q induced by a point defect. Once one observes that the number of unoccupied zero modes N can be related to Q through $N = -2Q$, one can concentrate the efforts into computing Q near a point defect of the position-dependent order parameter $\boldsymbol{\varphi}(\boldsymbol{x})$. We carry out this procedure within a gradient expansion for smoothly spatially varying fluctuations of the charge-conjugation-symmetric order parameter $\boldsymbol{\varphi}(\boldsymbol{x})$.[21]

The resulting expression is then applied to generic systems in one-dimensional space, the chiral p-wave superconductor in two-dimensional space, and to Dirac fermions in d-dimensional space. In one-dimensional space, we find that the number of unoccupied zero modes is related to the first Chern number. For the p-wave superconductor in two-dimensional space, it is related to the second Chern number. For the Dirac Hamiltonians in d-dimensional space, the number of zero modes is determined by the d-th Chern number. Therefore, we can establish in a logical and constructive way the relation between the number of zero modes induced by a point defect and topological invariants (the Chern numbers) in a number of cases.

There has been a resurgence of efforts dedicated to counting zero modes induced by defective order parameters in the condensed matter community.[36,37] Fukui and Fujiwara in Ref. 38 have revisited Dirac fermions in up to three spatial dimensions. They use the chiral anomaly to carry the counting. Our general counting formula reproduces their and previous results, and extends them to arbitrary d dimensions of space. An elegant formulation of the Dirac problem in d dimensions in the presence of an isotropic hedgehog has been carried out by Herbut.[39] (See also Freedman *et al.* in Ref. 40.) The number of zero modes should not depend on deformations away from the isotropic case, and this result is captured by our counting formula, which we can express as the d-th Chern number. Finally, Teo and Kane in Ref. 41 have studied defects of arbitrary dimensions r coupled to noninteracting fermionic quasiparticles with the help of the classification scheme of Schnyder *et al* for topological band insulators and superconductors.[42–44] They conjecture that the topological invariant for a given symmetry class is related to the number of zero modes attached to r-dimensional defects, but cannot make any direct connection between these two integer numbers. Our explicit construction provides this relation, although, only for specific examples and for point defects. We do not have yet a proof for generic Hamiltonians that have a momentum-dependent coupling to the order parameter.

In summary, we showed that one can count zero modes induced by point defects using a conserved quasi-particle charge present for any Hermitian mean-field Hamiltonian with charge-conjugation symmetry. This counting supports a direct relation between topological invariants and the number of zero modes bound to defective order parameters.

Acknowledgments

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Appendix A: Counting the zero modes with the conserved quasiparticle charge

In this Appendix, we review an identity that relates the total number of zero modes of any single-particle

Hamiltonian $\mathcal{H}$ with charge-conjugation symmetry to its Euclidean Green function $\mathcal{G}(\omega) := (\mathrm{i}\omega - \mathcal{H})^{-1}$. This identity appears implicitly in Ref. 2 and explicitly in Ref. 3.[4] We will only assume that the single-particle Hamiltonian $\mathcal{H}$ has the property that there exists a transformation $\mathcal{C}$ such that

$$\mathcal{C}^{-1}\,\mathcal{H}\,\mathcal{C} = -\mathcal{H}, \tag{A1}$$

whereby the transformation $\mathcal{C}$ is norm preserving, i.e., it can either be an unitary or an antiunitary transformation. We call $\mathcal{C}$ the operation of charge conjugation.

To simplify notation, we take the spectrum

$$\{0\,,\,\mathrm{sgn}(n)\,\varepsilon_{|n|}\,|\,n = \pm 1, \pm 2, \cdots\} \tag{A2}$$

of $\mathcal{H}$ to be discrete up to finite degeneracies of the eigenvalues. This is the case if $\mathcal{H}$ describes the single-particle physics of a lattice Hamiltonian. The spectral decomposition of $\mathcal{H}$ thus reads

$$\mathcal{H} = \sum_{n=\pm 1,\pm 2,\cdots} \mathrm{sgn}(n)\varepsilon_{|n|}|\psi_n\rangle\langle\psi_n|, \qquad \varepsilon_{|n|} > 0, \tag{A3}$$

with the single-particle orthonormal basis obeying

$$\begin{aligned} &\langle\psi_n|\psi_{n'}\rangle = \delta_{n,n'}, \qquad \langle\psi_n|\alpha\rangle = 0, \qquad \langle\alpha|\alpha'\rangle = \delta_{\alpha,\alpha'}, \\ &\mathbb{1} = \sum_{\alpha=1}^{N} |\alpha\rangle\langle\alpha| + \sum_{n=\pm 1,\pm 2,\cdots} |\psi_n\rangle\langle\psi_n|\,. \end{aligned} \tag{A4}$$

We have assumed the existence of N zero modes labeled by the index α. The relation

$$|\psi_n\rangle = \mathcal{C}|\psi_{-n}\rangle \tag{A5}$$

holds for any finite energy eigenstate labeled by the index $n = \pm 1, \pm 2, \cdots$ as a result of the charge-conjugation symmetry (A1).

On the lattice, we denote the value of the energy eigenfunctions at site i by

$$\psi_{\mathrm{sgn}(n)\varepsilon_{|n|,i}} := \langle i|\psi_n\rangle, \qquad n = \pm 1, \pm 2, \cdots, \tag{A6}$$

for the finite-energy eigenvalues and

$$\psi_{\alpha,i} := \langle i|\alpha\rangle, \qquad \alpha = 1, \cdots, N, \tag{A7}$$

for the N zero modes. For any two lattice sites i and j,

$$\begin{aligned} \delta_{i,j} =& \langle i|j\rangle \\ =& \langle i|\mathbb{1}|j\rangle \\ =& \sum_{n=-1,-2,\cdots} \psi^*_{-|n|,j}\psi_{-|n|,i} \\ &+ \sum_{n=-1,-2,\cdots} (\mathcal{C}\psi)^*_{-|n|,j}\,(\mathcal{C}\psi)_{-|n|,i} \\ &+ \sum_{\alpha=1}^{N} \psi^*_{\alpha,j}\psi_{\alpha,i}, \end{aligned} \tag{A8}$$

where we have used the completeness relation defined in Eq. (A4). When $i = j$, we find the local sum rule

$$1 = 2\sum_{n=-1,-2,\cdots} \psi^*_{-|n|,i}\psi_{-|n|,i} + \sum_{\alpha=1}^{N} \psi^*_{\alpha,i}\psi_{\alpha,i} \tag{A9}$$

owing to the fact that $\mathcal{C}$ is norm preserving.

We now assume that there exists a single-particle Hamiltonian

$$\begin{aligned} \mathcal{H}_0 =& \sum_{n=\pm 1,\pm 2,\cdots} \mathrm{sgn}(n)\varepsilon^0_{|n|}|\phi_n\rangle\langle\phi_n| \\ =& -\mathcal{C}^{-1}\,\mathcal{H}_0\,\mathcal{C} \end{aligned} \tag{A10}$$

such that $\mathcal{H}_0$ *does not support zero modes.* Evidently, the local sum rule

$$1 = 2\sum_{n=-1,-2,\cdots} \phi^*_{-|n|,i}\phi_{-|n|,i} \tag{A11}$$

also applies.

After subtracting Eq. (A11) from Eq. (A9), the local local sum rule

$$\begin{aligned} -2\sum_{n=-1,-2,\cdots} &\left(\psi^*_{-|n|,i}\psi_{-|n|,i} - \phi^*_{-|n|,i}\phi_{-|n|,i}\right) = \\ &\sum_{\alpha=1}^{N} \psi^*_{\alpha,i}\psi_{\alpha,i} \end{aligned} \tag{A12}$$

follows. In turn, after summing over all lattice sites and making use of the fact that the zero modes are normalized to one, the global sum rule

$$\frac{N}{2} = \sum_{n=-1,-2,\cdots}\sum_{i} \left(\phi^*_{-|n|,i}\phi_{-|n|,i} - \psi^*_{-|n|,i}\psi_{-|n|,i}\right) \tag{A13}$$

follows. The global sum rule is only meaningful in the thermodynamic limit after this subtraction procedure has been taken.

We now present the global sum rule (A13) with the help of Euclidean single-particle Green functions. We thus define the Euclidean single-particle Green functions

$$\mathcal{G}(\omega) := \frac{1}{\mathrm{i}\omega - \mathcal{H}} \tag{A14a}$$

and

$$\mathcal{G}_0(\omega) := \frac{1}{\mathrm{i}\omega - \mathcal{H}_0} \tag{A14b}$$

for any real-valued and non-vanishing ω. Next, we choose γ to be the contour in the complex ω plane that runs counterclockwise along the real axis $\omega \in \mathbb{R}$, avoids the origin $\omega = 0$ by an infinitesimal deformation into the upper complex plane $\mathrm{Re}\,\omega > 0$, and closes through a semi-circle in the very same upper complex plane (see

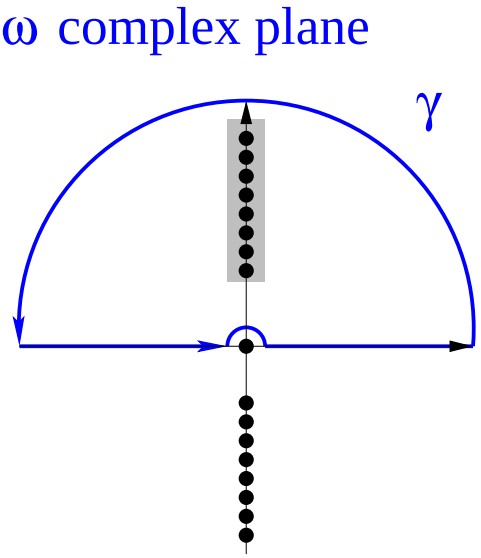

FIG. 3: (Color online) Definition of the integration contour γ that picks up the discrete negative energy eigenvalues of the single-particle Hamiltonian. The shaded box represents the Fermi sea. The filled circles represent the discrete energy eigenvalues. The contour γ avoids the zero mode.

Fig. 1). With the help of the residue theorem, it then follows that

$$\sum_{n=-1,-2,\cdots} \psi^*_{-|n|,i}\psi_{-|n|,i} = \int_\gamma \frac{\mathrm{d}\omega}{2\pi} \langle i\,|\mathcal{G}(\omega)|\,i\rangle \tag{A15a}$$

and

$$\sum_{n=-1,-2,\cdots} \phi^*_{-|n|,i}\phi_{-|n|,i} = \int_\gamma \frac{\mathrm{d}\omega}{2\pi} \langle i\,|\mathcal{G}_0(\omega)|\,i\rangle\,. \tag{A15b}$$

Equation (A13) can now be rewritten in the desired form

$$\frac{N}{2} = -\mathrm{Tr}\int_\gamma \frac{\mathrm{d}\omega}{2\pi}\left[\mathcal{G}(\omega)-\mathcal{G}_0(\omega)\right]. \tag{A16a}$$

Equation (A16a) can be expressed in terms of the local "lattice charge" ρ_i through

$$\begin{aligned}\frac{N}{2} &= -\sum_i \rho_i \equiv -Q,\\ \rho_i &:= \int_\gamma \frac{\mathrm{d}\omega}{2\pi}\langle i\,|[\mathcal{G}(\omega)-\mathcal{G}_0(\omega)]|\,i\rangle\,,\end{aligned} \tag{A16b}$$

or the local "continuum charge" through "charge" $\rho(\boldsymbol{r})$

$$\begin{aligned}\frac{N}{2} &= -\int \mathrm{d}^d\boldsymbol{r}\,\rho(\boldsymbol{r}) \equiv -Q,\\ \rho(\boldsymbol{r}) &:= \int_\gamma \frac{\mathrm{d}\omega}{2\pi}\langle \boldsymbol{r}\,|[\mathcal{G}(\omega)-\mathcal{G}_0(\omega)]|\,\boldsymbol{r}\rangle\,.\end{aligned} \tag{A16c}$$

Hermiticity of $\mathcal{H}$ and $\mathcal{H}_0$ guarantees that these "quasiparticle charge densities" obey a continuity equation, i.e., that their quasiparticle charge Q is conserved.

Appendix B: Time-reversal symmetry of $\mathcal{H}_d^{\mathrm{Dirac}}$

We are going to prove the existence of a time-reversal symmetry for the Dirac Hamiltonian defined by Eq. (3.28). To this end, we shall introduce an auxiliary Lagrangian $\mathcal{L}_{\mathrm{Dirac}}$ in $(2d+1)$-dimensional space and time.

By construction,[30]

$$\mathcal{L}^{\mathrm{Dirac}}_{2d+1}(x_1,\cdots,x_{2d},x_{2d+1}) := \sum_{\nu=1}^{2d+1}\Gamma_\nu\frac{\partial}{\partial x_\nu} \tag{B1}$$

can be interpreted as the free Euclidean Dirac Lagrangian in $(2d+1)$-dimensional Euclidean space $(\boldsymbol{x})$ and time (x_{2d+1}). Observe here that 2^d, where $d=[(2d+1)/2]$ is the smallest integer larger or equal to the real number $(2d+1)/2$, is the smallest dimensionality of an irreducible representation of a Dirac Hamiltonian in $2d$-dimensional space.[30] In the reminder of the argument, we adopt the summation convention over repeated indices.

The classical action

$$S = -\int \mathrm{d}^{2d+1}x\,\bar\psi_\alpha(x)\left(\Gamma_\nu\frac{\partial}{\partial x_\nu}\right)_{\alpha\beta}\psi_\beta(x) \tag{B2}$$

is invariant under the group $\mathrm{O}(2d+1)$ generated by

$$\Sigma_{\nu\nu'} := \frac{1}{2\mathrm{i}}[\Gamma_\nu,\Gamma_{\nu'}], \qquad 1\le\nu<\nu'\le 2d+1. \tag{B3}$$

Consequently, there must exist $(2d+1)$ unitary matrices $\mathcal{P}_\nu$ from the (complex) Clifford algebra such that

$$\mathcal{P}_\nu^{-1}\,\Gamma_{\nu'}\,\mathcal{P}_\nu = (-)^{\delta_{\nu,\nu'}}\Gamma_{\nu'}, \qquad \nu,\nu' = 1,\cdots,2d+1. \tag{B4}$$

Symmetry under reflection about any direction $\nu = 1,\cdots,2d+1$ in $(2d+1)$-dimensional space and time thus becomes

$$\begin{aligned}&\mathcal{L}^{\mathrm{Dirac}}_{2d+1}(\cdots,x_{\nu-1},x_\nu,x_{\nu+1},\cdots) =\\ &\quad\mathcal{P}_\nu^{-1}\,\mathcal{L}^{\mathrm{Dirac}}_{2d+1}(\cdots,+x_{\nu-1},-x_\nu,+x_{\nu+1},\cdots)\,\mathcal{P}_\nu.\end{aligned} \tag{B5}$$

There must also exist an auxiliary unitary matrix $\mathcal{T}_{\mathrm{aux}}$ from the (complex) Clifford algebra under which

$$\begin{aligned}&\mathcal{L}^{\mathrm{Dirac}}_{2d+1}(x_1,\cdots,x_{2d},x_{2d+1}) =\\ &\quad\mathcal{T}_{\mathrm{aux}}^{-1}\,\mathcal{L}^{*}_{\mathrm{Dirac}}(+x_1,\cdots,+x_{2d},-x_{2d+1})\,\mathcal{T}_{\mathrm{aux}}.\end{aligned} \tag{B6}$$

We are now ready to define the unitary matrix

$$\mathcal{T}_d := \mathcal{T}_{\mathrm{aux}}\mathcal{P}_{2d+1}\prod_{a=1}^{d}\mathcal{P}_a \tag{B7}$$

from the (complex) Clifford algebra. If K denotes complex conjugation, we interpret the operation $\mathcal{T}_d\mathsf{K}$ as implementing reversal of time on the Dirac Hamiltonian (3.28d). It is a symmetry, for

$$\mathcal{H}_d^{\mathrm{Dirac}} = \mathcal{T}_d^{-1}\,\mathcal{H}_d^{\mathrm{Dirac}\,*}\,\mathcal{T}_d. \tag{B8}$$

Appendix C: Zero modes of $\mathcal{H}_d^{\text{Dirac}}$

This Appendix provides intermediary steps for Sec. III C.

1. Dirac fermions in one-dimensional space

a. Definition

When $d = 1$, Eq. (3.28) simplifies to

$$\mathcal{H}_1^{\text{Dirac}}\big(\hat{p}, \boldsymbol{\phi}(x)\big) := \sigma_3 \hat{p} + \sigma_1 \phi_1(x) + \sigma_2 \phi_2(x). \tag{C1}$$

The ($R = 2$)-dimensional representation of the Clifford algebra is here generated from the Pauli matrices σ_1, σ_2, and σ_3. If the doublet of Higgs fields $\boldsymbol{\phi}_0$ is constant through one-dimensional space (x), the single-particle spectrum of Hamiltonian (C1) has a gap controlled by the Higgs components adding in quadrature,

$$\varepsilon_0^2(\boldsymbol{p}) = \boldsymbol{p}^2 + \boldsymbol{\phi}_0^2 = \boldsymbol{p}^2 + \phi_{0\,1}^2 + \phi_{0\,2}^2. \tag{C2}$$

The generator of the chiral symmetry of the Dirac Hamiltonian (C1) is

$$\mathcal{C}_{\text{ch}} = \sigma_2 \tag{C3a}$$

if

$$\phi_2 = 0 \tag{C3b}$$

everywhere in Euclidean space $x \in \mathbb{R}$.

The operation of time-reversal is implemented by

$$\mathcal{T}_1 \mathsf{K} := \sigma_1 \mathsf{K} \tag{C4}$$

where K denotes complex conjugation. It is a symmetry of the Dirac Hamiltonian (C1) for any Higgs configuration ϕ_1 and ϕ_2.

The operation of particle-hole exchange is implemented by

$$\mathcal{C}_{\text{ph}} := \mathcal{T}_1 \mathsf{K} \mathcal{C}_{\text{ch}} = -\mathrm{i}\sigma_3 \mathsf{K} \tag{C5}$$

and it is only a symmetry of the Dirac Hamiltonian (C1) provided $\phi_2 = 0$.

The discovery that this model supports zero modes was made by Jackiw and Rebbi in Ref. 5. Its relevance to the physics of polyacetylene was made by Su, Schrieffer, and Heeger in Refs. 6 and 7. In polyacetylene, the kinetic energy results from linearizing the dispersion around the two Fermi points of a single-band nearest-neighbor tight-binding model at half-filling in the left- and right-mover basis. The Higgs field $\phi_1 \equiv \varphi_1$ realizes a modulation of the nearest-neighbor hopping amplitude that is mediated by phonons. A Peierls transition opens up a single-particle electronic gap through the breaking of the translation symmetry by one lattice spacing down to a residual translation symmetry by two lattice spacings. A domain wall in φ_1 that interpolates between the two possible dimer ground states binds one zero mode per electronic spin (which we have ignored so far). The Higgs field ϕ_2 breaks the sublattice symmetry of the tight-binding model. The quasiparticle charge density induced by a domain wall with a non-vanishing charge-conjugation symmetry breaking ϕ_2 was computed by Goldstone and Wilczek in Ref. 31 and shown to vary continuously with ϕ_2. We are going to reproduce all these results using the adiabatic approximation (2.21).

b. Counting zero modes

We start from the expansion (2.18a) of the quasiparticle charge density which, for a Dirac Hamiltonian, is exact and consider the contribution from $n = 1$. The adiabatic approximation to the quasiparticle charge density is

$$\begin{aligned}\rho_{\text{adia}}(x) &= \int \frac{\mathrm{d}\omega \mathrm{d}p}{(2\pi)^2} \frac{2\epsilon_{\mathsf{ab}} \left(\partial_x \phi_{\mathsf{a}}\right)(x)\, \phi_{\mathsf{b}}(x)}{\left[\omega^2 + p^2 + \phi_1^2(x) + \phi_2^2(x)\right]^2} \\ &= \frac{\epsilon_{\mathsf{ab}}}{2\pi} \frac{\left(\partial_x \phi_{\mathsf{a}}\right)(x)\, \phi_{\mathsf{b}}(x)}{\left[\phi_1^2(x) + \phi_2^2(x)\right]}.\end{aligned} \tag{C6}$$

For a constant $\phi_2 > 0$ that breaks the conjugation symmetry, integration over one-dimensional Euclidean space (x) gives the adiabatic approximation to the conserved quasiparticle charge

$$\begin{aligned}Q_{\text{adia}}(\phi_2) &= \int_{\phi_1(x=-\infty)}^{\phi_1(x=+\infty)} \frac{\mathrm{d}\phi_1}{2\pi} \frac{\phi_2}{(\phi_1^2 + \phi_2^2)} \\ &= (2\pi)^{-1} \left(\arctan \frac{\phi_1(x)}{\phi_2}\right)\Bigg|_{x=-\infty}^{x=+\infty}.\end{aligned} \tag{C7}$$

The single domain wall with the asymptotic values

$$\phi_1(\pm\infty) = \mp\varphi_0, \qquad \varphi_0 > 0, \tag{C8}$$

here chosen in such a way that any zero mode is shifted in energy by $\phi_2 > 0$ *above* the chemical potential, induces the *negative* conserved quasiparticle charge

$$\lim_{\phi_2 \to 0^+} Q_{\text{adia}}(\phi_2) = -\frac{1}{2}. \tag{C9}$$

Having restored the charge-conjugation symmetry by removing ϕ_2 in Eq. (C9), the counting formula (2.12) can, in turn, be used to deliver the *positive* number of unoccupied zero modes

$$N = 1 \tag{C10}$$

bound to this single domain wall.

c. Chern number

Whereas the counting formula (2.12) relies on the charge-conjugation symmetry, the adiabatic approximation to the quasiparticle charge density does not. We are going to take advantage of this freedom to relate the conserved quasiparticle charge to *the first Chern number*.

To this end, we compactify the base space $x \in \mathbb{R}$ to the circle $x \in S^1$ by imposing periodic boundary conditions. We then parametrize the doublet of Higgs fields according to the polar decomposition

$$\boldsymbol{\phi}(x) := \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = m \begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \end{pmatrix}. \tag{C11}$$

We have thus compactified the target space $\boldsymbol{\phi}(x) \in \mathbb{R}^2$ to the unit circle $\theta(x) \in S^1$.

In the adiabatic approximation (2.21), the conserved quasiparticle charge is

$$Q_{\text{adia}} = \frac{1}{2(2\pi)^2} \int \mathrm{d}\omega \int_0^{2\pi} \mathrm{d}p \int_0^{2\pi} \mathrm{d}x \, (-\mathrm{i}) \frac{\partial\theta}{\partial x} \times \mathrm{tr} \left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial p} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \theta} \mathcal{G} - \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \theta} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial p} \mathcal{G} \right)_0 . \tag{C12}$$

The integrand can be written in a more compact and symmetric form by observing that

$$\left(\frac{\partial \mathcal{G}^{-1}}{\partial \omega} \right)_0 = \mathrm{i}, \qquad \mathrm{i}\, \mathcal{G}_0 = \left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial \omega} \right)_0, \tag{C13}$$

and by introducing the three-momentum

$$K_\nu \equiv (\omega, p, \theta) \in \mathbb{R} \times S^1 \times S^1. \tag{C14}$$

Equation (C12) is then nothing but *the first Chern number*,[25] for

$$Q_{\text{adia}} = \frac{-1}{24\pi^2} \int_{\mathbb{R}\times S^1 \times S^1} \mathrm{d}^3 K \epsilon_{\mu\nu\lambda} \mathrm{tr} \left(\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial K_\mu} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial K_\nu} \mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial K_\lambda} \right)_0 . \tag{C15}$$

We infer that the conserved quasiparticle charge induced by the adiabatic winding of the Higgs doublet around the circle takes integer values.

Moreover, the domain wall from Sec. C 1 is a half-winding of the unit circle S^1. More precisely, evaluation of the trace in the integrand of *the first Chern number* (C15) gives the adiabatic approximation to the conserved quasiparticle charge

$$\begin{aligned} Q_{\text{adia}} &= \frac{1}{2\pi} \int \mathrm{d}x \, \frac{\epsilon_{\mathsf{ab}} \phi_{\mathsf{a}} \partial_x \phi_{\mathsf{b}}}{|\boldsymbol{\phi}|^2} \\ &= \text{winding number in } \boldsymbol{\phi}. \end{aligned} \tag{C16}$$

The integrand is nothing but the space (x) and time (t) dependent quasiparticle charge density

$$j^0 := \frac{1}{2\pi|\boldsymbol{\phi}|^2} \epsilon_{\mathsf{ab}} \phi_{\mathsf{a}} \partial_x \phi_{\mathsf{b}} \tag{C17a}$$

that obeys the continuity equation[31]

$$\partial_\nu j^\nu = 0 \tag{C17b}$$

with

$$j^\nu = \frac{1}{2\pi\boldsymbol{\phi}^2} \epsilon^{\nu\nu'} \epsilon_{\mathsf{a}\mathsf{a}'} \phi_{\mathsf{a}} \partial_{\nu'} \phi_{\mathsf{a}'}. \tag{C17c}$$

2. Dirac fermions in two-dimensional space

a. Definition

When $d = 2$, Eq. (3.28) simplifies to

$$\mathcal{H}_2^{\text{Dirac}}\big(\hat{p}, \boldsymbol{\phi}(x)\big) := \alpha_1 \hat{p}_1 + \alpha_2 \hat{p}_2 + \beta_1 \phi_1 + \beta_2 \phi_2 + \beta_3 \phi_3. \tag{C18a}$$

The ($R = 4$)-dimensional representation of the Clifford algebra can be chosen to be generated from the five traceless and Hermitian matrices

$$\Gamma_\nu = (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \tag{C18b}$$

with

$$\alpha_1 := \sigma_3 \otimes \tau_1, \qquad \alpha_2 := \sigma_3 \otimes \tau_2, \tag{C18c}$$

and

$$\beta_1 := \sigma_1 \otimes \tau_0, \qquad \beta_2 := \sigma_2 \otimes \tau_0, \qquad \beta_3 := \sigma_3 \otimes \tau_3. \tag{C18d}$$

A second set of Pauli matrices τ_1, τ_2, and τ_1 has been introduced, together with the unit 2×2 matrices σ_0 and τ_0. If the triplet of Higgs field $\boldsymbol{\phi}_0$ is constant throughout two-dimensional space ($\boldsymbol{x}$), the single-particle spectrum of Hamiltonian (C18) has a gap controlled by the Higgs components adding in quadrature,

$$\varepsilon_0^2(\boldsymbol{p}) = \boldsymbol{p}^2 + \boldsymbol{\phi}_0^2 = \boldsymbol{p}^2 + \phi_{0\,1}^2 + \phi_{0\,2}^2 + \phi_{0\,3}^2. \tag{C19}$$

The generator of the chiral symmetry of the Dirac Hamiltonian (C18) is

$$\mathcal{C}_{\text{ch}} = \beta_3 = \sigma_3 \otimes \tau_3 \tag{C20a}$$

if

$$\phi_3 = 0 \tag{C20b}$$

everywhere in Euclidean space $\boldsymbol{x} \in \mathbb{R}^2$.

The operation of time-reversal is implemented by

$$\mathcal{T}_2\, \mathsf{K} := \sigma_1 \otimes \tau_1\, \mathsf{K} \tag{C21}$$

where K denotes complex conjugation. It is a symmetry of the Dirac Hamiltonian (C18) for any Higgs configuration ϕ_1, ϕ_2, and ϕ_3.

The operation of particle-hole exchange is implemented by

$$\mathcal{C}_{\text{ph}} := \mathcal{T}_2\, \mathsf{K}\, \mathcal{C}_{\text{ch}} = -\sigma_2 \otimes \tau_2\, \mathsf{K} \tag{C22}$$

and it is only a symmetry of the Dirac Hamiltonian (C18) provided $\phi_3 = 0$.

The discovery that this model supports zero modes (Majorana fermions) was made by Jackiw and Rossi in Ref. 33 within an interpretation of Hamiltonian (C18) as a relativistic superconductor. Weinberg shortly thereafter proved an index theorem in Ref. 34 for these zero modes. The effect on the induced conserved quasiparticle charge by a triplet of Higgs fields was investigated by Jaroszewicz in Ref. 32 (see also Refs. 45–47). It was proposed by Hou, Chamon, and Mudry in Ref. 8 that graphene could realize Hamiltonian (C18) with the Higgs doublet ϕ_1 and ϕ_2 responsible for a Kekulé bond-density-wave instability and the charge-conjugation-symmetry-breaking ϕ_3 responsible for a charge-density-wave instability (see also Refs. 10–12).

b. *Counting zero modes*

We start from the expansion (2.18a) of the quasiparticle charge density induced by a static triplet of Higgs fields $\boldsymbol{\phi}$ which, for a Dirac Hamiltonian, is exact. We compute first the contribution from $n = 1$. It vanishes. The adiabatic approximation to the quasiparticle charge density is in fact given by Eq. (2.22)

$$\begin{aligned}\rho_{\text{adia}}(\boldsymbol{x}) &= \int \frac{\mathrm{d}\omega \mathrm{d}^2\boldsymbol{p}}{(2\pi)^3} \frac{8\epsilon_{\mathsf{abc}} \left(\partial_1\phi_{\mathsf{a}}\right)(\boldsymbol{x}) \left(\partial_2\phi_{\mathsf{b}}\right)(\boldsymbol{x})\phi_{\mathsf{c}}(\boldsymbol{x})}{\left[\omega^2 + \boldsymbol{p}^2 + \boldsymbol{\phi}^2(\boldsymbol{x})\right]^3} \\ &= \frac{\epsilon_{\mathsf{abc}}}{4\pi} \frac{\left(\partial_1\phi_{\mathsf{a}}\right)(\boldsymbol{x}) \left(\partial_2\phi_{\mathsf{b}}\right)(\boldsymbol{x})\phi_{\mathsf{c}}(\boldsymbol{x})}{|\boldsymbol{\phi}(\boldsymbol{x})|^3}.\end{aligned} \tag{C23}$$

For a constant ϕ_3, the integration over $\boldsymbol{x}$ gives the adiabatic approximation to the conserved quasiparticle charge

$$Q_{\text{adia}}(\phi_3) = \frac{1}{4\pi} \int \mathrm{d}\Theta \int_0^{\Lambda_0} \mathrm{d}\rho\, \rho \frac{\phi_3}{\left(\rho^2 + \phi_3^2\right)^{3/2}} \tag{C24a}$$

where the parametrization

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}\rho(r)\cos\Theta(\theta) \\ \rho(r)\sin\Theta(\theta)\end{pmatrix} \tag{C24b}$$

is assumed for the charge-conjugation-symmetric doublet of Higgs fields with r and θ denoting the polar coordinates of $\boldsymbol{x} \in \mathbb{R}^2$. In the limit in which ϕ_3 tends to zero, the induced charge

$$Q_{\text{adia}}(\phi_3) = \frac{\text{sgn}(\phi_3)}{2} \times \text{winding number in } (\phi_1, \phi_2) \tag{C25}$$

follows. To compute the number of unoccupied zero modes N with the counting formula (2.12), we choose the sign of the charge-conjugation-symmetry-breaking ϕ_3 such that it shifts the zero mode in energy above the chemical potential, i.e., with the opposite sign to the winding number of the Higgs doublet (ϕ_1, ϕ_2). We then take the limit $\phi_3 \to 0$. If so, for a unit winding

$$N = 1. \tag{C26}$$

Observe that the number of zero modes (C24a) agrees with Weinberg's index theorem in Ref. 34 applied to a single vortex with unit vorticity.

c. *Chern number*

We use the notation $\mathcal{G} := (\mathrm{i}\omega - \mathcal{H}^{\text{Dirac}}_{d=2})^{-1}$ and compactify both space and the order-parameter space,

$$\boldsymbol{x} \in S^2, \qquad \boldsymbol{\phi}(\boldsymbol{\theta}) \in S^2 \subset \mathbb{R}^3, \tag{C27}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$ are the spherical coordinates on the two-sphere $S^2 \subset \mathbb{R}^3$.

With the general manipulations of Appendix C 3, it is then possible to write

$$Q_{\text{adia}} = \frac{-\mathrm{i}(2\pi)^2}{60} \int\limits_K \epsilon_{\nu_1\cdots\nu_5} \text{tr}_4 \left(\mathcal{G}\partial_{\nu_1}\mathcal{G}^{-1}\cdots\mathcal{G}\partial_{\nu_5}\mathcal{G}^{-1}\right)_0 \tag{C28a}$$

where we have introduced the family of indices ν

$$\nu_1, \cdots, \nu_5 = 1, \cdots, 5, \tag{C28b}$$

the momentum

$$K_\nu = (\omega, p_1, p_2, \theta_1, \theta_2), \tag{C28c}$$

and the domain of integration

$$\int\limits_K \equiv \int \frac{\mathrm{d}\omega}{2\pi} \int\limits_{\boldsymbol{p}\in S^2} \frac{\mathrm{d}\Omega_2(\boldsymbol{p})}{(2\pi)^2} \int\limits_{\boldsymbol{\theta}\in S^2} \frac{\mathrm{d}\Omega_2(\boldsymbol{\theta})}{(2\pi)^2}. \tag{C28d}$$

The "surface" element of the sphere S^2 is here denoted by $\mathrm{d}\Omega_2$. The subscript 0 refers to the semi-classical Green function (2.20). Equation (C28) is *the second Chern number*.[29] It takes integer values only.

3. Chern number for Dirac fermions in d-dimensional space

To prove Eq. (3.38), imagine that we integrate out the Dirac fermions in the background, not necessarily static, of the Higgs fields in Eq. (3.28d) subject to the constraint

$$\boldsymbol{\phi}(\boldsymbol{\theta}) \in S^d \subset \mathbb{R}^{d+1}, \tag{C29a}$$

where $\boldsymbol{\theta}$ are the polar coordinates of the d-sphere $S^d \subset \mathbb{R}^{d+1}$. The conserved current of the fermionic single-particle Dirac Hamiltonian must induce a conserved current j^ν_{adia} with $\nu = 0, 1, \cdots, d$ for the Higgs fields. Its

time-like component $j^0_{\rm adia}$ enters in the counting formula (2.12). Relativistic covariance, current conservation, dimensional analysis, and the constraint (C29a) all conspire to bring this current to the form

$$j^\nu_{\rm adia} \propto \epsilon^{\nu\nu_1\cdots\nu_d}\epsilon_{\mathsf{a}_1\cdots\mathsf{a}_d\mathsf{a}_{d+1}}\phi_{\mathsf{a}_{d+1}}\partial_{\nu_d}\phi_{\mathsf{a}_d}\cdots\partial_{\nu_1}\phi_{\mathsf{a}_1}. \quad \text{(C29b)}$$

Here, summation convention over the repeated indices

$$\begin{aligned}&\nu,\nu_1,\cdots,\nu_d = 0,1,\cdots,d,\\ &\mathsf{a}_1,\cdots,\mathsf{a}_d,\mathsf{a}_{d+1} = d+1,\cdots,2d+1,\end{aligned} \quad \text{(C29c)}$$

is implied. (Compared to our convention in the definition (3.28d) of the Dirac Hamiltonian, we have shifted the values taken by the family of indices $\mathsf{a} = d+1,\cdots,2d+1$ to stress that it differs from the family of indices $i = 1,\cdots,d$.)

If we compare Eq. (C29) with the gradient expansion (2.24), we deduce that the leading non-vanishing contribution to the gradient expansion (2.24) must be of order $n = d$ and, for a static Higgs background, given by

$$j^0_{\rm adia}(\boldsymbol{x}) = (-\mathrm{i})^d\mathcal{I}_{i_d\mathsf{a}_d\cdots i_1\mathsf{a}_1}\left(\partial_{i_d}\phi_{\mathsf{a}_d}\right)\cdots\left(\partial_{i_1}\phi_{\mathsf{a}_1}\right)(\boldsymbol{x}). \quad \text{(C30a)}$$

The summation convention over the two distinct families of indices

$$\begin{aligned}&i_1,\cdots,i_d = 1,\cdots,d,\\ &\mathsf{a}_1,\cdots,\mathsf{a}_d = d+1,\cdots,2d+1,\end{aligned} \quad \text{(C30b)}$$

is implied and the expansion coefficients are

$$\begin{aligned}\mathcal{I}_{i_d\mathsf{a}_d\cdots i_1\mathsf{a}_1} := -\,\mathrm{i}\int\frac{\mathrm{d}\omega}{2\pi}\int_{\boldsymbol{p}}\mathrm{tr}_R\Bigg[\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_{i_d}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_d}}\right)\cdots\\ \cdots\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_{i_1}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\phi_{\mathsf{a}_1}}\right)\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\omega}\right)\Bigg]_0(\omega,\boldsymbol{p}).\end{aligned} \quad \text{(C30c)}$$

Any permutation of the indices on the left-hand side is defined by the same permutation of the differentials in the trace of the right-hand side.

We are first going to prove that

$$\begin{aligned}Q_{\rm adia} :=& \int_{S^d}\mathrm{d}^d\boldsymbol{x}\, j^0_{\rm adia}(\boldsymbol{x})\\ =&\frac{(-)^{d(d-1)/2}(-\mathrm{i})^d}{d!}\int_{\boldsymbol{\theta}\in S^d}\epsilon_{i_d\mathsf{b}_d\cdots i_1\mathsf{b}_1}\mathcal{J}_{i_d\mathsf{b}_d\cdots i_1\mathsf{b}_1}(\boldsymbol{\theta})\end{aligned} \quad \text{(C31a)}$$

where the target space $S^d \subset \mathbb{R}^{d+1}$ of the order-parameter is parametrized by the d-independent spherical coordinates

$$\boldsymbol{\theta} \equiv (\theta_{d+1},\cdots,\theta_{2d}), \quad \text{(C31b)}$$

the integral

$$\int_{\boldsymbol{\theta}\in S^d} \equiv \int_{S^d}\mathrm{d}\Omega_d \quad \text{(C31c)}$$

with $\mathrm{d}\Omega_d$ the "surface" element of the d-sphere, and

$$\begin{aligned}\mathcal{J}_{i_d\mathsf{b}_d\cdots i_1\mathsf{b}_1} := -\,\mathrm{i}\int\frac{\mathrm{d}\omega}{2\pi}\int_{\boldsymbol{p}}\mathrm{tr}_R\Bigg[\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_{i_d}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\theta_{\mathsf{b}_d}}\right)\cdots\\ \cdots\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_{i_1}}\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\theta_{\mathsf{b}_1}}\right)\left(\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial\omega}\right)\Bigg]_0(\omega,\boldsymbol{p}).\end{aligned} \quad \text{(C31d)}$$

Again any permutation of the indices on the left-hand side is defined by the same permutation of the differentials in the trace of the right-hand side.

Proof. To evaluate the trace in the integrand (C30c), the semi-classical Green functions are re-massaged so as to bring all the Dirac matrices in the numerator,

$$\begin{aligned}\mathcal{G}_{\text{s-c}}(\omega,\boldsymbol{p},\boldsymbol{x}) :=& \frac{1}{\mathrm{i}\omega - \Gamma_i p_i - \Gamma_{\mathsf{a}}\phi_{\mathsf{a}}(\boldsymbol{x})}\\ =& -\frac{\mathrm{i}\omega + \Gamma_i p_i + \Gamma_{\mathsf{a}}\phi_{\mathsf{a}}(\boldsymbol{x})}{\omega^2 + \boldsymbol{p}^2 + \boldsymbol{\phi}^2(\boldsymbol{x})}.\end{aligned} \quad \text{(C32)}$$

Multiplying out all Green functions in the trace from the integrand in Eq. (C30c) yields in the numerator terms made of the product from $2d$ Γ-matrices, $2d+1$ Γ-matrices, ..., $2d+2j$ Γ-matrices, $2d+2j+1$ Γ-matrices, ..., $2d+2d$ Γ-matrices, and $2d+2d+1$ Γ-matrices. Any trace over an even number $2d+2j$ of Γ-matrices is odd under $\omega \to -\omega$ since it comes multiplied by the power $\omega^{2d+1-2j}$ in the numerator. Such a trace does not contribute to the ω integration since the denominator is an even function of ω. Any trace over an odd number $2d+2j+1$ of Γ-matrices is even under $\omega \to -\omega$ since it comes multiplied by the power $\omega^{2d+1-2j-1}$. Such a trace can only be non-vanishing if $\Gamma_1,\cdots,\Gamma_{2d+1}$ all appear in the trace and all an odd number of times.[30] For such traces, the Clifford algebra delivers another key identity in that

$$\mathrm{tr}_R\left(\Gamma_{2d+1}\Gamma_{\mu_1}\cdots\Gamma_{\mu_{2d}}\right) = R\,(+\mathrm{i})^d\epsilon_{\mu_1\cdots\mu_{2d}} \quad \text{(C33)}$$

if $\mu_j = 1,\cdots,2d$ for $j = 1,\cdots,2d$. Here, $R = 2^d$ and Γ_{2d+1} was defined in Eq. (3.31b). The coefficients (C30c) inherit the antisymmetry of Eq. (C33) in that they are fully antisymmetric under any exchange of the indices (C30b).

We need to overcome the fact that the ranges of i and a are unequal in cardinality. To this end, we change variables on the target space and introduce the spherical coordinates (C31b) of the target space $S^d \subset \mathbb{R}^{d+1}$. The adiabatic approximation (C30a) to the quasiparticle charge density becomes

$$j^0_{\rm adia}(\boldsymbol{x}) = (-\mathrm{i})^d\mathcal{J}_{i_d\mathsf{b}_d\cdots i_1\mathsf{b}_1}\left(\partial_{i_d}\theta_{\mathsf{b}_d}\right)\cdots\left(\partial_{i_1}\theta_{\mathsf{b}_1}\right)(\boldsymbol{x}). \quad \text{(C34)}$$

Now, it is the summation convention over the two distinct families of indices

$$\begin{aligned} &i_1, \cdots, i_d = 1, \cdots, d, \\ &\mathsf{b}_1, \cdots, \mathsf{b}_d = d+1, \cdots, 2d, \end{aligned} \tag{C35}$$

that replaces (C30b), whereby the expansion coefficients (C34) are related to the expansion coefficients (C30c) through the chain rule for differentiation, i.e.,

$$\mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} = \mathcal{I}_{i_d \mathsf{a}_d \cdots i_1 \mathsf{a}_1} \left(\frac{\partial \phi_{\mathsf{a}_d}}{\partial \theta_{\mathsf{b}_d}} \right) \cdots \left(\frac{\partial \phi_{\mathsf{a}_1}}{\partial \theta_{\mathsf{b}_1}} \right). \tag{C36}$$

Here, any permutation of the indices of $\mathcal{J}$ on the left-hand side is defined by the same permutation on the indices of $\mathcal{I}$ on the right-hand side.

By linearity, the antisymmetry (C33) thus carries over to

$$\mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} = \mathcal{N}_d \, \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \tag{C37}$$

where $\mathcal{N}_d$ is a normalization constant. We will not need the explicit dependence of the normalization $\mathcal{N}_d$. We will only make use of the fact that it obeys the identity

$$\mathcal{N}_d(\boldsymbol{\theta}) = \frac{1}{(d!)^2} \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1}(\boldsymbol{\theta}). \tag{C38}$$

This follows from contracting the Levi-Civita antisymmetric tensor $\epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1}$ with itself and observing that the two sets of indices i and b run over d distinct values each.

The Jacobian of the map from $\boldsymbol{x} \in S^d$ to $\boldsymbol{\theta} \in S^d \subset \mathbb{R}^{d+1}$ is

$$\begin{aligned} \left| \frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{x}} \right| &= \epsilon_{i_d \cdots i_1} \partial_{i_d} \theta_{2d} \cdots \partial_{i_1} \theta_{d+1} \\ &= \frac{1}{d!} \epsilon_{i_d \cdots i_1} \epsilon_{\mathsf{b}_d \cdots \mathsf{b}_1} \partial_{i_d} \theta_{\mathsf{b}_d} \cdots \partial_{i_1} \theta_{\mathsf{b}_1} \\ &= \frac{1}{d!} \epsilon_{i_d \cdots i_1 \mathsf{b}_d \cdots \mathsf{b}_1} \partial_{i_d} \theta_{\mathsf{b}_d} \cdots \partial_{i_1} \theta_{\mathsf{b}_1} \\ &= \frac{(-)^{d(d-1)/2}}{d!} \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \partial_{i_d} \theta_{\mathsf{b}_d} \cdots \partial_{i_1} \theta_{\mathsf{b}_1}. \end{aligned} \tag{C39}$$

The conserved quasiparticle charge then becomes

$$\begin{aligned} Q_{\text{adia}} &:= \int\limits_{\boldsymbol{x} \in S^d} j^0_{\text{adia}} \\ &= (-\mathrm{i})^{d+1} \int\limits_{\boldsymbol{x} \in S^d} \mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \partial_{i_d} \theta_{\mathsf{b}_d} \cdots \partial_{i_1} \theta_{\mathsf{b}_1} \\ &= (-\mathrm{i})^{d+1} \int\limits_{\boldsymbol{x} \in S^d} \mathcal{N}_d \, \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \partial_{i_d} \theta_{\mathsf{b}_d} \cdots \partial_{i_1} \theta_{\mathsf{b}_1} \\ &= (-\mathrm{i})^{d+1} (-)^{d(d-1)/2} d! \int\limits_{\boldsymbol{\theta} \in S^d} \mathcal{N}_d(\boldsymbol{\theta}) \\ &= \frac{(-)^{d(d-1)/2} (-\mathrm{i})^{d+1}}{d!} \int\limits_{\boldsymbol{\theta} \in S^d} \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1}(\boldsymbol{\theta}) \end{aligned} \tag{C40}$$

with the help of Eq. (C38) to reach the last equality. □

For convenience, we introduce another family of indices μ through

$$\begin{aligned} &\mu_1, \cdots, \mu_{2d} = 1, \cdots, 2d, \\ &i_1, \cdots, i_d = 1, \cdots, d, \\ &\mathsf{b}_1, \cdots, \mathsf{b}_d = d+1, \cdots, 2d. \end{aligned} \tag{C41}$$

For any tensor $\mathcal{J}_{\mu_1 \cdots \mu_{2d}}$ that reduces to the fully antisymmetric tensor $\mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1}$, we have the identity

$$\epsilon_{\mu_1 \cdots \mu_{2d}} \mathcal{J}_{\mu_1 \cdots \mu_{2d}} = \frac{(2d)!}{(d!)^2} \, \epsilon_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1} \mathcal{J}_{i_d \mathsf{b}_d \cdots i_1 \mathsf{b}_1}, \tag{C42}$$

since the contraction on the left-hand side of this equation yields a combinatorial factor of $(2d)!$ whereas the contraction on the right-hand side yields a combinatorial factor of $d! \times d!$, i.e., one $d!$ for the family i of indices and another $d!$ for the distinct family b of indices. In particular, we can choose

$$\mathcal{J}_{\mu_1 \cdots \mu_{2d}} := \int\limits_{\omega} \int\limits_{\boldsymbol{p} \in S^d} \mathrm{tr}_R \left(\mathcal{G} \partial_{\mu_1} \mathcal{G}^{-1} \cdots \mathcal{G} \partial_{\mu_{2d}} \mathcal{G}^{-1} \mathcal{G} \partial_\omega \mathcal{G}^{-1} \right)_0 \tag{C43}$$

where the subscript 0 refers to the semi-classical Green function (C32).

At last, we add the non-compact imaginary-time label with the introduction of the family ν of indices,

$$\begin{aligned} &\nu_1, \cdots, \nu_{2d+1} = 0, 1, \cdots, 2d, \\ &\mu_1, \cdots, \mu_{2d} = 1, \cdots, 2d, \\ &i_1, \cdots, i_d = 1, \cdots, d, \\ &\mathsf{b}_1, \cdots, \mathsf{b}_d = d+1, \cdots, 2d. \end{aligned} \tag{C44}$$

Define

$$\begin{aligned} &K_\nu := (\omega, p_1, \cdots, p_d, \theta_{d+1}, \cdots, \theta_{2d}), \\ &\mathcal{J}_{\nu_1 \cdots \nu_{2d+1}} := \int\limits_{\omega} \int\limits_{\boldsymbol{p} \in S^d} \mathrm{tr}_R \left(\mathcal{G} \partial_{\nu_1} \mathcal{G}^{-1} \cdots \mathcal{G} \partial_{\nu_{2d+1}} \mathcal{G}^{-1} \right)_0, \\ &\int\limits_K \equiv \int\limits_{\omega} \int\limits_{\boldsymbol{p} \in S^d} \int\limits_{\boldsymbol{\theta} \in S^d} \equiv \int \frac{\mathrm{d}\omega}{2\pi} \int\limits_{S^d} \frac{\mathrm{d}\Omega_d(\boldsymbol{p})}{(2\pi)^d} \int\limits_{S^d} \frac{\mathrm{d}\Omega_d(\boldsymbol{\theta})}{(2\pi)^d}. \end{aligned} \tag{C45a}$$

The "surface" element of S^d is here denoted by $\mathrm{d}\Omega_d$. It follows that

$$\begin{aligned} Q_{\text{adia}} = &\frac{(-)^{d(d-1)/2} (-\mathrm{i})^{d+1} d!}{(2d+1)!} (2\pi)^d \\ &\times \int\limits_K \epsilon_{\nu_1 \cdots \nu_{2d+1}} \mathrm{tr}_R \left(\mathcal{G} \partial_{\nu_1} \mathcal{G}^{-1} \cdots \mathcal{G} \partial_{\nu_{2d+1}} \mathcal{G}^{-1} \right)_0 \end{aligned} \tag{C45b}$$

thereby completing the proof of Eq. (3.38). This is *the d-th Chern number* in $(2d+1)$-dimensional Euclidean space and time.

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[20] For a BCS treatment of superconductivity, this second quantized Hamiltonian is *not* the mean-field second-quantized BCS Hamiltonian. In the latter case the components $\widehat{\Psi}_r^\dagger(\boldsymbol{x})$ are not all independent from the components $\widehat{\Psi}_r(\boldsymbol{x})$. However, this is the correct auxiliary second-quantized Hamiltonian needed to derive the global conserved quasiparticle (Bogoliubov) charge as shown explicitly in Ref. 16.

[21] This additive decomposition is very similar to the spin-wave approximation. It is only valid locally and fails to capture the non-trivial topology of a defect such as a vortex. However, we will remedy this difficulty by the introduction of an additional charge-symmetry-breaking component to the Higgs multiplet for which this additive decomposition becomes meaningful globally.

[22] The vanishing term $0 = \mathcal{G}_0(\omega) - \mathcal{G}_0(\omega)$ that arises from the order $n = 0$ in the geometric series (2.17) need not vanish anymore after performing the semi-classical substitution by which $\mathcal{G}_0(\omega) - \mathcal{G}_0(\omega) \to \mathcal{G}_{\text{s-c}}(\omega) - \mathcal{G}_0(\omega)$. However, the term $\int\limits_{\omega,\boldsymbol{p}} \text{tr}_R\left[\mathcal{G}_{\text{s-c}}(\omega,\boldsymbol{p},\boldsymbol{x}) - \mathcal{G}_0(\omega,\boldsymbol{p})\right]$ does not contribute to the quasiparticle charge density provided that the traces tr_R over the Euclidean Green functions (2.19) and (2.20) both yield odd functions of ω. This is verified as a consequence of $\mathcal{H}^2(\boldsymbol{p},\boldsymbol{\phi}_0)$ being proportional to the unit matrix and $\text{tr}_R\,\mathcal{H}(\boldsymbol{p},\boldsymbol{\phi}_0) = 0$ for all the Hamiltonians studied in this paper.

[23] The Taylor expansion of each Green function in Eq. (2.18c) is performed to the d-th order. However, the identity

$$\left(\partial^n_{i_1\cdots i_n}\mathcal{G}\right)_0 = (-)^n\left(\mathcal{G}\partial_{i_1}\mathcal{G}^{-1}\cdots\mathcal{G}\partial_{i_n}\mathcal{G}^{-1}\mathcal{G}\right)_0 + \text{ all permutations of } i_1,\cdots,i_n$$

(the Leibnitz rule for the differentiation of $\mathcal{G}\mathcal{G}^{-1}$ when $\mathcal{G}^{-1}$ is linear in momentum) in combination with the fact that the trace (C33) over the Dirac matrices is fully antisymmetric can be used to show that only the linear expansion of each Green function in Eq. (2.18c) matters.

[24] The m-sphere S^m is the set of point in $(m+1)$-dimensional Euclidean space a unit distance away from the origin.

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