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Immortal quantum correlation in quasiperiodic quasi-1D systems

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The prevailing view on long-range correlations is that they typically attenuate uniformly with distance and temperature, as most interactions either exhibit short-range dominance or decay following a power law. In contrast to this belief, this study demonstrates that the intricate interplay between quasiperiodicity and the quasi-1D nature of subbands can result in strong long-range coupling without attenuation, a phenomenon referred to as an *immortal interaction*. Exemplifying a periodically stacked Fibonacci chain, we uncover an *immortal interaction* with greatly enhanced, persistent long-range coupling. Using negativity, it is shown that this interaction creates stable entanglement that endures over long distances and remains robust at finite temperatures. Additionally, unconventional logarithmic scaling entanglement is revealed, deviating from the traditional area law. These findings offer quasiperiodic quasi-1D systems as a novel platform for sustaining stable entanglement across exceptionally long distances, even in the presence of finite temperature effects.

Introduction— Entanglement, a fundamental quantum correlation, has fueled a broad spectrum of applications that are central to the second quantum revolution[1, 2]. Over the past century, entanglement has propelled advancements across various scientific domains, including quantum computing[3], condensed matter physics[4–6], and optics[7, 8]. One of the most significant research emerging from the discovery of entanglement is quantum information processing. A prime example is quantum teleportation — a protocol that enables the transfer of an unknown quantum state using preexisting entanglement and a classical communication channel[9]. The effectiveness of such applications critically depends on the availability of long-range entanglement between the sending and receiving parties[10].

In condensed matter systems, long-range entanglement also carries significant importance. By analyzing their scaling behavior, it can distinguish different phases of matters and topological orders in the presence of strong correlation, such as quantum spin liquids[11–14]. Moreover, recent studies explore various experimental probes capable of measuring quantum entanglement in multipartite systems, with neutron scattering experiments being one method for extracting quantum Fisher information as an example[15–21]. Accordingly, both theoretical and experimental researchers are highly invested in preparation and control of long-range entanglement in condensed matter systems[22, 23]. In typical condensed matter systems, however, the entanglement between distant particles tends to decay uniformly since the interactions responsible for generating entanglement are generally short-range or follow a power-law decay, diminishing to negligible levels over experimentally accessible distances[23–25]. Hence, long-range entanglement is usually obscured by thermalization[17]. Thus, to effectively utilize long-range entanglement and explore strongly correlated physics, extremely low temperatures are necessary[26]. This requirement presents substantial challenges for exploring various applications of quantum correlations, including long-distance quantum teleportation and long-range entangled phases of matter.

Some slowly decaying long-range interactions like unscreened Coulomb interaction, have been expected as key drivers of long-distance entanglement in many-body quantum systems[23, 27, 28]. Such interactions can accumulate mutual information across a subsystem, leading to long-range entanglement which diverges logarithmically or follows a volume law, even in non-critical systems[29]. Various experimental platforms, such as cold atoms[30], Rydberg atom arrays[31], superconducting islands[32] and dipolar systems[33] have been explored for their ability to fine-tune interactions and enhance entanglement over long distances[34]. Utilizing these systems, however, is challenging since diminished interaction strength over long distances requires a large number of qubits to maintain coherence while serving as intermediaries between widely separated target qubits[27, 35]. This makes these systems prone to environmental disturbances like decoherence, limiting their practical use for long-range entanglement[36, 37]. Hence, finding a platform that can achieve stable long-range entanglement over practical distances despite environmental influences remains an unsolved task.

In this work, we study the quasiperiodic quasi-1D system, exemplified by a periodically stacked Fibonacci chain, and explore anomalously non-decaying long-range interactions and resulting unexpected characteristics of stable quantum correlations. This system exhibits quasiperiodicity along the x direction and periodicity along the orthogonal y direction (see Fig. 1). Such a structure, where quasiperiodicity and periodicity coexist, has been observed in certain materials, for instance, Al-Ni-Co alloy[38, 39]. Notably, the indirect spin interaction in this system is not only anomalously enhanced over long distances in the quasiperiodic x-direction but, even more strikingly, it exhibits no attenuation along

the periodic y-direction — an effect we refer to as the *immortal interaction*. This unique phenomenon stems from the interplay between the criticality inherent to the quasiperiodic structure and the quasi-1D nature of the system, where non-Bloch wavefunctions with critical characteristics form subbands due to stacking. These intertwined quantum effects give rise to unusually strong interactions that are exceptionally persistent and non-decaying. By employing entanglement negativity, we show that this interaction leads to very stable entanglement, even over long distances and with finite temperature. Furthermore, the immortal interaction enforces entanglement monogamy, leading to an unconventional logarithmic scaling contrast to the conventional area law. Our work presents a novel platform that maintains stable entanglement across vast distances and under finite temperature conditions, highlighting its unusual scaling properties.

Periodically stacked quasiperiodic chains— Let us consider identical quasiperiodic chains that are periodically stacked as shown in Fig.1. We assume that the number of quasiperiodic chains, L_y, is much smaller than the length of each quasiperiodic chain, L_x, thus a quasiperiodic quasi-1D system. To provide a concrete example, we consider the 1D Fibonacci quasicrystal as a specific instance of a quasiperiodic chain. However, we emphasize that our results are not limited to this specific case and can be observed in other quasiperiodic models near criticality. In the Fibonacci case, the system consists of two types of nearest-neighbor atomic distances, A and B, each corresponding to different hopping integrals, denoted $t_{\rm A}$ and $t_{\rm B}$, respectively[40]. The strength of quasiperiodicity is given by their ratio, $\kappa = |\log(t_{\rm A}/t_{\rm B})|$. We assume $t_{\rm A} \leq t_{\rm B}$. The infinitely long Fibonacci quasicrystal is generated by a successive substitution map, A $\rightarrow$ AB and B $\rightarrow$ A[40]. For clarity, we refer to the x direction as quasiperiodic and the y direction as periodic.

Let us consider the nearest neighbor tight-binding Hamiltonian Eq.(1) for the electronic state, along with localized spin-1/2 impurity spins positioned at a specific set of lattice sites, $\mathcal{D}$. The set $\mathcal{D}$ is comprised of the sites with different y positions at two x positions exhibiting strong long-range interactions at the given Fermi level of the quasiperiodic chain.

$$H = T_x + T_y + J_K \sum_{p\in\mathcal{D}} \vec{S}_p \cdot \vec{s}_p. \tag{1}$$

Here, T_x and T_y represent the nearest neighbor tight-binding Hamiltonian along x direction and y direction, respectively. In detail, $T_x = -\sum_{i,j,\sigma}(t_{i,i+1}c^\dagger_{i+1,j,\sigma}c_{i,j,\sigma} + h.c.)$ and $T_y = -t\sum_{i,j,\sigma}(c^\dagger_{i,j+1,\sigma}c_{i,j,\sigma} + h.c.)$, where $c^\dagger_{i,j,\sigma}$ and $c_{i,j,\sigma}$ are electron creation and annihilation operators for the site (i,j) and spin σ. Note that $T_x + T_y$ has sublattice symmetry with open boundary conditions we consider here. Given that the hopping integral along the y direction is $t = 1$, the average hopping integral along the x-direction is also set to be 1. $\vec{S}_p$ and $\vec{s}_p = \sum_{\sigma,\sigma'} c^\dagger_{p,\sigma}\vec{\sigma}_{\sigma,\sigma'}c_{p,\sigma'}/2$, where $\vec{\sigma}$ is the Pauli matrix vector represent the localized impurity spin-1/2 and the tight-binding electron spin at the p site, respectively. $J_K < 0$ is the local exchange coupling between the impurity and tight-binding electron spins, which is perturbatively small compared to the average hopping integrals, $|J_K| \ll t$. By integrating out the electronic degrees of freedom, one can get the long-range indirect exchange coupling between different impurity spins, $\vec{S}_p$ and $\vec{S}_q$, $J_{p,q}$, given by,

$$J_{p,q} = \frac{J_K^2}{4} \sum_{m,n,E_m\neq E_n} \mathcal{R}[\psi_m(p)\psi_m(q)^*\psi_n(q)\psi_n(p)^*] \times \frac{n_F(E_n - E_F) - n_F(E_m - E_F)}{E_n - E_m}. \tag{2}$$

Here, $\hbar = k_B = 1$, E_F is the Fermi level. $\mathcal{R}[x]$ is real part of x, respectively. n, m are indices of the eigenstates, and $\psi_n(p) = \langle p|n\rangle$ is the wave function of the energy eigenstate $|n\rangle$ whose energy is E_n at the p site. $n_F(x) = (1+\exp(\beta x))^{-1}$ is the Fermi distribution function, where $\beta = 1/\tau$ and τ is temperature[41]. Hence, $J_{p,q}$ in Eq.(2) has temperature dependence. In conventional crystals, the magnitude of $J_{p,q}$ decays according to a power-law, especially in 2D systems, $|J_{p,q}| \sim R^{-2}_{p,q}$, where $R_{p,q}$ is the distance between p and q sites[42].

The energy levels near the Fermi level play a crucial role in the long-range coupling described by Eq.(2)[42].

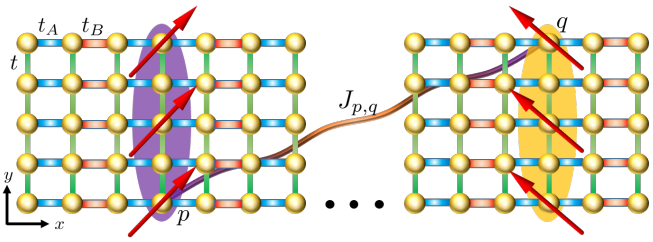


FIG. 1. Schematic illustration of the quasiperiodic quasi-1D system, consisting of periodically stacked quasiperiodic chains. Yellow balls represent the atomic sites. Red and blue bars represent different hopping integrals, t_A and t_B in each quasiperiodic chain along the x direction, respectively. Green bars represent uniform hopping integral, t along the y direction. Red arrows represent localized impurity spin moments that interact locally with the tight-binding electron spin. Tight-binding electrons in such quasiperiodic system mediate indirect long-range interaction between two localized impurity spins at the p and q sites, say $J_{p,q}$. As t_A deviates from t_B, $J_{p,q}$ is enhanced, even when the p and q sites are in different layers and are separated by a distance on the order of 10^3 sites. Violet and orange shaded regions are drawn for emphasizing two widely separated parties of the localized impurity spins. Each party contains $N = (L_y + 1)/2$ impurity spins, where L_y is the number of quasiperiodic chains. See the main text for details.

Specifically, when states near the Fermi level are not highly extended but instead exhibit critical behavior, the coupling strength between certain sites gets enhanced[41]. This effect is further amplified by the strong quasiperiodicity, which makes critical states concentrated on certain multiple widely separated sites, leading to an anomalous enhancement of the coupling strength between many pairs of sites that are far apart within each quasiperiodic layer[41]. Note that these strongly interacting sites can be identified by their local tiling patterns, independent of system size and boundary conditions[41]. Furthermore, since quasiperiodicity discretizes the electronic spectrum[40], only a portion of subbands originating from the periodic stacking would be dominant in Eq.(2). This leads to an anomalously strong long-range coupling that does not attenuate along the stacking direction, as we will discuss further.

Immortal interaction— Given a pair of x positions, say x_1 and x_2, that exhibits anomalously strong coupling within each quasiperiodic chain, we analyze $J_{p,q}$ for $p = (x_p, y_p)$ and $q = (x_q, y_q)$, where $x_{p(q)}$ are either x_1 or x_2 and $0 \leq y_{p(q)}/a_y < L_y$, where a_y is the unit length of the y-axis. Here, we consider the case of $|x_1 - x_2| \sim \mathcal{O}(10^3 a)$, where a is the average atomic distance in a Fibonacci chain. As a concrete example, we focus on the half-filled case with $E_F = 0$ under the sublattice symmetry. It is important to note that our results can be generalized to the cases with different Fermi levels.

Fig.2 illustrates $J_{p,q}$ for $\kappa = 0$ and $\kappa \neq 0$, as a function of $\Delta y/a_y = |y_p - y_q|/a_y$. For the perfectly periodic case with $\kappa = 0$, the coupling strength shows conventional attenuation. Thus, $J_{p,q}$ is significant only for the neighboring sites (See Fig.2 (b)). On the other hand, for quasiperiodic case with $\kappa \neq 0$, the $J_{p,q}$ interaction is substantial even between widely separated positions and does not diminish along the periodic stacking direction. In detail, $J_{p,q}$ shown in Fig.2 (a) oscillates with a period of $2a_y$ while maintaining nearly constant amplitudes, in contrast to the case of $\kappa = 0$ depicted in Fig.2 (b). It is surprising that the coupling strength does not diminish in the y direction, despite the periodic stacking of the chains. We will refer to this unexpected non-decaying interaction as the *immortal interaction*. As a graphical representation of connectivity, Fig.2 (c) shows how the immortal interaction, $J_{p,q}$, connects different sites p and q, where the solid lines of the same color represent the same $J_{p,q}$. Here, sites with negligible $J_{p,q}$ values are omitted.

To understand such immortal interaction, we first note that the electronic energy of tight-binding Hamiltonian can be expressed as the sum of the energies originating from T_x and T_y, respectively, because $[T_x, T_y] = 0$. For sufficiently large κ, $\langle n_x|x\rangle$ for $x = x_1$ or x_2 is highly concentrated near the zero energy, where $|n_x\rangle$ is the energy level of T_x in a Fibonacci chain by the definition of x_1 and x_2[41]. Thus, the energy contribution from T_y must

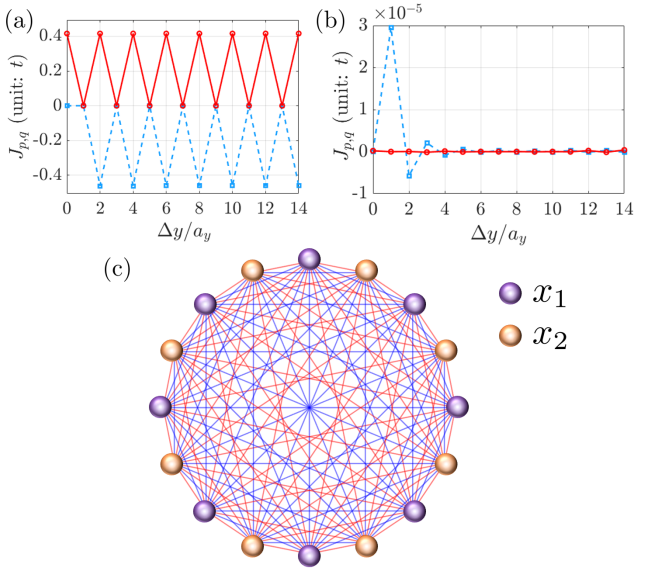


FIG. 2. Long-range coupling $J_{p,q}$ as a function of $\Delta y = |y_p - y_q|$ for (a) quasiperiodic case with $\kappa = 2.3$ and (b) periodic case with $\kappa = 0$, respectively. Red solid lines and blue dashed lines represent the cases of $x_p \neq x_q$ and $x_p = x_q$, respectively. $L_y = 15$ and $L_x = 1597$. $x_p = 305$ and $y_p = 0$. x_q is either 305 or 1292. Here, 305 and 1292 are specific examples of x_1 and x_2, which are strongly interacting long-distant sites in a Fibonacci chain. $|J_K/t| = 0.1$. $\beta^{-1} = 10^{-5}t$. (c) Schematic graphical representation of the immortal interaction in (a). Each colored (violet and orange) spheres represent sites of the same x-positions, x_1 and x_2 with different y positions, respectively. Sites with negligible interaction strength are omitted. Here, red and blue lines represent the couplings shown as the red solid line and the blue dashed line in (a), respectively.

also be zero in order for the total energy $T_x + T_y$ to be near the Fermi level. Note that due to the quasi-1D nature, the subbands sparsely occupy the energy spectrum. This effectively compresses the y degrees of freedom to $k_y a_y = \pi/2$, resulting in $T_y\,|k_y a_y = \pi/2\rangle = 0$[43]. Consequently, the immortal interaction oscillates with a period of $2a_y$. In detail, the immortal interaction is nearly zero for an odd $|y_p - y_q|/a_y$, while it is uniformly strong for an even $|y_p - y_q|/a_y$. We note that the immortal interaction occurs only for odd L_y, where T_y admits the zero energy. Thus, one can conclude the interplay of quasiperiodicity and the subband characteristics of quasi-1D system leads to a highly restricted phase space of electronic states in determining the indirect spin interaction. This results in the emergence of a long-range, non-decaying phenomenon, *immortal interaction*.

Anomalous entanglement driven by immortal interactions — The immortal interaction is expected to produce unique entanglement properties. Note that conventional decaying interactions typically lead to area law scaling of entanglement, which is proportional to the boundary area between subsystems[27]. Thus, the long-range en-

tanglement decreases rapidly with increasing temperature or distance[22, 27]. While, the immortal interaction can modify these conventional properties of entanglement without requiring intermediary qubits, owing to its anomalously enhanced strength and the monogamy of entanglement.

To compute the entanglement between impurity spins using the thermal density matrix at finite temperature, $\rho(\beta) = e^{-\beta H_{\rm loc}}/{\rm Tr}(e^{-\beta H_{\rm loc}})$, we adopt negativity as the measure of thermal entanglement. Here, $H_{\rm loc} = \sum_{p\neq q} J_{p,q}\vec{S}_p \cdot \vec{S}_q$ is the effective Hamiltonian for the impurity spins. We group impurity spins into two parties based on their x-positions, x_1 and x_2 that admit antiferromagnetic couplings between parties while ferromagnetic couplings within each party, as shown in Fig.2 (a). Two parties, separated by a mesoscopic distance, are represented by violet and orange shaded regions in Fig.1. Since the immortal interaction oscillates as a function of Δy with a period of $2a_y$, $N = (L_y+1)/2$ impurity spins in each party interact via the immortal couplings with a large κ and an odd L_y. Fig.3 (a) provides a graphical representation of how $2N$ spins, belonging to two parties, are coupled through immortal interactions, with an example for $N = 5$. Note that there are three different immortal couplings. The negativity of $\rho(\beta)$ is given by $\mathcal{N}(\rho(\beta)) = \log_2(||\rho(\beta)^{\Gamma_1}||)$, where Γ_1 denotes the partial transpose operation with respect to the x_1 subsystem, and $||\cdot||$ represents the trace norm.

Fig.3 illustrates the negativity as functions of κ, N and temperature. First, when the quasiperiodicity is sufficiently strong, entanglement between two widely separated parties can persist even at finite temperatures. Specifically, Fig.3 (b) shows that for small κ, the negativity quickly disappears as temperature increases even at low temperature regime, whereas for large κ, the negativity remains robust even at higher temperatures. Hence, strong quasiperiodicity enables us to utilize long-range quantum correlations at higher temperatures.

Next, we find the unusual scaling behavior of negativity under conditions of sufficiently strong quasiperiodicity. Fig.3 (c) shows that for sufficiently large values of κ, the negativity grows logarithmically with N, deviating from the conventional area law where it would typically increase linearly with N[22, 43]. In detail, for given $\beta < \infty$, the negativity is either constantly zero (small κ) or logarithmically grows (large κ). When κ is small, we have $\mathcal{N}(\rho(\beta)) = 0$ due to the thermalization. Whereas, for large κ, we find logarithmic scaling given by

$$\mathcal{N}(\rho(\beta)) \sim \log_2(N+1). \tag{3}$$

Particularly, for $\kappa \gg 0$, $\mathcal{N}(\rho(\beta)) = \log_2(N+1)$. Note that for given finite temperature, the immortal couplings with $\kappa \gg 0$ are strong compared to β^{-1}. Thus, the density matrix is approximated by the pure ground state, $|\Psi_G\rangle\langle\Psi_G|$ of $H_{\rm loc}$. Here, $H_{\rm loc}$ can be rewritten in terms

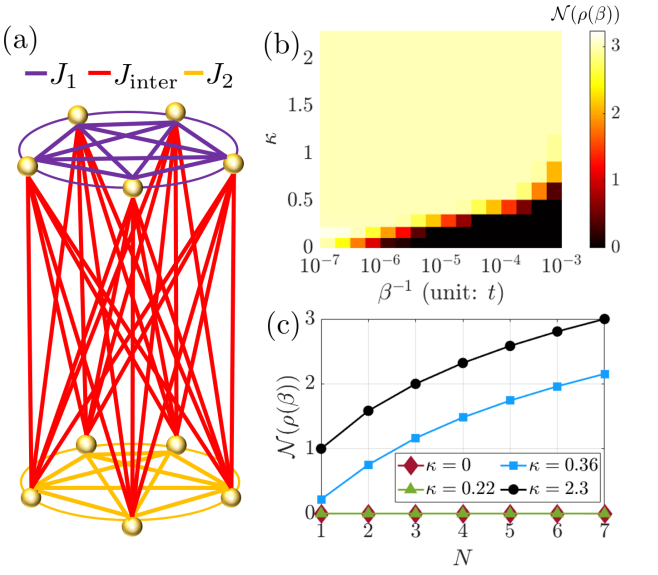


FIG. 3. Entanglement characteristics induced by immortal interactions. (a) Schematic graphical representation of the immortal interactions between two parties, highlighted as violet and orange shaded regions in Fig.1, exemplified for $N = 5$. Violet, orange lines represent the couplings within each party, J_1, J_2, respectively, while red lines represent the couplings between parties, $J_{\rm inter}$. Here, J_1 and J_2 need not be identical for general cases. (b) Negativity between two parties as functions of temperature and quasiperiodicity strength for $N = 7$. The negativity remains constant at 3 for $\kappa > 0.5$, even at finite temperature $\beta^{-1} = 10^{-4}t$. (c) Negativity as a function of the size of each party, N for $\beta^{-1} = 10^{-5}t$ and different values of κ. Black circle and blue square curves show logarithmic scaling. Particularly, the black circle curve is given by $\mathcal{N}(\rho(\beta)) = \log_2(N+1)$. Here, $L_x = 1597$. $x_1 = 305$ and $x_2 = 1292$. $|J_K/t| = 0.1$.

of the immortal couplings as,

$$\begin{aligned} H_{\rm loc}(\kappa \gg 0) =& (J_1 - J_{\rm inter})\vec{S}^2_{1,\rm tot} + (J_2 - J_{\rm inter})\vec{S}^2_{2,\rm tot} \\ &+ J_{\rm inter}\vec{S}^2_{\rm tot} + {\rm const}, \end{aligned} \tag{4}$$

where $J_1, J_2 < 0$ are the ferromagnetic immortal coupling within each parties, and $J_{\rm inter} > 0$ is the antiferromagnetic immortal coupling between parties. $\vec{S}_{1,\rm tot}$, $\vec{S}_{2,\rm tot}$ are total spins of each party, and $\vec{S}_{\rm tot} = \vec{S}_{1,\rm tot} + \vec{S}_{2,\rm tot}$. Now, $H_{\rm loc}$ possesses SU(2) symmetry, and the four operators $H_{\rm loc}, \vec{S}^2_{\rm tot}, \vec{S}^2_{1,\rm tot}$ and $\vec{S}^2_{2,\rm tot}$ are all commute with each other. Hence, $|\Psi_G\rangle = |S_1 = N/2, S_2 = N/2, S = 0\rangle$, where S_1, S_2 and S are total spin quantum numbers defined by $\vec{S}^2_{1,\rm tot}, \vec{S}^2_{2,\rm tot}$ and $\vec{S}^2_{\rm tot}$, respectively. Thus, the negativity is given by $\mathcal{N}(|\Psi_G\rangle\langle\Psi_G|) = \log_2(N+1)$.

The logarithmic scaling of negativity is a consequence of the monogamy of entanglement induced by the immortal interactions[44]. Since the immortal interaction with sufficiently large κ remains constant regardless of the y-distances, N spins in one party compete equally to form

singlets with spins in the other party. This causes a significant enhancement in the monogamy of entanglement, which in turn suppresses the growth of entanglement. As a result, the scaling behavior of negativity follows a logarithmic pattern as Eq.(3), deviating from the conventional area law. Therefore, the immortal interaction offers a novel approach to realizing an unconventional phase of matter, characterized by this distinct entanglement scaling behavior.

Conclusion— In summary, we propose quasiperiodic quasi-1D systems as a novel platform for realizing anomalous long-range interactions. Using periodically stacked Fibonacci chains as an example, the indirect interaction between two spins, even when separated by mesoscopic or macroscopic distances, is shown to be significantly enhanced and, more strikingly, does not decay along the stacking direction — a phenomenon referred to as the *immortal interaction*. First, the long-range entanglement is stable with respect to unavoidable thermal fluctuation effect. Hence, such systems would be a key player to develop a stable platform of a wide range of quantum engineering such as quantum teleportation and quantum computing. Second, the entanglement induced by the immortal interaction grows logarithmically since it enhances the effect of the monogamy of entanglement. Thus, the immortal interaction also enables us to reach novel long-range entangled phase of matters[14, 45, 46]. To measure this long-range entanglement, negativity is one example as we suggested, which can be quantified through various quantum state tomography techniques[47]. Notably, our unconventional logarithmic scaling of entanglement allows for an efficient matrix product state representation with a polynomial bond dimension, making tomography feasible[43, 48]. Beyond negativity, there are also other experimentally accessible indirect entanglement measures, like quantum Fisher information[15, 16], which can be also applicable to our systems.

Our findings can be examined in a variety of distinct systems, such as quasiperiodic alloy materials[40, 49], artificially engineered photonic crystals and optical lattices with cold atoms[50, 51]. The anomalous immortal interaction between magnetic impurities on certain alloy surfaces like Al-Ni-Co alloy is anticipated to be observed experimentally using STM-ESR measurements[52, 53]. Furthermore, in artificial platforms like photonic crystals or optical lattices, the strength of quasiperiodicity can be flexibly manipulated, for instance, by tuning the dielectric constant or the lattice potential[50, 54]. This flexibility would make the anomalous behavior of long-range correlations in these platforms more pronounced. In addition, our physics also extend beyond the current scope, offering potential applications even in quasiperiodic quasi-2D systems, such as heterostructures and multilayer van der Waals materials with twisting angles that induce quasiperiodicity[55, 56]. These discoveries pave the way for designing exotic quantum phases, stronger long-range interactions, and robust quantum entanglement in quantum materials.

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