



This is the accepted manuscript made available via CHORUS. The article has been published as:

Unconventional Josephson supercurrent diode effect induced by chiral spin-orbit coupling

Andreas Costa, Osamu Kanehira, Hiroaki Matsueda, and Jaroslav Fabian

Phys. Rev. B **111**, L140506 — Published 17 April 2025

DOI: [10.1103/PhysRevB.111.L140506](https://doi.org/10.1103/PhysRevB.111.L140506)

Unconventional Josephson Supercurrent Diode Effect Induced by Chiral Spin-Orbit Coupling

Andreas Costa,^{1,*} Osamu Kanehira,² Hiroaki Matsueda,^{2,3} and Jaroslav Fabian¹

¹*Institute for Theoretical Physics, University of Regensburg, 93040 Regensburg, Germany*

²*Department of Applied Physics, Tohoku University, Sendai 980-8579, Japan*

³*Center for Science and Innovation in Spintronics, Sendai 980-8577, Japan*

(Dated: February 16, 2025)

Chiral materials lacking mirror symmetry can exhibit unconventional spin-orbit fields, including fully momentum-aligned radial Rashba fields as seen in twisted van-der-Waals homobilayers. We theoretically study Cooper-pair transfer in superconductor/ferromagnet/superconductor Josephson junctions with crossed (tangential and radial) interfacial Rashba fields. We find that their interplay leads to what we call the unconventional supercurrent diode effect (SDE), where supercurrent rectification occurs even with collinear (w.r.t. the current) barrier magnetization, not possible for conventional spin-orbit fields. This SDE, distinct from conventional Rashba-induced effects on Cooper-pair momenta, arises from the spin precession in the magnetic barrier. We propose it as a sensitive probe of chiral spin textures.

Introduction. The interplay of spin-orbit coupling (SOC) and magnetism is essential for spintronics applications [1, 2], enabling transport magnetoanisotropies [3, 4], tunneling Hall effects [5, 6], or spin-orbit torque in magnetic tunnel junctions [7]. Particularly sensitive probes are superconducting junctions [8, 9], in which superconducting coherence can significantly amplify this interplay [10–14] and lead, e.g., to triplet pairing [8, 15–21].

In Josephson junctions formed by Al/InAs heterostructures, the interfacial Rashba field induces supercurrent rectification in the presence of an in-plane magnetic field perpendicular to the current direction [22–36]. This supercurrent diode effect (SDE) has been observed in a variety of systems including superconducting superlattices [37], twisted bi- and tri-layer graphene [38–40], van-der-Waals heterostructures [41–44], or topological materials [25, 45–47]. Detecting the SDE in twisted multilayer high-temperature superconductors [48–52] indicates unconventional (e.g., *d*-wave-like) superconducting pairing.

The common argument for the SDE is the formation of Cooper pairs with a finite center-of-mass momentum due to the distorted Fermi surface in the presence of SOC, such as Rashba [53, 54] and Dresselhaus [55], and an in-plane magnetic field [30, 56–60]. This results in marked phase asymmetries of the Andreev states [61] and anomalous φ_0 -phase shifts [62–80] in the current–phase relations (CPRs) of the individual transverse channels. The interference of multiple channels in wide junctions—each with slightly different φ_0 —leads to a strongly distorted total CPR and a sizable Josephson SDE [22, 23, 28, 29, 34, 81, 82]. Recent theoretical works [83, 84] investigating supercurrents through a chiral quantum dot argue for an SDE even without SOC and finite Cooper-pair momentum.

Rashba and Dresselhaus spin-orbit fields in magnetic junctions are well studied, particularly in III–V semiconductors [85, 86] like InAs being used in Al/InAs/Al Josephson junctions [22, 23]. However, the recently discovered chiral spin textures in topological materials [87–93] and predictions of

purely radial Rashba SOC in twisted heterostructures [94, 95] offer ways for controlling magnetotransport, as shown by large-scale simulations of quantum focusing [36].

In this paper, we investigate Josephson junctions with a magnetic barrier and interfacial regions that feature crossed conventional (CR) and radial Rashba (RR) fields in the plane transverse to the tunneling. While it is expected that a magnetization in the plane of the spin-orbit fields leads to the SDE, for the crossed fields we find a marked nonreciprocal critical current for the magnetization of the barrier collinear with the transport direction and perpendicular to the spin-orbit fields. We term this effect *unconventional SDE (USDE)*. The mechanism for the USDE is polarity- and field-orientation-dependent precession of the electron spins, conditioned by the interfacial spin-orbit fields, by the magnetization of the tunneling barrier finally resulting in different transmission probabilities for left- and right-propagating electrons. The USDE is different from the commonly considered finite-momentum Cooper-pair generation in the conventional SDE, which requires the magnetization (or an external magnetic field) to lie in the plane of the Rashba field (if the magnetization is perpendicular, as in our case, the Cooper pairs have nominally zero momentum in each electrode). Our numerical model calculations illustrate that the USDE is sizable already at weak radial Rashba coupling, reproducing all symmetries (tunabilities) w.r.t. the out-of-plane magnetization orientation and Rashba angle that are predicted by our spin-precession picture.

Theoretical model. We consider an epitaxially grown, ballistic, vertical S/F/S Josephson junction whose semi-infinite *s*-wave superconducting electrodes (S; spanning $z < 0$ and $z > d$) are weakly coupled by a ferromagnet (F; spanning $0 < z < d$) and that hosts ultrathin tunneling barriers at both F/S interfaces, as illustrated in Fig. 1. The interface at $z = 0$ could, e.g., consist of a van-der-Waals monolayer to induce—apart from scalar potential scattering—CR SOC, while a twisted heterostructure or another chiral material induces an unconventional RR component, quantified by the twist-angle-dependent [94] Rashba angle θ_R , at the $z = d$ interface. A possible experimental realization could be based on NbSe₂ as superconducting electrodes [42] and a twisted heterostructure of the quasi-2D van-der-Waals magnet FGT (Fe_{3–*x*}GeTe₂) [96] serving as magnetic link and inducing crossed Rashba fields

* Corresponding author: andreas.costa@physik.uni-regensburg.de

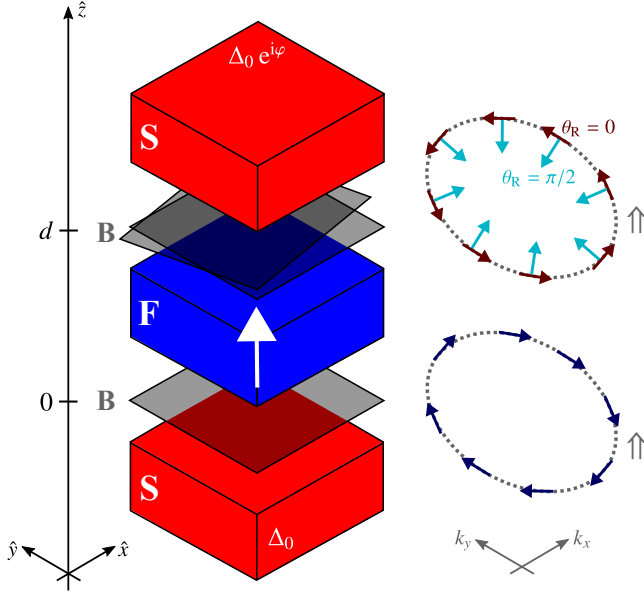


FIG. 1. Sketch of the vertical S/F/S Josephson junction using C_{2v} principal crystallographic orientations $\hat{x} \parallel [110]$, $\hat{y} \parallel [\bar{1}10]$, and $\hat{z} \parallel [001]$; $\Delta_0(\varphi)$ indicates the superconducting gap (phase difference) and the magnetization direction (white arrow) of the F with length d is aligned along $+\hat{z}$ (out-of-plane). The two-dimensional monolayer barrier (B) at $z = 0$ induces CR SOC (dark-blue arrows, shown along the spin-up Fermi surface $\uparrow$), while the twisted bilayer B at $z = d$ results in crossed CR (dark-red) and RR (turquoise) SOC depending on the Rashba angle θ_R .

due to its lacking mirror symmetry.

Solving the stationary Bogoljubov–de Gennes (BdG) [97] equation $\hat{\mathcal{H}}_{\text{BdG}}\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$, with the corresponding BdG Hamiltonian

$$\hat{\mathcal{H}}_{\text{BdG}} = \begin{bmatrix} \hat{\mathcal{H}}_e & \hat{\Delta}_S(z) \\ \hat{\Delta}_S^\dagger(z) & \hat{\mathcal{H}}_h \end{bmatrix}, \quad (1)$$

yields the general scattering states $\Psi(\mathbf{r})$ for incoming spin-polarized electronlike and holelike quasiparticles of excitation energy E , from which we obtain the Andreev-reflection amplitudes. These coefficients provide the input to numerically compute the CPRs $I = I(\varphi)$ through the Green's function-based Furusaki–Tsukada approach [98], as outlined in the Supplemental Material (SM) [99]. Assuming equal Fermi levels μ and effective masses m throughout the junction, the effective Hamiltonian for unpaired electrons reads as $\hat{\mathcal{H}}_e = [-\hbar^2/(2m)\nabla^2 - \mu]\hat{\sigma}_0 - (\Delta_{\text{XC}}/2)\Theta(z)\Theta(d-z)(\hat{\mathbf{m}} \cdot \hat{\boldsymbol{\sigma}}) + \hat{\mathcal{H}}_B$, its holelike counterpart as $\hat{\mathcal{H}}_h = -\hat{\sigma}_y\hat{\mathcal{H}}_e^*\hat{\sigma}_y$, and the S pairing potential is approximated [100] by $\hat{\Delta}_S(z) = \Delta_0 \tanh(1.74\sqrt{T_c/T} - 1)[\Theta(-z) + e^{i\varphi}\Theta(z-d)]\hat{\sigma}_0$ with the zero-temperature gap $\Delta_0 = 2.5 \text{ meV} \approx 10^{-3}\mu$, the ratio between temperature T and critical temperature T_c set to $T/T_c = 0.1$, and the phase difference φ . The unit vector $\hat{\mathbf{m}} = [\cos(\Theta)\cos(\Phi), \cos(\Theta)\sin(\Phi), \sin(\Theta)]^T$ describes the magnetization direction inside the F with exchange splitting Δ_{XC} in terms of the out-of-plane angle Θ (w.r.t. the interfaces) and in-plane azimuthal angle Φ ($\Phi = 0.5\pi$ in all numerical cal-

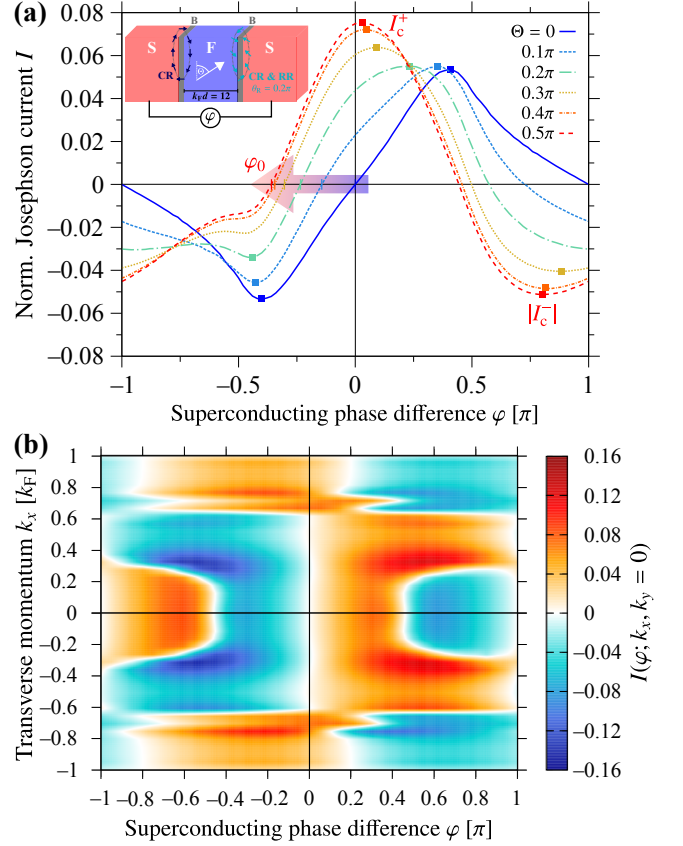


FIG. 2. (a) CPRs $I(\varphi)$ for the effective F length $k_F d = 12$, Rashba SOC $\lambda_R = 2m\alpha/\hbar^2 = 1$, Rashba angle $\theta_R = 0.2\pi$, and indicated out-of-plane magnetization angles Θ ; the polarity-dependent critical currents I_c^+ and I_c^- are indicated by squares, and the φ_0 -shifts by ticks along the φ -axis. (b) Resolved CPRs $I(\varphi; k_x, k_y = 0)$ for transverse channels with momenta $k_x \in [-k_F, k_F]$ (while $k_y = 0$) and out-of-plane magnetization $\Theta = 0.5\pi$. The total CPR in (a)—dashed red curve—follows from integrating out k_x and k_y ; white regions indicate the φ_0 -phase shifts (CPR zero crossings). The current is always given in multiples of $\pi\Delta_0 G_S/e$; $G_S = Ae^2 k_F^2/(2\pi\hbar)$ is Sharvin's conductance of a point contact with cross-section A .

culations), while $\hat{\sigma}_0$ ($\hat{\sigma}_i$) indicates the 2×2 identity (i th Pauli spin) matrix and $\hat{\boldsymbol{\sigma}} = [\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z]^T$. The ultrathin barriers of height (width) V_B (d_B) are described in the delta-like form $\hat{\mathcal{H}}_B = V_B d_B \hat{\sigma}_0 [\delta(z) + \delta(z-d)] + \hat{\boldsymbol{\Omega}}(\mathbf{k}_{\parallel}) \cdot \hat{\boldsymbol{\sigma}}$, with the Rashba spin-orbit field $\hat{\boldsymbol{\Omega}}(\mathbf{k}_{\parallel}) = \hat{\boldsymbol{\Omega}}(k_x, k_y) = \alpha[k_y, -k_x, 0]\delta(z) + \alpha[-\sin(\theta_R)k_x - \cos(\theta_R)k_y, \cos(\theta_R)k_x - \sin(\theta_R)k_y, 0]\delta(z-d)$ accounting for CR (crossed CR and RR) SOC of strength α at the $z = 0$ ($z = d$) interface.

Numerical results. To analyze our numerical results, we introduce the dimensionless parameters $P = (\Delta_{\text{XC}}/2)/\mu [k_F d]$ for the spin polarization [effective length] of the F, while $Z = 2mV_B d_B/(\hbar^2 k_F)$ and $\lambda_R = 2m\alpha/\hbar^2$ quantify the interfacial barrier and Rashba strengths. We consider a weak F described by $P = 0.4$ and $k_F d = 12$ (corresponding to a length of a few nm depending on the Fermi level), as well as high interfacial transparencies of $\tau = [1 + (Z/2)^2]^{-1} = 80\%$ (equivalently $Z = 1$) [101] and realistic Rashba SOC $\lambda_R = 1$ [10, 13].

Figure 2(a) illustrates a generic case: the CPRs $I(\varphi)$ in the presence of CR at the $z = 0$ and *crossed* Rashba fields with Rashba angle $\theta_R = 0.2\pi$ at the $z = d$ interface, with variable F magnetization from the in-plane to the positive out-of-plane orientation. If the magnetization is fully aligned in the interfacial plane ($\Theta = 0$), the CPR is strictly point-symmetric w.r.t. zero phase difference and the critical-current amplitudes are polarity-independent, i.e., $I_c^+ = |I_c^-| \equiv I_c(\Theta = 0)$. With increasing Θ , a finite out-of-plane component $m_z \neq 0$ is imprinted on the magnetization and the electron spins precess in the F as explained in the next section. The precession w.r.t. the radial Rashba field leads to polarity-dependent transmission probabilities, manifesting themselves in polarity-dependent critical-current amplitudes $I_c^+ \neq |I_c^-|$ as a direct signature of the USDE. Furthermore, the CPRs acquire intrinsic φ_0 -shifts—such that $I(\varphi) \propto \sin(\varphi - \varphi_0)$ in the simplest case—effectively shifting their zero crossings and inflection points to finite phase differences [28, 29]; $|\varphi_0|$ monotonically increases with increasing $|\Theta|$ and becomes maximal for fully perpendicular magnetization ($\Theta = 0.5\pi$). The amplitudes of both φ_0 [reaching about 0.4π ; see Fig. 3(a)] and the USDE [see Fig. 3(b)] are sizable already at rather small Rashba angles (and weak spin polarization).

Anticipating our spin-precession picture discussed below reveals a fundamental difference between the USDE and the conventional SDE, in agreement with our numerical results. While the conventional SDE results from a superposition of multiple channels' individual CPRs—all with slightly different φ_0 -shifts—that finally distorts the total CPR such that the critical currents have different amplitudes (and while the critical currents of the individual channels are polarity-independent) [28, 29], the *USDE occurs already in single-channel junctions* as the direction-dependent transmission probabilities directly produce nonreciprocal CPRs in all channels with finite transverse momentum $|\mathbf{k}_\parallel| \neq 0$ (at $|\mathbf{k}_\parallel| = 0$, SOC vanishes). This explains the larger efficiency of the USDE and the observation that its amplitudes are not directly connected to those of the φ_0 -shifts—contrary to the conventional SDE in which sudden jumps of φ_0 close to current-reversing $0-\pi$ -like transitions are tightly bound to peaks in the SDE efficiency [28, 29]. For better illustration, the color map in Fig. 2(b) shows the channel-resolved (as a function of k_x ; $k_y = 0$ for simplicity) CPRs $I(\varphi; k_x, k_y = 0)$. Inspecting the color scale, we conclude that I_c^+ and $|I_c^-|$ are indeed (slightly) different in magnitude for individual k_x -channels (i.e., the colors indicating maximal-amplitude currents are slightly asymmetric around $\varphi = 0$ at $|k_x| \neq 0$), while the φ_0 -shifts are mostly caused by channels with large $|k_x|$.

To quantify the USDE, we extract the amplitudes of the polarity-dependent critical currents I_c^+ and $|I_c^-|$ from the CPRs to compute the critical-current difference $\Delta I_c = I_c^+ - |I_c^-|$ as the figure of merit of the SDE. The critical currents are presented as functions of the out-of-plane magnetization angle Θ and for the same Rashba angle as before ($\theta_R = 0.2\pi$) in Fig. 3(a). Note that $I_c^+[m_z \propto -\sin(\Theta)] = |I_c^-|(-m_z)$, as anticipated from our spin-precession picture below, holds. While I_c^+ monotonically increases with a more dominant out-of-plane $+\hat{z}$ -magnetization ($0 < \Theta \leq 0.5\pi$)—which could be a possible

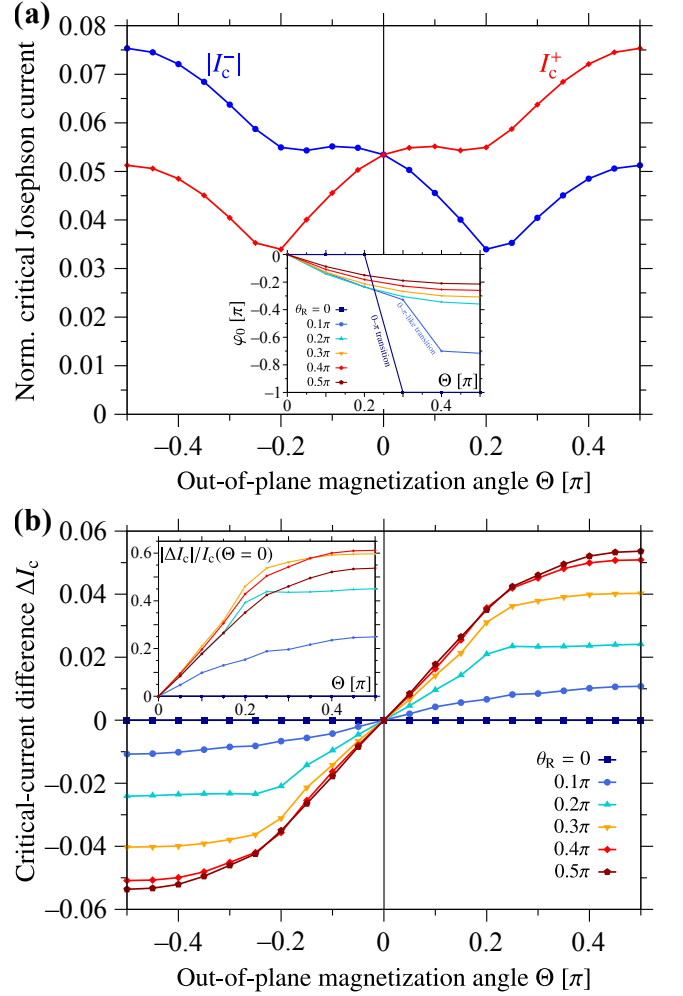


FIG. 3. (a) Polarity-dependent critical-current amplitudes I_c^+ and $|I_c^-|$ as functions of the out-of-plane magnetization angle Θ and for the same parameters as in Fig. 2(a). Inset: φ_0 -shifts vs. Θ for indicated Rashba angles θ_R . (b) Dependence of the critical-current difference $\Delta I_c = I_c^+ - |I_c^-|$ on Θ for indicated θ_R ; currents are normalized as in Fig. 2. Inset: $|\Delta I_c|$ normalized to the polarity-independent critical current $I_c(\Theta = 0)$ for in-plane magnetization.

signature of a more sizable triplet supercurrent induced by RR SOC— $|I_c^-|$ initially decreases, drops into a dip at $\Theta \approx 0.2\pi$, and finally increases as well. The reason for the nonmonotonic $|I_c^-|$ -dependence is the remainder of a current-reversing $0-\pi$ -(like) transition, which is most pronounced if both interfaces induce CR SOC ($\theta_R = 0$)—see also Refs. [11] and [99]—and gets strongly suppressed by a RR component ($\theta_R > 0$) that favors a $0 < |\varphi_0| < \pi$ -shift instead of a constant π -shift; the $0-\pi$ -(like) transitions emerge as sudden φ_0 -jumps [102] in the dark- and light-blue curves ($\theta_R = 0$ and $\theta_R = 0.1\pi$) in the inset of Fig. 3(a).

The corresponding dependence of the critical-current difference ΔI_c on Θ is illustrated in Fig. 3(b) tuning the spin-orbit field at the $z = d$ interface from purely CR ($\theta_R = 0$) to purely RR ($\theta_R = 0.5\pi$) SOC. While ΔI_c scales (nearly perfectly) sinusoidally with Θ at large Rashba angles ($\theta_R = 0.4\pi$ and

$\theta_R = 0.5\pi$), the aforementioned $0-\pi$ -(like) transitions may still cause reminiscent deviations at smaller θ_R -values—such as a suddenly emerging steeper increase at small $|\Theta|$ followed by a saturation into the maximal $|\Delta I_c|$ already at rather small $|\Theta|$, as seen, e.g., in the turquoise curve for $\theta_R = 0.2\pi$. In agreement with our spin-precession picture below, the maximal $|\Delta I_c|$ is reached at fully perpendicular magnetization. Moreover, $|\Delta I_c|$ increases monotonically with the Rashba angle θ_R (the maximal $|\Delta I_c|$ at $\Theta = 0.5\pi$ increases nearly sinusoidally with θ_R [99]), indicating that a maximal asymmetry between the SOC—fully CR at one and fully RR at the other interface—is most effective to maximize the USDE. The sign reversal of ΔI_c when reversing the out-of-plane magnetization direction reflects again that $I_c^+[m_z \propto -\sin(\Theta)] = |I_c^-(-m_z)|$. As pointed out above, and contrary to the conventional SDE, a large USDE does not necessarily coincide with sizable $|\varphi_0|$ -CPR shifts. Figure 3 confirms this expectation, as the absolute SDE measure $|\Delta I_c|$ increases with increasing θ_R , while the corresponding $|\varphi_0|$ simultaneously even decrease as a result of the initially present and with increasing θ_R quickly suppressed $0-\pi$ -(like) transitions [103].

To give a relative estimate of the USDE, we normalize $|\Delta I_c|$ to the polarity-independent critical current for in-plane (along $+\hat{y}$) magnetization $I_c(\Theta = 0)$ in Fig. 3(b) [104]. Note that $I_c(\Theta = 0)$ itself depends on θ_R , which explains the nonmonotonic dependence of $|\Delta I_c|/I_c(\Theta = 0)$ on θ_R . Relative SDE efficiencies beyond 20 % at small RR SOC and reaching maxima of about 60 % are sizable, also compared to the conventional SDE [28, 29].

Physical picture. To trace the origin of the USDE in our Josephson junction, we analyze the transmission probabilities of spin-polarized electrons through the junction in the presence of CR SOC at both interfaces (the SOC field at the $z = d$ interface is aligned oppositely owing to hybridization), as illustrated in Figs. 4(a) and 4(b). For simplicity, we focus on a single transverse channel with transverse momentum $\mathbf{k}_\parallel = [-k_F, 0]^T$ ($k_F = \sqrt{2m\mu}/\hbar$ is the Fermi wave vector), noting that a similar mechanism applies to all other channels.

The SOC at the $z = 0$ interface polarizes the spins of the electrons traveling along the $+\hat{z}$ -direction in-plane along $\pm\hat{y}$. When entering the F, the spins are exposed to the magnetization $m_z > 0$ pointing along the $+\hat{z}$ -(out-of-plane)-direction—i.e., parallel to the electrons' propagation direction—and precess counter-clockwise in the plane parallel to the interfaces. Arriving at the $z = d$ interface, the precessing spins will effectively see the second spin-orbit field rotated by the angle ϕ_{CR} determining their transmission probability $\mathcal{T}_{CR}^{\rightarrow} \propto \cos^2(\phi_{CR}/2)$. As this mechanism applies analogously to spin-up and spin-down electrons, the Cooper-pair-forming electrons sequentially tunnel from the left ($z < 0$) into the right ($z > d$) superconductor in the same way, and the corresponding critical Josephson current $|I_c^-(m_z > 0)|$ must also scale with $\mathcal{T}_{CR}^{\rightarrow}$. For electrons propagating along the opposite $-\hat{z}$ -direction, the tunneling picture is similar (i.e., $\mathcal{T}_{CR}^{\leftarrow} = \mathcal{T}_{CR}^{\rightarrow}$), suggesting that the positive critical Josephson current is equal in magnitude, $I_c^+(m_z > 0) = |I_c^-(m_z > 0)|$. If the magnetization direction is reversed ($m_z < 0$), the spins precess also oppositely (clockwise) but still enclose the same angle ϕ_{CR}

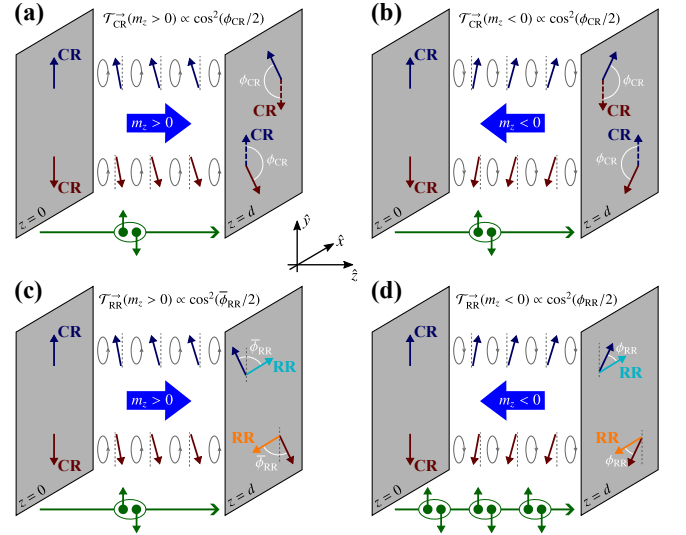


FIG. 4. (a) Spin-resolved electron tunneling (incident from the left) through the F link (center) with magnetization $m_z > 0$. The spins polarized along $\pm\hat{y}$ by CR SOC at the $z = 0$ interface precess counter-clockwise in-plane inside the F such that the CR field at the $z = d$ interface looks effectively rotated by the angle ϕ_{CR} ; the transmission probability is $\mathcal{T}_{CR}^{\rightarrow}(m_z > 0)$. (b) Reversing the magnetization ($m_z < 0$) results in the same tunneling picture with transmission probability $\mathcal{T}_{CR}^{\rightarrow}(m_z < 0) = \mathcal{T}_{CR}^{\rightarrow}(m_z > 0)$. The critical-current amplitudes—illustrated by dark-green Cooper pairs—are equal for magnetizations parallel and antiparallel to the current. (c) and (d) In the presence of RR SOC at the $z = d$ interface, the spin-precession angles $\bar{\phi}_{RR} \neq \phi_{RR}$, as well as the transmission probabilities $\mathcal{T}_{RR}^{\rightarrow}(m_z > 0) \neq \mathcal{T}_{RR}^{\rightarrow}(m_z < 0)$, depend on the relative orientation between magnetization and current. The critical currents become polarity-dependent (Josephson USDE).

w.r.t. the spin-orbit field at the $z = d$ interface. The critical-current amplitudes are consequently not affected and we expect $I_c^+(m_z < 0) = |I_c^-(m_z < 0)| = I_c^+(m_z > 0) = |I_c^-(m_z > 0)|$, meaning that the SDE is absent if both spin-orbit fields are conventional; the same arguments rule out the SDE in the case that both spin-orbit fields are radial.

Let us now consider *crossed* Rashba fields. Suppose that the chiral spin texture is exhibited by the $z = d$ interface. As shown in Figs. 4(c) and 4(d), and in strong contrast to the previous case, the angles between the precessing spins and the radial spin-orbit are different for magnetizations along $+\hat{z}$ (angle $\bar{\phi}_{RR}$) and $-\hat{z}$ (angle $\phi_{RR} \neq \bar{\phi}_{RR}$). Therefore, the transmission probabilities $\mathcal{T}_{RR}^{\rightarrow}(m_z > 0) \propto \cos^2(\bar{\phi}_{RR}/2) \neq \mathcal{T}_{RR}^{\rightarrow}(m_z < 0) \propto \cos^2(\phi_{RR}/2)$ along the $+\hat{z}$ -direction, and the resulting critical currents $|I_c^-(m_z > 0)| \neq |I_c^-(m_z < 0)|$, depend on the magnetization direction (parallel or antiparallel) w.r.t. the current. Moreover, $\bar{\phi}_{RR}$ and ϕ_{RR} are interchanged when the electrons travel instead from right to left (along the $-\hat{z}$ -direction), entailing $I_c^+(m_z > 0) = |I_c^-(m_z < 0)| \neq I_c^+(m_z < 0) = |I_c^-(m_z > 0)|$, i.e., transport becomes nonreciprocal and the Josephson USDE occurs. The microscopic origin of the USDE—spin-precession-induced polarity- and field-orientation-dependent transmission probabilities—is thus well distinct from the in-plane Cooper-pair momentum being responsible for the con-

ventional SDE.

Alternative realization in lateral junctions. As an alternative platform for the USDE, we consider a *two-dimensional lateral* S/F/S Josephson junction with in-plane CR in one and RR SOC in the other S, and the F magnetization aligned perpendicular to the plane. The electron Hamiltonian of this junction in tight-binding formulation—entering the BdG Hamiltonian in Eq. (1)—is given by $\hat{\mathcal{H}}_e = \hat{\mathcal{H}}_{S,L} + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{S,R}$, with

$$\begin{aligned} \hat{\mathcal{H}}_{S,L} = & -t \sum_{\langle i,j \rangle, \sigma} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} - \mu \sum_{j,\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} - \sum_j \left[\Delta_0 \hat{c}_{j,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger + \text{h.c.} \right] \\ & + i\alpha \sum_{\mu'=x,y} \sum_{\langle i,j \rangle_{\mu'}} \sum_{\alpha',\beta'} (\vec{n}_{\mu'}^{\text{CR}} \cdot \vec{\sigma})_{\alpha'\beta'} \hat{c}_{i,\alpha'}^\dagger \hat{c}_{j,\beta'}, \quad (2) \\ \hat{\mathcal{H}}_{S,R} = & -t \sum_{\langle i,j \rangle, \sigma} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} - \mu \sum_{j,\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} - \sum_j \left[\Delta_0 e^{i\varphi} \hat{c}_{j,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger + \text{h.c.} \right] \\ & + i\alpha \cos(\theta_R) \sum_{\mu'=x,y} \sum_{\langle i,j \rangle_{\mu'}} \sum_{\alpha',\beta'} (\vec{n}_{\mu'}^{\text{CR}} \cdot \vec{\sigma})_{\alpha'\beta'} \hat{c}_{i,\alpha'}^\dagger \hat{c}_{j,\beta'} \\ & + i\alpha \sin(\theta_R) \sum_{\mu'=x,y} \sum_{\langle i,j \rangle_{\mu'}} \sum_{\alpha',\beta'} (\vec{n}_{\mu'}^{\text{RR}} \cdot \vec{\sigma})_{\alpha'\beta'} \hat{c}_{i,\alpha'}^\dagger \hat{c}_{j,\beta'}, \quad (3) \end{aligned}$$

and

$$\hat{\mathcal{H}}_F = -t \sum_{\langle i,j \rangle, \sigma} \hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} - \mu \sum_{j,\sigma} \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} - m_z \sum_{j,\sigma} \sigma \hat{c}_{j,\sigma}^\dagger \hat{c}_{j,\sigma} \quad (4)$$

representing the single-electron Hamiltonians in the left S ($x < 0$), right S ($x > N_F$), and F ($0 < x < N_F$), respectively; $\hat{c}_{i,\sigma}^{(\dagger)}$ annihilates (creates) a spin- σ electron [$\sigma = (-)1$ for a spin (down) up] at lattice site i and $\langle i, j \rangle$ indicates nearest-neighbor hoppings. The hopping amplitude, chemical potential, Rashba strength (angle), *s*-wave superconducting pairing potential, and the amplitude of the exchange field along the $\hat{z} \parallel [001]$ -direction are denoted by t , μ , α (θ_R), Δ_0 , and m_z , whereas $\vec{n}_x^{\text{CR}} = (0, 1, 0)$, $\vec{n}_y^{\text{CR}} = (-1, 0, 0)$ and $\vec{n}_x^{\text{RR}} = (-1, 0, 0)$, $\vec{n}_y^{\text{RR}} = (0, -1, 0)$ are the Rashba vectors for CR and RR SOC, accordingly. After numerically implementing this tight-binding BdG description in the KWANT Python package [105], we apply a generalized Green's function technique [106, 107] to obtain the Josephson CPRs; see the SM [99] for more details. All calculations are performed for a junction with semi-infinite S leads, $N_F = 4$ lattice sites in the F link, and $N_y = 100$ sites along the transverse $\hat{y}$ -direction; the other system parameters are $\mu = -t$, $\alpha = 0.4t$, $\Delta_0 = 0.1t$, $k_B T = 0.01t$, and $t = \hbar^2/(2ma^2) = 238$ meV (or equivalently lattice spacing $a = 0.4$ nm if m is the free-electron mass).

The calculated critical-current difference ΔI_c for a lateral junction hosting *crossed* Rashba fields ($\theta_R = 0.25\pi$), shown in Fig. 5, points to qualitatively similar physics as discussed in the vertical case with the USDE emerging at finite out-of-plane exchange fields $m_z > 0$ [99]. The maximal USDE (maximal $|\Delta I_c|$) occurs when approaching the half-metallic limit ($|m_z| \rightarrow \mu$)—contrary to vertical S/F/S Josephson junctions in which ΔI_c oscillates as a function of the (effective) length $k_F d$ and spin polarization P of the F link as discussed in the SM [99]—and the pronounced sign reversal of ΔI_c indicates a current-reversing $0-\pi$ -like transition [28]. The inset of Fig. 5 illustrates the computed spin-resolved Fermi surfaces inside the F link, which provide no signatures of finite-momentum Cooper pairing (i.e., no relative shifts or displace-

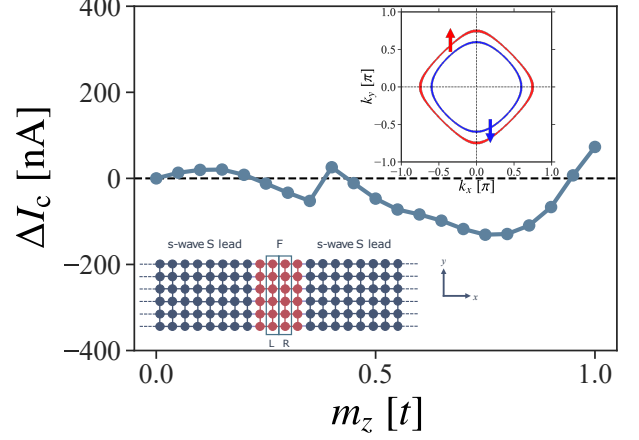


FIG. 5. Dependence of the critical-current difference $\Delta I_c = I_c^+ - |I_c^-|$ of the lateral S/F/S junction on the out-of-plane magnetization strength m_z (perpendicular to the system plane) at Rashba angle $\theta_R = 0.25\pi$; other parameters are given in the text. The insets show a schematical illustration of the lateral junction in the tight-binding picture (bottom) and the computed momentum-space spin-up ($\uparrow$; red) and spin-down ($\downarrow$; blue) Fermi surfaces in the F link (top).

ments of the Fermi surfaces) and therefore emphasize again that the physical mechanism of the USDE is well distinct from the conventional (finite-momentum-pairing) SDE [30, 56–60].

Conclusions. In summary, we predicted the Josephson USDE that emerges—contrary to the yet-studied conventional SDE—from the interplay of CR and RR SOC with the out-of-plane magnetization in vertical (3D) and lateral (2D) S/F/S junctions. In-plane precessions of the current-carrying electrons' spins in the F link trigger different transmission probabilities for left- and right-propagating electrons, as well as for magnetizations parallel and antiparallel to the current, manifesting in nonreciprocal transport and polarity-dependent critical currents. After performing numerical model calculations and analyzing various system parameters, we elaborated more on the qualitative spin-precession picture and deduced its most relevant ramifications on the USDE. We numerically quantified the USDE together with the intrinsic φ_0 -CPR shifts, unraveling that the amplitudes of both do not necessarily coincide; while the conventional SDE requires the superposition of many transverse channels with distinct φ_0 to obtain a sizable SDE, the spin-precession mechanism produces the USDE already for a single channel independent of its $|\varphi_0|$. The efficiency of the USDE is experimentally widely tunable through knobs like the magnetization orientation, Rashba angle, and the thickness or spin polarization of the F [99].

ACKNOWLEDGMENTS

A.C. and J.F. gratefully acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-IDs 454646522; 314695032. O.K. acknowledges support from GP-Spin at Tohoku University and

- [1] I. Žutić, J. Fabian, and S. Das Sarma, Spintronics: Fundamentals and applications, *Rev. Mod. Phys.* **76**, 323 (2004).
- [2] J. Fabian, A. Matos-Abiague, C. Ertler, P. Stano, and I. Žutić, Semiconductor Spintronics, *Acta Phys. Slovaca* **57**, 565 (2007).
- [3] J. Moser, A. Matos-Abiague, D. Schuh, W. Wegscheider, J. Fabian, and D. Weiss, Tunneling Anisotropic Magnetoresistance and Spin-Orbit Coupling in Fe/GaAs/Au Tunnel Junctions, *Phys. Rev. Lett.* **99**, 056601 (2007).
- [4] A. Matos-Abiague and J. Fabian, Anisotropic tunneling magnetoresistance and tunneling anisotropic magnetoresistance: Spin-orbit coupling in magnetic tunnel junctions, *Phys. Rev. B* **79**, 155303 (2009).
- [5] A. Matos-Abiague and J. Fabian, Tunneling Anomalous and Spin Hall Effects, *Phys. Rev. Lett.* **115**, 056602 (2015).
- [6] V. V. Rylkov, S. N. Nikolaev, K. Y. Chernoglazov, V. A. Demin, A. V. Sitnikov, M. Y. Presnyakov, A. L. Vasiliev, N. S. Perov, A. S. Vedenev, Y. E. Kalinin, V. V. Tugushev, and A. B. Granovsky, Tunneling anomalous Hall effect in nanogranular CoFe-B-Al-O films near the metal-insulator transition, *Phys. Rev. B* **95**, 144202 (2017).
- [7] T. Gao and K. Ando, *Chapter 1 – Spin-orbit torques*, edited by E. Brück, Handbook of Magnetic Materials, Vol. 29 (Elsevier, 2020) pp. 1–55.
- [8] M. Eschrig, Spin-polarized supercurrents for spintronics, *Phys. Today* **64**(1), 43 (2011).
- [9] J. Linder and J. W. A. Robinson, Strong odd-frequency correlations in fully gapped Zeeman-split superconductors, *Sci. Rep.* **5**, 15483 (2015).
- [10] P. Högl, A. Matos-Abiague, I. Žutić, and J. Fabian, Magnetoanisotropic Andreev Reflection in Ferromagnet-Superconductor Junctions, *Phys. Rev. Lett.* **115**, 116601 (2015); *Phys. Rev. Lett.* **115**, 159902(E) (2015).
- [11] A. Costa, P. Högl, and J. Fabian, Magnetoanisotropic Josephson effect due to interfacial spin-orbit fields in superconductor/ferromagnet/superconductor junctions, *Phys. Rev. B* **95**, 024514 (2017).
- [12] A. Costa, A. Matos-Abiague, and J. Fabian, Skew Andreev reflection in ferromagnet/superconductor junctions, *Phys. Rev. B* **100**, 060507(R) (2019).
- [13] I. Martínez, P. Högl, C. González-Ruano, J. P. Cascales, C. Tiu-san, Y. Lu, M. Hehn, A. Matos-Abiague, J. Fabian, I. Žutić, and F. G. Aliev, Interfacial Spin-Orbit Coupling: A Platform for Superconducting Spintronics, *Phys. Rev. Appl.* **13**, 014030 (2020).
- [14] T. Vezin, C. Shen, J. E. Han, and I. Žutić, Enhanced spin-triplet pairing in magnetic junctions with *s*-wave superconductors, *Phys. Rev. B* **101**, 014515 (2020).
- [15] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Long-Range Proximity Effects in Superconductor-Ferromagnet Structures, *Phys. Rev. Lett.* **86**, 4096 (2001).
- [16] A. F. Volkov, F. S. Bergeret, and K. B. Efetov, Odd Triplet Superconductivity in Superconductor-Ferromagnet Multilayered Structures, *Phys. Rev. Lett.* **90**, 117006 (2003).
- [17] R. S. Keizer, S. T. B. Goennenwein, T. M. Klapwijk, G. Miao, G. Xiao, and A. Gupta, A spin triplet supercurrent through the half-metallic ferromagnet CrO₂, *Nature (London)* **439**, 825 (2006).
- [18] K. Halterman, P. H. Barsic, and O. T. Valls, Odd Triplet Pairing in Clean Superconductor/Ferromagnet Heterostructures, *Phys. Rev. Lett.* **99**, 127002 (2007).
- [19] M. Eschrig and T. Löfwander, Triplet supercurrents in clean and disordered half-metallic ferromagnets, *Nat. Phys.* **4**, 138 (2008).
- [20] K. Sun and N. Shah, General framework for transport in spin-orbit-coupled superconducting heterostructures: Nonuniform spin-orbit coupling and spin-orbit-active interfaces, *Phys. Rev. B* **91**, 144508 (2015).
- [21] A. Costa and J. Fabian, Superconducting triplet pairings and anisotropic magnetoresistance effects in ferromagnet/superconductor/ferromagnet double-barrier junctions, *Phys. Rev. B* **104**, 174504 (2021).
- [22] C. Baumgartner, L. Fuchs, A. Costa, S. Reinhardt, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, P. E. Faria Junior, D. Kochan, J. Fabian, N. Paradiso, and C. Strunk, Supercurrent rectification and magnetochiral effects in symmetric Josephson junctions, *Nat. Nanotechnol.* **17**, 39 (2022).
- [23] C. Baumgartner, L. Fuchs, A. Costa, J. Picó-Cortés, S. Reinhardt, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, P. E. Faria Junior, D. Kochan, J. Fabian, N. Paradiso, and C. Strunk, Effect of Rashba and Dresselhaus spin-orbit coupling on supercurrent rectification and magnetochiral anisotropy of ballistic Josephson junctions, *J. Phys. Condens. Matter* **34**, 154005 (2022).
- [24] K.-R. Jeon, J.-K. Kim, J. Yoon, J.-C. Jeon, H. Han, A. Cottet, T. Kontos, and S. S. P. Parkin, Zero-field polarity-reversible Josephson supercurrent diodes enabled by a proximity-magnetized Pt barrier, *Nat. Mater.* **21**, 1008 (2022).
- [25] B. Pal, A. Chakraborty, P. K. Sivakumar, M. Davydova, A. K. Gopi, A. K. Pandeya, J. A. Krieger, Y. Zhang, M. Date, S. Ju, N. Yuan, N. B. M. Schröter, L. Fu, and S. S. P. Parkin, Josephson diode effect from Cooper pair momentum in a topological semimetal, *Nat. Phys.* **18**, 1228 (2022).
- [26] B. Turini, S. Salimian, M. Carrega, A. Iorio, E. Strambini, F. Giazotto, V. Zannier, L. Sorba, and S. Heun, Josephson Diode Effect in High-Mobility InSb Nanoflags, *Nano Lett.* **22**, 8502 (2022).
- [27] B. Zhang, Z. Li, V. Aguilar, P. Zhang, M. Pendharkar, C. Dempsey, J. S. Lee, S. D. Harrington, S. Tan, J. S. Meyer, M. Houzet, C. J. Palmstrom, and S. M. Frolov, Evidence of ϕ_0 -Josephson junction from skewed diffraction patterns in Sn-InSb nanowires, *SciPost Phys.* **18**, 013 (2025).
- [28] A. Costa, C. Baumgartner, S. Reinhardt, J. Berger, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, J. Fabian, D. Kochan, N. Paradiso, and C. Strunk, Sign reversal of the Josephson inductance magnetochiral anisotropy and $0-\pi$ -like transitions in supercurrent diodes, *Nat. Nanotechnol.* **18**, 1266 (2023).
- [29] A. Costa, J. Fabian, and D. Kochan, Microscopic study of the Josephson supercurrent diode effect in Josephson junctions based on two-dimensional electron gas, *Phys. Rev. B* **108**, 054522 (2023).
- [30] A. Banerjee, M. Geier, M. A. Rahman, C. Thomas, T. Wang, M. J. Manfra, K. Flensberg, and C. M. Marcus, Phase Asymmetry of Andreev Spectra from Cooper-Pair Momentum, *Phys. Rev. Lett.* **131**, 196301 (2023).
- [31] D. Kochan, A. Costa, I. Žhumagulov, and I. Žutić, Phenomeno-

- logical Theory of the Supercurrent Diode Effect: The Lifshitz Invariant, arXiv:2303.11975 (2023).
- [32] S. Banerjee and M. S. Scheurer, Altermagnetic superconducting diode effect, *Phys. Rev. B* **110**, 024503 (2024).
- [33] T. Kokkeler, I. Tokatly, and F. S. Bergeret, Nonreciprocal superconducting transport and the spin Hall effect in gyrotropic structures, *SciPost Phys.* **16**, 055 (2024).
- [34] S. Reinhardt, T. Ascherl, A. Costa, J. Berger, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, J. Fabian, D. Kochan, C. Strunk, and N. Paradiso, Link between supercurrent diode and anomalous Josephson effect revealed by gate-controlled interferometry, *Nat. Commun.* **15**, 4413 (2024).
- [35] B. Scharf, D. Kochan, and A. Matos-Abiague, Superconducting diode effect in quantum spin Hall insulator based Josephson junctions, *Phys. Rev. B* **110**, 134511 (2024).
- [36] W.-H. Kang, M. Barth, A. Costa, A. Garcia-Ruiz, A. Mreńca-Kolasińska, M.-H. Liu, and D. Kochan, Magnetotransport and Spin-Relaxation Signatures of the Radial Rashba and Dresselhaus Spin-Orbit Coupling in Proximitized Graphene, *Phys. Rev. Lett.* **133**, 216201 (2024).
- [37] F. Ando, Y. Miyasaka, T. Li, J. Ishizuka, T. Arakawa, Y. Shiota, T. Moriyama, Y. Yanase, and T. Ono, Observation of superconducting diode effect, *Nature (London)* **584**, 373 (2020).
- [38] J. Díez-Mérida, A. Díez-Carlón, S. Y. Yang, Y. M. Xie, X. J. Gao, J. Senior, K. Watanabe, T. Taniguchi, X. Lu, A. P. Higginbotham, K. T. Law, and D. K. Efetov, Symmetry-broken Josephson junctions and superconducting diodes in magic-angle twisted bilayer graphene, *Nat. Commun.* **14**, 2396 (2023).
- [39] J.-X. Lin, P. Siriviboon, H. D. Scammell, S. Liu, D. Rhodes, K. Watanabe, T. Taniguchi, J. Hone, M. S. Scheurer, and J. I. A. Li, Zero-field superconducting diode effect in small-twist-angle trilayer graphene, *Nature Physics* **18**, 1221 (2022).
- [40] H. D. Scammell, J. I. A. Li, and M. S. Scheurer, Theory of zero-field superconducting diode effect in twisted trilayer graphene, *2D Materials* **9**, 025027 (2022).
- [41] H. Wu, Y. Wang, Y. Xu, P. K. Sivakumar, C. Pasco, U. Filippozzi, S. S. P. Parkin, Y.-J. Zeng, T. McQueen, and M. N. Ali, The field-free Josephson diode in a van der Waals heterostructure, *Nature (London)* **604**, 653 (2022).
- [42] L. Bauriedl, C. Bäuml, L. Fuchs, C. Baumgartner, N. Paulik, J. M. Bauer, K.-Q. Lin, J. M. Lupton, T. Taniguchi, K. Watanabe, C. Strunk, and N. Paradiso, Supercurrent diode effect and magnetochiral anisotropy in few-layer NbSe₂, *Nat. Commun.* **13**, 4266 (2022).
- [43] J. Yun, S. Son, J. Shin, G. Park, K. Zhang, Y. J. Shin, J.-G. Park, and D. Kim, Magnetic proximity-induced superconducting diode effect and infinite magnetoresistance in a van der Waals heterostructure, *Phys. Rev. Res.* **5**, L022064 (2023).
- [44] J.-K. Kim, K.-R. Jeon, P. K. Sivakumar, J. Jeon, C. Koerner, G. Woltersdorf, and S. S. P. Parkin, Intrinsic supercurrent non-reciprocity coupled to the crystal structure of a van der Waals Josephson barrier, *Nat. Commun.* **15**, 1120 (2024).
- [45] B. Lu, S. Ikegaya, P. Burset, Y. Tanaka, and N. Nagaosa, Tunable Josephson Diode Effect on the Surface of Topological Insulators, *Phys. Rev. Lett.* **131**, 096001 (2023).
- [46] P.-H. Fu, Y. Xu, S. A. Yang, C. H. Lee, Y. S. Ang, and J.-F. Liu, Field-effect Josephson diode via asymmetric spin-momentum locking states, *Phys. Rev. Appl.* **21**, 054057 (2024).
- [47] S. Fracassi, S. Traverso, N. Traverso Ziani, M. Carrega, S. Heun, and M. Sasseti, Anomalous supercurrent and diode effect in locally perturbed topological Josephson junctions, *Appl. Phys. Lett.* **124**, 242601 (2024).
- [48] O. Can, T. Tummuru, R. P. Day, I. Elfimov, A. Damascelli, and M. Franz, High-temperature topological superconductivity in twisted double-layer copper oxides, *Nat. Phys.* **17**, 519 (2021).
- [49] S. Y. F. Zhao, X. Cui, P. A. Volkov, H. Yoo, S. Lee, J. A. Gardener, A. J. Akey, R. Engelke, Y. Ronen, R. Zhong, G. Gu, S. Plugge, T. Tummuru, M. Kim, M. Franz, J. H. Pixley, N. Poccia, and P. Kim, Time-reversal symmetry breaking superconductivity between twisted cuprate superconductors, *Science* **382**, 1422 (2023).
- [50] S. Ghosh, V. Patil, A. Basu, Kuldeep, A. Dutta, D. A. Jangade, R. Kulkarni, A. Thamizhavel, J. F. Steiner, F. von Oppen, and M. M. Deshmukh, High-temperature Josephson diode, *Nat. Mater.* **23**, 612 (2024).
- [51] P. A. Volkov, Étienne Lantagne-Hurtubise, T. Tummuru, S. Plugge, J. H. Pixley, and M. Franz, Josephson diode effects in twisted nodal superconductors, *Phys. Rev. B* **109**, 094518 (2024).
- [52] Y. Yerin, S.-L. Drechsler, A. A. Varlamov, M. Cuoco, and F. Giazotto, Supercurrent rectification with time-reversal symmetry broken multiband superconductors, *Phys. Rev. B* **110**, 054501 (2024).
- [53] Y. A. Bychkov and E. I. Rashba, Oscillatory effects and the magnetic susceptibility of carriers in inversion layers, *J. Phys. C* **17**, 6039 (1984).
- [54] Y. A. Bychkov and E. I. Rashba, Properties of a 2D electron gas with lifted spectral degeneracy, *Pis'ma Zh. Eksp. Teor. Fiz.* **39**, 66 (1984); *JETP Lett.* **39**, 78 (1984).
- [55] G. Dresselhaus, Spin-Orbit Coupling Effects in Zinc Blende Structures, *Phys. Rev.* **100**, 580 (1955).
- [56] A. Daido, Y. Ikeda, and Y. Yanase, Intrinsic superconducting diode effect, *Phys. Rev. Lett.* **128**, 037001 (2022).
- [57] N. F. Q. Yuan and L. Fu, Supercurrent diode effect and finite-momentum superconductors, *Proceedings of the National Academy of Sciences* **119**, e2119548119 (2022).
- [58] J. J. He, Y. Tanaka, and N. Nagaosa, A phenomenological theory of superconductor diodes, *New Journal of Physics* **24**, 053014 (2022).
- [59] S. Ilić and F. S. Bergeret, Theory of the Supercurrent Diode Effect in Rashba Superconductors with Arbitrary Disorder, *Phys. Rev. Lett.* **128**, 177001 (2022).
- [60] M. Davydova, S. Prembabu, and L. Fu, Universal Josephson diode effect, *Science Advances* **8**, eabo0309 (2022).
- [61] A. F. Andreev, Electron Spectrum of the Intermediate State of Superconductors, *Zh. Eksp. Teor. Fiz.* **49**, 655 (1966); *J. Exp. Theor. Phys.* **22**, 455 (1966).
- [62] E. V. Bezuglyi, A. S. Rozhavsky, I. D. Vagner, and P. Wyder, Combined effect of Zeeman splitting and spin-orbit interaction on the Josephson current in a superconductor–two-dimensional electron gas–superconductor structure, *Phys. Rev. B* **66**, 052508 (2002).
- [63] I. V. Krive, L. Y. Gorelik, R. I. Shekhter, and M. Jonson, Chiral symmetry breaking and the Josephson current in a ballistic superconductor–quantum wire–superconductor junction, *Low Temp. Phys.* **30**, 398 (2004).
- [64] A. Buzdin, Direct Coupling Between Magnetism and Superconducting Current in the Josephson φ_0 Junction, *Phys. Rev. Lett.* **101**, 107005 (2008).
- [65] A. A. Reynoso, G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Anomalous Josephson Current in Junctions with Spin Polarizing Quantum Point Contacts, *Phys. Rev. Lett.* **101**, 107001 (2008).
- [66] R. Grein, M. Eschrig, G. Metalidis, and G. Schön, Spin-Dependent Cooper Pair Phase and Pure Spin Supercurrents in Strongly Polarized Ferromagnets, *Phys. Rev. Lett.* **102**, 227005 (2009).
- [67] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Anoma-

- lous Josephson Current through a Spin-Orbit Coupled Quantum Dot, *Phys. Rev. Lett.* **103**, 147004 (2009).
- [68] J.-F. Liu and K. S. Chan, Relation between symmetry breaking and the anomalous Josephson effect, *Phys. Rev. B* **82**, 125305 (2010).
- [69] J.-F. Liu and K. S. Chan, Anomalous Josephson current through a ferromagnetic trilayer junction, *Phys. Rev. B* **82**, 184533 (2010).
- [70] J.-F. Liu, K. S. Chan, and J. Wang, Anomalous Josephson current through a ferromagnet–semiconductor hybrid structure, *Journal of the Physical Society of Japan* **80**, 124708 (2011).
- [71] A. A. Reynoso, G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Spin-orbit-induced chirality of Andreev states in Josephson junctions, *Phys. Rev. B* **86**, 214519 (2012).
- [72] T. Yokoyama, M. Eto, and Y. V. Nazarov, Josephson Current through Semiconductor Nanowire with Spin–Orbit Interaction in Magnetic Field, *Journal of the Physical Society of Japan* **82**, 054703 (2013).
- [73] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, Anomalous Josephson current, incipient time-reversal symmetry breaking, and Majorana bound states in interacting multilevel dots, *Phys. Rev. B* **88**, 144515 (2013).
- [74] K. Shen, G. Vignale, and R. Raimondi, Microscopic Theory of the Inverse Edelstein Effect, *Phys. Rev. Lett.* **112**, 096601 (2014).
- [75] T. Yokoyama, M. Eto, and Y. V. Nazarov, Anomalous Josephson effect induced by spin-orbit interaction and Zeeman effect in semiconductor nanowires, *Phys. Rev. B* **89**, 195407 (2014).
- [76] F. Konschelle, I. V. Tokatly, and F. S. Bergeret, Theory of the spin-galvanic effect and the anomalous phase shift φ_0 in superconductors and Josephson junctions with intrinsic spin-orbit coupling, *Phys. Rev. B* **92**, 125443 (2015).
- [77] D. B. Szombati, S. Nadj-Perge, D. Car, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Josephson φ_0 -junction in nanowire quantum dots, *Nature Physics* **12**, 568 (2016).
- [78] A. Assouline, C. Feuillet-Palma, N. Bergeal, T. Zhang, A. Motaghizadeh, A. Zimmers, E. Lhuillier, M. Eddrie, P. Atkinson, M. Aprili, and H. Aubin, Spin-Orbit induced phase-shift in Bi₂Se₃ Josephson junctions, *Nature Communications* **10**, 126 (2019).
- [79] W. Mayer, M. C. Dartailh, J. Yuan, K. S. Wickramasinghe, E. Rossi, and J. Shabani, Gate controlled anomalous phase shift in Al/InAs Josephson junctions, *Nature Communications* **11**, 212 (2020).
- [80] E. Strambini, A. Iorio, O. Durante, R. Citro, C. Sanz-Fernández, C. Guarcello, I. V. Tokatly, A. Braggio, M. Rocci, N. Ligato, V. Zannier, L. Sorba, F. S. Bergeret, and F. Giazotto, A Josephson phase battery, *Nature Nanotechnology* **15**, 656 (2020).
- [81] N. Lotfizadeh, W. F. Schiela, B. Pekerten, P. Yu, B. H. Elfeky, W. M. Strickland, A. Matos-Abiague, and J. Shabani, Superconducting diode effect sign change in epitaxial Al-InAs Josephson junctions, *Commun. Phys.* **7**, 120 (2024).
- [82] S. Patil, G. Tang, and W. Belzig, The Andreev-Ising-Josephson Diode, *arXiv:2411.04061* (2024).
- [83] Q. Cheng and Q.-F. Sun, Josephson diode based on conventional superconductors and a chiral quantum dot, *Phys. Rev. B* **107**, 184511 (2023).
- [84] D. Debnath and P. Dutta, Gate-tunable Josephson diode effect in Rashba spin-orbit coupled quantum dot junctions, *Phys. Rev. B* **109**, 174511 (2024).
- [85] I. V. Rozhansky and N. S. Averkiev, Manifestation of spin-orbit interaction in tunneling between two-dimensional electron layers, *Phys. Rev. B* **77**, 115309 (2008).
- [86] I. V. Rozhansky, N. S. Averkiev, and E. Lähderanta, Resonant tunneling between two-dimensional layers accounting for spin-orbit interaction, *Phys. Rev. B* **93**, 195405 (2016).
- [87] B. Bradlyn, J. Cano, Z. Wang, M. G. Vergniory, C. Felser, R. J. Cava, and B. A. Bernevig, Beyond Dirac and Weyl fermions: Unconventional quasiparticles in conventional crystals, *Science* **353**, aaf5037 (2016).
- [88] N. B. M. Schröter, D. Pei, M. G. Vergniory, Y. Sun, K. Manna, F. de Juan, J. A. Krieger, V. Süss, M. Schmidt, P. Dudin, B. Bradlyn, T. K. Kim, T. Schmitt, C. Cacho, C. Felser, V. N. Strocov, and Y. Chen, Chiral topological semimetal with multifold band crossings and long Fermi arcs, *Nat. Phys.* **15**, 759 (2019).
- [89] M. Sakano, M. Hirayama, T. Takahashi, S. Akebi, M. Nakayama, K. Kuroda, K. Taguchi, T. Yoshikawa, K. Miyamoto, T. Okuda, K. Ono, H. Kumigashira, T. Ideue, Y. Iwasa, N. Mitsuishi, K. Ishizaka, S. Shin, T. Miyake, S. Murakami, T. Sasagawa, and T. Kondo, Radial Spin Texture in Elemental Tellurium with Chiral Crystal Structure, *Phys. Rev. Lett.* **124**, 136404 (2020).
- [90] C. Mera Acosta, L. Yuan, G. M. Dalpian, and A. Zunger, Different shapes of spin textures as a journey through the Brillouin zone, *Phys. Rev. B* **104**, 104408 (2021).
- [91] F. Calavalle, M. Suárez-Rodríguez, B. Martín-García, A. Johansson, D. C. Vaz, H. Yang, I. V. Maznichenko, S. Osttanin, A. Mateo-Alonso, A. Chuvilin, I. Mertig, M. Gobbi, F. Casanova, and L. E. Hueso, Gate-tuneable and chirality-dependent charge-to-spin conversion in tellurium nanowires, *Nat. Mater.* **21**, 526 (2022).
- [92] D. Gosálbez-Martínez, A. Crepaldi, and O. V. Yazyev, Diversity of radial spin textures in chiral materials, *Phys. Rev. B* **108**, L201114 (2023).
- [93] J. A. Krieger, S. Stolz, I. Robredo, K. Manna, E. C. McFarlane, M. Date, B. Pal, J. Yang, E. B. Guedes, J. H. Dil, C. M. Polley, M. Leandersson, C. Shekhar, H. Borrmann, Q. Yang, M. Lin, V. N. Strocov, M. Caputo, M. D. Watson, T. K. Kim, C. Cacho, F. Mazzola, J. Fujii, I. Vobornik, S. S. P. Parkin, B. Bradlyn, C. Felser, M. G. Vergniory, and N. B. M. Schröter, Weyl spin-momentum locking in a chiral topological semimetal, *Nat. Commun.* **15**, 3720 (2024).
- [94] T. Frank, P. E. F. Junior, K. Zollner, and J. Fabian, Emergence of radial Rashba spin-orbit fields in twisted van der Waals heterostructures, *Phys. Rev. B* **109**, L241403 (2024).
- [95] G. Menichetti, L. Cavicchi, L. Lucchesi, F. Taddei, G. Iannaccone, P. Jarillo-Herrero, C. Felser, F. H. L. Koppens, and M. Polini, Giant chirality-induced spin polarization in twisted transition metal dichalcogenides, *arXiv:2312.09169* (2023).
- [96] H. Iturriaga, L. M. Martinez, T. T. Mai, A. J. Biacchi, M. Augustin, A. R. Hight Walker, M. F. Sanad, S. T. Sreenivasan, Y. Liu, E. J. G. Santos, C. Petrovic, and S. R. Singamaneni, Magnetic properties of intercalated quasi-2D Fe₃-xGeTe₂ van der Waals magnet, *npj 2D Mater. Appl.* **7**, 56 (2023).
- [97] P. G. De Gennes, *Superconductivity of Metals and Alloys* (Addison Wesley, Redwood City, 1989).
- [98] A. Furusaki and M. Tsukada, De Josephson effect and Andreev reflection, *Solid State Commun.* **78**, 299 (1991).
- [99] See the Supplemental Material—including Refs. [11, 30, 56–60, 97, 98, 105–107]—at [link] for more details.
- [100] C. W. J. Beenakker, Random-matrix theory of quantum transport, *Rev. Mod. Phys.* **69**, 731 (1997).
- [101] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion, *Phys. Rev. B* **25**, 4515 (1982).

- [102] The φ_0 -shifts are defined as the superconducting phase difference at which the Josephson energy $E_J = \int_0^\varphi d\phi I(\phi)$ is minimized.
- [103] As another cross-check of our simulations, we reversed the orientation of the RR field (considering, e.g., $\theta_R = -0.5\pi$ instead of $\theta_R = 0.5\pi$) and obtained a sign change of ΔI_c similarly to reversing the out-of-plane magnetization m_z . This observation agrees well with our spin-precession picture, in which flipping the RR-SOC orientation interchanges the precession angles $\bar{\phi}_{RR}$ and ϕ_{RR} , and acts therefore analogously to reversing m_z .
- [104] Note that we need to define a different relative measure for the USDE efficiency than for the conventional SDE in lateral Josephson junctions [28], in which the effect is induced by an applied in-plane magnetic field and $|\Delta I_c|$ is typically compared to the critical current after turning off the field. Regarding the USDE studied here, the exchange splitting in the F is always present in the experiment and the polarity-independent critical currents at spin polarization $P = 0$ cannot be measured in the same sample. We therefore normalize $|\Delta I_c|$ to the polarity-independent critical current $I_c(\Theta = 0)$, which is experimentally accessible when rotating the magnetization into the plane.
- [105] C. W. Groth, M. Wimmer, A. R. Akhmerov, and X. Waintal, Kwant: a software package for quantum transport, *New J. Phys.* **16**, 063065 (2014).
- [106] A. Furusaki, DC Josephson effect in dirty SNS junctions: Numerical study, *Physica B: Condensed Matter* **203**, 214 (1994).
- [107] K. Zuo, V. Mourik, D. B. Szombati, B. Nijholt, D. J. van Woerkerkom, A. Geresdi, J. Chen, V. P. Ostroukh, A. R. Akhmerov, S. R. Plissard, D. Car, E. P. A. M. Bakkers, D. I. Pikulin, L. P. Kouwenhoven, and S. M. Frolov, Supercurrent Interference in Few-Mode Nanowire Josephson Junctions, *Phys. Rev. Lett.* **119**, 187704 (2017).