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Non-perturbative decay of bipartite discrete time crystals

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We study prethermal time-crystalline order in periodically driven quantum Ising models on disorder-free decorated lattices. Using a tensor network ansatz for the state which reflects the geometry of a unit cell of the lattice, we show through finite entanglement scaling that the system has an exponentially long-lived subharmonic response in the thermodynamic limit, which decays non-perturbatively in deviations from a perfect periodic drive. The resulting prethermal discrete time crystal is not only stable to imperfections in the transverse field, but also exhibits a bipartite rigidity to generic perturbations in the longitudinal field. We call this state a bipartite discrete time crystal and reveal a rich prethermal phase diagram, including multiple regions of bipartite time-crystalline order, uniform time-crystalline order and thermalization, with boundaries depending delicately on the topology of the decorated lattice. Our results thus uncover a variety of time crystals which may be realized on current digital quantum processors and analog quantum simulators.

Introduction. — The stable crystalline structure of solid materials is a striking example of spontaneous symmetry breaking, as the ground state violates spatial translational invariance in the formation of a periodic lattice. The temporal counterpart of this phenomenon, a state which evolves periodically in time, constitutes a breaking of time translational invariance which has been coined a *time crystal* [1, 2]. While the spontaneous breaking of a continuous time symmetry by an isolated quantum system in equilibrium was proven to be impossible [3–5], out-of-equilibrium states which break a discrete time translational invariance can be realized in Floquet systems, where an external drive modulates the governing Hamiltonian over a period τ , $\mathcal{H}(t) = \mathcal{H}(t + \tau)$ [6–8]. In this case, the discrete time symmetry imposed by the periodic drive is broken when an observable O does not stabilize at stroboscopic times $t = n\tau$, but instead exhibits long-lived periodicity at a lower frequency than that of the Hamiltonian,

$$\langle O(t) \rangle = \langle O(t + \tilde{n}\tau) \rangle, \quad (\tilde{n} > 1). \quad (1)$$

When stable to generic local perturbations, such symmetry-broken states are called *discrete time crystals* (DTC) [9–11] due to the spontaneous emergence of a subharmonic order parameter $\langle O \rangle$. The primary obstacle for constructing long-lived DTCs is the absorption of energy injected by the Floquet drive, which causes the system to relax to an infinite temperature equilibrium state. With the exception of nonergodic dynamics in quantum-scarred eigenstates [12–15] and systems exhibiting many-body localization due to disorder [16–23], linear external fields [24–27] or gauge symmetries [28–30], discrete time crystals in non-integrable many-body quantum systems are a transient phenomenon with a finite *prethermal* [31–35] lifetime.

Prethermal time-crystalline order was notably shown to have a lifetime scaling exponentially with the driving frequency [36–38], and has since been studied extensively in a wide variety of systems [39–52]. Among the mechanisms proposed to extend DTCs beyond the limiting case $\tau \rightarrow 0$ [11, 53] is the presence of a confining potential, which, in analogy to the confinement of quarks in QCD [54, 55], causes magnetic domain walls to form bound states as a result of the energy cost associated with the growth of the domain [56–58]. The consequently inhibited spread of correlations was found to delay the onset of thermalization [59–61], and shown to allow for long-lived time-crystalline order in the case of confinement by an external field in a one-dimensional Ising chain [62]. Since symmetry-broken states of ferromagnetic models in higher dimensions generically exhibit confinement due to the growth of a domain wall with the size of the domain itself [61, 63], disorder-free DTCs have been predicted to occur in the two-dimensional square Ising model [45], and were recently observed on a digital quantum processor with a decorated hexagonal geometry [49]. In the latter system, both Floquet dynamics and non-thermal dynamics caused by confinement of excitations have been accurately reproduced through tensor network state (TNS) simulations [61, 64].

We build upon these successes to study the emergence, stability and lifetime of time-crystalline order in the Floquet Ising model on generic decorated lattice structures. By employing a tensor network reflecting the unit cell of an infinite realization of the lattice, optimized with belief propagation, we directly access the thermodynamic limit. Supported by state vector simulations on finite graphs, this will lead us to introduce *bipartite* discrete time crystals and construct a prethermal phase diagram hosting several DTC regimes determined by the lattice topology.

Model and approach. — We consider the Ising model in the presence of both transverse and longitudinal

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magnetic fields, described by the Hamiltonian

$$\mathcal{H} = h_x \sum_j X_j + h_z \sum_j Z_j - J \sum_{\langle jk \rangle} Z_j Z_k, \quad (2)$$

where X and Z are the spin-1/2 Pauli operators. Both the interaction strength J and the fields h_x and h_z are taken to be uniform, while the summations j and $\langle jk \rangle$ run over all sites and nearest-neighbor links of the underlying decorated lattice. We define a decorated lattice as one in which each of the edges of a lattice with vertices of uniform coordination number z is decorated with an additional vertex, with the example of the decorated hexagonal lattice shown in Fig. 1(a). A distinguishing feature of this geometry is its bipartite structure, comprising two distinct sublattices of vertices with degree 2 and $z > 2$. We will refer to these sublattices as the A - and B -lattices, respectively. A discrete time symmetry is introduced by the periodically driven dynamics illustrated in Fig. 1(b), in which unitary time evolution over the period τ of a single Floquet cycle is generated by successive application of the three contributions to the Hamiltonian for a time $\tau/3$. Setting $\hbar = 1$ and defining the rotation operator of an operator O as $R_O(\theta) = e^{-i\theta O/2}$, we can write this Floquet unitary as

$$U(\tau) = \left[\prod_j R_X^j(\theta_x) \right] \left[\prod_j R_Z^j(\theta_z) \right] \left[\prod_{\langle jk \rangle} R_{ZZ}^{jk}(\theta_J) \right], \quad (3)$$

where the rotation angles are related to the Hamiltonian and the Floquet period τ as $\theta_J = -2J\tau/3$, $\theta_x = 2h_x\tau/3$ and $\theta_z = 2h_z\tau/3$.

To compute the dynamics induced by $U(\tau)$ on a given decorated lattice in the thermodynamic limit, we use the BP-iTNS ansatz [61, 64, 65]. Specifically, we simulate the dynamics induced by the propagator in Eq. (3) on the $K_{2,z}$ complete bipartite graph [66]. This geometry corresponds to a single unit cell of the decorated lattice with periodic boundaries. We use a tensor network state (TNS) whose structure matches that of the $K_{2,z}$ graph. The application of two-site gates and the measurement of expectation values is done via belief propagation (BP), which works by identifying vectorized message tensors as a rank-1 approximation for the environments in the tensor network [64, 67–70]. These BP-iTNS results are exactly equivalent [61, 64, 65] to those obtained by simulating the dynamics of the full decorated lattice in the thermodynamic limit with a TNS matching the infinite lattice and optimized with BP. The error made by the BP approximation decreases exponentially with the size of the smallest loop in the system, making it especially suitable for the considered geometries due to the fact the decorations double the smallest loop size in the original lattice. Within the limitations set by the bond dimension of the iTNS and the loop corrections neglected by the BP approximation, the BP-iTNS ansatz allows us to access the dynamics $U(\tau)$ on any decorated lattice structure in the thermodynamic limit. We picture this ansatz for the

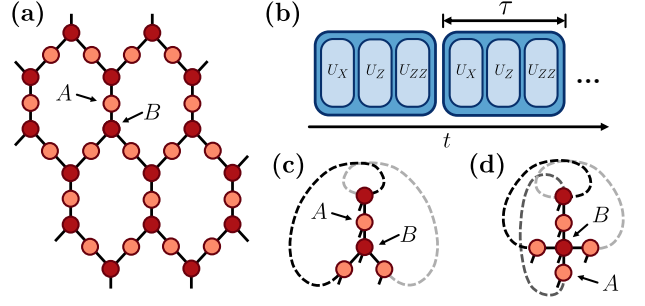


FIG. 1. **Geometry and dynamics.** (a) The two-dimensional decorated hexagonal lattice consists of two sublattices of sites with two (A , light) and three (B , dark) nearest neighbors. (b) Unitary sequence of two Floquet cycles. (c) The complete bipartite graph $K_{2,3}$ as the BP-iTNS unit cell of a decorated hexagonal lattice, with periodic boundaries (dashed lines). (d) The $K_{2,4}$ unit cell of a Lieb lattice.

decorated hexagonal ($z = 3$) and decorated square (Lieb) lattice ($z = 4$) in Fig. 1(c-d).

Time-crystalline order. — We consider the system initialized in the symmetry-broken product state $|\uparrow\rangle^{\otimes(z+2)}$ and subject to an effective confining potential through interactions, taken to be maximally entangling two-qubit rotations $\theta_J = -\pi/2$ at a vanishing longitudinal field $\theta_z = 0$. For a transverse field $\theta_x = \pi$, the dynamics consist of R_{ZZ} interactions alternating with a π -flip of all qubits around the X -axis, resulting in a perfect time-crystalline order of the magnetization with a subharmonic period 2τ ,

$$\langle \psi(n\tau) | Z | \psi(n\tau) \rangle = (-1)^n. \quad (4)$$

For any imperfect drive $\theta_x \neq \pi$, the system leaves the Z -polarized state and eventually thermalizes. However, as the stroboscopic dynamics in the limit $\theta_x \rightarrow \pi$ are governed to lowest order by a static Ising Hamiltonian, confinement dramatically slows down the absorption of energy and generation of entanglement [61], delaying thermalization during a prethermal time-crystalline stage.

Figure 2(a-c) shows the time evolution of the magnetization $\langle Z \rangle$ in the A -sublattice of a decorated hexagonal lattice subject to Floquet dynamics at three different transverse field drives $\theta_x/\pi = \{0.65, 0.75, 0.85\}$. For rotation angles deviating significantly from the perfect π -periodic drive as in Fig. 2(a), $\langle Z \rangle$ dephases rapidly as the system relaxes to an infinite temperature state. Conversely, Fig. 2(c) reveals a long-lived subharmonic response in the limit $\theta_x \rightarrow \pi$, signaling the emergence of a discrete time crystal. A crossover between these dynamical regimes of DTC order and thermalization was recently observed in error-mitigated quantum simulations on IBM's Heron architecture [49]. We develop a more exhaustive picture of this crossover directly in the thermodynamic limit by plotting in Fig. 2(d) the mean oscillation amplitude $\overline{Z}(\theta_x, \chi)$, defined as the average of $|\langle Z(n\tau) \rangle|$ during the first $t/\tau = 100$ cycles of the Floquet unitary (3). In agreement with the dynamics illus-

trated in Fig. 2(a-c), the amplitude reaches zero as θ_x approaches $\pi/2$, and converges to unity in the limit $\theta_x \rightarrow \pi$. In the latter regime, the insensitivity of the magnetization to the bond dimension χ of the employed BP-iTNS implies that entanglement growth is suppressed, consistent with the presence of an effective confining potential. For intermediary values of θ_x , a slow convergence with respect to χ indicates that the system becomes highly entangled as it crosses over into the thermalizing regime.

The scaling of the mean magnetization $\bar{Z}$ with the bond dimension allows us to predict the time-crystalline dynamics at an unbounded entanglement entropy $S_E \propto \ln(\chi)$ by extrapolating our results at finite bond dimension to the limit $\ln(\chi) \rightarrow \infty$. As illustrated in the inset of Fig. 2(d), we find that $\bar{Z}(\theta_x, \chi)$ for any θ_x scales with the bond dimension as

$$\bar{Z}(\theta_x, \chi) = \frac{\zeta(\theta_x)}{\ln(\chi)} + \bar{Z}_\infty(\theta_x). \quad (5)$$

On the condition that the slope $\zeta(\theta_x)$ of this entanglement scaling is finite, the quantity $\bar{Z}_\infty(\theta_x)$ corresponds to the prethermal magnetization in the limit $\chi \rightarrow \infty$, indicated by the full black lines in Fig. 2(a-d). The slope $\zeta(\theta_x)$ signifies the dependence of $\bar{Z}$ on the bond dimension, and hence serves as a measure of entanglement production during the dynamics. This quantity is plotted in Fig. 2(e) as a function of the inverse perturbation $\pi/|\theta_x - \pi|$ to a π -periodic drive, for $\bar{Z}$ evaluated over a varying number t/τ of Floquet cycles. From this relation, the growth of entanglement is found to reach a maximum at a drive field which converges at late times to $\theta_x^*/\pi \approx 0.73$, providing an estimate of the crossover between DTC and rapidly thermalizing dynamics in agreement with the range $\theta_x^*/\pi \sim 0.7 - 0.8$ inferred from quantum simulation [49]. Moreover, Fig. 2(e) reveals that the entanglement production in the DTC regime scales as

$$\zeta(\theta_x \rightarrow \pi) \sim \exp\{-\gamma(t)/|\theta_x - \pi|\}, \quad (6)$$

indicating that the decay of time-crystalline order is non-perturbative in the deviation $|\theta_x - \pi|$ from a perfect π -periodic drive. Finally, the time evolution of the exponent $\gamma(t)$ provides an estimate for the lifetime of the prethermal DTC. The drift of $\gamma(t)$ to zero in the limit $t \rightarrow \infty$, seen in Fig. 2(e), indicates that the production of entanglement entropy eventually results in thermalization for any imperfect drive $\theta_x < \pi$ at late times. Nevertheless, the inset shows that this decay of $\gamma(t)$ occurs at most logarithmically in time, suggesting a DTC lifetime which grows exponentially in $1/|\theta_x - \pi|$. More precisely, the relation $-\gamma(t) \propto \ln t$ implies that the entanglement production (6) at late times in the DTC regime is of the sublinear form $\zeta(\theta_x, t) \sim t^{1/|\theta_x - \pi|}$, in stark contrast to the linear (volume law) growth of entanglement entropy which characterizes rapid thermalization.

Bipartite rigidity. — A comparable decay of the order parameter and a crossover value $\theta_x^*/\pi \sim 0.6 - 0.8$ were previously inferred through matrix product state

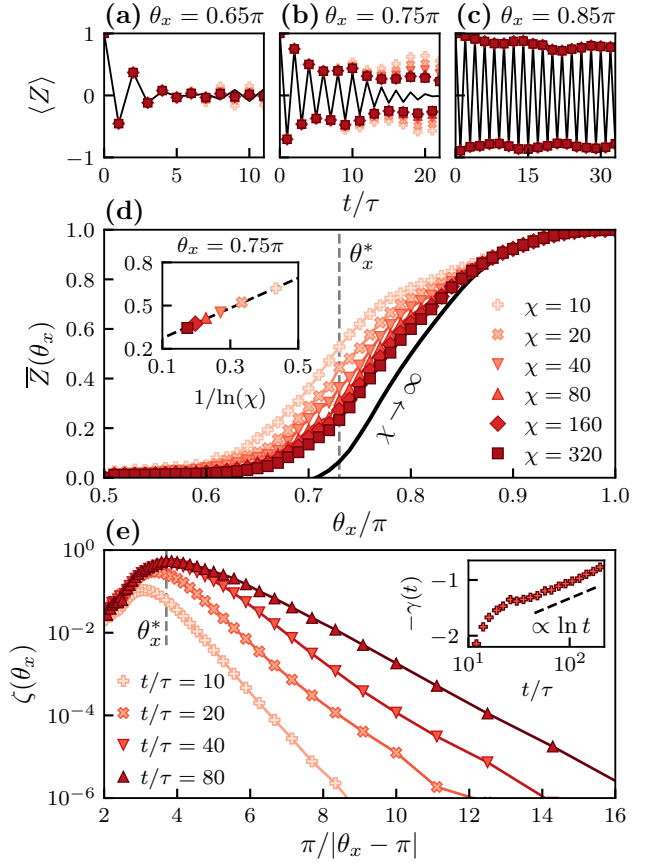


FIG. 2. **Time-crystalline order.** (a-c) Evolution of the magnetization on an infinite decorated hexagonal lattice for a periodic transverse field drive in the thermalizing (a), crossover (b) and DTC regime (c), evaluated at the bond dimensions shown in (d). (d) Mean magnetization in the thermodynamic limit as a function of θ_x , averaged over the first 100 cycles at different bond dimensions. The inset illustrates for $\theta_x = 0.75\pi$ the extrapolation $1/\ln(\chi) \rightarrow 0$, resulting in the full black line in panels (a-d). The crossover value θ_x^* inferred from a finite entanglement scaling is shown by the grey dashed line in (d). (e) Slope of the extrapolation (5) for different number of Floquet cycles, exhibiting exponential decay with $1/|\theta_x - \pi|$ in the DTC regime. The inset shows the evolution of the exponent defined in Eq. (6).

(MPS) simulations on finite square lattices [45], indicating that prethermal subharmonic oscillations are not specific to decorated lattice structures. However, true time-crystalline order requires, besides a robustness of the subharmonic response to an imperfect periodic drive, a self-stabilizing rigidity with respect to generic local perturbations [10]. We therefore extend the Floquet unitary (3) to include a finite longitudinal field $\theta_z > 0$. Where the interference of two driving fields will cause a generic system to exhibit a beating, a system in a stable time-crystalline phase retains its subharmonic response.

Figure 3(a) shows the response of $\langle Z \rangle$ on both sublattices of a decorated hexagonal lattice to a transverse drive of $\theta_x = 0.85\pi$ within the time-crystalline regime,

perturbed by a longitudinal field rotation $\theta_z = 0.4\pi$. This configuration was reported in Ref. [49] to cause a beating in the global magnetization, signalling the emergence of a discrete time quasi-crystal (DTQC) [71–73]. However, we find that the different connectivity of qubits in either sublattice results in qualitatively different dynamics. Whereas all qubits in the *A*-lattice exhibit a beating frequency manifesting as an envelope to the subharmonic response, a similar breaking of DTC order into quasi-crystalline dynamics is absent in the *B*-lattice, which is insensitive to longitudinal perturbations. This bipartite rigidity persists for all values of $\theta_z \in [0, \pi]$, as demonstrated in Figs. 3(b-d) through the power spectrum of the magnetization dynamics,

$$S_\omega = \frac{\tau}{t} \sum_{n=1}^{t/\tau} e^{-in\omega\tau} \langle Z(n\tau) \rangle. \quad (7)$$

While the peak of the subharmonic response at $\omega\tau = \pi$ remains unperturbed by the longitudinal field in the *B*-sublattice, the instability in the *A*-lattice causes a splitting into two peaks separated by $\delta\omega \propto \theta_z$. Since both sublattices exhibit reversible oscillatory dynamics, the entanglement entropy remains low. As shown by the crosses in Fig. 3(a) and dashed lines in Fig. 3(b-d), this property allows us to observe all qualitative features even from the evolution of a product state ($\chi = 1$), suggesting that the bipartite response is not caused by intricate many-body dynamics, but is instead a necessary feature of the decorated lattice topology.

In the Supplemental Material, we crosscheck the validity of our BP-iTNS method with exact state vector simulations of complete regular graphs with decorated edges, demonstrate the absence of rigidity in the corresponding undecorated lattices, and show that bipartite rigidity is not found in decorated square lattices ($z = 4$). Being a feature unique to systems with a bipartite topology, we thus define a *bipartite DTC* as a nonequilibrium state of matter in which the inhibition of entanglement growth results in a robust subharmonic response in an extensive part of, but not all of, the system.

Prethermal phase diagram. — The decorated lattice topology not only causes a bipartite rigidity to perturbations, but determines the presence of time-crystalline order in the transverse field Ising model even at $\theta_z = 0$. To illustrate this, we extend the result of Fig. 2(d) to arbitrary interaction strengths θ_J and construct a prethermal phase diagram in the (θ_x, θ_J) parameter space. To distinguish the rigid subharmonic response characteristic of a DTC from both rapid thermalization and enveloped oscillations emblematic of unstable periodicity, we define the spectral power density $|S_\omega^{A/B}|^2$ at the subharmonic peak $\omega\tau = \pi$ as a prethermal DTC order parameter in either sublattice of the bipartite geometry. For the *A*- and *B*-lattices of a decorated hexagonal graph, this gives rise to the diagrams shown in Fig. 4(a-b).

In the absence of interactions ($\theta_J = 0$), a nonzero DTC order parameter $|S_{\pi/\tau}|^2$ appears in both lattices only

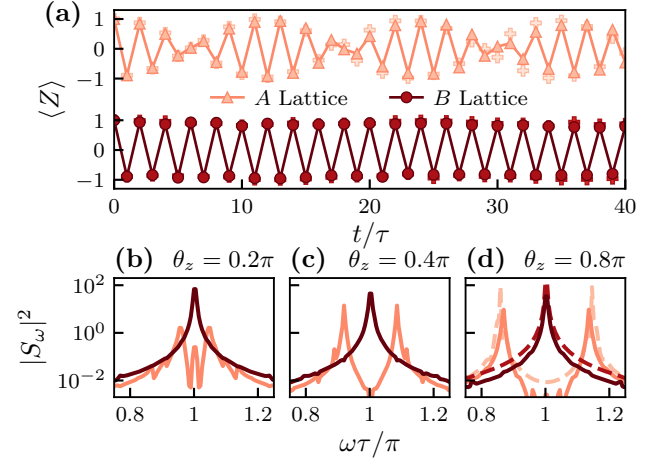


FIG. 3. **Bipartite rigidity.** (a) Time evolution of $\langle Z \rangle$ on the *A*- (top) and *B*-sublattice (bottom) of a decorated hexagonal lattice for $(\theta_J, \theta_x, \theta_z) = (-0.5\pi, 0.85\pi, 0.4\pi)$, calculated in the thermodynamic limit at $\chi = 160$. (b-d) Power spectra (7) of the magnetization in the first $t/\tau = 400$ cycles, for $\theta_x = 0.85\pi$ and $\theta_z/\pi = \{0.2, 0.4, 0.8\}$. The crosses in (a) and dashed lines in (d) indicate the corresponding result for $\chi = 1$.

for a perfect periodic drive $\theta_x = \pi$, as the system lacks any mechanism stabilizing the response to an imperfect drive. The onset of Ising interactions ($\theta_J < 0$) enables self-stabilization of the subharmonic response to increasingly perturbed π -flips, resulting in a finite range of θ_x for which the system exhibits stable DTC order (shaded in red). This region, which grows linearly with $|\theta_J|$ in both sublattices, signals the emergence of an interaction-stabilized DTC phase. This behavior is consistent with the presence of a confining potential that inhibits rapid thermalization induced by a weak transverse field [61]. Crucially, stable DTC order at weak interactions does not decay gradually as perturbations $|\theta_x - \pi|$ increase. Instead, it falls off abruptly in a sharp crossover between time-crystalline order and thermalization.

For the decorated hexagonal lattice in Fig. 4(a-b), the stabilizing effect of interactions stagnates around $\theta_J \approx -\pi/3$. Beyond this point, the range of perturbations $|\theta_x - \pi|$ to which the time crystal is robust shrinks until the eventual disappearance of DTC order in the node at $\theta_J = -2\pi/3$, followed by a revival at larger interaction strengths. This disappearance and recurrence of DTC order as a function of θ_J is a general feature of decorated lattice structures, as seen from the *B*-sublattice order parameter $|S_{\pi/\tau}^B|^2$ in the thermodynamic limit of a Lieb lattice ($z = 4$) and decorated 5-regular graph ($z = 5$) shown in Fig. 4(c-d). In particular, nodes of vanishing DTC order occur at values $\theta_J = -2n\pi/z$, with $n \in \mathbb{N}$ and z the connectivity of the *B*-lattice. This property is readily explained by the fact that the $R_{ZZ}^{jk}(\theta_J)$ interaction terms in Eq. (3) effectively rotate each of the involved qubits by an angle θ_J around the *Z*-axis. Hence, the total rotation of a z -connected qubit in the *B*-lattice

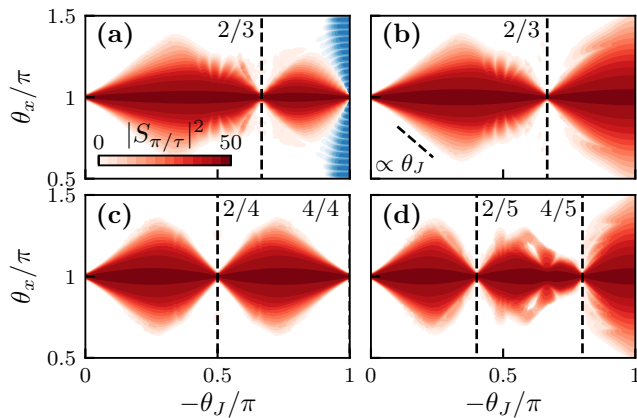


FIG. 4. **Prethermal phase diagrams.** (a-b) Spectral power density $|S_\omega|^2$ at the subharmonic peak $\omega\tau = \pi$ in the A - (a) and B -sublattice (b) of the Floquet Ising model ($\theta_z = 0$) in the decorated hexagonal lattice, evaluated over 50 Floquet cycles in the limit $\chi \rightarrow \infty$. Red regions indicate the presence of DTC order, while the blue region in (a) represents the spectral density of the largest non-subharmonic peak, indicating quasi-crystalline enveloped oscillations. (c-d) Subharmonic spectral power density $|S_{\pi/\tau}|^2$ in the B -sublattice of a Lieb lattice (c) and decorated 5-regular graph (d).

interacting with its z neighbors in a single Floquet cycle is $z\theta_J$. At $\theta_J = -2n\pi/z$ this reduces to $-2n\pi$, reintroducing the instability of a non-interacting model at $\theta_J = 0$. Following the same reasoning, qubits in the A -sublattice of any decorated lattice have two nearest neighbors and thus experience an effective rotation of $2\theta_J$. As seen in Fig. 4(a), this results in the disappearance of DTC order in the A -sublattice at interaction strengths $\theta_J = -n\pi$.

However, while nodes of the B -lattice occur also in the phase diagram of the A -lattice and thus destabilize DTC order in the entire system, the A -lattice node at $\theta_J = -\pi$ results in a regime of bipartite DTC order: qubits in the B -lattice [Fig. 4(b)] exhibit a robust subharmonic response in the limit $\theta_J \rightarrow -\pi$, whereas the A -lattice [Fig. 4(a)] acquires a quasi-crystalline beating similar to the dynamics in Fig. 3. This is shown by the blue-shaded amplitude of the largest non-subharmonic peak in the spectral density $|S_\omega^A|^2$ in Fig. 4(a). Finally, we note that regimes of bipartite DTC order are not observed in Lieb lattices with even coordination number $z = 4$. This follows from the counting argument above, according to which the B -lattice exhibits a DTC node at $\theta_J = -n\pi/2$, shown in Fig. 4(c). The coincidence of the DTC node $\theta_J = -\pi$ with that of the A -lattice prevents the emergence of a bipartite DTC phase witnessed in decorated hexagonal and 5-regular graphs.

Conclusion. — Through tensor network simulations in the thermodynamic limit, we have studied discrete time crystals in periodically driven Ising models on decorated lattices. We have established the existence of a

prethermal phase of time-crystalline order which is exponentially long-lived and decays in a non-perturbative fashion from small deviations from the ideal π -periodic drive. Additionally, we have found that the sensitivity to a longitudinal field, observed on a finite decorated hexagonal lattice in Ref. [49], manifests itself in only one of the two sublattices comprising a generic decorated graph structure. The latter observation implies the existence of a state of bipartite time-crystalline order, in which an extensive part of a bipartite system resisting thermalization displays subharmonic dynamics robust to generic local perturbations. Finally, we have constructed a prethermal DTC phase diagram for the specific cases of a decorated hexagonal lattice, Lieb lattice and decorated 5-regular graph, uncovering a decay and resurrection of both uniform and bipartite prethermal DTC order at interaction strengths determined by the topology of the graph.

The bipartite discrete time crystals we have introduced bear a resemblance to the notion of boundary time crystals (BTC) [74–76] and topological Floquet phases of edge states [50, 77, 78]. In contrast to these phenomena, bipartite DTCs as introduced in this work are distinguished by their extensive scaling with system size. As such, we anticipate experimental realizations of this phenomenon not only on current digital quantum processors [48, 77, 79–83], but also on analog quantum simulators with periodically driven Ising interactions on non-trivial topologies. Possible candidates for such studies include atomic systems with tunable lattice structures [84–86] and condensed matter platforms such as spin impurities in diamond [22, 73]. However, further theoretical research is required to elucidate the physical mechanisms underpinning bipartite time-crystallinity, as well as to investigate whether the phenomenology presented here can be extended to the breaking of a continuous time symmetry in driven-dissipative systems [87–90] with a bipartite geometry.

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