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# Spin current generation due to differential rotation

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We study nonequilibrium spin dynamics in differentially rotating systems, deriving an effective Hamiltonian for conduction electrons in the comoving frame. In contrast to conventional spin current generation mechanisms that require vorticity, our theory describes spins and spin currents arising from differentially rotating systems regardless of vorticity. We demonstrate the generation of spin currents in differentially rotating systems, such as liquid metals with Taylor-Couette flow. Our alternative mechanism will be important in the development of nanomechanical spin devices.

*Introduction.*— The physics of spin currents, initially introduced as the flow of Fermi particles [1], and the diffusion of spin magnetic moments [2, 3] and Ising spins [4], has long attracted the interest of researchers. Generating and controlling spin currents is a key challenge in spintronics [5], involving mechanisms like the spin Hall effect [6–10], spin pumping [11–15], spin Seebeck effect [16], spin accumulation at ferromagnet/nonmagnet interfaces [17, 18], and Edelstain effect [19–21]. Advances in nanofabrication now enable mechanical motions in materials for spin transport [22–38]. The gyromagnetic effect [39–54], involving angular momentum interconversion between mechanical rotation and spin, is crucial as it enables spin current generation without strong spin-orbit coupling [55–60].

The gyromagnetic effect, initially understood through the conservation of angular momentum [39–41, 61], is now recognized as originating from spin-rotation coupling, $H_{\rm sr} = -\boldsymbol{s}\cdot\boldsymbol{\Omega}$, where $\boldsymbol{s}$ is the spin and $\boldsymbol{\Omega}$ is the angular velocity of rigid rotation, derived from the Dirac equation [62]. This coupling is analogous to the Zeeman effect, with angular velocity acting as an effective magnetic field on the spin. Theoretical and experimental work shows that inhomogenieties of the angular velocity can generate a spin current via the Stern-Gerlach type effect [63–65], using vorticity gradients in liquid metal [55, 57–59] and surface acoustic waves [28, 56, 60, 66, 67]. In these studies, a spin-vorticity coupling, $H_{\rm sv} = -\boldsymbol{s}\cdot\boldsymbol{\omega}$, was explored by replacing constant angular velocity $\boldsymbol{\Omega}$ in $H_{\rm sr}$ with vorticity $\boldsymbol{\omega} = (1/2)\boldsymbol{\nabla}\times\boldsymbol{v}$. However, in addition to the spin-vorticity coupling, differential rotation $\boldsymbol{\Omega}(\boldsymbol{r}) = \boldsymbol{r}\times\boldsymbol{v}/r^2$ offers a new method for localizing the interaction. This overlooked coupling may enhance our understanding of spin transport driven by non-uniform rotation.

In this study, we investigate the non-equilibrium spin dynamics in differentially rotating systems within a microscopic theory. By mapping into a comoving frame, we construct an effective Hamiltonian for conduction electrons in these systems, demonstrating the emergence of effective gauge fields. Furthermore, we derive microscopic expressions for the spin density and spin current of conduction electrons driven by these emergent gauge fields. The mechanism of spin current generation proposed in the present paper is based on the fundamental principles of quantum mechanics, without interjecting phenomenological arguments.

Although our mechanism applies to general differentially rotating systems, we present specific examples of experiments that may verify our proposal. By applying our mechanism to a liquid metal and a non-magnetic metallic cantilever as examples of differentially rotating systems, we estimate the concrete amount of the spin current. In particular, we show that even in cases such as Taylor-Couette flow where the vorticity-gradient is zero, spin currents can be generated due to the differential rotation. Consequently, we uncover mechanisms of angular momentum transfer that have not been captured by traditional frameworks.

*Emergent Gauge Fields in Comoving Frame.*—We consider the free electron system subject to momentum scattering and spin-orbit scattering due to the impurities. In the inertial laboratory frame the Hamiltonian is given by

$$\hat{H}' = \int d^3x \hat{\psi}^\dagger(\boldsymbol{x})\Big\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 - \epsilon_F + V'_{\rm imp}(\boldsymbol{x},t) + \lambda_{\rm so}\boldsymbol{\sigma}\cdot[\boldsymbol{\nabla}V'_{\rm imp}(\boldsymbol{x},t)\times(-i\hbar\boldsymbol{\nabla})]\Big\}\hat{\psi}(\boldsymbol{x}), \quad (1)$$

where $\hat{\psi}(\boldsymbol{x})$ is the electron field operator, $\epsilon_F$ is the Fermi energy, $\boldsymbol{\sigma} = (\sigma^x,\sigma^y,\sigma^z)$ are the Pauli matrices, and $\lambda_{\rm so}$ is the strength of the spin-orbit interaction. The third term represents the impurity scattering and the fourth term represents the spin-orbit scattering. Here, $V'_{\rm imp}(\boldsymbol{x},t) = \sum_j u(\boldsymbol{x}-\boldsymbol{r}'_j(t))$ is the total impurity potential, where $u(\boldsymbol{x}-\boldsymbol{r}'_j(t))$ is a single impurity potential due to the $j$-th impurity located at the position $\boldsymbol{r}'_j(t)$. It is worth noting that the electrons are subject to the moving impurities because we suppose the total system

is differentially rotating. To characterize the differential rotation of the system, we introduce a rotation angle $\Phi(\boldsymbol{x},t)$ around the $z$-axis, which is chosen as a rotation axis. When we take a cylindrical coordinate system, the coordinate transformation from the laboratory frame $\boldsymbol{r}'=(r',\varphi',z')$ to the rotating frame $\boldsymbol{r}=(r,\varphi,z)$ can be written as $r=r'$, $z=z'$, and $\varphi=\varphi'-\Phi(\boldsymbol{r}',t)$. Note that $\Phi$ is independent of $\varphi$, i.e., $\partial_\varphi\Phi=0$ because of axisymmetry. Supposing $\Phi(\boldsymbol{x},t)=0$ at an initial time $t=0$, the position of the $j$-th impurity at $t$ is given by $\boldsymbol{r}'_j(t)=\mathcal{R}_z[\Phi(\boldsymbol{r}_j,t)]\boldsymbol{r}_j$, where $\mathcal{R}_z$ denotes rotation around the $z$-axis and $\boldsymbol{r}_j$ is the position at $t=0$.

Now, we define a generator of the differential rotation with angle $\Phi(\boldsymbol{x},t)$ as

$$\hat{Q}_\Phi(t)=\int d^3x\Phi(\boldsymbol{x},t)\hat{\psi}^\dagger(\boldsymbol{x})J^z\hat{\psi}(\boldsymbol{x}), \tag{2}$$

where $J^z$ is the total angular momentum operator acting on coordinates and spin space as $J^z=-i\hbar\partial_\varphi+\hbar\sigma^z/2$. Note that $J^z$ and $\Phi(\boldsymbol{x},t)$ are commutative. For an arbitrary state vector in the laboratory frame $|\Psi'(t)\rangle$, the state vector in the rotating frame is given by

$$|\Psi(t)\rangle=\exp\left[\frac{i}{\hbar}\hat{Q}_\Phi(t)\right]|\Psi'(t)\rangle. \tag{3}$$

The Schrödinger equation in the laboratory frame, $i\hbar\partial_t|\Psi'(t)\rangle=\hat{H}'|\Psi'(t)\rangle$, yields

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle&=(e^{i\hat{Q}_\Phi/\hbar}\hat{H}'e^{-i\hat{Q}_\Phi/\hbar}-\hat{Q}_{\partial_t\Phi})|\Psi(t)\rangle\\ &=\hat{H}_T|\Psi(t)\rangle, \end{aligned} \tag{4}$$

where $\hat{Q}_{\partial_t\Phi}=\int d^3x\partial_t\Phi(\boldsymbol{x},t)\hat{\psi}^\dagger(\boldsymbol{x})J^z\hat{\psi}(\boldsymbol{x})$ and $\hat{Q}_\Phi$ commute because of $\partial_\varphi\Phi=0$. The Hamiltonian $H_T$ which governs dynamics in the rotating frame. The density operator in the rotating frame, $\hat{\rho}(t)$, is given by $\hat{\rho}(t)=e^{i\hat{Q}_\Phi/\hbar}\hat{\rho}'(t)e^{-i\hat{Q}_\Phi/\hbar}$, where $\hat{\rho}'(t)$ is the density operator in the laboratory frame. The time evolution of $\hat{\rho}(t)$ is determined by $i\hbar\partial_t\hat{\rho}(t)=[\hat{H}_T,\hat{\rho}(t)]$. Assuming that the single impurity potential $u(\boldsymbol{x})$ is isotropic and its typical range $a$, such that $u(\boldsymbol{x})\simeq 0$ for $|\boldsymbol{x}|\gg a$, is much smaller than a typical scale of the gradient of the differential rotation, i.e., $a|\boldsymbol{\nabla}\Phi|\ll 1$, the Hamiltonian in the rotating frame can be rewritten as

$$\begin{aligned} \hat{H}_T=\int d^3x\hat{\psi}^\dagger(\boldsymbol{x})\Big\{&\frac{(-i\hbar\boldsymbol{\nabla}-\boldsymbol{A}_sJ^z)^2}{2m}-A_{s,0}J^z-\epsilon_F+V_{\rm imp}(\boldsymbol{x})\\ &+\lambda_{\rm so}\boldsymbol{\sigma}\cdot[\boldsymbol{\nabla}V_{\rm imp}(\boldsymbol{x})\times(-i\hbar\boldsymbol{\nabla}-\boldsymbol{A}_sJ^z)]\Big\}\hat{\psi}(\boldsymbol{x}), \end{aligned} \tag{5}$$

where the time and spatial derivatives of the rotation angle are denoted by

$$A_{s,\mu}(\boldsymbol{x},t)=\Big(\partial_t\Phi(\boldsymbol{x},t),\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)\Big)\quad(\mu=0,x,y,z). \tag{6}$$

We call $A_{s,\mu}(\boldsymbol{x},t)$ "emergent gauge field" in this paper. In the rotating frame, the effects of the differential rotation are represented by the emergent gauge fields, whereas the impurity potential given by $V_{\rm imp}(\boldsymbol{x})=\sum_j u(\boldsymbol{x}-\boldsymbol{r}_j)$ does not depend on time under the assumption $a|\boldsymbol{\nabla}\Phi|\ll 1$.

*Setup.*—We present the Fourier representation of the total Hamiltonian in the rotating frame to facilitate calculations: $\hat{H}_T=\hat{H}_0+\hat{H}_{\rm imp}+\hat{H}_{\rm so}+\hat{H}'(t)$, where $\hat{H}'(t)$ is the contribution of the emergent gauge field, and we treat it as a perturbation. The first term $\hat{H}_0=\sum_{\boldsymbol{k}}\epsilon_{\boldsymbol{k}}\hat{\psi}^\dagger_{\boldsymbol{k}}\hat{\psi}_{\boldsymbol{k}}$ represents the kinetic term, where $\epsilon_{\boldsymbol{k}}=\hbar^2k^2/2m-\epsilon_F$ is the kinetic energy, and $\hat{\psi}_{\boldsymbol{k}}$ is the Fourier component of the electron annihilation operator. The second and third terms describe the momentum scattering and the spin-orbit scattering due to the impurities, respectively. These are expressed as $\hat{H}_{\rm imp}=\sum_{\boldsymbol{k}\boldsymbol{k}'}V_{\boldsymbol{k}-\boldsymbol{k}'}\hat{\psi}^\dagger_{\boldsymbol{k}}\hat{\psi}_{\boldsymbol{k}'}$ and $\hat{H}_{\rm so}=i\hbar\lambda_{\rm so}\sum_{\boldsymbol{k}\boldsymbol{k}'}V_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{k}\times\boldsymbol{k}')\cdot\hat{\psi}^\dagger_{\boldsymbol{k}}\boldsymbol{\sigma}\hat{\psi}_{\boldsymbol{k}'}$, where $V_{\boldsymbol{k}}$ denotes the Fourier component of the impurity potential $V_{\rm imp}(\boldsymbol{x})$. We assume a short-range impurity potential, i.e., $u(\boldsymbol{x}-\boldsymbol{r}_j)=u_{\rm i}\delta(\boldsymbol{x}-\boldsymbol{r}_j)$, where $u_{\rm i}$ is the strength of the impurity potential defined by $u_{\rm i}=\int d^3xu(\boldsymbol{x})$ in general. While, the perturbed part, denoted by $\hat{H}'(t)=\hat{H}_s+\mathcal{O}(L^z)$ with $L^z=-i\hbar\partial_\varphi$ being the orbital angular momentum, represents the effect of the emergent gauge fields. The Hamiltonian $\hat{H}_s$ incorporates the electron spin, given by

$$\begin{aligned} \hat{H}_s=&-\frac{\hbar^2}{2m}\sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}}\hat{\psi}^\dagger_{\boldsymbol{k}_+}\left(\boldsymbol{k}\sigma^z\delta_{\boldsymbol{k}\boldsymbol{k}'}-\frac{1}{2}\boldsymbol{A}_{s,\boldsymbol{k}-\boldsymbol{k}'}\right)\hat{\psi}_{\boldsymbol{k}'_-}\cdot\boldsymbol{A}_s(\boldsymbol{q})\\ &-\frac{\hbar}{2}\hat{s}(\boldsymbol{q})A_{s,0}(\boldsymbol{q}), \end{aligned} \tag{7}$$

where $\boldsymbol{A}_{s,\boldsymbol{q}}$ is the Fourier component of the emergent gauge fields, and $\boldsymbol{k}_\pm=\boldsymbol{k}\pm\boldsymbol{q}/2$ are defined.

To define the spin-current operator, we consider the temporal modulation of the $z$-polarized spin density, $\partial_t\hat{s}(\boldsymbol{q})=-i\boldsymbol{q}\cdot\hat{\boldsymbol{j}}_s(\boldsymbol{q})+\hat{\mathcal{T}}_{\boldsymbol{q}}$, where $\hat{s}(\boldsymbol{q})=\sum_{\boldsymbol{k}}\hat{\psi}^\dagger_{\boldsymbol{k}_-}\sigma^z\hat{\psi}_{\boldsymbol{k}_+}$ is the spin-density operator, and $\hat{\mathcal{T}}_{\boldsymbol{q}}$ describes the spin torque due to the spin-orbit interaction of the impurities. The spin-current density operator polarized in the $z$-direction is defined by $\hat{\boldsymbol{j}}_s(\boldsymbol{q})=\sum_{\boldsymbol{k}\boldsymbol{k}'}\hat{\psi}^\dagger_{\boldsymbol{k}'_-}\boldsymbol{j}_{s,\boldsymbol{k}'\boldsymbol{k}}\hat{\psi}_{\boldsymbol{k}_+}$, where the matrix elements $\boldsymbol{j}_{s,\boldsymbol{k}'\boldsymbol{k}}$ is given by

$$\boldsymbol{j}_{s,\boldsymbol{k}'\boldsymbol{k}}=\delta_{\boldsymbol{k}'\boldsymbol{k}}\boldsymbol{v}_{\boldsymbol{k}}\sigma^z+\lambda_{\rm so}V_{\boldsymbol{k}'-\boldsymbol{k}}[\boldsymbol{e}_z\times(\boldsymbol{k}'-\boldsymbol{k})]-\frac{\hbar\boldsymbol{A}_{s,\boldsymbol{k}'-\boldsymbol{k}}}{2m}, \tag{8}$$

where $\boldsymbol{v}_{\boldsymbol{k}}=\hbar\boldsymbol{k}/m$ is the velocity and $\boldsymbol{e}_z$ is the unit vector in $z$-direction.

*Calculation of Spin Current.*—We now compute the spin current induced by the emergent gauge fields. The statistical averages of the spin density and spin current are given by

$$\langle\hat{j}_\mu(\boldsymbol{q},\omega)\rangle=\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi i}\sum_{\boldsymbol{k}\boldsymbol{k}'}{\rm Tr}\left[j_{s\mu,\boldsymbol{k}'\boldsymbol{k}}G^<_{\boldsymbol{k}_+,\boldsymbol{k}'_-}(\epsilon_+,\epsilon_-)\right], \tag{9}$$

where $\epsilon_\pm=\epsilon\pm\omega/2$, $j_{s0,\boldsymbol{k}'\boldsymbol{k}}=\sigma^z\delta_{\boldsymbol{k}'\boldsymbol{k}}$, and the trace is taken for the spin space. Here, the four-vector $\hat{j}_\mu=(\hat{s},\hat{\boldsymbol{j}}_s)$ represents the spin density and

spin current operators. The function $G^<_{\boldsymbol{k}_+,\boldsymbol{k}'_-}(\epsilon_+,\epsilon_-)$ is the lesser component of the nonequilibrium path-ordered Green function, defined by $G_{\boldsymbol{k},\boldsymbol{k}'}(t,t') = -i\langle T_K \hat{\psi}_{\boldsymbol{k}_+}(t)\hat{\psi}^\dagger_{\boldsymbol{k}'_-}(t')\rangle$, where $T_K$ is a path-ordering operator, $\hat{\psi}(t) = \hat{U}^\dagger(t)\hat{\psi}(t)\hat{U}(t)$ is the Heisenberg representation with $\hat{U}(t) = T\exp[-(i/\hbar)\int_{-\infty}^t \hat{H}_T(\tau)d\tau]$ and $T$ being time-ordering operator, and $\langle\cdots\rangle = \mathrm{tr}(\hat{\rho}\cdots)$ is the expectation value with the density operator $\hat{\rho}$.

Assuming that the characteristic energy scales of the momentum scattering and the spin-orbit scattering due to the impurities are much smaller than the Fermi energy, i.e., $n_{\rm i}u_{\rm i} \ll \epsilon_F$ and $\hbar^2\lambda_{\rm so}^2 k_F^4 \ll 1$, we treat them in the Born approximation. With the uniformly random distribution of impurities, we perform the average of their positions to obtain the retarded/advanced Green function: $g^{r/a}_{\boldsymbol{k}}(\epsilon) = 1/(\epsilon - \epsilon_{\boldsymbol{k}} \pm i\hbar\gamma)$, where $\hbar\gamma = \pi n_{\rm i}u_{\rm i}^2\nu_0(1 + 2\hbar^2\lambda_{\rm so}^2k_F^4/3)$ is the damping constant calculated with the density of state per spin at Fermi level $\nu_0 = mk_F/2\pi^2\hbar^2$. We assume that $\hbar\gamma \ll \epsilon_F$. This condition is well-satisfied when $u_{\rm i}\nu_0 \lesssim 1$.

The spin-current density in linear response to the emergent gauge fields is expressed as $\langle\hat{j}_\mu(\boldsymbol{q},\omega)\rangle = K_{\mu\nu}(\omega)A_{s,\nu}(\boldsymbol{q},\omega)$, where $K_{\mu\nu}(\omega)$ is the response function. It is presumed that the time and spatial variation of the differential rotation are much slower than the electron mean-free path $l = v_F\tau$ and momentum relaxation time $\tau = 1/2\gamma$, respectively, i.e., $l|\boldsymbol{\nabla}\Phi| \ll 1$ and $\tau|\partial_t\Phi| \ll 1$, where $v_F = \hbar k_F/m$ is the Fermi velocity and $k_F = \sqrt{2m\epsilon_F/\hbar^2}$ is the Fermi wavenumber. In terms of Fourier space, conditions $lq \ll 1$ and $\tau\omega \ll 1$ hold. By including the ladder vertex corrections due to the impurities, and using the relations $\hbar v_F q/2 \ll \hbar\gamma \ll \epsilon_F$ and $\hbar\omega/2 \ll \hbar\gamma \ll \epsilon_F$, the response function is calculated as

$$K_{\mu\nu}(\omega) = \delta_{\mu0}\delta_{\nu0}\frac{\hbar\sigma_0}{2e^2D} + i\omega\frac{\hbar}{4\pi}\sum_{\boldsymbol{k}} v_{\boldsymbol{k},\mu}\,\mathrm{Tr}\big[\sigma^z g^r_+\sigma^z(v_{\boldsymbol{k},\nu} + \Lambda^s_\nu)g^a_-\big], \quad (10)$$

where $\sigma_0 = n_e e^2\tau/m$ is the Drude conductivity with $n_e = 4\epsilon_F\nu_0/3$ being the number density of the electrons and $e(>0)$ being the elementary charge, and $D = v_F^2\tau/3$ is the diffusion constant. We set $v_{\boldsymbol{k},0} = 1$ and $v_{\boldsymbol{k},i} = \hbar k_i/m$. Here, $\Lambda^s_\nu$ describes the three-point vertex corrections, and $g^{r/a}_\pm = g^{r/a}_{\boldsymbol{k}_\pm}(\pm\omega/2)$ are specified. The first term of the response function represents the spin susceptibility for the rigid rotation [68], known as the Barnett effect.

Performing a straightforward calculation, we derive the rotation-induced spin density and spin current:

$$\langle\hat{s}(\boldsymbol{q},\omega)\rangle = -i\omega\frac{\hbar\sigma_0}{2e^2D}\frac{\tau_s^{-1}}{Dq^2 - i\omega + \tau_s^{-1}}\Phi(\boldsymbol{q},\omega), \quad (11)$$

$$\langle\hat{\boldsymbol{j}}_s(\boldsymbol{q},\omega)\rangle = i\omega\frac{\hbar\sigma_0}{2e^2}\frac{\tau_s^{-1}}{Dq^2 - i\omega + \tau_s^{-1}}i\boldsymbol{q}\Phi(\boldsymbol{q},\omega), \quad (12)$$

where $\tau_s = 9\tau/8\hbar^2\lambda_{\rm so}^2k_F^4$ is the spin-relaxation time. Combining these results, we obtain Fick's law, $\boldsymbol{j}_s = -D\boldsymbol{\nabla}s$. This implies that our spin current is a diffusive flow produced by the gradient of the spin density, in which the impurity scattering governs the diffusion.

Now, we focus on long-term dynamics such that time scales are longer than the period of the rotation, $\omega \lesssim \Omega$. In metals, the spin relaxation time and the spin diffusion length are much shorter than the typical scale of the period and that of the spatial variation for the differential rotation, respectively (i.e., $\Omega^{-1} \gg \tau_s$ and $|\boldsymbol{\nabla}\Omega/\Omega|^{-1} \gg l_s$, where $l_s = \sqrt{D\tau_s}$). Thus, the relations $Dq^2, \omega \ll \tau_s^{-1}$ are satisfied, and the rotation-induced spin current reduces to the following form in the real space:

$$\boldsymbol{j}_s(\boldsymbol{x},t) = -\frac{\hbar\sigma_0}{2e^2}\boldsymbol{\nabla}\partial_t\Phi(\boldsymbol{x},t). \quad (13)$$

In addition, the rotation-induced spin density reduces to

$$s(\boldsymbol{x},t) = \frac{\hbar\sigma_0}{2e^2D}\partial_t\Phi(\boldsymbol{x},t), \quad (14)$$

which is the Barnett effect generalized to differential rotations. The susceptibility given by $\hbar\sigma_0/2e^2D$ is identical to that of the Barnett effect for rigid rotations. These results suggest that the spin density and current are polarized along the rotation axis and the spin current is driven in the direction of the spatial gradient of the angular velocity. By contrast, if the spin relaxation is so slow that $\tau_s^{-1} \ll \omega \lesssim \Omega$, the spin density (11) as well as the spin current (12) vanish, which implies that the spin relaxation is necessary to generate the spin current and spin density. Despite this fact, the magnitude of the spin current (13) is independent of the spin relaxation time.

The absence of $\tau_s$ from the long-term dynamics of the spin density and the spin current is explained as follows. In the response function (10), the first term that originates from the spin-rotation coupling $A_{s,0}J^z$ in (5) is principal, while the other terms including the spin-orbit coupling are suppressed by $\tau_s\omega \ll 1$. This means that the spin density is determined only by the susceptibility of the Barnett effect and the angular velocity. The gradient of this spin density produces the spin current due to the diffusion caused by the impurity scattering, as shown. Thus, the spin density and current are independent of $\tau_s$.

The spin-orbit interaction contributes only to the transient process that is necessary to drive the system to the final steady state, but it does not contribute to long-term dynamics. Indeed, for $\omega \gtrsim \Omega$, (11) and (12) provide the following spin transport equation:

$$\frac{\partial s}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_s = -\frac{s}{\tau_s} + \frac{\hbar\sigma_0}{2e^2D\tau_s}\partial_t\Phi, \quad (15)$$

which describes the transient process with a time-scale $\omega \simeq \tau_s^{-1}$. We expect to obtain similar diffusive spin currents as long as there is an interaction producing a transient process satisfying $\Omega \ll \tau_s^{-1} \ll \tau^{-1}$, not necessarily that presented here.

*Taylor-Couette Flow.*—As an explicit example, let us consider a two-dimensional steady flow with concentric

circular streamlines. In this case, the flow velocity is parallel to the $\varphi$-direction, $\boldsymbol{v} = (0, v_\varphi, 0)$, satisfying the following Navier-Stokes equation: $\partial_r^2 v_\varphi + (\partial_r v_\varphi)/r - v_\varphi/r^2 = 0$. (The detailed derivation is given in Supplementary Material [69].) The general solution is $v_\varphi = c_1/r + c_2 r$ with integration constants $c_1$ and $c_2$ determined by boundary conditions. The first term represents irrotational flow, while the second term represents rigid-rotation flow. We consider the two infinitely long coaxial cylinders of radii $r_1$ and $r_2$ ($r_2 > r_1 > 0$), and the inner and outer cylinders are rotating at constant angular velocities $\Omega_1$ and $\Omega_2$, respectively. Under these boundary conditions, $v_\varphi(r_1) = r_1\Omega_1$ and $v_\varphi(r_2) = r_2\Omega_2$, the constants are obtained as $c_1 = (\Omega_1 - \Omega_2) r_1^2 r_2^2/(r_2^2 - r_1^2)$ and $c_2 = (\Omega_2 r_2^2 - \Omega_1 r_1^2)/(r_2^2 - r_1^2)$. This concentric steady flow, known as the Taylor-Couette flow [70], induces the steady differential rotation with angular velocity $\Omega(r) = c_1/r^2 + c_2$, leading to the generation of spin current (see Fig. 1(a)):

$$\boldsymbol{j}_s(r) = \boldsymbol{e}_r \frac{\hbar\sigma_0}{e^2} \frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \frac{\Omega_1 - \Omega_2}{r^3}, \tag{16}$$

where $\boldsymbol{e}_r$ being the unit vector in the $r$-direction. Notably, since the vorticity in this system is constant, $\boldsymbol{\nabla} \times \boldsymbol{v} = 2c_2 \boldsymbol{e}_z$, the conventional spin currents owing to the spin-vorticity coupling, which require the vorticity gradient [55, 64] or time-dependent vorticity [63], do not appear. On the other hand, our theory predicts the generation of spin current even in vorticity-free cases $c_2 = 0$.

To estimate the magnitude of the spin current, we assume that the radii of the two cylinders are much larger than the gap between them $d = r_2 - r_1$, i.e., $r_1, r_2 \gg d$, and only the outer cylinder is rotating, $\Omega_1 = 0$ and $\Omega_2 \neq 0$, for simplicity. Under this assumption, the spin current is approximated as $j_s = \hbar\sigma_0\Omega_2/2e^2 d$. We consider (Ga,In)Sn as the fluid with the electric conductivity $\sigma_0 = 3.26 \times 10^6 (\Omega \cdot \mathrm{m})^{-1}$ [58]. Set $d \sim 1\mu$m and $\Omega_2 \sim 10^2$kHz, the magnitude of the spin current in charge current units is estimated as $e j_s \sim 1.07 \times 10^2 \mathrm{A{\cdot}m^{-2}}$.

*Torsional Oscillation of Cantilever.*—As another example, we focus on the torsional oscillation of a cantilever [42, 44], wherein one of the ends is securely fixed while external forces are exerted on the opposite end. These forces induce only a twisting motion in the cantilever, not bending or other deformations. In this case, the angular velocity of the system varies along the rotation axis rather than the radial direction. This position-dependent rotation induces relative motion of the impurities in the cantilever, and our analysis applies even to such a setting.

The distortion angle $\varphi(z,t)$ of the cantilever dictates the subsequent equation of motion: $C\partial_z^2\varphi = \rho_m I \partial_t^2 \varphi$, where $C$ is the torsional rigidity, $\rho_m$ is the mass density, and $I$ is the moment of inertia of the cross-section about its center of mass. By solving the equation of motion under the boundary conditions $\varphi(0,t) = 0$ and $\partial_z\varphi(l,t) = 0$ and considering the initial conditions $\varphi(l,0) = \varphi_0$ and $\partial_t\varphi(z,0) = 0$, we derive the solution

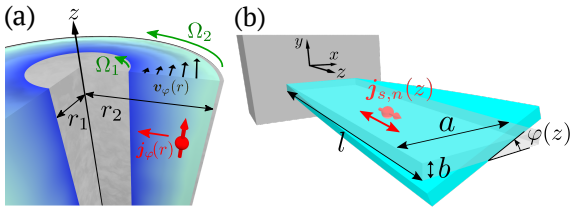



FIG. 1. Schematic illustration showing the generation of spin current due to (a) the Taylor-Couette flow in a liquid metal and (b) torsional motion of a cantilever.

as $\varphi_n(z,t) = \varphi_0 \sin k_n z \cos\omega_n t$, where $k_n = (2n-1)\pi/2l$ and $\omega_n = v k_n$ with the integer $n \geq 1$ and the velocity $v = \sqrt{C/\rho_m I}$. Plugging $\varphi$ into $\Phi$ in (13), we find the spin current, driven by the $n$-th torsional oscillation of cantilever, flows along the $z$-direction as given by (see Fig. 1(b))

$$\boldsymbol{j}_{s,n}(z,t) = \boldsymbol{e}_z \frac{\hbar\sigma_0\varphi_0 v}{2e^2} k_n^2 \cos k_n z \sin\omega_n t. \tag{17}$$

The mechanism under investigation in this study represents a universal phenomenon, irrespective of material choice, and fundamentally distinct from the previous theory [71] that focus solely on magnetic materials.

Finally, we estimate the magnitude of the spin current driven by the torsional oscillation. For a plate-shaped cantilever with width $a$, thickness $b$ and length $l$, the quantities $C$ and $I$ are calculated as $C \simeq \mu a b^3/3$ and $I \simeq a^3 b/12$ ($a \gg b$) with Lamé constant $\mu$. The magnitude of the total spin current in charge current units is denoted by $J_n = e a b j_{s,n}(0,0)$, while that attributed to the first torsional oscillation mode is given by $J_1 = (\pi^2\hbar\sigma_0\varphi_0/4e)(b/l)^2\sqrt{\mu/\rho_m}$. We consider that the cantilever is composed of copper with weak spin-orbit interaction. By using the charge conductivity $\sigma_0 = 6.45\times10^7\,\Omega^{-1}\mathrm{m}^{-1}$, the Lamé constant $\mu = 48.3\,$GHz and the mass density $\rho_m = 8.96 \times 10^3\,\mathrm{kg/m^3}$, the total spin current is estimated as $J_1 \sim 0.15\,\mu$A for $\varphi_0 \sim 0.01$ and $b/l = 1/4$.

*Conclusion and discussion.*—We have proposed a novel mechanism for spin current generation in differentially rotating systems. This mechanism produces spin currents even in vorticity-free systems and differs from the known mechanisms based on the spin-vorticity coupling. Our result is not phenomenological, but is derived from a microscopic theory of non-equilibrium spin dynamics. This is a natural extension of the Barnett effect in rigidly rotating systems to non-uniform rotating systems. It removes the previous limitation of spin current generation which was restricted to systems with vorticity, and broadens the potential for spin devices using a wider range of spatially non-uniform mechanical motions. In addition, it would provide a deeper understanding of the ubiquitous gyromagnetic effect, both from a theoretical and an experimental point of view.

In more detail, distinctions between our mechanism and those proposed in other literature [55, 63, 64] can be

found in the source terms that violate the conservation of the spin density, where the spin current is proportional to the gradient of those source terms. In our case, the spin transport equation is given in (15) and the source term is proportional to $\Omega$, the angular velocity of the orbital rotation around a fixed axis. On the other hand, the source term given in [63] and those in [55, 64] are proportional to $\partial_t\tilde{\omega}$ and $\zeta\tilde{\omega}$, respectively, where $\tilde{\omega}$ is the vorticity and $\zeta$ is a phenomenological parameter. The difference between our proposal and that of [63] can be detected experimentally, for the torsional oscillation, from the frequency dependence of the spin current. We can distinguish our mechanism from those of [55, 64] experimentally based on its dependence on phenomenological parameters (for details, see supplementary material). Moreover, an irrotational flow, in which the vorticity is zero, can prevent the spin current generation due to the spin-vorticity coupling. Creating an irrotational flow akin to a differentially rotating system allows us to detect spin currents specific to our mechanism. One may achieve such flow in the Taylor-Couette flow by taking appropriate boundary conditions that realize $c_2 = 0$ with $c_1 \neq 0$. Furthermore, we can distinguish the mechanisms even in the case of $c_2 \neq 0$. Since the vorticity in this system is uniform and time-independent, neither the source term in [63] nor that in [55, 64] can contribute to the spin current. As a result, the Taylor-Couette flow can generate the spin current in our mechanism but not in the other ones. Exploring the implications of our new mechanism in various experiments would be valuable for both fundamental physics and next-generation devices, such as integrating liquid metal fluids with micro-electromechanical systems (MEMS) [72] to utilize electrical and spin degrees of freedom.

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