



This is the accepted manuscript made available via CHORUS. The article has been published as:

Krylov fractality and complexity in generic random matrix ensembles

Budhaditya Bhattacharjee and Pratik Nandy

Phys. Rev. B **111**, L060202 — Published 6 February 2025

DOI: [10.1103/PhysRevB.111.L060202](https://doi.org/10.1103/PhysRevB.111.L060202)

Krylov fractality and complexity in generic random matrix ensembles

Budhaditya Bhattacharjee ^{1,*} and Pratik Nandy ^{2,3,†}

¹*Center for Theoretical Physics of Complex Systems,
Institute for Basic Science (IBS), Daejeon 34126, Republic of Korea*

²*Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics,
Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

³*RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS), Wako, Saitama 351-0198, Japan*

Krylov space methods provide an efficient framework for analyzing the static and dynamical aspects of quantum systems, with tridiagonal matrices playing a key role. Despite their importance, the behavior of such matrices from chaotic to integrable states, transitioning through an intermediate phase, remains unexplored. We aim to fill this gap by considering the properties of the tridiagonal matrix elements and the associated basis vectors for appropriate random matrix ensembles. We utilize the Rosenzweig-Porter model as our primary example, which hosts a fractal phase in addition to the ergodic and localized regimes. We discuss the characteristics of the matrix elements and basis vectors across these three (ergodic, fractal, and localized) regimes and introduce Krylov space tools to identify the transition points. The exact expressions of the Lanczos coefficients of such tridiagonal matrices are provided in terms of q -logarithmic function across the full parameter regime. The numerical results are corroborated with analytical reasoning for certain features of the Krylov spectra. Additionally, we investigate the Krylov state (spread) complexity within these regimes, showcasing the efficacy of our methods in pinpointing these transitions.

Introduction: A large extent of our understanding of quantum systems stems from the study of random matrices that closely model the generators of the dynamics of the system [1]. Pioneering conjectures by Bohigas-Giannoni-Schmidt [2] and Berry-Tabor [3] have laid the groundwork for understanding level statistics [4, 5], while random matrix theories have emerged as potent models for elucidating black hole physics [6]. Moreover, the transitions between chaotic and integrable phases in quantum systems have been a focal point of research, with the eigenspectrum of the Hamiltonian providing valuable insights into these transitions [7–10].

This manuscript addresses phase transitions within a prototypical random matrix ensemble, employing Krylov space techniques [11–13]. These techniques have gained prominence due to their simplicity and wide applicability; see [14–28] for the initial explorations and [13] for a detailed review and the references therein. By transforming the Hamiltonian into a tridiagonal form, one analyzes its elements, known as Lanczos coefficients, which comprise the Krylov spectrum [11, 12]. Such techniques have been employed in a few particular cases in different phases [29–32]. However, the behavior across the transition regime remains unexplored. We address such transition through Krylov space tools in the Rosenzweig-Porter model (RP) [33–36] which is particularly intriguing as it exhibits a spectral fractal phase, bridging the gap between ergodic and localized regimes. We propose approximate *Ansätze* that characterizes the statistical behavior of the Lanczos coefficients during the transitional phases. The *static* inverse participation ratio (IPR) [37] of Krylov vectors is introduced to assess the degree of localization, uncovering the IPR’s ability to differentiate among the three

distinct phases. We also investigate the complexity associated with the spread of a Gibbs state through these phases, identifying specific features that sharply delineate phase boundaries.

Rosenzweig-Porter model: The Rosenzweig-Porter (RP) model [33] is given by the $N \times N$ Hamiltonian [38]

$$H = A + \frac{1}{N^{\gamma/2}} B, \quad (1)$$

where A is a diagonal matrix, drawn from a normal distribution with zero mean and unit variance, i.e., $\langle a_{ii} \rangle^2 = 1$, and B is a matrix from Gaussian Orthogonal Ensemble (GOE), drawn from a normal distribution with zero mean (for both diagonal and off-diagonal elements) and variance $\langle b_{ii} \rangle^2 = 1$ and $\langle b_{ij} \rangle^2 = 1/2$ for $i \neq j$, respectively. Here $\gamma \geq 0$ is a parameter that effectively controls the dominant behavior of A and B . It gives rise to the corresponding delocalized or ergodic ($\gamma \in [0, 1]$), fractal ($\gamma \in [1, 2]$), and localized ($\gamma \in [2, \infty)$) phases, which leave imprints on the eigenspectrum of H . In the large N limit, one can obtain the $\langle r \rangle$ value statistics of the mid-spectrum [39]. Specifically, one obtains the Wigner-Dyson distribution for $\gamma = [0, 2]$ while the Gaussian distribution appears in the $\gamma = (2, \infty)$ limit [40]; see Supplemental Material (SM1) [41] for further details. On the other hand, the fractal phase is characterized by the *fractal dimension* $\mathcal{D}$, defined through the scaling of the Inverse Participation Ratio (IPR) as $\text{IPR} = \sum_n |\langle n | \psi \rangle|^4 \sim N^{-\mathcal{D}}$ [42]. Here $|\psi\rangle$ is a state in the eigenbasis and $|n\rangle$ is a computational basis element, and the overlap captures the spread of the eigenstate in the computational basis. The ergodic and localized regime correspond to constant fractal dimensions, i.e.,

$\mathcal{D} = 1$ and $\mathcal{D} = 0$, respectively, while in the fractal phase, the fractal dimension linearly decreases as $\mathcal{D} = 2 - \gamma$ [8].

Tridiagonalization and Krylov Spectrum: To investigate the various phases of the RP model, we employ the Krylov spectrum of the Hamiltonian as our primary set of tools. The Hamiltonian is transformed into a tridiagonal matrix on the Krylov basis through the application of the Lanczos algorithm [43, 44]

$$H|K_n\rangle = b_n|K_{n-1}\rangle + a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle. \quad (2)$$

Here, the sets $\{a_n, b_n\}$ are referred to as the Lanczos coefficients, and the set $\{|K_n\rangle\}$ constitutes the Krylov basis, which together forms the Krylov spectrum [12]; see [13] for a detailed review. For numerical purposes, it is often useful to consider the Hessenberg decomposition [45], which uses the Householder reflections [46] instead of the Lanczos algorithm. This produces identical results for Lanczos coefficients, subject to some redefinition of Krylov basis vectors. Curiously, the density of the states (DOS) can be well approximated from the statistics of the Lanczos coefficients $a(x)$ and $b(x)$ where $x = n/N$ in the large N limit as [47, 48]

$$\rho(E) = \int_0^1 dx \frac{\Theta(4b(x)^2 - (E - a(x))^2)}{\pi \sqrt{4b(x)^2 - (E - a(x))^2}}. \quad (3)$$

Here $\Theta(z)$ is the Heaviside theta function such that $\Theta(z) = 1$ for $z \geq 0$ and $\Theta(z) = 0$ for $z < 0$. Given the Lanczos coefficients, the Krylov wavefunctions $\psi_n(t)$ satisfy the following differential equation:

$$i\partial_t \psi_n(t) = b_n \psi_{n-1}(t) + a_n \psi_n(t) + b_{n+1} \psi_{n+1}(t). \quad (4)$$

which is the Schrödinger equation in the ‘‘Krylov chain’’ governed by the Hamiltonian (1), and the time-evolved state is expressed in the Krylov basis $|K_n\rangle$ with the amplitude $\psi_n(t)$. The Krylov state (spread) complexity is defined by the average position on the Krylov chain [12]

$$K_S(t) = \sum_n n |\psi_n(t)|^2. \quad (5)$$

The initial state plays an important role in governing the dynamics. In principle, one can start with any state, including the thermofield double (TFD) state or thermal Gibbs state, in which case the spread complexity shows distinct behavior: early quadratic growth followed by linear growth until it reaches a peak. The presence of the peak is crucial for chaotic systems and differs from integrable systems where such a peak is absent [12, 28]. At late times, the spread complexity saturates to a plateau, depending on the dimension of the Hilbert space.

Statistics of Lanczos coefficients: In a manner akin to the conventional eigenspectrum analysis, we focus on the behavior of the Lanczos coefficients for the Hamiltonian (1) throughout all phases, including the critical points of

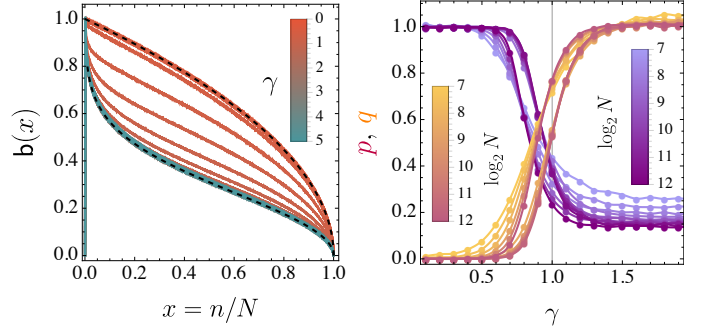


FIG. 1. (Left) The Lanczos coefficients for the RP model with $N = 4096$ (with 100 ensemble averages) as a function of $x = n/N$, where n is the index of the Lanczos coefficient. The black dashed lines are the fitted results given by (6) (top) and (8) (bottom) at the two extremes. (Right) The behavior of the fitting parameters u and v as functions of γ and N .

fractal and localization transitions. We choose the initial state as $(1, 0, 0, \dots, 0)^T$. However, the choice of the initial state does not affect the *averaged* or *mean* of the Lanczos spectrum *in the bulk*, given a sufficient number of ensemble averages of the Hamiltonian is taken. This is because the averaged bulk Lanczos coefficients depend solely on the density of states of the Hamiltonian. However, even after a large number of ensemble averages, the edge coefficients, i.e., $n \sim o(1)$ coefficients, may exhibit a slight dependence on the choice of the initial state in small system sizes. This effect diminishes as the system size increases.

Figure 1 shows the behavior of the *rescaled* Lanczos coefficients in the RP model as a function of γ . As the largest eigenvalue scales with system size N , selecting larger system sizes will increase the largest eigenvalue, making the Lanczos coefficients dependent on N . For example, in the case of GUE, $b_n \sim \sqrt{N}$ with $n \sim o(1)$ if the variance of GUE matrices is taken as unity. Conversely, if the variances are taken as $1/\sqrt{N}$, then $b_n \sim 1$ when $n \sim o(1)$, and does not scale with system size. To ensure this fact, we choose appropriate variances for the random matrices so that the Lanczos coefficients are expressed in a system size-independent manner.

We observe a transition in the behavior of the Lanczos coefficients as γ transits from the ergodic to the localized phase. The behavior in the ergodic phase (where the matrix is full GOE) is known analytically [47, 49] and given by

$$b_{\gamma=0}(x) \equiv b_{\text{GOE}}(x) \propto \sqrt{1-x}. \quad (6)$$

The symmetry of the energy in the DOS requires $a(x) = 0$ upon the ensemble average of the Hamiltonian. In the localized phase $\gamma \gtrsim 2$, the nature of the function is previously unknown. In this paper, we ascertain that such a form is given by

$$b_{\gamma \gtrsim 2}(x) \equiv b_P(x) = \xi \sqrt{d} \text{ nib}(x, d), \quad (7)$$

where $d = \log_2 N$ and N being the matrix dimension. Here “nib” stands for the inverse of the shifted binomial function $\text{bin}(x, d) = 2^{-d} \binom{d}{d(1/2-x)}$ [49]. Such structure has been previously studied in the context of nuclear spectra [49, 50], yet remains largely unknown. The inverse binomial is ill-defined at the origin, while away from $x \sim 0$, it can be approximated to a simpler form

$$\text{b}_P(x) \sim \sqrt{-\frac{\xi^2}{2} \ln x} \Leftrightarrow \rho(E) = \frac{1}{\sqrt{2\pi\xi}} e^{-E^2/(2\xi^2)}, \quad (8)$$

with corresponding Gaussian DOS of zero mean and variance ξ^2 , obtained through (3); see SM2 [41]. Note that the expressions for the Lanczos coefficients are only valid in bulk and the edge corrections in $n \sim O(1)$ should be determined from the moments of the DOS [48]. We numerically adjust the factor ξ to make the analytical expression (8) coincide with the value of x where the numerical $\text{b}(x)$ is maximum. We find excellent agreement with the numerical results with $\xi \simeq 1/2$, shown by the black dashed line (green region) in Fig. 1 (left).

Given the Lanczos coefficients in two extreme limits, we *propose* the following *Ansatz* to interpolate across the phases in the following form:

$$\text{b}_\gamma(x)^2 = p \left(\frac{1 - x^{1-q}}{1 - q} \right) = -p \ln_q x, \quad (9)$$

given by the q -logarithm, introduced by Tsallis [51]. It reduces to the usual logarithm for $q \rightarrow 1$, i.e., in the Poisson limit. Here, $p \equiv p(\gamma, N)$ and $q \equiv q(\gamma, N)$ weakly depend on the system size N . We scale the coefficients appropriately to lie between zero and unity. Importantly, (9) reduces to (6) and (8) at $q \rightarrow 0$ and $q \rightarrow 1$ limits, which are proxy for the $\gamma \rightarrow 0$ and $\gamma \gtrsim 2$ limits, respectively. The corresponding limits of p are $p \simeq 1$ and $p \simeq \xi^2/2$. Note that the *Ansatz* (9) is not unique, and an alternate *Ansatz* is discussed in the SM3 [41].

Figure 1 (right) shows the dependence of γ and N on p and q . The parameter $p \simeq 1$ in the ergodic regime, while close to the ergodic to fractal transition, starts to decrease, eventually saturating at $p \simeq \xi^2/2 \simeq 1/8$, which is an excellent agreement with the ξ extracted from Fig. 1 (right). Conversely, the coefficient q vanishes in the ergodic regime and eventually saturates at unity in the deep localized regime. The saturation values for p and q weakly depend on N while converging to an asymptotic value in the large N limit. This behavior signals a transition from ergodic to fractal properties close to $\gamma = 1$, where the parameters p and q are comparable. We compute the relative “goodness of fit” as $\varepsilon = \frac{\sqrt{\Delta p + \Delta q} \times 10^2}{\min(p-1, q-1)+1}$ in SM2 [41]. This relative error is always $< 10\%$, supporting the *Ansatz* (9) as a good analytical form of the Lanczos coefficients across the full parameter regime of γ . Results for another quantity, the log-variance of Lanczos coefficients, are presented in SM4 [41].

Curiously, the DOS corresponding to (9) can be obtained by solving the integral equation (3), which is valid in the bulk of the spectrum. We find

$$\rho_\gamma(E) = \sqrt{\frac{1}{4\pi p(1-q)}} \frac{\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{2} + \frac{1}{1-q})} \left(1 - \frac{E^2}{4p}(1-q)\right)^{\frac{1+q}{2(1-q)}}. \quad (10)$$

In the particular case of $q \rightarrow 0$ and $q \rightarrow 1$ limits, (10) reduces to the semicircle law and the Gaussian distribution (8), respectively, while (10) holds across the full spectrum. It is normalized $\int_{-z}^z \rho_\gamma(E) dE = 1$ with $z = 2\sqrt{p/(1-q)}$. Both the limits suggest the effective variance as p , consistent with the observations in [49, 50]. Moments of the above distribution can be straightforwardly computed, see SM5 [41].

Behavior in the deep localized regime - analytic arguments: A notable aspect of the RP model in its tridiagonal form is the persistence of non-zero off-diagonal elements even in the deeply localized regime, as evidenced by Fig. 1. At first glance, this seems counterintuitive, especially when considering that at high values of γ , the primary contribution should theoretically arise from the diagonal matrix A . Yet, our numerical findings indicate that the off-diagonal elements remain finite and are of comparable magnitude to those found in the ergodic regime, a phenomenon we explore below.

We first examine the response of an orthogonal transformation to the RP model [52]. For a Hamiltonian characterized by the distribution with $\langle H_{ij} \rangle = 0$ and $\langle H_{ij}^2 \rangle = \alpha \delta_{ij} + \beta(1 - \delta_{ij})$, an orthogonal transformation via a matrix C keeps the expectation value unchanged $\langle \tilde{H}_{ij} \rangle = 0$ while transforming the variance to $\langle \tilde{H}_{ij}^2 \rangle = \beta(1 + \delta_{ij}) + (\alpha - 2\beta) \sum_l c_{li}^2 c_{lj}^2$; see SM6 [41] for details. Here $\tilde{H}$ denotes the transformed Hamiltonian, and c_{ij} represents the elements of C . Although this matrix can be arbitrary, we limit it to a special orthogonal matrix M that executes the Householder transformation on H . The elements of M can be approximated using the distribution of H_{ij} [41]. The variance of the distribution of the elements of the matrix $\tilde{H}$ after a single orthogonal transformation is given by

$$\begin{aligned} \langle \tilde{H}_{ij}^2 \rangle &\sim \beta(\delta_{i1} + \delta_{j1}) + \beta(1 - \delta_{ij}) \\ &+ \frac{3}{N^2} (\alpha - 2\beta)(1 - \delta_{ij})(1 - \delta_{i1}) + (\alpha - \beta)\delta_{ij}(1 - \delta_{i1}) \\ &+ \frac{1}{N} (\alpha - 2\beta)(\delta_{i1}\delta_{j1} + 2\delta_{i1} - 4\delta_{ij} + 2\delta_{ij}\delta_{i1}). \end{aligned} \quad (11)$$

These expressions result from a single step of the orthogonal transformation, assuming $\alpha, \beta \neq 0$. Cases where $\alpha = 0$ or $\beta = 0$ are considered separately. The complete tridiagonalization process involves N such transformations, and these expressions are applied recursively to determine the final distribution of the Lanczos coefficients. The coefficients $\text{b}(x)$ correspond to the norm

of the off-diagonal terms $\tilde{H}_{1j}$ at each step. We argue that the distribution of these off-diagonal terms is approximately Gaussian by the central limit theorem, or the “law of large numbers”. This allows the norm to be estimated by the mean of the Nakagami distribution [53] with parameters N and $\langle \tilde{H}_{1j}^2 \rangle$.

It is, thus, evident that although β approaches zero deep in the localized regime, the full $\tilde{H}_{1j}$ remains non-zero due to the α/N contribution (with $\alpha = 1$ for the RP model). This contribution has $1/N$ suppression, but this reduction is offset when calculating the norm of all $\tilde{H}_{1j}$ (with $\langle \tilde{H}_{1j}^2 \rangle = \beta + \frac{2}{N}(\alpha - 2\beta)$) which introduces a $\sqrt{N}$ factor. Consequently, the overall suppression in the deeply localized regime is on the order of $N^{-1/2}$, persisting even at very high γ values. This accounts for the collapse of Lanczos coefficients onto a single curve in the deeply localized regime, as shown in Fig. 1.

Krylov IPR: To gain further insight, we address the extension of localization in Krylov space by computing the Inverse Participation Ratio (IPR) for the Krylov subspace. Instead of taking the usual eigenvectors, this amounts to considering the Krylov vectors corresponding to an initial localized state. It is widely understood that for quantum mechanical systems, Krylov space localization agrees with real space localization. The eigenvector IPR has been extensively studied in the RP model [54]. We extend this study to the Krylov IPR and determine the appropriate scaling that allows us to distinguish between the three phases of the RP model. We define the Krylov IPR as the overlap between the computational basis state and the Krylov state

$$\text{IPR}_K^\ell(\varphi_k) \equiv \sum_{n=1}^N |\langle n | \varphi_k \rangle|^{2\ell}, \quad (12)$$

where $|\varphi_k\rangle$ stands for the k^{th} Krylov vector and $|n\rangle$ is a computational basis element. The logarithm of the $\ell = 1$ case is often known as the *participation entropy*.

The Krylov state $|\varphi_k\rangle$ can be expressed in terms of the Hamiltonian eigenstates $|\psi_m\rangle$ as $|\varphi_k\rangle = \sum_{m=1}^N \eta_m^k |\psi_m\rangle$. The coefficients η_m^k satisfy a three-term recursion relation; see SM7 [41] with additional reference [55]. Thus, IPR_K^ℓ can be further rewritten as $\text{IPR}_K^\ell(\varphi_k) = \sum_{n=1}^N \left| \sum_{m=1}^N \eta_m^k s_n^m \right|^{2\ell}$ where $s_n^m = \langle n | \psi_m \rangle$. The usual IPR for the eigenstate $|\psi_k\rangle$ is then obtained by simply setting $\eta_m^k = \delta_m^k$ i.e., the usual eigenstate IPR information is encoded in the Krylov IPR. In SM7 [41], the overlap of Krylov vectors and eigenstates is also investigated. This indicates that the Krylov IPR can serve as a robust indicator for both localization and multifractality [35]. Physically, this corresponds to the spread of eigenstates on the computational basis and the spread of the Krylov vectors in the eigenbasis. Note that our definition of Krylov IPR is *static* and reflects the localization with respect to the γ in comparison to the *dynamic* IPR

defined in [56].

The detailed behavior of Krylov IPR is discussed in the SM7 [41]. Here, we are interested in the scaling of the IPR, which is much more apparent for $k = N-1$ than $k = N/2$, i.e., the middle of the Krylov spectrum. The results are presented in Fig. 2, and we compare them with the $\ell = 1$ case in SM7 [41]. The IPR is higher in the localized regime compared to the ergodic and fractal phases. The scaling exponent is obtained by fitting the linear decrease $\text{IPR}_K^2(\varphi_{N-1}) \sim N^{-D_2}$ in the fractal region. We observe the following features of the scaling exponent

$$\mathcal{D}_2(\gamma) \sim \begin{cases} 1 & \gamma \leq 1, \\ 2 - \gamma & 1 < \gamma \leq 2, \\ 0 & \gamma > 2. \end{cases} \quad (13)$$

Up to the numerical and the finite-size effects, $\mathcal{D}_2(\gamma) = \text{constant} > 0$ for $\gamma \leq 1$ (ergodic phase). In the fractal phase, it linearly decreases $\mathcal{D}_2(\gamma) \sim -\alpha\gamma + \chi > 0$ (where α and χ are some positive constants) for $1 < \gamma \leq 2$ (fractal phase) while $\mathcal{D}_2(\gamma) \sim 0$ for $\gamma > 2$ (localized phase), indicating strong localization.

The underlying reason behind the strong sensitivity of φ_{N-1} (or any φ_k close to $k = N$) to γ comes from the Lanczos algorithm. In the context of matrix diagonalization, the Lanczos algorithm converges from both extreme ends, making it particularly effective for finding the smallest or largest few eigenvalues and eigenvectors. However, to obtain the eigenvalues and eigenvectors in the middle of the spectrum, the algorithm must effectively span the entire basis. Since φ_k is generated by the application of H^k on the initial state followed by the subtraction of the contribution of φ_{k-1} and φ_{k-2} , the effect of the full Lanczos spectrum is encoded in the Krylov vectors towards the end of the spectrum. For example, by considering the overlap of all Krylov vectors with the mid-spectrum eigenstate, one finds that the larger the value of k , the more information about the Lanczos spectrum is encoded in this overlap coefficient. Since we have seen that the full Lanczos spectrum is sensitive to the transition, it is desirable to utilize a Krylov vector that encodes the full spectrum. This also explains why φ_{N-1} is highly sensitive to mid-spectrum eigenstates.

Spread complexity: The rate of spreading of the information in different phases can be captured by the spread complexity associated with the Hamiltonian (1), initializing with the maximally entangled infinite-temperature TFD state. As depicted in Fig. 3, the spread complexity exhibits distinct behaviors across three regimes: ergodic (purple), fractal (red), and localized (teal). Notably, pronounced peaks are observable in the ergodic and fractal regimes, contrasting their absence in the localized regime. This disparity suggests that the emergence of peaks, and consequently a linear slope, may be attributed to the level-repulsion among the eigenvalues [28, 47, 57, 58].

The saturation value of spread complexity always con-

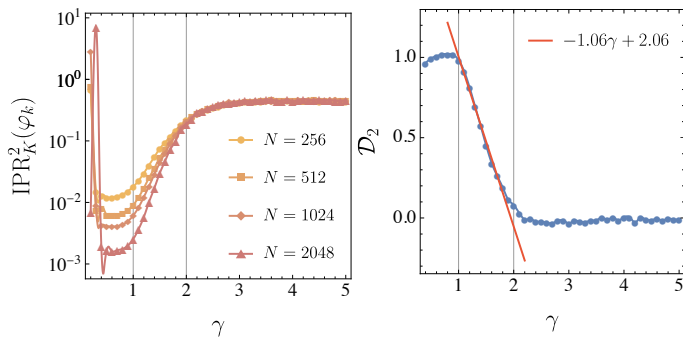


FIG. 2. (Left) Scaling of the IPR_K^2 for φ_{N-1} for different values of N as a function of γ . The scaling with system size is nearly unaffected by γ . The numerical instability at low γ arises due to the finite size of the system, which reduces with increasing system size. (Right) Scaling exponent $\mathcal{D}_2$ of IPR_K^2 for φ_{N-1} as a function of γ . In both cases, the exponent is nearly constant in the ergodic regime and decreases linearly in the fractal regime to settle at zero in the localized regime.

verges to a value of $\bar{K}_S/N = (N-1)/(2N) \approx 0.5$ [28]. In contrast, the peak value of spread complexity exhibits a gradual decline as we transition into the localized regime. In the latter case, the peak is notably absent, rendering the peak value synonymous with the saturation value. Analogously, the peak time of spread complexity demonstrates a clear differentiation in growth across the three regimes. This is marked by transitions at the critical points of $\gamma = 1$ and $\gamma = 2$, highlighting the unique dynamics within each regime.

Conclusion and outlook: In this study, we investigate the Rosenzweig-Porter (RP) model, a random matrix model that exhibits extended, fractal, and localized phases. Our exploration utilizes static and dynamic measures within the Krylov subspace to illuminate the phase transitions among these three distinct states. We present a semi-analytic expression for the Lanczos coefficients that interpolates between the three phases. We introduce the Krylov Inverse Participation Ratio (IPR) and a Krylov fractal dimension, which accurately indicate the two critical transition points. Notably, the *edge-spectrum* Krylov vectors exhibit stronger sensitivity to fractality, in contrast to those near the center of the spectrum. Dynamic metrics, such as the spread complexity of Krylov states, also reflect these transitions.

Our findings affirm that both static and dynamic Krylov subspace probes are attuned to the transitions from ergodicity through fractality to localization. These probes present robust alternatives to eigenspectrum analysis. This work bridges a significant gap in the literature on Krylov techniques, especially by providing an analytic expression of Lanczos coefficients across the full parameter regime, thereby establishing a standard for detecting transitions from chaotic to non-chaotic states in physical systems via Krylov methodologies [13].

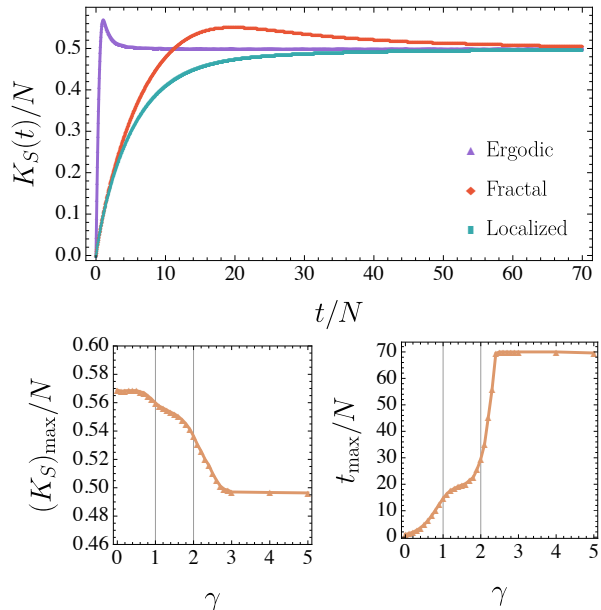


FIG. 3. (Top) Spread complexity for the RP model (1) in different regimes, initialized by a TFD state at infinite temperature. (Bottom, left) The variation of the peak value of the spread complexity, and (bottom, right) the peak time in different phases of the Hamiltonian (1), initialized by a TFD state at infinite temperature. The system size is $N = 500$, variance $\sigma^2 = 1/N$ with 800 Hamiltonian realizations.

An immediate extension of this method could be the utilization of these probes in systems undergoing localization transitions. The behavior of the DOS (10) shares similar features in two extreme limits of the spin-glass [59] or the double-scaled SYK model [60] (see SM for some initial attempts). It would be intriguing to assess how the proposed *Ansatz* (9) compares to the tridiagonal elements found in the models with corresponding q -normal distribution [59, 60]. This seems to be illuminating if one computes the moments of the Hamiltonian (1) using the chord diagrams [61], where the A and B matrices provide the intersecting and the non-intersecting chords. Future research might explore such transitions within various matrix ensembles and different diagonal distributions [38]. We anticipate that the overarching characteristics will align with our findings. These probes could also be applied to non-Hermitian extensions within the 38-fold symmetry classes [62], particularly those lacking a fractal phase in generic complex diagonal Hamiltonians [63]. In generic systems, extracting the exact form of the Lanczos coefficients across the full spectrum with a known DOS is also unknown. We sketch a general methodology in SM5 that may benefit such endeavors.

Note added: While finishing this manuscript, we became aware of an upcoming work [64] that also obtains the same logarithmic function (8) in the Gaussian limit.

Acknowledgements: We thank Tanay Pathak for the

initial collaboration and feedback. We also acknowledge valuable discussions with Alexei Andreanov, Adolfo del Campo, Barbara Dietz, Anatoly Dymarsky, András Grabarits, Ivan Khaymovich, Rafael Molina, Horacio Pastawski, Anatoli Polkovnikov, Dario Rosa, Lea Santos, Ruth Shir, Luca Tessieri, and Zhuo-Yu Xian. B.B. thanks the organizers of the International Workshop of Disorder Systems (Salamanca) 2024 and the University of Luxembourg for their hospitality during the final stages of the preparation of this manuscript. P.N. thanks the University of Kentucky and the Berkeley Center for Theoretical Physics, University of California, Berkeley, for hosting him through the Adopting Sustainable Partnerships for Innovative Research Ecosystem (ASPIRE) program of the Japan Science and Technology Agency (JST), Grant No. JPMJAP2318, during the final stages of the work. B.B. acknowledges financial support from the Institute for Basic Science (IBS) in the Republic of Korea through Project No. IBS-R024-D1. The work of P.N. is supported by the Japan Society for the Promotion of Science (JSPS) Grants-in-Aid for Transformative Research Areas (A) “Extreme Universe” No. JP21H05190.

* budhadityab@ibs.re.kr

† pratik@yukawa.kyoto-u.ac.jp

- [1] M.L. Mehta, *Random Matrices* (Academic Press, 1991).
- [2] O. Bohigas, M. J. Giannoni, and C. Schmit, “Characterization of chaotic quantum spectra and universality of level fluctuation laws,” *Phys. Rev. Lett.* **52**, 1–4 (1984).
- [3] M. V. Berry and M. Tabor, “Closed orbits and the regular bound spectrum,” *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **349**, 101–123 (1976).
- [4] Eugene P. Wigner, “Characteristic Vectors of Bordered Matrices With Infinite Dimensions,” *Annals of Mathematics* **62**, 548–564 (1955).
- [5] Freeman J. Dyson, “Statistical Theory of the Energy Levels of Complex Systems. I,” *Journal of Mathematical Physics* **3**, 140–156 (1962).
- [6] Jordan S. Cotler, Guy Gur-Ari, Masanori Hanada, Joseph Polchinski, Phil Saad, Stephen H. Shenker, Douglas Stanford, Alexandre Streicher, and Masaki Tezuka, “Black Holes and Random Matrices,” *JHEP* **05**, 118 (2017), [Erratum: *JHEP* **09**, 002 (2018)].
- [7] Maxim Olshanii, Kurt Jacobs, Marcos Rigol, Vanja Dunjko, Harry Kennard, and Vladimir A. Yurovsky, “An exactly solvable model for the integrability–chaos transition in rough quantum billiards,” *Nature Communications* **3**, 641 (2012).
- [8] V E Kravtsov, I M Khaymovich, E Cuevas, and M Amini, “A random matrix model with localization and ergodic transitions,” *New Journal of Physics* **17**, 122002 (2015).
- [9] Adway Kumar Das, Anandamohan Ghosh, and Ivan M. Khaymovich, “Absence of mobility edge in short-range uncorrelated disordered model: Coexistence of localized and extended states,” *Phys. Rev. Lett.* **131**, 166401 (2023).
- [10] Adway Kumar Das, Anandamohan Ghosh, and Ivan M. Khaymovich, “Robust non-ergodicity of ground state in the β ensemble,” *Phys. Rev. B* **109**, 064206 (2024).
- [11] Daniel E. Parker, Xiangyu Cao, Alexander Avdoshkin, Thomas Scaffidi, and Ehud Altman, “A Universal Operator Growth Hypothesis,” *Phys. Rev. X* **9**, 041017 (2019).
- [12] Vijay Balasubramanian, Pawel Caputa, Javier M. Magan, and Qingyue Wu, “Quantum chaos and the complexity of spread of states,” *Phys. Rev. D* **106**, 046007 (2022).
- [13] Pratik Nandy, Apollonas S. Matsoukas-Roubeas, Pablo Martínez-Azcona, Anatoly Dymarsky, and Adolfo del Campo, “Quantum Dynamics in Krylov Space: Methods and Applications,” (2024), [arXiv:2405.09628 \[quant-ph\]](https://arxiv.org/abs/2405.09628).
- [14] J. L. F. Barbón, E. Rabinovici, R. Shir, and R. Sinha, “On The Evolution Of Operator Complexity Beyond Scrambling,” *JHEP* **10**, 264 (2019).
- [15] Anatoly Dymarsky and Alexander Gorsky, “Quantum chaos as delocalization in Krylov space,” *Phys. Rev. B* **102**, 085137 (2020).
- [16] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, “Operator complexity: a journey to the edge of Krylov space,” *JHEP* **06**, 062 (2021).
- [17] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, “Krylov localization and suppression of complexity,” *JHEP* **03**, 211 (2022).
- [18] Anatoly Dymarsky and Michael Smolkin, “Krylov complexity in conformal field theory,” *Phys. Rev. D* **104**, L081702 (2021).
- [19] Pawel Caputa, Javier M. Magan, and Dimitrios Patramanis, “Geometry of Krylov complexity,” *Phys. Rev. Res.* **4**, 013041 (2022).
- [20] Budhaditya Bhattacharjee, Xiangyu Cao, Pratik Nandy, and Tanay Pathak, “Krylov complexity in saddle-dominated scrambling,” *JHEP* **05**, 174 (2022).
- [21] Niklas Hörnedal, Nicoletta Carabba, Apollonas S. Matsoukas-Roubeas, and Adolfo del Campo, “Ultimate Speed Limits to the Growth of Operator Complexity,” *Commun. Phys.* **5**, 207 (2022).
- [22] Aranya Bhattacharya, Pratik Nandy, Pingal Pratyush Nath, and Himanshu Sahu, “Operator growth and Krylov construction in dissipative open quantum systems,” *JHEP* **12**, 081 (2022).
- [23] Chang Liu, Haifeng Tang, and Hui Zhai, “Krylov complexity in open quantum systems,” *Phys. Rev. Res.* **5**, 033085 (2023).
- [24] Budhaditya Bhattacharjee, Samudra Sur, and Pratik Nandy, “Probing quantum scars and weak ergodicity breaking through quantum complexity,” *Phys. Rev. B* **106**, 205150 (2022).
- [25] Budhaditya Bhattacharjee, Xiangyu Cao, Pratik Nandy, and Tanay Pathak, “Operator growth in open quantum systems: lessons from the dissipative SYK,” *JHEP* **03**, 054 (2023).
- [26] Alexander Avdoshkin, Anatoly Dymarsky, and Michael Smolkin, “Krylov complexity in quantum field theory, and beyond,” *JHEP* **06**, 066 (2024).
- [27] Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, and Mitsuhiro Nishida, “Krylov complexity in free and interacting scalar field theories with bounded power spectrum,” *JHEP* **05**, 226 (2023).
- [28] Johanna Erdmenger, Shao-Kai Jian, and Zhuo-Yu Xian, “Universal chaotic dynamics from Krylov space,” *JHEP*

- 08**, 176 (2023).
- [29] Gastón F. Scialchi, Augusto J. Roncaglia, and Diego A. Wisniacki, “Integrability-to-chaos transition through the krylov approach for state evolution,” *Phys. Rev. E* **109**, 054209 (2024).
 - [30] Mohsen Alishahiha and Mohammad Javad Vasli, “Thermalization in Krylov Basis,” (2024), [arXiv:2403.06655 \[quant-ph\]](#).
 - [31] Heiko Georg Menzler and Rishabh Jha, “Krylov delocalization/localization across ergodicity breaking,” *Phys. Rev. B* **110**, 125137 (2024).
 - [32] Khen Cohen, Yaron Oz, and De-liang Zhong, “Complexity measure diagnostics of ergodic to many-body localization transition,” *Phys. Rev. B* **110**, L180101 (2024).
 - [33] Norbert Rosenzweig and Charles E. Porter, “repulsion of energy levels” in complex atomic spectra,” *Phys. Rev.* **120**, 1698–1714 (1960).
 - [34] Alexander Altland, Martin Janssen, and Boris Shapiro, “Perturbation theory for the Rosenzweig-Porter matrix model,” *Phys. Rev. E* **56**, 1471–1475 (1997).
 - [35] Ivan M. Khaymovich and Vladimir Kravtsov, “Dynamical phases in a “multifractal” Rosenzweig-Porter model,” *SciPost Physics* **11**, 045 (2021).
 - [36] M. Pino, J. Tabanera, and P. Serna, “From ergodic to non-ergodic chaos in Rosenzweig–Porter model,” *J. Phys. A: Math. Theor.* **52**, 475101 (2019).
 - [37] Ferdinand Evers and Alexander D. Mirlin, “Anderson transitions,” *Rev. Mod. Phys.* **80**, 1355–1417 (2008).
 - [38] Davide Venturelli, Leticia F. Cugliandolo, Grégory Schehr, and Marco Tarzia, “Replica approach to the generalized Rosenzweig-Porter model,” *SciPost Phys.* **14**, 110 (2023).
 - [39] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, “Distribution of the ratio of consecutive level spacings in random matrix ensembles,” *Phys. Rev. Lett.* **110**, 084101 (2013).
 - [40] Alexander Altland, Martin Janssen, and Boris Shapiro, “Perturbation theory for the rosenzweig-porter matrix model,” *Phys. Rev. E* **56**, 1471–1475 (1997).
 - [41] See Supplemental Materials for details.
 - [42] Wouter Buijsman and Yevgeny Bar Lev, “Circular Rosenzweig-Porter random matrix ensemble,” *SciPost Phys.* **12**, 082 (2022).
 - [43] Cornelius Lanczos, “An iteration method for the solution of the eigenvalue problem of linear differential and integral operators,” *Journal of research of the National Bureau of Standards* **45**, 255–282 (1950).
 - [44] V.S. Viswanath and G. Müller, *The Recursion Method: Application to Many Body Dynamics*, Lecture Notes in Physics Monographs (Springer Berlin Heidelberg, 1994).
 - [45] G. Golub, S. Nash, and C. Van Loan, “A hessenberg-schur method for the problem $ax + xb = c$,” *IEEE Transactions on Automatic Control* **24**, 909–913 (1979).
 - [46] Alston S. Householder, “Unitary Triangularization of a Nonsymmetric Matrix,” *J. ACM* **5**, 339–342 (1958).
 - [47] Vijay Balasubramanian, Javier M. Magan, and Qingyue Wu, “Tridiagonalizing random matrices,” *Phys. Rev. D* **107**, 126001 (2023).
 - [48] Vijay Balasubramanian, Javier M. Magan, and Qingyue Wu, “Quantum chaos, integrability, and late times in the Krylov basis,” (2023), [arXiv:2312.03848 \[hep-th\]](#).
 - [49] A. P. Zuker, L. Waha Ndeuna, F. Nowacki, and E. Caulier, “Canonical form of hamiltonian matrices,” *Phys. Rev. C* **64**, 021304 (2001).
 - [50] R. A. Molina, A. P. Zuker, A. Relaño, and J. Retamosa, “Spectral statistics of hamiltonian matrices in tridiagonal form,” *Phys. Rev. C* **71**, 064317 (2005).
 - [51] Constantino Tsallis, “What are the numbers that experiments provide,” *Química Nova* **17**, 468–471 (1994).
 - [52] Ioana Dumitriu and Alan Edelman, “Matrix models for beta ensembles,” *Journal of Mathematical Physics* **43**, 5830–5847 (2002).
 - [53] Minrou Nakagami, “The m-distribution—a general formula of intensity distribution of rapid fading,” in *Statistical Methods in Radio Wave Propagation* (Pergamon, 1960) pp. 3–36.
 - [54] Tilen Čadež, Dillip Kumar Nandy, Dario Rosa, Alexei Andreanov, and Barbara Dietz, “The rosenzweig–porter model revisited for the three wigner–dyson symmetry classes,” *New Journal of Physics* **26**, 083018 (2024).
 - [55] James Holehouse, “Closed-form solution of a general three-term recurrence relation: applications to Heun functions and social choice models,” (2023), [arXiv:2302.04187 \[physics.soc-ph\]](#).
 - [56] Pedro H. S. Bento, Adolfo del Campo, and Lucas C. Céleri, “Krylov complexity and dynamical phase transition in the quenched Lipkin-Meshkov-Glick model,” *Phys. Rev. B* **109**, 224304 (2024).
 - [57] Hugo A. Camargo, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, and Mitsuhiro Nishida, “Spectral and Krylov complexity in billiard systems,” *Phys. Rev. D* **109**, 046017 (2024).
 - [58] Hugo A. Camargo, Kyoung-Bum Huh, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, and Mitsuhiro Nishida, “Spread and spectral complexity in quantum spin chains: from integrability to chaos,” *JHEP* **08**, 241 (2024).
 - [59] László Erdős and Dominik Schröder, “Phase transition in the density of states of quantum spin glasses,” *Mathematical Physics, Analysis and Geometry* **17**, 441–464 (2014).
 - [60] Micha Berkooz, Prithvi Narayan, and Joan Simon, “Chord diagrams, exact correlators in spin glasses and black hole bulk reconstruction,” *JHEP* **08**, 192 (2018).
 - [61] We thank Zhuo-Yu Xian for this comment.
 - [62] Kohei Kawabata, Ken Shiozaki, Masahito Ueda, and Masatoshi Sato, “Symmetry and topology in non-hermitian physics,” *Phys. Rev. X* **9**, 041015 (2019).
 - [63] Giuseppe De Tomasi and Ivan M. Khaymovich, “Non-hermitian rosenzweig-porter random-matrix ensemble: Obstruction to the fractal phase,” *Phys. Rev. B* **106**, 094204 (2022).
 - [64] Debarghya Chakraborty, Anatoly Dymarsky, and Refat Ismail, *work in progress*.