



This is the accepted manuscript made available via CHORUS. The article has been published as:

Topological triviality of flat Hamiltonians

Pratik Sathe and Rahul Roy

Phys. Rev. B **111**, L041105 — Published 9 January 2025

DOI: [10.1103/PhysRevB.111.L041105](https://doi.org/10.1103/PhysRevB.111.L041105)

Topological Triviality of Flat Hamiltonians

Pratik Sathe^{1,2,*} and Rahul Roy^{2,†}

¹*Theoretical Division, T-4, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

²*Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy,
University of California Los Angeles, Los Angeles, CA 90095*

(Dated: December 19, 2024)

Landau levels play a key role in theoretical models of the quantum Hall effect. Each Landau level is degenerate, flat and topologically non-trivial. Motivated by Landau levels, we study tight-binding Hamiltonians whose energy levels are all flat. We demonstrate that in two dimensions, for such Hamiltonians, the flat bands must be topologically trivial. To that end, we show that the projector onto each flat band is necessarily strictly local. Our conclusions do not need the assumption of lattice translational invariance.

Introduction.— Topological phases of matter are a cornerstone of modern condensed matter physics. Beginning with the discovery of the quantum Hall effects (QHE) [1, 2], topology has played an important role in understanding and predicting novel phases of matter ranging from topological insulators and superconductors to topologically ordered phases of matter [3, 4].

The Hall conductivity of a filled Landau level or a Bloch band is proportional to a topological invariant called the Chern number [5–7]. Much like Landau levels, a Bloch band that is flat or dispersionless and which has a non-zero Chern number can host fractional quantum Hall phases when subject to interactions. Such systems are known as fractional Chern insulators (FCIs) and hold significant experimental appeal since they have the potential to exhibit fractional QHE at small (or even zero) magnetic fields and high temperatures [8]. Consequently, a substantial amount of research has focused on the study of topological flat-band models (see Refs. [9–11] for reviews).

Hamiltonians with flat bands are also interesting because of numerous other phenomena they exhibit [12]. The effects of interactions are pronounced in flat-band Hamiltonians, and hence they serve as a platform to explore novel correlated phases of matter. In addition to FCIs, some prominent examples include unconventional superconductivity in bilayer graphene [13–16], flat-band superfluidity [17, 18] and flat-band ferromagnetism [19].

Various no-go theorems restrict the possibility of topological flat bands. Strict localization of a band projector, a condition more restrictive than band flatness, was shown to imply a vanishing Hall conductance [20]. Further, if a strictly local (SL) Hamiltonian has a flat band, then it must have a Chern number of zero [21, 22] (see also Ref. [23]). These results apply to Hamiltonians with lattice translational invariance (LTI). It has also been shown that local commuting projector Hamiltonians cannot exhibit the QHE [24, 25]. The non-interacting limit of such a model has a flat band which is spanned by

an orthogonal basis of compactly supported wavefunctions [26, 27].

Non-interacting Hamiltonians that exclusively have flat bands, which we call flat Hamiltonians or all-bands-flat Hamiltonians, have also attracted significant attention. Flat Hamiltonians can exhibit unconventional behavior when interactions are introduced, such as many-body localization without LTI [28, 29], non-linear caging [30, 31] and transport via two-particle bound states [32, 33]. Furthermore, the spectrum of a flat Hamiltonian is similar to Landau levels since all the energies are flat and degenerate. Consequently, topological flat Hamiltonians that have SL hoppings seem to be particularly promising as platforms for fractional QHE. Indeed, many of the initial proposals for FCIs were Hamiltonians with all bands being approximately flat [34–37].

Naively, no-go theorems from Refs. [21, 22] would suggest that flat Hamiltonians should be topologically trivial. However, flat Hamiltonians *can* be topological [38], in apparent contradiction to these theorems. We identify the exotic conditions required for such models to be topological. To that end, we prove some unique properties of SL flat Hamiltonians and establish a no-go theorem. First, we show that the projection operators associated with all the flat bands in a SL flat Hamiltonian are also SL, regardless of lattice dimension or LTI. Next, we show that the spectrum of such a Hamiltonian is unchanged with twisted boundary conditions. Using properties unique to SL projectors, we show that each flat band in a flat Hamiltonian has a Chern number of zero, regardless of LTI. The no-go theorem is stated below.

Theorem. *Consider a SL tight-binding Hamiltonian defined on a 2d system of size $L_x \times L_y$ cells with periodic boundary conditions. Let n denote the number of distinct energies and R denote the maximum hopping range of the Hamiltonian. If $L_{x,y} \geq 3nR$, then the Chern number associated with each energy is 0.*

Within the context of flat Hamiltonians with LTI, the inequality above is broadly satisfied. However, topological flat Hamiltonians must break the condition above (as exemplified by the models in Ref. [38]).

While our focus is on flat Hamiltonians, we emphasize that our conclusions go beyond flat Hamiltonians.

* psathe@lanl.gov

† rroy@physics.ucla.edu

Specifically, we also show that if a band or energy has a SL projector, then its Chern number is zero, independent of the flatness of the remaining bands.

Notation and Setup.— We consider 2d tight-binding Hamiltonians with orbitals $|\mathbf{r}, \alpha\rangle$ where $\mathbf{r} = (x, y)$ denotes cell position and $\alpha = 1, \dots, N$ denotes different orbitals in a unit cell. We will restrict our discussion to systems that lie on a torus (i.e. those that satisfy periodic boundary conditions). Without loss of generality, we consider square lattices of size $L_x \times L_y$ with lattice constant 1.

The concepts of strict localization will play an important role in our arguments. A wavefunction is compactly supported if it has non-zero support only on a finite region of the lattice. An operator $\mathcal{O}$ in an infinite system is said to be strictly local (SL) if $\langle \mathbf{r}, \alpha | \mathcal{O} | \mathbf{r}', \beta \rangle = 0$ (for any α, β) whenever $|\mathbf{r} - \mathbf{r}'| > R$ for some finite number R . We call the smallest value of R for which this is true the maximum hopping range of the operator. For finite sized systems, this condition is satisfied by every operator. Therefore, we propose the following definition for finite systems on a torus: an operator is called SL if it does not connect any two cells that are maximally separated from each other. This is equivalent to the conditions $2R + 2 \leq L_{x/y}$.

Let us now define flat Hamiltonians. We recall that in an infinite sized system, the spectrum of a Hamiltonian with LTI consists of a finite number of continuous bands following Bloch's theorem. The energy eigenvalues of a Hamiltonian on a finite system are necessarily discrete but are still arranged in the form of bands if the Hamiltonian has LTI. If a Hamiltonian with LTI has only flat bands, then we call it a flat Hamiltonian. Thus, each band has a degeneracy equal to the system size. In the case of Hamiltonians without LTI, we consider those that have a similar property of possessing highly degenerate energy levels. In a slight abuse of notation, we will refer to the distinct energy levels of such Hamiltonians as flat bands even in the absence of LTI. Regardless of LTI, all our results are derived for Hamiltonians and system sizes which satisfy $3nR \leq L_{x,y}$, where n denotes the number of distinct energies or flat bands, and R denotes the maximum hopping range of H [39].

Each energy eigenspace of a Hamiltonian may be associated with an orthogonal projector, i.e., a Hermitian operator P which satisfies $P^2 = P$. The locality properties of projectors associated with energy eigenspaces will form an important part of our subsequent discussion.

We will also find it useful to use a simple kind of dimensional reduction. Specifically, any 2d SL operator can always be regarded as a 1d SL operator by grouping together all the lattice sites in every row (or column) into a single 1d lattice site. For example, a 2d next-nearest-neighbor (NNN)-hopping projector P (i.e. one with $R = \sqrt{2}$) is also a nearest-neighbor (NN)-hopping projector (with $R = 1$) on a 1d lattice with one truncated dimension.

We will find it useful to define ‘hopping matrices’ for

1d SL projectors. Specifically, we define P_i^x for a SL projector P as follows:

$$P_i^x := \langle x + i | P | x \rangle. \quad (1)$$

The number of rows or columns in this matrix is equal to the number of orbitals in each cell. Such hopping matrices can also be defined for 2d SL projectors using the dimensional reduction described above.

Flat Hamiltonians have strictly local projectors.— We will now show that if a SL Hamiltonian is flat, then all the projectors are also SL. Furthermore, if the maximum hopping range of the Hamiltonian is R and the number of distinct energies is n , then the maximum hopping range of each projector is less than or equal to nR .

To that end, consider a Hamiltonian H with energy eigenvalues $E_1 < E_2 < \dots < E_n$. Let the corresponding projectors be denoted by $P_1, P_2, \dots, P_n$. Without loss of generality, all the energies can be chosen to be non-zero. Taking the m^{th} power of H , we get

$$H^m = \sum_{i=1}^n E_i^m P_i, \quad (2)$$

for all integers $m \geq 1$. This follows from the fact that $P_i P_j = P_j P_i = P_i \delta_{i,j}$ for all i, j .

For any m , H^m has a maximum hopping range of no more than mR . Therefore, each of $H, \dots, H^n$ has vanishing matrix elements between any two locations separated by a distance greater than nR . For concreteness, consider orbitals α and β at any two cells $\vec{r}_1$ and $\vec{r}_2$ respectively, such that $|\vec{r}_1 - \vec{r}_2| > nR$. Let $j \equiv (\vec{r}_1, \alpha)$ and $k \equiv (\vec{r}_2, \beta)$ label the orbitals. Equating the (j, k) matrix elements of both sides of equations (2) for $m = 1, \dots, n$ gives us

$$\underbrace{\begin{pmatrix} E_1 & E_2 & \dots & E_n \\ E_1^2 & E_2^2 & \dots & E_n^2 \\ \vdots & & & \vdots \\ E_1^n & E_2^n & \dots & E_n^n \end{pmatrix}}_M \begin{pmatrix} (P_1)_{j,k} \\ (P_2)_{j,k} \\ \vdots \\ (P_n)_{j,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3)$$

The only solution of (3) is the zero vector since

$$\det M = \prod_{i=1}^n E_i \prod_{1 \leq j < k \leq n} (E_j - E_k) \neq 0,$$

where the second product series is the Vandermonde determinant. Let p_i denote the maximum hopping range of projector P_i . We have thus shown that $p_i \leq nR$ for each i . We note that all the conclusions so far are valid independent of the system dimensionality and regardless of LTI, even though we will focus on 2d systems below.

For systems of infinite size and finite n , all the projectors are thus SL. However, for finite-sized systems, the projectors may not be SL (in the sense defined above for finite-sized systems) if nR is comparable to the system size. We note that $q := \max\{R, p_1, \dots, p_n\}$ is such that

$q \leq nR$. All the projectors are SL if $2q + 2 \leq L_{x/y}$. This is therefore guaranteed if $2nR + 2 \leq L_{x/y}$.

In the subsequent sections, the following observation will be useful. If we group the unit cells so that $q \times q$ primitive cells form a supercell, then the projectors and the Hamiltonian are NNN-hopping operators.

Spectrum is unchanged upon flux insertion.— The spectrum of a Hamiltonian generally changes when magnetic fluxes are threaded through the two holes of a torus as shown in Fig. 1. However, for flat Hamiltonians which have SL projectors (and which further satisfy $L_{x,y} \geq 3q$ and $L_{x,y} \geq 2q + 2$), the spectrum is unchanged even after threading any amount of flux, as shown below.

Let us first consider the case when only ϕ_x is inserted, so that H changes to $H(\phi_x)$ following Peierls substitution [40]. We choose a gauge for the substitution that corresponds to twisted boundary conditions (TBCs) [6], so that only those hopping elements that cross the boundary at $x = L_x$ are modified. In other words, every hopping elements that connects a cell with an x -coordinate in $\{L_x - R + 1, \dots, L_x\}$ to a cell with an x -coordinate in $\{1, 2, \dots, R\}$ is multiplied by a phase of $e^{-2\pi i \phi_x / \Phi_0}$, while hoppings in the reverse direction get multiplied by $e^{2\pi i \phi_x / \Phi_0}$. ($\Phi_0 = h/e$ is the magnetic flux quantum.) The rest of the hopping elements are unchanged in the substitution.

Similarly, we can implement TBCs for all the P_i as well, if they are SL. Denoting the post-transformation operators by $P_i(\phi_x)$ s, we note that

$$H(\phi_x) = \sum_{i=1}^N E_i P_i(\phi_x). \quad (4)$$

While it is not immediately obvious, this equation is actually a spectral decomposition of $H(\phi_x)$ when $L_x \geq 3q$ [41].

This fact can also be used to obtain a spectral decomposition for the Hamiltonian $H(\phi_x, \phi_y)$, the Hamiltonian with both the fluxes are inserted, if $L_y \geq 3q$ and $L_y \geq 2q + 2$. The key idea is to treat $H(\phi_x)$ as the ‘initial Hamiltonian’ and repeat the steps above. Specifically, we obtain $H(\phi_x, \phi_y)$ by applying TBCs to $H(\phi_x)$ along $\hat{y}$. Transforming the projectors similarly, so that $P_i(\phi_x) \rightarrow P_i(\phi_x, \phi_y)$, we get

$$H(\phi_x, \phi_y) = \sum_{i=1}^N E_i P_i(\phi_x, \phi_y). \quad (5)$$

Applying the proof from the Supplementary Material [41] again, we conclude that (5) is also a spectral decomposition. Thus, flat Hamiltonians have the interesting property of having an unchanging spectrum if $L_{x,y} \geq 2q + 2$ and $L_{x,y} \geq 3q$. Both these conditions are satisfied if $L_{x,y} \geq 3nR$ and $L_{x,y} \geq 2nR + 2$.

Many-Body Chern Number.— We now turn to the computation of the Chern number. Let us recall that in the absence of interactions, the transverse conductivity

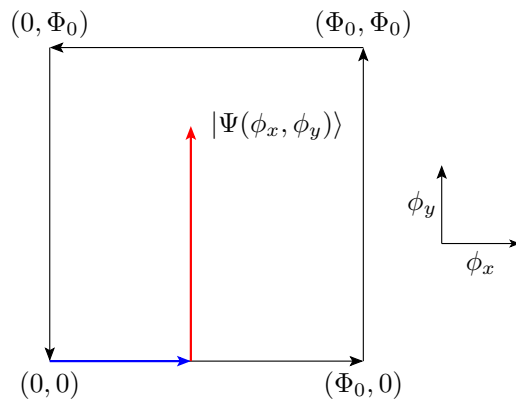


FIG. 1. Periodic boundary conditions are imposed on the system, so that it lies on a torus. Two magnetic fluxes ϕ_x and ϕ_y are then inserted through the two holes of the torus.

of a system with a Hamiltonian that has LTI is proportional to the TKNN invariant [5], i.e. the Chern number of the occupied Bloch bands. More generally, the transverse conductivity is proportional to the Chern number of the (gapped) many-body ground state as a function of twist angles or equivalently, the fluxes $\phi_{x,y}$ [6]. This many-body Chern number is expressed as

$$C = \frac{1}{2\pi} \int_0^{\Phi_0} \int_0^{\Phi_0} d\phi_x d\phi_y \nabla \times i \langle \Psi | \nabla_{\phi} \Psi \rangle, \quad (6)$$

where $|\Psi(\phi_x, \phi_y)\rangle$ is the many-body ground state of $H(\phi_x, \phi_y)$. Even for finite systems, the transverse conductivity has been shown to be proportional to this Chern number up to exponentially small errors in the system size [42, 43]. We can straightforwardly generalize (6) in order to define a Chern number for an isolated flat band or energy in a flat Hamiltonian. Specifically, we use expression Eq. (6) but with $|\Psi\rangle$ taken to be the ‘flat band wavefunction’, i.e. the many-body wavefunction corresponding to only the flat band states being fully occupied, with the rest of the energies being unoccupied. We will henceforth use this definition.

Many-body wavefunctions as a function of the fluxes.— Let us now compute the Chern number of a flat band with energy E and projector P . We assume that $L_{x,y} \geq 3nR$ so that the spectrum of a flat Hamiltonian is unaffected by the fluxes as shown above. Thus, for any ϕ_x, ϕ_y , while the state of the system corresponding to that energy level being fully filled is unique, the corresponding wavefunction $|\Psi(\phi_x, \phi_y)\rangle$ has a phase ambiguity. We will now provide a prescription that fixes the phase to obtain a global expression for $|\Psi(\phi_x, \phi_y)\rangle$ that can then be used to compute the Chern number.

We will start with a brief outline of our approach. First, we obtain a particular type of orthogonal basis of single-particle wavefunctions spanning the energy E subspace when no fluxes are inserted. The Slater determinant then yields $|\Psi\rangle$ at zero flux. Next, we consider the case when only ϕ_x is inserted, so that P trans-

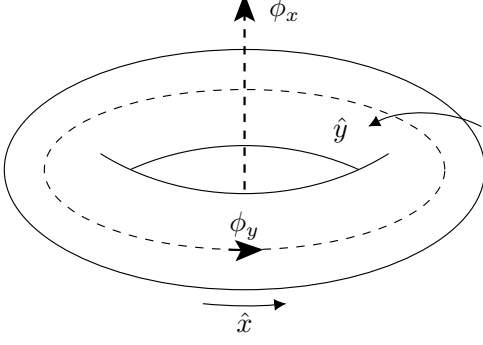


FIG. 2. The (ϕ_x, ϕ_y) parameter space with $\phi_x, \phi_y \in [0, \Phi_0]$. For any pair of values (ϕ_x, ϕ_y) , wavefunction $|\Psi(\phi_x, \phi_y)\rangle$ is obtained in two steps.

forms to $P(\phi_x)$. We show that wavefunctions obtained for $\phi_x = 0$, when modified appropriately, are eigenstates of $P(\phi_x)$ and thus span E at flux ϕ_x . The corresponding Slater determinant yields $|\Psi(\phi_x)\rangle$, which by construction is shown to be a smooth function of ϕ_x . In the last step, we insert flux ϕ_y in addition to ϕ_x . Specifically, for each value of ϕ_x , we follow a similar procedure to obtain a smooth family of wavefunctions as a function of ϕ_y . This finally results in a global expression for $|\Psi(\phi_x, \phi_y)\rangle$.

For the remainder of the manuscript we will work within the size $q \times q$ supercell representation described previously. Thus, H and all P_i s are then NNN operators in this representation, and we have a supercell grid of a size of at least 3×3 since we assumed $L_{x,y} \geq 3nR$. The wavefunctions obtained in this representation can always be expressed in the original primitive cell representation at the end.

The first step involves obtaining the flat band many-body wavefunction before inserting ϕ_x . First, we demote the y index to an orbital index so that P can be regarded as a NN projector on a 1d lattice with position coordinate x . Next, we will use the property that the image of a 1d SL projector can always be spanned by an orthogonal basis of compactly-supported wavefunctions [26, 27]. Specifically, we use the construction of generalized Wannier functions presented in Ref. [27] for 1d NN projectors. The wavefunctions so obtained are of two types: ‘monomers’ and ‘dimers’. A monomer has support on only one cell, while a dimer has support only on two consecutive cells. (In the original 2d picture, each such wavefunction is strictly localized along $\hat{x}$ but is possibly delocalized along $\hat{y}$.)

Monomer-dimer bases have an important property which we call the ‘support property’. Specifically, whenever two wavefunctions from such a basis have non-zero support at a common cell, say x_0 , then their supports at x_0 are also mutually orthogonal.

We note further that a monomer-dimer bases are not

unique. For instance, multiplying any monomer or dimer in a given basis by a phase itself results in a different basis. Less trivial possibilities arise whenever any P_1^x hopping matrix [defined in (1)] has degenerate singular values; see Ref. [27]. However, for our purpose, it suffices to choose any valid monomer-dimer basis for P . For convenience, let us denote the chosen basis by a set $\mathcal{B}$. We obtain the flat band many-body wavefunction $|\Psi\rangle$ at zero flux by computing the Slater determinant of $\mathcal{B}$.

Next, we insert flux ϕ_x so that P now transforms to $P(\phi_x)$. The hopping matrices of P that connect $x = 0$ and $x = 1$ get transformed, with the rest being unaffected. The substitution can be summarized as:

$$P_1^0 \xrightarrow{\phi_x} e^{-2\pi i \phi_x / \Phi_0} P_1^0 \quad (7)$$

and $P_{-1}^1 \xrightarrow{\phi_x} e^{2\pi i \phi_x / \Phi_0} P_{-1}^1$.

While $\mathcal{B}$ is not an orthonormal basis for the image of $P(\phi_x)$, a simple modification of it does span the image. Consider two subsets of $\mathcal{B}$, namely $\mathcal{C}$ which consists only of those dimers that have support at both $x = 0$ and $x = 1$, and $\mathcal{U}$ that consists of all other wavefunctions from $\mathcal{B}$, so that $\mathcal{B} = \mathcal{U} \cup \mathcal{C}$. It is easy to see that every element of $\mathcal{U}$ is also an eigenvector of $P(\phi_x)$ (with an eigenvalue of 1), since $P(\phi_x)$ is the same as P except for hopping elements between $x = 0$ and $x = 1$. However, the elements of $\mathcal{C}$ are not eigenvectors of $P(\phi_x)$. They can however be modified as follows. Consider a wavefunction $|w\rangle \in \mathcal{C}$. Let

$$|w\rangle = \alpha |x=0, \psi_0\rangle + \beta |x=1, \psi_1\rangle, \quad (8)$$

for some support vectors $|\psi_{0,1}\rangle$. We now define $|w_{\phi_x}\rangle$ as follows:

$$|w_{\phi_x}\rangle := \alpha |x=0, \psi_0\rangle + e^{-i2\pi \phi_x / \Phi_0} \beta |x=1, \psi_1\rangle. \quad (9)$$

Using (7) and the fact that $P|w\rangle = |w\rangle$, we conclude that $P(\phi_x)|w_{\phi_x}\rangle = |w_{\phi_x}\rangle$. Let us denote by $\mathcal{C}_{\phi_x}$ the set obtained after modifying every element of $\mathcal{C}$ in this way, and define $\mathcal{B}_{\phi_x} := \mathcal{U} \cup \mathcal{C}_{\phi_x}$. Then, using the support property of monomer-dimer bases, we furthermore conclude that $\mathcal{B}_{\phi_x}$ is an orthogonal basis (and actually a monomer-dimer basis) for the image of $P(\phi_x)$. For any value of ϕ_x , the Slater determinant of $\mathcal{B}_{\phi_x}$ thus yields $|\Psi(\phi_x)\rangle$. Clearly, $|\Psi(\phi_x)\rangle$ is a smooth function of ϕ_x . Furthermore, since $\mathcal{B}_{\phi_x=\Phi_0}$ is the same as $\mathcal{B}_{\phi_x=0}$, we also find that $|\Psi(\phi_x=0)\rangle = |\Psi(\phi_x=\Phi_0)\rangle$.

Having obtained $|\Psi(\phi_x)\rangle$, we will now discuss the last step, i.e. obtaining the many-body wavefunction when ϕ_y is also inserted. To that end, for any value of ϕ_x , we first set $H(\phi_x)$ to be the ‘initial’ flat Hamiltonian and then apply the entire procedure discussed above, but for the flux ϕ_y . [We thus obtain monomer-dimer states for each band projector $P_i(\phi_x, \phi_y)$, with the states being strictly localized along $\hat{y}$.] The Slater determinant of these wavefunctions, denoted $|\tilde{\Psi}_{\phi_x}(\phi_y)\rangle$, corresponds to the flat band at those flux values. Since the many-body state corresponding to the flat band is unique for any flux values, $|\Psi(\phi_x)\rangle$

and $|\tilde{\Psi}_{\phi_x}(\phi_y = 0)\rangle$ must be the same up to a phase, so that $|\Psi(\phi_x)\rangle = e^{i\theta_{\phi_x}}|\tilde{\Psi}_{\phi_x}(\phi_y = 0)\rangle$ for some $\theta_{\phi_x} \in \mathbb{R}$. Finally we define $|\Psi(\phi_x, \phi_y)\rangle := e^{i\theta_{\phi_x}}|\tilde{\Psi}_{\phi_x}(\phi_y)\rangle$.

We find that by construction, for all values of ϕ_x and ϕ_y ,

$$\begin{aligned} |\Psi(\phi_x, 0)\rangle &= |\Psi(\phi_x, \Phi_0)\rangle, \\ \text{and } |\Psi(0, \phi_y)\rangle &= |\Psi(\Phi_0, \phi_y)\rangle. \end{aligned} \quad (10)$$

Using Stokes' theorem, we can write (6) as

$$C = \frac{1}{2\pi} \oint_{\Gamma} i \langle \Psi | \nabla_{\phi} \Psi \rangle \cdot d\phi, \quad (11)$$

where Γ is the boundary $(0, 0) \rightarrow (\Phi_0, 0) \rightarrow (\Phi_0, \Phi_0) \rightarrow (0, \Phi_0) \rightarrow (0, 0)$.

From (10), we conclude that $C = 0$. We note that the proof of the Chern triviality also extends to an energy that has a SL projector, regardless of the dispersiveness of the rest of the spectrum, a notion that is not always clear or applicable in the absence of LTI.

Conclusions.— We showed that all bands of a two dimensional strictly local (SL) flat Hamiltonian have a Chern number of zero. To that end, we showed that each band in such a system is described by a SL projector. We also showed that the spectrum of such Hamiltonians is unchanged even after flux threading in a toroidal geometry. We demonstrated all our results without

the requirement of lattice-translational invariance (LTI) thereby going beyond existing no-go theorems concerning flat bands. Furthermore, we clarified the role of a subtle condition that allows for a topological flat Hamiltonian in finite systems.

An important step of our proof involved showing that each band projector is strictly local. Since projection operators are equal-time Green's functions in non-interacting systems, generalizations to interacting cases might be possible. Our no-go theorem was proven for system sizes that are greater than a lower bound expressible in terms of the number of energies and maximum hopping range of the Hamiltonian. It would be interesting to obtain an improved lower bound. Due to the topology-localization correspondence, the existence of localized Wannier functions follows for flat Hamiltonians. Exploring no-go theorems in the presence of crystallographic symmetries, such as with fragile topology [44], would constitute an interesting direction for future work.

Acknowledgements.— We thank Adrian Culver for providing helpful comments on the manuscript. P.S. and R.R. acknowledge financial support from the University of California Laboratory Fees Research Program funded by the UC Office of the President (UCOP), grant number LFR-20-653926. P.S. acknowledges financial support from the Center for Quantum Science and Engineering Fellowship (UCLA) and the Bhaumik Graduate Fellowship (UCLA).

-
- [1] K. v. Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance, *Physical Review Letters* **45**, 494 (1980).
 - [2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Two-dimensional magnetotransport in the extreme quantum limit, *Physical Review Letters* **48**, 1559 (1982).
 - [3] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, *Reviews of Modern Physics* **82**, 3045 (2010).
 - [4] X.-G. Wen, Colloquium: Zoo of quantum-topological phases of matter, *Reviews of Modern Physics* **89**, 041004 (2017).
 - [5] D. J. Thouless, M. Kohmoto, M. Nightingale, and M. Den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, *Phys. Rev. Lett.* **49**, 405 (1982).
 - [6] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized Hall conductance as a topological invariant, *Physical Review B* **31**, 3372 (1985).
 - [7] J. E. Avron and R. Seiler, Quantization of the hall conductance for general, multiparticle schrödinger hamiltonians, *Physical Review Letters* **54**, 259 (1985).
 - [8] Y. Xie, A. T. Pierce, J. M. Park, D. E. Parker, E. Khalaf, P. Ledwith, Y. Cao, S. H. Lee, S. Chen, P. R. Forrester, K. Watanabe, T. Taniguchi, A. Vishwanath, P. Jarillo-Herrero, and A. Yacoby, Fractional Chern insulators in magic-angle twisted bilayer graphene, *Nature* **600**, 439 (2021).
 - [9] S. A. Parameswaran, R. Roy, and S. L. Sondhi, Fractional quantum Hall physics in topological flat bands, *Comptes Rendus Physique* **14**, 816 (2013).
 - [10] E. J. Bergholtz and Z. Liu, Topological flat band models and fractional Chern insulators, *International Journal of Modern Physics B* **27**, 1330017 (2013).
 - [11] Z. Liu and E. J. Bergholtz, Recent developments in fractional chern insulators, in *Reference Module in Materials Science and Materials Engineering* (Elsevier, 2023).
 - [12] D. Leykam, A. Andreanov, and S. Flach, Artificial flat band systems: from lattice models to experiments, *Advances in Physics: X* **3**, 1473052 (2018).
 - [13] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, *Nature* **556**, 43 (2018).
 - [14] G. E. Volovik, Graphite, Graphene, and the Flat Band Superconductivity, *JETP Letters* **107**, 516 (2018).
 - [15] X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, Superconductors, orbital magnets and correlated states in magic-angle bilayer graphene, *Nature* **574**, 653 (2019).

- [16] M. Yankowitz, S. Chen, H. Polshyn, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, Tuning superconductivity in twisted bilayer graphene, *Science* **363**, 1059 (2019).
- [17] S. Peotta and P. Törmä, Superfluidity in topologically nontrivial flat bands, *Nature Communications* **6**, 8944 (2015).
- [18] F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Topology-Bounded Superfluid Weight in Twisted Bilayer Graphene, *Physical Review Letters* **124**, 167002 (2020).
- [19] H. Tasaki, From Nagaoka's Ferromagnetism to Flat-Band Ferromagnetism and Beyond: An Introduction to Ferromagnetism in the Hubbard Model, *Progress of Theoretical Physics* **99**, 489 (1998).
- [20] R. Bezrukavnikov and A. Kapustin, Localization Properties of Chern Insulators, *Arnold Mathematical Journal* **5**, 15 (2019).
- [21] L. Chen, T. Mazaheri, A. Seidel, and X. Tang, The impossibility of exactly flat non-trivial Chern bands in strictly local periodic tight binding models, *Journal of Physics A: Mathematical and Theoretical* **47**, 152001 (2014).
- [22] N. Read, Compactly supported Wannier functions and algebraic K-theory, *Phys. Rev. B* **95**, 115309 (2017).
- [23] J.-W. Rhim and B.-J. Yang, Classification of flat bands according to the band-crossing singularity of Bloch wave functions, *Physical Review B* **99**, 045107 (2019).
- [24] A. Kapustin and L. Fidkowski, Local Commuting Projector Hamiltonians and the Quantum Hall Effect, *Communications in Mathematical Physics* **373**, 763 (2020).
- [25] C. Zhang, M. Levin, and S. Bachmann, Vanishing Hall conductance for commuting Hamiltonians, *Physical Review B* **105**, L081103 (2022), <https://link.aps.org/doi/10.1103/PhysRevB.105.L081103>.
- [26] P. Sathe, F. Harper, and R. Roy, Compactly supported Wannier functions and strictly local projectors, *Journal of Physics A: Mathematical and Theoretical* **54**, 335302 (2021).
- [27] P. Sathe and R. Roy, Compact Wannier functions in one dimension, *Physical Review B* **109**, 155150 (2024), <https://link.aps.org/doi/10.1103/PhysRevB.109.155150>.
- [28] C. Danieli, A. Andreanov, and S. Flach, Many-body flatband localization, *Physical Review B* **102**, 041116 (2020).
- [29] Y. Kuno, T. Orito, and I. Ichinose, Flat-band many-body localization and ergodicity breaking in the Creutz ladder, *New Journal of Physics* **22**, 013032 (2020).
- [30] C. Danieli, A. Andreanov, T. Mithun, and S. Flach, Nonlinear caging in all-bands-flat lattices, *Physical Review B* **104**, 085131 (2021).
- [31] G. Gligorić, P. P. Beličev, D. Leykam, and A. Maluckov, Nonlinear symmetry breaking of Aharonov-Bohm cages, *Physical Review A* **99**, 013826 (2019).
- [32] M. Tovmasyan, S. Peotta, L. Liang, P. Törmä, and S. D. Huber, Preformed pairs in flat Bloch bands, *Physical Review B* **98**, 134513 (2018).
- [33] G. Pelegrí, A. M. Marques, V. Ahufinger, J. Mompart, and R. G. Dias, Interaction-induced topological properties of two bosons in flat-band systems, *Physical Review Research* **2**, 033267 (2020).
- [34] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Fractional Quantum Hall States at Zero Magnetic Field, *Physical Review Letters* **106**, 236804 (2011).
- [35] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Nearly Flatbands with Nontrivial Topology, *Physical Review Letters* **106**, 236803 (2011).
- [36] E. Tang, J.-W. Mei, and X.-G. Wen, High-Temperature Fractional Quantum Hall States, *Physical Review Letters* **106**, 236802 (2011).
- [37] F. Wang and Y. Ran, Nearly flat band with Chern number $C=2$ on the dice lattice, *Physical Review B* **84**, 241103 (2011).
- [38] T. Scaffidi and S. H. Simon, Exact solutions of fractional Chern insulators: Interacting particles in the Hofstadter model at finite size, *Physical Review B* **90**, 115132 (2014).
- [39] We also require that the Hamiltonian satisfy $L_{x,y} \geq 2nR + 2$ to ensure that all the projectors are SL. This is always true when $L_{x,y} \geq 3nR$ (except if nR is 0 or 1).
- [40] R. Peierls, Zur Theorie des Diamagnetismus von Leitungselektronen, *Zeitschrift für Physik* **80**, 763 (1933).
- [41] See Supplemental Material at [URL will be inserted by publisher].
- [42] M. B. Hastings and S. Michalakis, Quantization of Hall Conductance for Interacting Electrons on a Torus, *Communications in Mathematical Physics* **334**, 433 (2015).
- [43] K. Kudo, H. Watanabe, T. Kariyado, and Y. Hatsugai, Many-Body Chern Number without Integration, *Physical Review Letters* **122**, 146601 (2019).
- [44] H. C. Po, H. Watanabe, and A. Vishwanath, Fragile Topology and Wannier Obstructions, *Physical Review Letters* **121**, 126402 (2018).