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Filling constraints on translation invariant dipole conserving systems

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Systems with conserved dipole moment have drawn considerable interest in light of their realization in recent experiments on tilted optical lattices. An important question for such systems is delineating the conditions under which they admit a unique gapped ground state that is consistent with all symmetries. Here, we study one-dimensional translation-invariant lattices that conserve $U(1)$ charge and $\mathbb{Z}_L$ dipole moment, where discreteness of the dipole symmetry is enforced by periodic boundary conditions, with L the system size. We show that in these systems, a symmetric, gapped, and non-degenerate ground state requires not only integer charge filling, but also a fixed value of the dipole filling, while other fractional dipole fillings enforce either a gapless or symmetry-breaking ground state. In contrast with prior results in the literature, we find that the dipole filling constraint depends both on the charge filling as well as the system size, emphasizing the subtle interplay of dipole symmetry with boundary conditions. We support our results with numerical simulations and exact results.

Introduction: Quantum many-body systems that conserve charge as well as higher multipole moments are an exciting frontier in the study of strongly correlated quantum matter. While the multipolar (or polynomial shift) symmetries that result in higher moment conservation laws recently elicited interest given their close kinship with “fractonic” phases [1, 2], the intriguing phenomenology that results from such symmetries has endowed them with a life of their own in condensed matter, high energy, and quantum information [3–9]. Recent experimental realizations of dipole conserving dynamics (up to exponentially long prethermal timescales) [10–13] has further fueled the burgeoning interest in understanding the ground state(s) of dipole conserving quantum systems [14–22].

In dipole conserving systems, single particle hopping is forbidden, and dynamics occurs through correlated hopping processes involving two (or more) particles. This leads to striking dynamical behavior, including anomalously slow dynamics [23–27], Hilbert space fragmentation [28–34], and unconventional hydrodynamic universality classes [35–41]. However, the inherently interacting nature of correlated hopping makes determining the ground state physics of *generic* dipole conserving systems challenging. Further complicating matters, many 1d dipole conserving models studied have extra unconventional symmetries (due to fragmentation [30, 33]), that can significantly alter the nature of the ground states [30, 31].

It is thus desirable to obtain non-perturbative results constraining the low-energy spectrum, which rely only on the manner in which microscopic symmetries act on the many-body Hilbert space and not on specifics of

any Hamiltonian. One such result is the Lieb-Schultz-Mattis (LSM) theorem [42] which, along with its generalizations [43–46], provides an example of minimal microscopic data constraining the universal long-wavelength physics of lattice spin systems. LSM-type theorems now exist for various internal and spatial symmetries [47–63]; the role of spatial symmetries in these theorems underlies their deep connection with crystalline symmetry protected topological (cSPT) phases [64–77]. For example, in charge-conserving translation invariant 1d systems, a gapped, symmetric, and non-degenerate ground state is permitted only when the filling per unit cell is an integer¹, which labels distinct “weak” SPT phases that are distinct from the trivial phase only in the presence of translation symmetry [64].

Establishing similar constraints is subtle for multipole symmetries, which intertwine non-trivially with spatial symmetries, and *cannot* be considered internal symmetries [2]. Crucially, the symmetry itself depends sensitively on boundary conditions: in 1d translation invariant systems, dipole symmetry is a continuous $U(1)$ symmetry on an infinite lattice but a discrete $\mathbb{Z}_L$ symmetry on a closed L -site ring [5], a distinction which was ignored in earlier considerations of LSM constraints for dipole symmetry [79, 80].² Here, we show that this distinction must be carefully accounted for, and we obtain a non-perturbative filling constraint on the ground state(s) of 1d dipole conserving, translation invariant systems with

¹ We emphasize that the filling constraint is not an anomaly [78].

² This subtlety is absent for subsystem symmetries, for which Refs. [79, 80] also derived LSM constraints.

periodic boundary conditions (PBC). For bosonic/integer spin systems at integer charge filling we show that a symmetric, gapped, and non-degenerate ground state is permitted at a single dipole filling (whose precise value depends on the charge filling and system size), where in PBC the dipole filling is well-defined only modulo an integer. At other dipole fillings, the ground state must be either gapless or symmetry-breaking. We establish similar constraints for fermionic/half-integer spin systems, and support our claims through exact solutions and numerical analysis.

Global symmetries: We consider arbitrary 1d lattice systems with local interactions invariant under the following symmetries: $\mathbb{Z}_L^T$ lattice translation, $U(1)_c$ charge conservation, and $\mathbb{Z}_L^D$ discrete dipole symmetry. The dipole symmetry is forced to be discrete here since we consider a finite lattice with L sites (where L is arbitrarily large) with PBC, in contrast to previous works that treat it as a continuous $U(1)$ symmetry.

The operator $\hat{Q} = \sum_{j=1}^L \hat{n}_j$ measures the total $U(1)_c$ charge, where $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ is the on-site number operator and the creation/annihilation operators $\hat{b}_j^\dagger, \hat{b}_j$ obey canonical commutation (anti-commutation) relations for bosonic (fermionic) systems. For spin- s degrees of freedom (d.o.f.'s), $\hat{Q} = \sum_{j=1}^L \hat{S}_j^z$, where $\hat{S}_j^z$ is the spin- s operator at site j . The global $U(1)_c$ symmetry is generated by $e^{i\alpha\hat{Q}}$, which sends $\hat{b}_j \rightarrow e^{i\alpha}\hat{b}_j$ under conjugation; $\alpha \in [0, 2\pi)$ is a continuous periodic variable due to charge conservation. $\hat{T} = e^{ia\hat{P}}$ is the generator of translation symmetry, which conjugates $\hat{b}_j \rightarrow \hat{b}_{j+1}$, where $\hat{P}$ is the total momentum operator and a is the lattice constant (which we set equal to 1).

On the translation invariant infinite chain, the dipole operator is defined as $\hat{Q}_D = \sum_{j=1}^L j\hat{n}_j$, leading to the $U(1)$ dipole symmetry considered in Refs. [79, 80] i.e., the symmetry of the form $e^{i\beta\hat{Q}_D}$ with $\beta \in [0, 2\pi)$ (analogously, for spins, $\hat{Q}_D = \sum_{j=1}^L j\hat{S}_j^z$). Hereinafter we restrict to $s \in \mathbb{Z}$, with the half-integer case discussed in the Supplemental Material (SM) [81] (see also Refs. [82–92] therein). On a finite ring, however, $\hat{Q}_D$ is incompatible with PBC, required for translation invariance, and the symmetry generator for the $\mathbb{Z}_L^D$ dipole symmetry is:

$$\mathbb{Z}_L^D \text{ dipole : } \hat{D} = e^{i\frac{2\pi}{L}\hat{Q}_D}. \quad (1)$$

Under conjugation by $\hat{D}$, we have $\hat{b}_j \rightarrow e^{i\frac{2\pi}{L}j}\hat{b}_j$. Hence only operators $e^{i\beta\hat{Q}_D}$ with β a multiple of $2\pi/L$ are symmetry operators: this constraint comes from demanding single-valuedness of this phase rotation under translation by L for all $U(1)_c$ sectors; hence the “dipole charge” Q_D is well-defined only modulo L . We emphasize that the discreteness of dipole symmetry is enforced by the boundary conditions, and β would be quantized even for a dis-

ordered dipole conserving system on a finite ring [31].³ There is an origin-dependence of the dipole operator and its filling fraction, and here we fix it to a lattice site, which at integer charge filling fixes the dipole filling fraction modulo 1. Our main result, however, is independent of this choice [81].

Symmetry algebra: While the charge $\hat{Q}$ commutes with $\hat{T}$, the dipole $\hat{D}$ obeys the algebra $\hat{T}\hat{D} = \exp(-i2\pi\hat{Q}/L)\hat{D}\hat{T}$ [31, 93]. We restrict our attention to systems with fixed charge so that $\hat{Q}$ can be replaced by its eigenvalue Q ; $\hat{D}$ and $\hat{T}$ then satisfy:

$$\hat{T}\hat{D} = \exp(-i2\pi\nu)\hat{D}\hat{T}, \quad (2)$$

where $\nu = Q/L$ is the charge filling fraction.

At fixed integer filling, the $\hat{T}$ and $\hat{D}$ commute, and the full symmetry group is $G = \mathbb{Z}_L^T \times \mathbb{Z}_L^D \times U(1)_c$. In contrast, at fixed *fractional* charge filling, the symmetry group is $G_\nu = (\mathbb{Z}_L^T \times \mathbb{Z}_L^D) \rtimes_\nu U(1)_c$ where $\rtimes_\nu$ indicates that the symmetry algebra is a central extension⁴ of $\mathbb{Z}_L^T \times \mathbb{Z}_L^D$ by the $U(1)_c$ filling fraction. The resemblance with the magnetic translation group [94, 95] in charge conserving (i.e., $U(1)_c$ symmetric) systems is not accidental; the thin-torus limit of 2d fractional quantum Hall systems results in 1d dipole conserving models [96–99].

A symmetric, gapped, and unique ground state is not permitted at fractional charge filling since the dipole algebra Eq. (2) does not admit one-dimensional representations when $\nu \notin \mathbb{Z}$. In fact, *every* translation invariant eigenstate of any dipole conserving Hamiltonian can be shown to be (at least) q -fold degenerate for $\nu = p/q$ (p, q co-prime) [93]. As a corollary, we obtain the constraint that a symmetric, gapped, and non-degenerate ground state is only allowed at integer charge fillings $\nu \in \mathbb{Z}$, where for gapped states the different integer labels correspond to distinct weak SPT phases in 1D protected by $U(1)_c$ and $\mathbb{Z}_L^T$ [61, 64]. For half-integer spins, the constraint is modified: a unique, symmetric, and gapped ground state can only be obtained when $\nu \in \mathbb{Z} + 1/2$ [81].

Given the above charge filling constraints, it is natural to ask whether the *dipole* filling ν_D , defined as the dipole moment per unit cell, constrains the low-energy physics and leads to observable physical consequences. Note that arguments based on adiabatically threading dipolar flux cannot be directly applied here due to discreteness of the dipole symmetry. With PBC, the dipole filling is only defined modulo 1 due to the exponentiated form of the dipole operator $\hat{D}$ in Eq. (1), i.e., its eigenvalues are $e^{i2\pi\nu_D}$. To clearly delineate the role of dipole

³ Though continuous dipole symmetry could emerge in the $Q = 0$ sector [80], we consider systems with exact dipole symmetry here.

⁴ Since the commutator between $\hat{T}$ and $\hat{D}$ formally involves an operator, the symmetry group is not centrally extended when the total charge is *not* fixed [5]; we do not consider this here.

filling, we will henceforth work at fixed integer charge filling $\nu \in \mathbb{Z}$, where there are no charge filling constraints, and show that the answer to the above is affirmative.

Background dipole gauge fields: To study the constraints imposed by dipole symmetry, we couple the system to background gauge fields on the lattice (see SM [81] for the continuum limit) and consider the effect of a large gauge transformation of the dipole symmetry. We promote the global symmetry transformation Eq. (1) to a local one: $\hat{b}_j \rightarrow e^{i\alpha_j} \hat{b}_j$. Under such an arbitrary, site-dependent local phase rotation, the minimal-ranged charge and dipole conserving kinetic term $\hat{b}_{j-1}(\hat{b}_j^\dagger)^2 \hat{b}_{j+1}$ ($\hat{S}_{j-1}^-(\hat{S}_j^+)^2 \hat{S}_{j+1}^-$ for spins with $s \geq 1$) transforms as:

$$\hat{b}_{j-1}(\hat{b}_j^\dagger)^2 \hat{b}_{j+1} \rightarrow e^{-i(2\alpha_j - \alpha_{j+1} - \alpha_{j-1})} \hat{b}_{j-1}(\hat{b}_j^\dagger)^2 \hat{b}_{j+1}, \quad (3)$$

and similarly for its Hermitian conjugate.⁵

To render the theory gauge invariant, each minimal term is coupled to a single background gauge field A_j , leading to the gauge invariant cluster hopping term $e^{-iA_j} \hat{b}_{j-1}(\hat{b}_j^\dagger)^2 \hat{b}_{j+1}$, where $A_j \equiv A_j + 2\pi$ is a compact gauge field that transforms under gauge transformations as $A_j \rightarrow A_j - 2\alpha_j + \alpha_{j+1} + \alpha_{j-1}$, cancelling the phase shift in Eq. (3). For a Hamiltonian that is not simply a sum of minimal terms, the minimal coupling procedure is discussed in the SM [81]. Since $\alpha_{j+L} - \alpha_j \in 2\pi\mathbb{Z}$, the Wilson line $\hat{W} = e^{i\phi}$ is gauge-invariant, with $\phi = \sum_j A_j$ the net phase accrued by a dipole around the circle.⁶

We will be interested in large gauge transformations which shift $\phi \rightarrow \phi + 2\pi$. Phase rotations of the form $\alpha_j = a_0 + ja_1$ generate the global $U(1)_c$ (via a_0) and $\mathbb{Z}_L^D$ symmetries (via a_1) respectively, and do not change the value of the gauge field. The large gauge transformation thus involves terms quadratic in j ⁷:

$$\alpha_j = \frac{j(j-L)}{2} a_2, \quad a_2 \in \frac{2\pi}{L} \mathbb{Z}. \quad (4)$$

Requiring $\alpha_{j+L} = \alpha_j \pmod{2\pi}$ forces $a_2 \in (2\pi/L)\mathbb{Z}$. The gauge transformation Eq. (4) shifts $A_j \rightarrow A_j + a_2$ (preserving translation invariance of the gauge field), and changes $\delta\phi = -La_2 = -2\pi$ (for the minimal $a_2 = 2\pi/L$).

Large gauge transformations and translations: We now study the interplay between large gauge transformations and the generator of translation symmetry. Since the operator $\exp(i \sum_j \hat{n}_j \alpha_j)$ generates the phase rotation

$\hat{b}_k \rightarrow e^{i\alpha_k} \hat{b}_k$, the gauge transformations (4) are generated by the flux insertion operator

$$\hat{O} = \exp \left(i \sum_{j=1}^L \frac{j(j-L)}{2} a_2 \hat{n}_j \right); \quad a_2 = \frac{2\pi m}{L} \quad (5)$$

where for spins, we set $\hat{n}_j \rightarrow \hat{S}_j^z$. Being a large gauge transformation, this operator is unitary, and does not change the many-body spectrum.

A striking feature of the large gauge transformations Eq. (5) in dipole conserving models is their non-trivial commutation relation with translation symmetry at general charge and dipole fillings:

$$\hat{T} \hat{O} = \exp \left(\frac{i\pi m}{L} [-2\hat{Q}_D + \hat{Q}(1+L)] \right) \hat{O} \hat{T}, \quad (6)$$

Evaluated on a translation invariant state with fixed integer charge filling $\nu \in \mathbb{Z}$, we see that the flux insertion changes the many-body momentum unless

$$-\nu_D + \nu \frac{L+1}{2} \in \mathbb{Z}. \quad (7)$$

At even integer charge filling ν , this condition requires an integer dipole filling fraction ν_D ; at odd integer ν it requires half-integer (integer) ν_D when L is even (odd).

Eq. (7) constrains the dipole filling fractions at which 1d dipole conserving, translation invariant systems can have gapped, non-degenerate ground states. To see this for bosonic and integer-spin dipole conserving systems, we fix ν to be even, and consider a state $|\Psi_P\rangle$ with many-body momentum P i.e., $\hat{T} |\Psi_P\rangle = e^{iPa} |\Psi_P\rangle$ (we restore the lattice constant a here for clarity). Eq. (6) gives

$$\hat{T} (\hat{O} |\Psi_P\rangle) = e^{-2\pi i \nu_D} e^{iPa} (\hat{O} |\Psi_P\rangle). \quad (8)$$

The state $\hat{O} |\Psi_P\rangle$ is then an eigenstate of $\hat{T}$ with many-body momentum $Pa - 2\pi\nu_D$. Suppose that $|\Psi_0\rangle$ is the ground state of a translation invariant dipole conserving Hamiltonian. Since $\hat{O}$ is a product of single-site unitary operators (equivalently, a quantum circuit of depth 1), the theorem of Ref. [100] applies, which states that if there exists a symmetric finite-depth local unitary (FDLU) that boosts a state's momentum to a different value $\pmod{2\pi}$, then the state is necessarily long-range entangled. Hence $|\Psi_0\rangle$ is necessarily symmetry-breaking or gapless when $\nu_D \neq 0 \pmod{1}$. A similar analysis for odd-integer charge fillings shows that the obstruction occurs for any dipole filling $\nu_D \neq 1/2$ ($\nu_D \neq 0$) for L even (odd). For instance, in the thin-torus limit of bosonic QH

⁵ For spinless fermions (or spin-1/2 hard core bosons), the minimal kinetic term has additional sublattice symmetry [31]; our results apply once this symmetry is explicitly broken [81].

⁶ For $U(1)_c$ charge symmetry, α_j is continuous and there is no gauge invariant closed line operator associated with charge hopping. If we Higgs this symmetry from $U(1)$ to $\mathbb{Z}_L$, there is a second gauge invariant line operator $W_D = e^{i \sum_j j A_j}$ [5].

⁷ This form ensures that choosing $a_2 \in (2\pi/L)\mathbb{Z}$ is sufficient to ensure the periodicity $\alpha_{j+L} = \alpha_j \pmod{2\pi}$.

⁸ Eq. (6) gives an independent momentum shift at fractional dipole filling even for $\nu \neq \mathbb{Z}$ but we restrict to $\nu \in \mathbb{Z}$ to highlight the role of dipole symmetry.

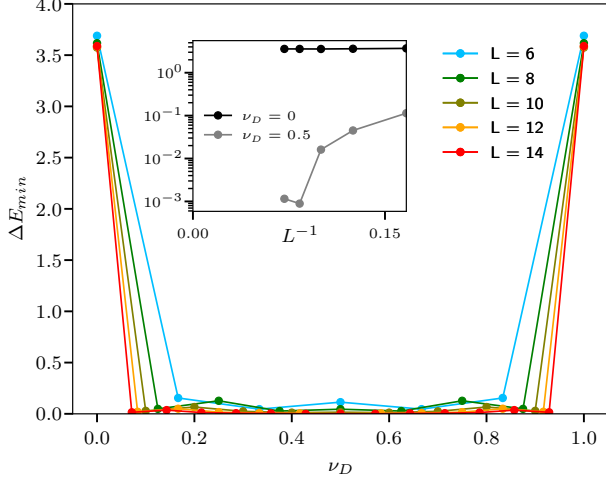


FIG. 1: (Color online) Plot of the gap of the spin-1 model Eq. (9) within the $\nu = 0$ sector as a function of dipole filling ν_D for various system sizes. Dipole fillings $\nu_D = 0 (\equiv 1)$ show a clear gap, whereas the other fillings appear gapless. The inset shows the gap scaling with inverse system size at dipole fillings $\nu_D = 0$ and $\nu_D = 0.5$, showing evidence for a gap and an absence of a gap respectively. Data shown for the Hamiltonian Eq. (9) with parameters $(J_3, J_4, V_0, V_1, V_2) = (1, 0.3, 1.3, 1.5, -1.2)$.

at $\nu = 2$, our constraint rules out Mott insulating states for any $\nu_D \neq 0 \pmod{1}$, but allows e.g., charge-density-wave states (consistent with results in Ref. [101]).

This analysis also applies to fermions [81], though for systems with one fermionic d.o.f per unit cell, the integer charge filling constraint only permits the empty lattice ($\nu = 0$) or the trivial atomic insulator ($\nu = 1$) as translation invariant gapped ground states, and fractional dipolar fillings cannot be realized. We extend our analysis to half-integer spin systems in the SM [81], where we again find that the allowed values of the dipole filling ν_D at which a symmetric, gapped, and non-degenerate ground state is permitted are specified by Eq. (7). In the SM [81], we also extend our constraint to systems where the charge is only defined mod N . Our analysis can also be generalized to systems with higher multipole moment conservation, since the crucial ingredient—a symmetric FDLU that has non-trivial commutation with translations—is furnished by the corresponding uniform large gauge transformation.

Explicit Examples: We now discuss the implications of the above constraint. Note first that translation-invariant product states of the form of $|s \cdots s\rangle$, where s denotes the spin, are always gappable and have $\nu = s$ and $\nu_D = s(L+1)/2 \pmod{1}$. For L odd and any integer s , or for L even and even integer s , $\nu_D = 0$, while for L even and odd integer s , we see that $\nu_D = 1/2$, consistent with our results. A similar condition can be derived for half-integer spins.

Next, we explore the ground states in the family of

interacting spin-1 Hamiltonians

$$\begin{aligned} \hat{H} = & J_3 \sum_j \left(\hat{S}_{j-1}^- (\hat{S}_j^+)^2 \hat{S}_{j+1}^- + h.c. \right) \\ & + J_4 \sum_j \left(\hat{S}_{j-1}^- \hat{S}_j^+ \hat{S}_{j+1}^+ \hat{S}_{j+2}^- + h.c. \right) \\ & + V_0 \sum_j \hat{S}_j^z + V_1 \sum_j (\hat{S}_j^z)^2 + V_2 \sum_j \hat{S}_j^z \hat{S}_{j+1}^z, \end{aligned} \quad (9)$$

where $\hat{S}_j^\pm$ are spin-1 raising and lowering operators, $\hat{S}_j^z$ measures the spin, and we use PBC. This model has been extensively studied in the context of Hilbert space fragmentation [28, 30, 33, 102]. Note that since the only states at $\nu = \pm 1$ are the fully polarized states $|\pm \cdots \pm\rangle$, which are eigenstates of the Hamiltonian (and do satisfy the dipolar constraint), it is not meaningful to discuss a gap in these sectors. It remains to show that there are no gappable states at non-zero dipole fillings when $\nu = 0$.

Numerical explorations of this model for various Hamiltonian parameters suggest that it is gapless within the $\nu = 0$ sector when $\nu_D \neq 0$. Fig. 1 shows the gap as a function of ν_D for a representative set of parameters, and also illustrates that $\nu_D = 0$ is gappable. To analytically support these results, we consider Eq. (9) when $J_4 = 0$, where this Hamiltonian exhibits strong Hilbert space fragmentation and allows explicit solutions within certain Krylov subspaces. At $\nu = 0$, the Hilbert space sector with $\nu_D = 1/2$ contains a closed Krylov subspace generated by the state $|+ - + - \cdots + -\rangle$ for even L . This is the largest Krylov subspace for $\hat{H}$ and contains the ground state for a range of coupling strengths [30]. In the SM [81], we show that the Hamiltonian within this subspace is a spin-1/2 $U(1)$ conserving model (e.g., XX or XXZ) with coupling parameters specified by the parameters of $\hat{H}$. In this description, the obstruction to constructing a unique, symmetric, and gapped ground state at $(\nu, \nu_D) = (0, 1/2)$ stems from the conventional LSM theorem, which forbids such a state for spin-1/2 models. This restriction applies to *all* models that preserve the above Krylov subspace [when the latter contains the ground state, which we numerically observe to be the case across a large portion of the parameter space of Eq. (9)], which can be systematically derived [33]. Hence this solvable limit is also consistent with the filling constraints. Similar models and arguments can be written down for higher integer spin systems, as we discuss in [81].

A second approach to searching for additional gapped states is to use Matrix Product States (MPS). In [81], we present constraints that an injective MPS – i.e., an MPS that can be realized as unique ground states of a local parent Hamiltonian – with charge and dipole symmetry must satisfy. We show that these constraints cannot be satisfied by an MPS of bond dimension 2 at fillings that fail to satisfy Eq. (7), providing further evidence supporting our result.

Discussion: We have established dipole-filling enforced constraints for 1d translation invariant dipole conserving systems which prohibit the existence of symmetric, gapped, and non-degenerate ground states except when the dipole filling takes certain special values. Our analysis highlights the importance of lattice-scale effects for symmetries (such as dipole symmetry) that intertwine non-trivially with translations, leading to differences in the allowed large gauge transformations on the lattice and in the continuum [81]. Crucially, the constraints obtained by first working on a finite chain and then considering the limit $L \rightarrow \infty$ differ from those obtained by analyzing the infinite-volume limit directly: as we have shown here, the former accurately predicts a symmetric, gapped, and unique ground state at $\nu_D = 1/2$ (for certain values of ν and L), whereas the latter ignores the subtlety with boundary conditions and incorrectly prohibits such a state at any fractional dipole filling [79]. Similar subtleties with taking the thermodynamic limit in the presence of translation symmetry were recently observed in Ref. [78] and, as our work highlights, become more pronounced in the presence of multipole symmetries. Our results invite further study of higher-moment conserving systems in higher dimensions and on arbitrary crystalline lattices [103], where their interplay with spatial symmetries will likely lead to additional filling-enforced constraints. It would also be interesting to study how our filling constraints relate to anomalies for dipole symmetry and to identify the 2d dipole SPTs that host the anomalous 1d theories on their boundaries.

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