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Bipartite mutual information in classical many-body dynamics

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Information theoretic measures have helped to sharpen our understanding of many-body quantum states. As perhaps the most well-known example, the entanglement entropy (or more generally, the bipartite mutual information) has become a powerful tool for characterizing the dynamical growth of quantum correlations. By contrast, although computable, the bipartite mutual information (MI) is almost never explored in classical many-particle systems; this owes in part to the fact that computing the MI requires keeping track of the evolution of the full probability distribution, a feat which is rarely done (or thought to be needed) in classical many-body simulations. Here, we utilize the MI to analyze the spreading of information in 1D elementary cellular automata (CA). Broadly speaking, we find that the behavior of the MI in these dynamical systems exhibits a few different types of scaling that roughly correspond to known CA universality classes. Of particular note is that we observe a set of automata for which the MI converges parametrically slowly to its thermodynamic value. We develop a microscopic understanding of this behavior by analyzing a two-species model of annihilating particles moving in opposite directions. Our work suggests the possibility that information theoretic tools such as the MI might enable a more fine-grained characterization of classical many-body states and dynamics.

Conventionally, quantum many-body states are distinguished by the behavior of their local correlation functions. The last decades have seen the language of quantum information used to further sharpen this distinction by allowing for measures of non-separability that quantify global correlations. Perhaps the most well-known example of such a measure is the bipartite mutual information (for a pure quantum state, this is equivalent to the bipartite entanglement entropy [1]) between two halves of a many-body system. On the one hand, in the context of quantum quenches, the dynamics of the entanglement entropy provide a particularly useful tool for distinguishing between ergodic and many-body localized systems [2]. On the other hand, in the context of equilibrium states, the scaling of the entanglement entropy with system size N , immediately enables a distinction between gapped quantum ground states (area law) [3], conformal critical states (logarithmic scaling) [4], and highly excited thermal states (volume law) [5, 6].

For both quantum and classical systems, computing the bipartite mutual information (MI) requires manipulating objects that are exponentially large in the number of particles: wavefunctions for quantum systems and probability distributions for classical ones [7]. In the quantum setting, one is taught that this exponential burden is somewhat unavoidable, while in classical systems, it is often perceived as unjustified since the expectation value of few-body observables can be accurately computed via simple Monte Carlo sampling [8]. However, the success of information theoretic tools such as the MI for characterizing quantum systems raises the question: Can they provide novel insights into classical many-body systems that justify their exponential complexity?

In this letter, we address this question in the context of perhaps the most paradigmatic class of dynamical systems—elementary cellular automata (CA). These consist of a one-dimensional array of bits that evolve under simple updates rules depending on their nearest neighbors [9]. The 256 possible elementary CA were originally classified by Wolfram into four distinct “universality” classes [10]. Working within this

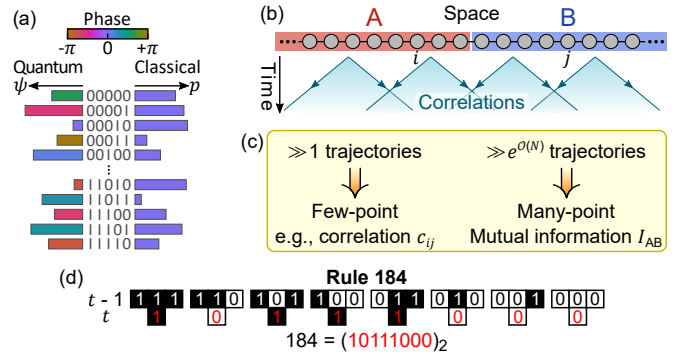


FIG. 1. Many-body information spreading. (a) The state of a many-body system is specified by exponentially many coefficients, one per each state in the configuration space. For quantum systems, they are complex and constitute the wavefunction ψ . For classical systems, they are real and constitute the probability distribution p . (b) We are interested in the study of correlation spreading in locally interacting many-body systems. (c) Few-point observables, e.g., two-point correlation functions, can be accurately estimated within Monte Carlo methods sampling a large number $\gg 1$ of classical trajectories. Many-point information diagnostics such as the bipartite MI, however, depend heavily on the details of the probability distribution p , requiring exponentially large resources for its computation. (d) Elementary CA rules are specified by associating the digits of the binary representation of the rule number to the 8 possible combinations of the 3 nearest bits at previous time (here shown for rule 184).

set, for each rule, we compute the bipartite MI and characterize its scaling with time and system size. We identify a few universal scaling behaviors of the MI, that roughly correspond to Wolfram’s universality classes: for class I rules, the asymptotic MI decays exponentially as a function of system size; for class II rules it saturates to a plateau; for both class III and class IV rules, it grows with system size. Perhaps most interestingly, within class II rules, we find a subset of *critical* rules for which the asymptotic MI is reached extremely slowly with increasing the system size. We uncover the microscopic origin of this behavior by analytically solving a two-annihilating-particle model to which these rules map. Our work establishes

a framework for exploring the MI in classical many-body dynamical systems, and hints at the possibility that such measures may provide new ways to distinguish classical many-body states.

As a toy model for classical dynamics, we focus here on CA. These are discrete, grid-based systems consisting of a collection of simple computational units, or cells, which evolve over discrete time steps according to a set of rules based on the states of neighboring cells. First introduced by von Neumann and Ulam in the 1940s [11], CAs have since evolved into a versatile tool for studying complex dynamic processes across diverse fields, from physics [12–14], to traffic modelling [15, 16], biology [17], ecological dynamics [18–21], and the social sciences [22–24]. The vast landscape of CA can be divided in universality classes. Among the various existing classifications [25], the most celebrated is arguably Wolfram’s [10]. This distinguishes 4 classes of CAs: I – for which most state evolve into an absorbing state; II – leading to simple repetition patterns; III – yielding chaos; IV – generating complex structures and conjectured to be able of universal computing.

Let us in particular focus on elementary CA, in which each site $i = 1, 2, \dots, N$ can take value $\sigma_i \in \{0, 1\}$, and with updates depending on the nearest neighbors at previous time, $\sigma_i(t+1) = f([\sigma_{i-1}, \sigma_i, \sigma_{i+1}](t))$. Periodic boundary conditions are assumed, $\sigma_{N+1} = \sigma_1$. The function f specifies a rule, namely a look up table associating 0 or 1 to each of the $2^3 = 8$ possible states of $[\sigma_{i-1}, \sigma_i, \sigma_{i+1}]$, Fig. 1(d). Of the resulting $2^8 = 256$ possible rules, some are equivalent upon left-right mirroring, $[\sigma_{i-1}, \sigma_i, \sigma_{i+1}] \rightarrow [\sigma_{i+1}, \sigma_i, \sigma_{i-1}]$, negation, $[\sigma_{i-1}, \sigma_i, \sigma_{i+1}] \rightarrow [\bar{\sigma}_{i-1}, \bar{\sigma}_i, \bar{\sigma}_{i+1}]$, or both, $[\sigma_{i+1}, \bar{\sigma}_i, \bar{\sigma}_{i-1}]$. We shall henceforth focus on the 88 rules that are *not* equivalent under these transformations [26].

We denote $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_N]$ the binary state of the system, and σ its decimal representation. Each rule is associated to a transition matrix $T \in [0, 1]^{2^N}$ with $T_{\sigma', \sigma} = 1$ if and only if the rule transforms σ in σ' , and $T_{\sigma', \sigma} = 0$ otherwise. Consider now a probability distribution p over the 2^N configurations, namely with entries p_σ yielding the probability of the configuration σ . At initial time $t = 0$ we assume that the bits are uncorrelated and take values 1 and 0 with probabilities q and $1 - q$, respectively, yielding

$$p_\sigma(0) = \prod_i q^{\sigma_i} (1 - q)^{1 - \sigma_i}. \quad (1)$$

Unless otherwise specified, we will focus on the equiprobable initial ensemble, $q = 0.5$, for which each state occurs with equal probability $p_\sigma = 2^{-N}$. Starting from such a structureless initial distribution, the system can dynamically self-organize under the CA evolution [9]. The probability distribution is evolved applying the transition matrix T as $p(t) = T^t p(0)$.

Let us denote $\sigma_l = [\sigma_1, \sigma_2, \dots, \sigma_{N/2}]$ and $\sigma_r = [\sigma_{N/2+1}, \sigma_{N/2+2}, \dots, \sigma_N]$ the left and right halves of the system, respectively. The respective marginal distributions read

$p_{\sigma_l} = \sum_{\sigma_r} p_\sigma$ and $p_{\sigma_r} = \sum_{\sigma_l} p_\sigma$. The bipartite MI reads

$$I = S_l + S_r - S, \quad (2)$$

where $S_{l,r} = -\sum_{\sigma_{l,r}} p_{\sigma_{l,r}} \log_2 p_{\sigma_{l,r}}$ are the marginal entropies of the two halves and $S = -\sum_{\sigma} p_\sigma \log_2 p_\sigma$ is the total entropy.

The bipartite MI generally depends on time. To compute its asymptotic value, we square the transition matrix N times and jump to exponentially large times $t = 2^N$, when transients are extinguished and the system has reached the long-time limit cycles or absorbing states (because no cycle can last more than 2^N steps). After jumping to $t = 2^N$, we evolve the system for t_{av} time steps, over which we perform the time average of I , an operation that we will henceforth implicitly assume when plotting $I(N)$ [27].

We compute the MI numerically with no further approximation, that is, exactly up to rounding errors. By explicitly dealing with the exponentially large configuration spaces for a finite N , our study falls under the umbrella of “finite CA”. These have never been studied with respect to the MI and, to the best of our knowledge, have never been pushed beyond $N = 16$ [28]. Making use of translational symmetry, namely the fact that $p_\sigma = p_{\sigma'}$ if σ' is a translation of σ , we can effectively reduce the size of the configuration space to $\sim 2^N/N$, and reach a remarkably large system size of $N = 36$ [29]. Achieving such system size is indeed key to gain confidence in assessing the scaling properties of global information measures such as the MI.

Elementary Cellular Automata—In terms of MI, the simplest rules are those that, for *all* possible states σ , either evolve towards the same state or act as a shift, leading, trivially, to $I = 0$. Among the 88 non-equivalent elementary CA, these are rules 0, 8, 15, 51, 170, and 204, which we will henceforth neglect. Beyond these, the simplest elementary CA are those of class I, namely the CA for which most states evolve towards the same absorbing state. These rules can be thought of as strongly dissipative, and their dynamics having an absorbing fixed point. Correlations are attributed to the states that do not get absorbed. The number of such states decays exponentially in time and so does the MI, $I(t) \sim e^{-t/\tau}$, before saturating to an exponentially small asymptotic value, $\sim e^{-N/l}$, due to the few configurations that never get absorbed. The same behavior holds for different initial filling fractions q , and the decay constant l depends on both the rule and q . Among the 88 non-equivalent elementary CA, this behavior holds for rules 32, 40, 128, 136, 160, and 168, see Fig. 2(a). For rule 32, 128, and 136 we compute the MI analytically [29], finding $I \propto N e^{-N/l}$ with $l = \frac{-2}{\log q(1-q)}$ for rule 32, and $l = \frac{-1}{\log q}$ for rules 128 and 136. For rules 40, 160, and 168 we fit $I \sim e^{N/l}$ and find $l \approx 2.63, 3.29$, and 5.50 , respectively.

Next in complexity are class II CA, that lead to simple stable or periodic structures. Most of the elementary rules belong to this class, including rules 29, 37, 43, and 184, instances of whom we show in Fig. 2(b). For these rules we find that the

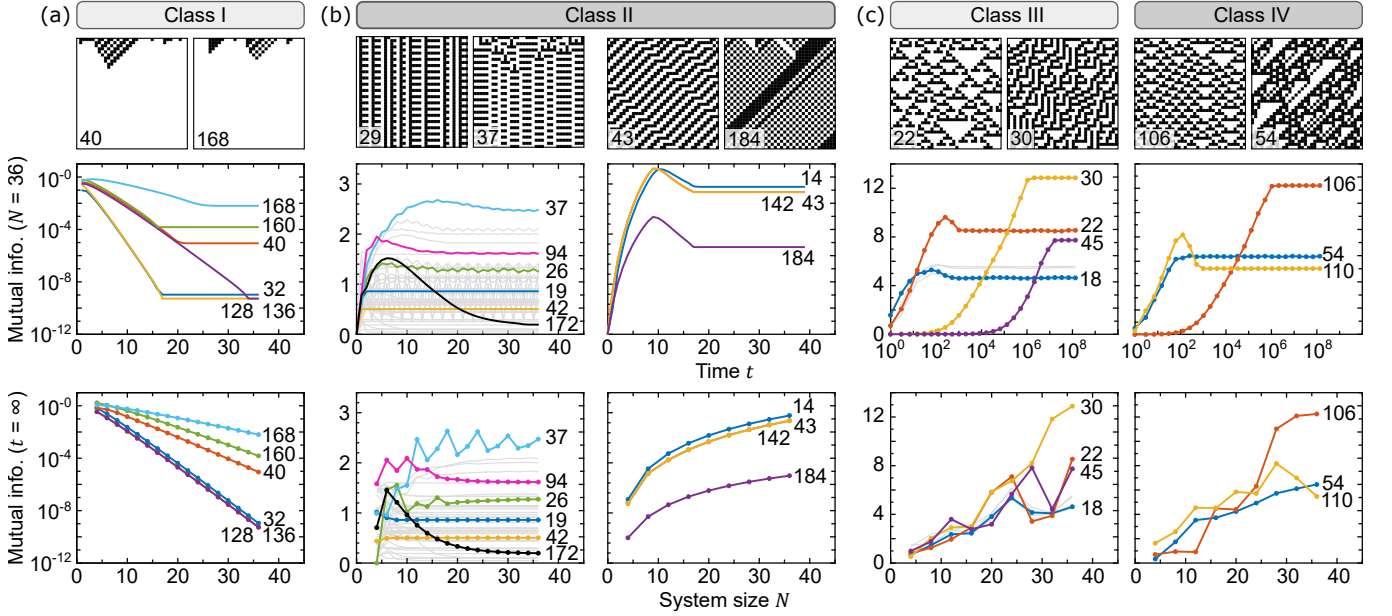


FIG. 2. **Bipartite mutual information in elementary cellular automata.** The bipartite MI is computed exactly across the space of elementary CA. Single instances of the dynamics generated by the CA are shown at the top (the horizontal and vertical direction are for space and time, respectively). The dynamics of the MI, $I(t)$, is shown for $N = 36$ in the middle. The scaling of its asymptotic value with system size, $I(N)$, is shown at the bottom. (a) For class I CA, all the states evolve towards the same absorbing state, except an exponentially small subset of them. Correspondingly, the MI quickly decays to an exponentially small MI. (b) Class II CA lead to simple repeating patterns, and the MI quickly relaxes to a finite value in the thermodynamic limit. This limit is reached exponentially fast with N for most rules, but very slowly for rules 14, 43, 142, and 184 (on the right). These are characterized by particles of two species, that move in opposite directions and annihilate when meeting, as shown in the instances of rules 43 and 184. (c) Class III and IV CA lead to chaotic and complex evolution, respectively. Compared to class II rules, the MI that they generate takes much longer to equilibrate, has generally larger values, and appears to grow with the system size.

MI generally has a finite limit $I \rightarrow \text{const}$ for $N \rightarrow \infty$. This is reached exponentially fast in N for most rules, but much more slowly for rules 14, 43, 142, and 184. These rules are indeed special. Their common feature are particles of two types that move in opposite directions and annihilate when meeting, see the instances of rules 43 and 184 in Fig. 2(b). The spreading and annihilation of the particles underpins the competition between generation and destruction of the MI, explaining the observed maximum in its dynamics, e.g., at time $t \sim 10$ for the shown $N = 36$. In the next subsection we will develop a quantitative understanding of how this particle structure impacts the spreading of information, explain the observed behavior of the MI, and argue that the asymptotic MI converges in the thermodynamic limit $N \rightarrow \infty$. These rules cease to be special for $q \neq 0.5$, for which exponential convergence of the MI is again found. Note: the results for some pairs or rules, e.g., 43 and 142, overlap exactly, making them indistinguishable in Fig. 2. This can be understood from the rules symmetries [29].

Third and last, in Fig. 2(c) we inspect non-additive rules of class III, that are chaotic, and of class IV, that generate complex structures and have been conjectured to be able of universal computation [10] (for some rules, such conjecture has been proved [30–32]). Compared to class I and II automata, these rules generate extended structures (e.g., triangles), yielding

long-range correlations and a large bipartite MI. The asymptotic MI is reached more slowly, in some cases in an exponentially large time $t \sim e^N$ (rules 30, 45, 106), due to the exponentially long trajectories in configuration space that the system has to exhaust before ultimately settling onto limit cycles. As for the scaling of the asymptotic bipartite MI with the system size N , while it is hard to make ultimate statements for $N \rightarrow \infty$, numerics suggest that the MI should grow with N . Additive class III rules are special with respect to MI, see the outlook and [29].

Annihilating particles model—To understand why, within class II CA, rules 14, 43, 142, and 183 behave differently, we now construct a closely related model, and solve it exactly. In these rules, two species of particles move in opposite directions over a stable background, and annihilate when meeting [33, 34]. For instance, in rule 184 the stable background is the checkerboard pattern, and black (white) particles are associated to sequences of two or more consecutive black (white) bits. Fig. 3(a) introduces the same two-particle species model, but discards the stable background and defines the particles at the level of a single site. An instance of the resulting dynamics is shown in Fig. 3(b). It is convenient to recast the model into one of a random walk, where time is played by the site index j , and space is played by the imbalance $N_{\text{black}} - N_{\text{white}}$: each black (white) particle increases (decreases) the imbalance by

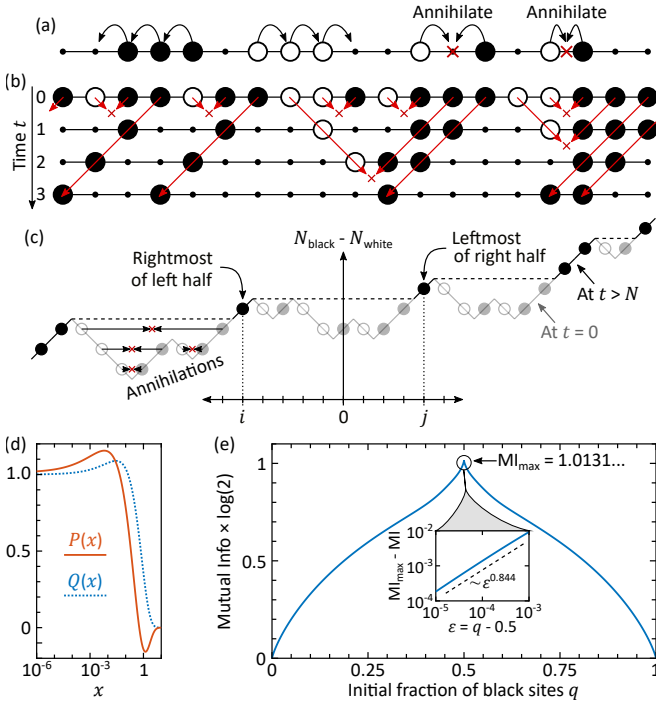


FIG. 3. **Two annihilating species model.** (a) Particles of two species, black and white, hop on a one dimensional lattice in opposite directions. When particles of different type face each other, they annihilate. (b) Example of dynamics. Initially each site is occupied by either a black particle, with probability q , or a white one, with probability $1 - q$. Black (white) particles hop to the left (right) at each time step. At long time, only particles of the more populous species survive, in a configuration that no longer evolves (except from translating). (c) The problem can be mapped to a random walk for the imbalance $N_{\text{black}} - N_{\text{white}}$ between black and white particles. At $t = 0$, black (white) particles are like a step up (down) in the random walk. At $t = \infty$, the valleys of the random walk are “annihilated out”, and only the black particles with a lower random walk to their left survive (assuming $q > 0.5$ and $N = \infty$). The first particles at the left and right of a chosen interface are tagged i and j , respectively. (d) Functions $P(x)$ and $Q(x)$ in Eq. (4). (e) MI between left and right halves at equilibrium and for $N \rightarrow \infty$.

1. At initial time $t = 0$, let’s assume that each site is occupied by a black or a white particle with probability q and $1 - q$, respectively. As time evolves and particles move, the valleys of the random walk are annihilated out, Fig. 3(c). Eventually, a monotonous random walk is left, upwards if $q > 0.5$, and downwards if $q < 0.5$. We shall henceforth consider the former case.

The MI across an interface only depends on the location of the two particles that are the closest to the interface. Calling $i = 0, 1, \dots$ and $j = 1, 2, \dots$ their distances from the interface [see Fig. 3(c)], and p_{ij} the respective probability distribution, we get $I = S_i + S_j - S_{ij}$, with $p_i = \sum_j p_{ij}$, $S_i = -\sum_i p_i \log_2 p_i$, $S_j = -\sum_j p_j \log_2 p_j$, and $S_{ij} = -\sum_{ij} p_{ij} \log_2 p_{ij}$. At long time, site i has black particle if the random walk starting in i and going left never crosses the origin. The probability of this event coincides with the probability of not return of a random walk, namely $2q - 1$ [35].

Given that i has a particle, j has the next particle if the random walk from i , towards the right, gets in j one step higher than in i , for the first time. The probability of this happening is equal to $(1 - q)^{-1} f_{i+j+1}$, with $f_{2k} = \binom{2k}{k} \frac{(1-q)^k q^k}{2^{2k-1}}$ the first return probability of a random walk and $f_{2k+1} = 0$ [35]. We thus get $p_{ij} = \frac{2q-1}{2(1-q)} f_{i+j+1}$ if $i + j$ is odd, and $p_{ij} = 0$ otherwise (see [29] for details).

Let us introduce an integer parameter $m \gg 1$. Computing the marginal p_i involves an infinite sum over j . The idea is to perform the sum exactly for $i + j < m$, and to turn it into an integral for $i + j > m$, for which a Stirling approximation yields $f_{2k} \approx \frac{1}{2\sqrt{\pi}} \frac{e^{-k/\xi}}{k^{3/2}}$, with $\xi = -[\log(4(1-q)q)]^{-1}$ a characteristic length that diverges as $\sim |q - 0.5|^{-2}$ for $q \rightarrow 0.5$. In this way, we get

$$p_i \approx \frac{2q-1}{2-2q} \left[\sum_{r=2+i}^m f_r + \frac{1}{\sqrt{\xi}} g\left(\frac{\max(i, m)}{2\xi}\right) \right], \quad (3)$$

where $g(x) = \frac{e^{-x}}{\sqrt{\pi x}} - \text{erfc}(\sqrt{x})$. The same logic applies when computing the entropies S_i and S_{ij} . After a tedious but straightforward calculation we obtain

$$\begin{aligned} I_e &\approx -\log\left(\frac{2q-1}{2-2q}\right) \\ &+ \frac{2q-1}{2-2q} \sum_{i=1}^{m-1} [(r-1)f_r \log f_r - 2g_r \log g_r] \\ &+ \frac{2q-1}{2-2q} \sqrt{\xi} \left[P\left(\frac{m}{2\xi}\right) - \log(\sqrt{\xi}) Q\left(\frac{m}{2\xi}\right) \right], \end{aligned} \quad (4)$$

where $I_e = \log 2 \times I$ is the MI in natural base, $g_r = \sum_{k=1+r}^{m-1} f_k + \frac{1}{\sqrt{\xi}} g\left(\frac{m}{2\xi}\right)$, and the functions P and Q are shown in Fig. 3(d), see [29] for details. This calculation holds for $q > 0.5$, but the results are specular upon replacing $q \rightarrow 1 - q$. The resulting MI is plotted in Fig. 3(e). In the limit $\epsilon = q - 0.5 \rightarrow 0$ we get $|I_e(\epsilon) - I_{\max}| \sim \epsilon^{0.844}$, with $I_{\max} = J(0) \approx 1.0131$. Note that at $q = 0.5$ an extra bit of information adds to the MI, because by looking at one half of the system one learns whether particles in the other half are black or white (whereas for $q \neq 0.5$ this is determined a priori).

The two-particle model clarifies the origin of the special behaviours of rules 14, 43, 142, and 184. The exchange of information across the half cut happens via the particles that are closest to it. For $q \neq 0.5$ these are localized nearby the interface, $p_i \sim e^{-i/\xi}$. The localization length ξ however diverges as $q \rightarrow 0.5$. Interestingly, the divergence of ξ does not make the MI diverge, but it makes it singular, and makes convergence from finite-size scaling very slow, as in Fig. 2. This highlights an intriguing comparison between classical and quantum systems. The quantum MI in conformal critical states slowly diverges as $\sim \log N$. The classical models considered here lack the conformal symmetry, but still have scale invariance, leading to a “missed divergence”, namely slow convergence, of the MI.

Discussion and summary—In conclusion, we have demonstrated that bipartite MI can be a useful tool for investigating information spreading in classical many-body dynamics. Scanning the space of elementary CA, we have shown that a few classes of behaviors of the MI exist, roughly reflecting Wolfram’s classification of CA. A slow convergence to a finite MI in the thermodynamic limit has been observed for some rules at criticality, a condition that we explored by exactly solving a model of two counter-moving and annihilating particles. We should emphasize that ours is not a strict classification, as most classifications of CA are not (including Wolfram’s): some rules exhibit ambiguous behavior, which could be attributed to the limited system sizes that we were able to reach, or that could be intrinsic to the rules themselves. More details on such rules, together with complementary results and technical derivations, are presented in the accompanying Supplementary Information [29].

Looking forward, we envision a number of possible directions to explore, including the so-called measurement induced phase transition [36–43] in classical many-body systems. Such transitions are normally studied for systems in which information-erasing measurements compete with time-reversible dynamics, that instead preserve and spread information [7]. Understanding the fate of the measurement-induced transitions of the MI for classical dynamics in which information spreading, dissipation, and measurements interplay represents an intriguing avenue for future exploration, that could be explored with elementary class III and IV CA. A second open problem regards the scaling of the MI with system size for class III and IV CA. To target this question, further research could attempt the exact computation of the MI for additive rules (e.g., rules 60, 90, and 150), for which analytical results are available [44, 45] and for which numerics suggest that the MI takes particularly simple (integer) values [29].

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- [1] The quantum MI reads $I_{A,B} = S_A + S_B - S_{AB}$, where S denotes the Von Neumann Entropy. For pure states, $S_{AB} = 0$, whereas $S_A = S_B$ is the entanglement entropy, so the quantum MI is twice the entanglement entropy.
 - [2] J. H. Bardarson, F. Pollmann, and J. E. Moore, Unbounded growth of entanglement in models of many-body localization, *Physical review letters* **109**, 017202 (2012).
 - [3] J. Eisert, M. Cramer, and M. B. Plenio, Colloquium: Area laws for the entanglement entropy, *Reviews of modern physics* **82**, 277 (2010).
 - [4] P. Calabrese and J. Cardy, Entanglement entropy and conformal

- field theory, *Journal of physics a: mathematical and theoretical* **42**, 504005 (2009).
- [5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Reviews of modern physics* **80**, 517 (2008).
- [6] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Colloquium: Many-body localization, thermalization, and entanglement, *Reviews of Modern Physics* **91**, 021001 (2019).
- [7] A. Pizzi, D. Malz, A. Nunnenkamp, and J. Knolle, Bridging the gap between classical and quantum many-body information dynamics, *Physical Review B* **106**, 214303 (2022).
- [8] M. E. Newman and G. T. Barkema, *Monte Carlo methods in statistical physics* (Clarendon Press, 1999).
- [9] S. Wolfram, Statistical mechanics of cellular automata, *Reviews of modern physics* **55**, 601 (1983).
- [10] S. Wolfram, Universality and complexity in cellular automata, *Physica D: Nonlinear Phenomena* **10**, 1 (1984).
- [11] J. v. Neumann, *Theory of self-reproducing automata*, Edited by Arthur W. Burks (1966).
- [12] G. Y. Vichniac, Simulating physics with cellular automata, *Physica D: Nonlinear Phenomena* **10**, 96 (1984).
- [13] T. Toffoli, Cellular automata as an alternative to (rather than an approximation of) differential equations in modeling physics, *Physica D: Nonlinear Phenomena* **10**, 117 (1984).
- [14] B. Denby, Neural networks and cellular automata in experimental high energy physics, *Computer Physics Communications* **49**, 429 (1988).
- [15] K. Nagel and M. Schreckenberg, A cellular automaton model for freeway traffic, *Journal de physique I* **2**, 2221 (1992).
- [16] S. Maerivoet and B. De Moor, Cellular automata models of road traffic, *Physics reports* **419**, 1 (2005).
- [17] G. B. Ermentrout and L. Edelstein-Keshet, Cellular automata approaches to biological modeling, *Journal of theoretical Biology* **160**, 97 (1993).
- [18] P. Hogeweg, Cellular automata as a paradigm for ecological modeling, *Applied mathematics and computation* **27**, 81 (1988).
- [19] H. Balzter, P. W. Braun, and W. Köhler, Cellular automata models for vegetation dynamics, *Ecological modelling* **107**, 113 (1998).
- [20] I. Karafyllidis and A. Thanailakis, A model for predicting forest fire spreading using cellular automata, *Ecological Modelling* **99**, 87 (1997).
- [21] B. S. Soares-Filho, G. C. Cerqueira, and C. L. Pennachin, Dinamica—a stochastic cellular automata model designed to simulate the landscape dynamics in an amazonian colonization frontier, *Ecological modelling* **154**, 217 (2002).
- [22] J. M. Epstein and R. Axtell, *Growing artificial societies: social science from the bottom up* (Brookings Institution Press, 1996).
- [23] A. Nowak and M. Lewenstein, Modeling social change with cellular automata, in *Modelling and simulation in the social sciences from the philosophy of science point of view* (Springer, 1996) pp. 249–285.
- [24] R. Hegselmann and A. Flache, Understanding complex social dynamics: A plea for cellular automata based modelling, *Journal of Artificial Societies and Social Simulation* **1**, 1 (1998).
- [25] M. Vispoel, A. J. Daly, and J. M. Baetens, Progress, gaps and obstacles in the classification of cellular automata, *Physica D: Nonlinear Phenomena* **432**, 133074 (2022).
- [26] S. Wolfram *et al.*, *A new kind of science*, Vol. 5 (Wolfram media Champaign, IL, 2002).
- [27] We find that $t_{av} = 10$ is enough to guarantee a good convergence. If we find that p has period $T < t_{av}$, we simply consider the average of the MI over one such periods.

- [28] S. Wolfram, Theory and applications of cellular automata, World Scientific (1986).
- [29] See supplemental material.
- [30] M. Cook *et al.*, Universality in elementary cellular automata, *Complex systems* **15**, 1 (2004).
- [31] M. Cook, A concrete view of rule 110 computation, arXiv preprint arXiv:0906.3248 (2009).
- [32] T. Neary and D. Woods, P-completeness of cellular automaton rule 110, in *Automata, Languages and Programming: 33rd International Colloquium, ICALP 2006, Venice, Italy, July 10-14, 2006, Proceedings, Part I* 33 (Springer, 2006) pp. 132–143.
- [33] J. Krug and H. Spohn, Universality classes for deterministic surface growth, *Physical Review A* **38**, 4271 (1988).
- [34] N. Boccara, J. Nasser, and M. Roger, Particlelike structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular-automaton rules, *Physical Review A* **44**, 866 (1991).
- [35] W. Feller, *An introduction to probability theory and its applications, Volume 2*, Vol. 81 (John Wiley & Sons, 1991).
- [36] B. Skinner, J. Ruhman, and A. Nahum, Measurement-induced phase transitions in the dynamics of entanglement, *Physical Review X* **9**, 031009 (2019).
- [37] Y. Li, X. Chen, and M. P. A. Fisher, Measurement-driven entanglement transition in hybrid quantum circuits, *Physical Review B* **100**, 134306 (2019).
- [38] Y. Bao, S. Choi, and E. Altman, Theory of the phase transition in random unitary circuits with measurements, *Physical Review B* **101**, 104301 (2020).
- [39] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Quantum error correction in scrambling dynamics and measurement-induced phase transition, *Physical Review Letters* **125**, 030505 (2020).
- [40] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. W. Ludwig, Measurement-induced criticality in random quantum circuits, *Physical Review B* **101**, 104302 (2020).
- [41] A. Zabalo, M. J. Gullans, J. H. Wilson, S. Gopalakrishnan, D. A. Huse, and J. H. Pixley, Critical properties of the measurement-induced transition in random quantum circuits, *Physical Review B* **101**, 060301(R) (2020).
- [42] M. Ippoliti, M. J. Gullans, S. Gopalakrishnan, D. A. Huse, and V. Khemani, Entanglement phase transitions in measurement-only dynamics, *Physical Review X* **11**, 011030 (2021).
- [43] J. Willsher, S.-W. Liu, R. Moessner, and J. Knolle, Measurement-induced phase transition in a classical, chaotic many-body system, arXiv preprint arXiv:2203.11303 (2022).
- [44] O. Martin, A. M. Odlyzko, and S. Wolfram, Algebraic properties of cellular automata, *Communications in mathematical physics* **93**, 219 (1984).
- [45] H. Akin, On the measure entropy of additive cellular automata f, *Entropy* **5**, 233 (2003).