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Dipolar Weyl semimetals

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In Weyl semimetals, Weyl points act as monopoles and antimonopoles of the Berry curvature, with a monopole-antimonopole pair producing a net zero Berry flux. When inversion symmetry is preserved, the two-dimensional (2D) planes that separate a monopole-antimonopole pair of Weyl points carry quantized Berry flux. In this work, we introduce a class of symmetry-protected Weyl semimetals which host monopole-antimonopole pairs of Weyl points that generate a dipolar Berry flux. Thus, both monopolar and dipolar Berry fluxes coexist in the Brillouin zone, which results in two distinct types of topologically non-trivial planes separating the Weyl points, carrying either a quantized monopolar or a quantized dipolar flux. We construct a topological invariant – the staggered Chern number – to measure the latter, and employ it to topologically distinguish between various Weyl points. Finally, through a minimal two-band model, we investigate physical signatures of bulk topology, including surface Fermi arcs, zero-energy hinge states, and response to insertion of a π -flux vortex.

I. INTRODUCTION

Weyl semimetals (WSMs) constitute the most well-known class of topological semimetals which has guided the exploration of other topological semimetallic phases^{1–21}. The fact that band-crossing points or Weyl points (WPs) are monopoles of the Berry curvature is one of the most remarkable features of WSMs^{1,22}. Arguably, inversion-symmetric WSMs offer the simplest realization of three-dimensional topological semimetals. In these systems, the two-dimensional planes separating a pair of WPs, carrying opposite Berry-monopole charges, are classified as Chern insulators. The Weyl points, respectively, act as source and sink of the quantized Berry flux passing through the Chern planes. Furthermore, the edge states supported by each Chern insulating layer stack up to give rise to the chiral Fermi-arc states on the surface of WSMs. These notions have been generalized by the discovery of higher-order WSMs, where all topologically non-trivial planes are not Chern insulators, and both Fermi arc surface states and hinge-localized zero modes are present at crystal terminations^{23–26}.

Recently, Nelson *et al.*²⁷ have shown that the topological critical point separating Hopf and ordinary insulators realizes a Berry *dipole*, which asymptotically act as both a source and a sink of Berry curvature. In contrast to an WP, the net Berry flux penetrating a Gaussian surface (GS) enclosing a Berry dipole vanishes. Instead, the flux is staggered on the GS, with its sign determined by the orientation of the dipole relative to local normals on the GS. Therefore, in a hypothetical semimetal, hosting at least a pair of Berry dipoles, one may expect that the planes separating the dipoles would be threaded by a staggered or dipolar Berry flux. Are these “dipolar planes” topologically non-trivial, and, importantly, does the notion of Berry dipoles lead to hitherto unexplored classes of WSMs?

In this Letter, we introduce a class of symmetry-protected WSMs, coined “dipolar Weyl semimetals”, where Chern, dipolar, and ordinary insulating planes coexist, as summarized in Fig. 1a. The WPs in such WSMs result from splitting Berry dipoles, which distinguishes dipolar WSMs from both conventional and higher-order WSMs (see Fig. 1b). We construct two band models for describing dipolar WSMs, and show that dipolar planes support a quantized, but staggered, Berry flux, as exemplified by Fig. 1c. Therefore, the WPs occurring at the boundary between dipolar and Chern insulators are distinguished from that at the bound-

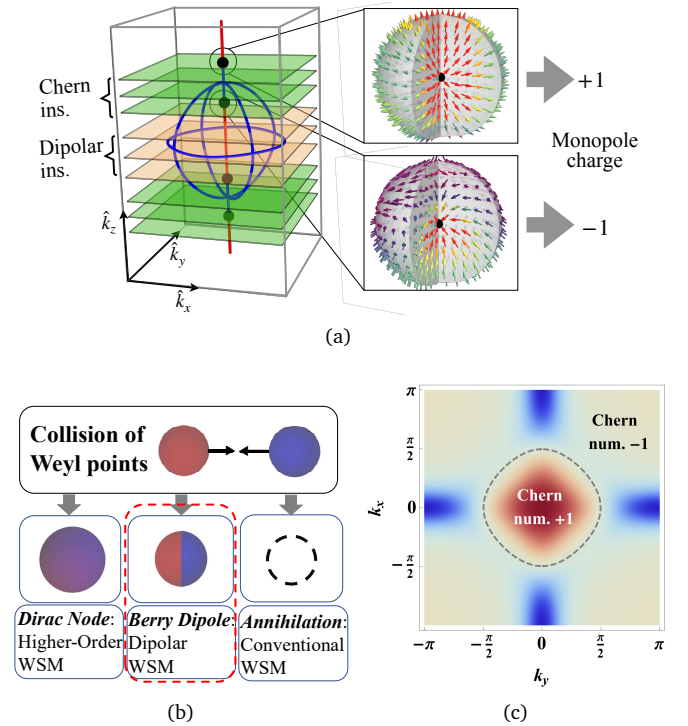


FIG. 1. Properties of the dipolar Weyl semimetal (WSM) phase. (a) Four Weyl points (WPs) (black spheres) occur on the rotation-axis, with alternating monopole charges. The sign is deduced by visualizing the Berry curvature, $\mathcal{B}$, in the vicinity of the WPs [right]. Depending on their location relative to the WPs, the 2D planes layered along $\hat{k}_z$ are either dipolar (red), Chern (green), or ordinary (unmarked) insulators. The red [blue] curves indicate points in the Brillouin zone that map to $\hat{n} = \vec{n}/|\vec{n}| = (0, 0, 1)$ [(0, 0, -1)], where $\hat{n}$ is defined in Eq. (1). The intersection of these curves along the rotation axis marks the locations of band-inversion (black spheres). (b) In dipolar- (higher-order²³) WSMs, there exist pairs of WPs with monopole-charge ± 1 whose collision leads to a Berry dipole (Dirac point). In conventional WSMs, any monopole-antimonopole pair of WPs annihilate upon collision. (c) On the dipolar insulating planes the distribution of $\mathcal{B}_z$ (color profile) is such that two Chern insulators, with opposite Chern numbers, are embedded within the same plane. In Eq. (4), we define the notion of “staggered Chern number” to succinctly capture this pattern.

ary between Chern and ordinary insulators. Remarkably, on surface-terminations perpendicular to the separation between WPs, both chiral Fermi arc states and hinge-localized zero-modes are realized. In order to characterize the bulk

topology, we determine the response of the bulk states to π -flux vortex insertions, and applied magnetic fields. In spite of its similarity to higher-order WSMs^{23,24}, we note that dipolar WSMs require only two bands, such that its bulk topology is completely determined by the bands that cross at the WPs.

II. MODEL AND PHASE DIAGRAM

We consider two-band models of WSMs protected by a combination of n -fold improper rotational (or roto-reflection) symmetry $\mathcal{S}_m^z = \mathcal{M}_z \circ \mathcal{C}_m^z$, and two anti-unitary mirror symmetries ($\mathcal{M}_1$ and $\mathcal{M}_2$). Here, $\mathcal{M}_z$ is the mirror operator that sends $z \rightarrow -z$, and $\mathcal{C}_m^z$ generates m -fold rotations about the $\hat{z}$ -axis. We begin with the single-particle Hamiltonian

$$H(\mathbf{k}) = \sum_{j=1}^3 n_j(\mathbf{k}) \sigma_j, \quad (1)$$

where σ_j is the j -th Pauli matrix, $n_1 = 2(u_4 u_1 + u_3 u_2)$, $n_2 = 2(u_4 u_2 - u_3 u_1)$, and $n_3 = u_4^2 + u_3^2 - u_1^2 - u_2^2$. We require (u_1, u_2) [u_3 and u_4] to transform under an E [B and A , respectively] representation of $\mathcal{C}_m^z$, and $\{u_1, u_2, u_4\}$ [u_3] to be even [odd] under $\mathcal{M}_z$. Consequently, (n_1, n_2) [n_3] transform under an E [A] representation of $\mathcal{S}_m^z$. Since n_j 's do not yet have the most general symmetry-allowed form, we modify them as $n_j \rightarrow n_j + \nu_j$, where ν_j 's will be considered as symmetry-allowed perturbations. Under $\mathcal{S}_m^z$, the Hamiltonian transforms covariantly,

$$\mathcal{S}_m^z H(\mathbf{k}) (\mathcal{S}_m^z)^{-1} = H(\mathbf{k}') = U^\dagger H(\mathbf{k}) U, \quad (2)$$

where $\mathbf{k}' = (\mathfrak{R}_m \mathbf{k}_\perp, -k_z)$ with $\mathbf{k}_\perp = (k_x, k_y)$, $\mathfrak{R}_m$ implements m -fold rotation about $\hat{k}_z$, and $U = \exp\{-\frac{i\pi}{m} \sigma_3\}$. Therefore, $[H(\mathbf{k}), \mathcal{S}_m^z] = 0$ at high-symmetry points (HSPs) that satisfy $\mathbf{k}' \equiv \mathbf{k}$. Because of the constraints placed on u_j 's and ν_j 's by $\mathcal{S}_m^z$, $n_{j=1,2}$ must vanish both at the HSPs and along the rotation-axis. The remaining component, n_3 , is finite in general, and it may change sign between a pair of HSPs only if $u_4^2 + \nu_3$ does not have a fixed sign throughout the Brillouin zone (BZ)²⁸. While the roto-reflection symmetry guarantees a parameter window where the bands cross along the rotation axis, the anti-unitary mirror symmetries protect quantized 1D polarizations along orthogonal high-symmetry axes.

For concreteness, we focus on $m = 4$, and choose

$$\begin{aligned} \frac{u_1}{t_p} &= \sin k_x; & \frac{u_2}{t_p} &= \sin k_y; & \frac{u_3}{t_d} &= \sin k_z (\cos k_y - \cos k_x) \\ u_4 &= t_s \{\Delta - (\cos k_x + \cos k_y + \gamma \cos k_z)\}; \\ \nu_1 &= 2t_s t_p \delta_\perp \sin k_x; & \nu_2 &= 2t_s t_p \delta_\perp \sin k_y; & \nu_3 &= -t_s^2 \delta_3^2. \end{aligned} \quad (3)$$

Here, $\{t_p, t_d, t_s, \Delta, \delta_\perp, \delta_z, \gamma\}$ are model parameters. We note that, at the critical points obtained by setting all $\nu_j = 0$, the Hamiltonian reduces to a form that may be obtained from a model of Dirac semimetal (DSM) with a $\mathbb{Z}_2$ -chiral symmetry, $h(\mathbf{k}) = \vec{u} \cdot \vec{\Gamma}$ with Γ_j 's being a set of four mutually anti-commuting matrices, by following the ‘‘Hopf mapping’’ procedure in Ref.²⁹. The anti-unitary mirror symmetries act as $\mathcal{M}_1 H(\mathbf{k}) \mathcal{M}_1^{-1} \equiv H^*(-k_+, k_-, k_z) = e^{-i\frac{\pi}{4} \sigma_3} H(\mathbf{k}) e^{i\frac{\pi}{4} \sigma_3}$ and $\mathcal{M}_2 H(\mathbf{k}) \mathcal{M}_2^{-1} \equiv H^*(k_+, -k_-, k_z) = e^{i\frac{\pi}{4} \sigma_3} H(\mathbf{k}) e^{-i\frac{\pi}{4} \sigma_3}$, where $k_\pm = \frac{1}{\sqrt{2}}(k_x \pm k_y)$. For simplicity, henceforth, we set

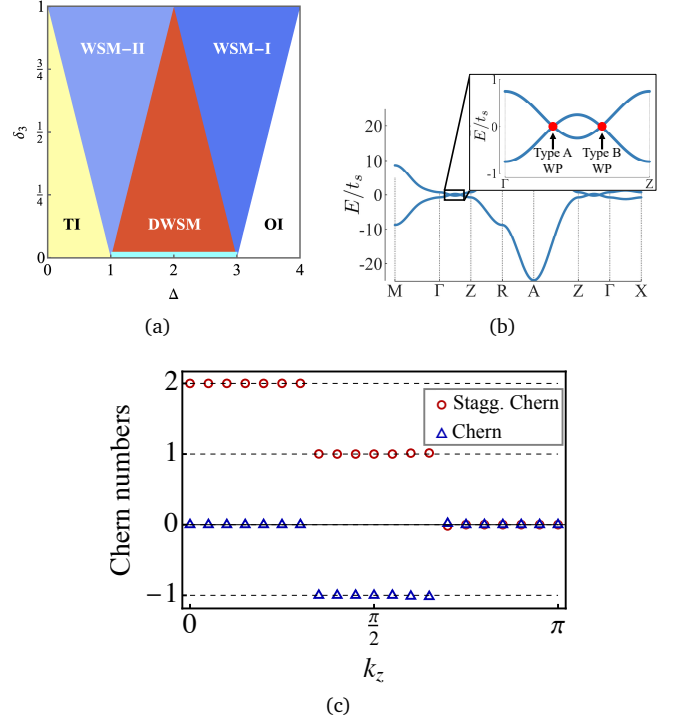


FIG. 2. Characterization of the phases supported by $H(\mathbf{k})$. (a) Phase diagram. A dipolar Weyl semimetal (‘DWSM’) phase exists in the range $(1 + \delta_3) < \Delta < (3 - \delta_3)$, while conventional WSM phases (‘WSM-I’ and ‘WSM-II’) are found when $\delta_3 > |\Delta - 1|$ and $\delta_3 > |\Delta - 3|$. Here, ‘TI’ (‘OI’) indicate topological (ordinary) insulating phases. At $\delta_3 = 0$ and $1 < \Delta < 3$ the Weyl points in the DWSM merge on either side of the k_z axis, thereby forming Berry dipoles [c.f. Fig. 1(b)]. These critical points are marked in cyan. (b) Band structure in the DWSM phase along high-symmetry paths in a tetragonal Brillouin zone. Four Weyl points occur along the $\Gamma - Z$ axis. These are classified into two categories (‘A’ and ‘B’) based on the type of 2D insulating planes they separate [see (c)]. (c) Distribution of the Chern and staggered Chern numbers carried by (k_x, k_y) -planes, as a function of k_z .

$(\gamma, \delta_\perp) = (1, 0)$ and $\Delta, \delta_3 \geq 0$, and note that relaxing these constraints does not qualitatively alter our conclusions.

In the BZ of a tetragonal lattice, $[H, \mathcal{S}_4^z] = 0$ at $\Gamma \equiv (0, 0, 0)$, $Z \equiv (0, 0, \pi)$, $M \equiv (\pi, \pi, 0)$, and $A \equiv (\pi, \pi, \pi)$ points. Bands invert with respect to $\mathcal{S}_4^z$ around the Γ (Z) point for $\delta_3 > |\Delta - 3|$ ($\delta_3 > |\Delta - 1|$). As shown in Fig. 2a, in these regions conventional Weyl semimetallic phases are realized. Since n_3 obtains the same sign at *all* HSPs for $(1 + \delta_3) < \Delta < (3 - \delta_3)$, no band-inversion would be detected by comparing the eigenvalues of $\mathcal{S}_4^z$ at these points. Thus, it may appear that in this parameter regime the system is topologically trivial. This is false, however, because along the $\Gamma - Z$ line we identify *two* locations of band-inversion with respect to $\mathcal{S}_4^z$ at non-HSPs, as shown in Fig. 1a. We refer to these points as ‘‘hidden’’ band inversions, since their existence cannot be deduced by consulting the HSPs alone. What are the ramifications of such ‘‘hidden’’ band inversions?

III. DIPOLAR WEYL SEMIMETAL

In d -dimensions, non-trivial topology can be deduced from the texture of $\hat{n}(\mathbf{k})$. In particular, $\hat{n}(\mathbf{k})$ maps the d -dimensional BZ (T^d) to the 2-sphere (S^2) as a function of $\mathbf{k}$. Valuable insights are obtained by consulting the set of

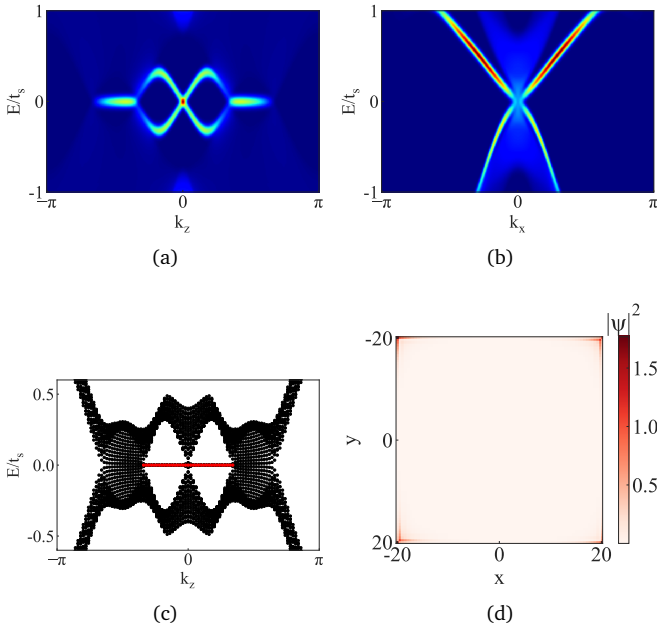


FIG. 3. Spectra of surface and hinge localized states. (a) Spectral density on the (100) surface as a function of k_z . Zero energy Fermi arcs are present between projections of the Weyl-points on the same side of the k_z axis. The central region supports a dispersive, non-degenerate, Dirac cone-like feature centered at the $\bar{\Gamma}$ -point of the surface Brillouin zone. (b) Spectra on (001) surface as a function of k_x fixing $k_y = 0$, displaying a Dirac cone without a band-crossing point. (c) Spectrum from exact diagonalization with periodic boundary condition only along $\hat{z}$. The dispersionless mid-gap states (marked in red) are localized at the corners of the (x, y) -planes, shown in (d), thereby forming dispersionless hinge modes.

points in the BZ which get mapped to a single point on S^2 by $\hat{n}(\mathbf{k})$ [i.e. the preimages $\hat{n}(\mathbf{k})$]. In analogy to Hopf insulators²⁹, the preimages of individual points on S^2 are 1D curves in the three-dimensional BZ for the dipolar WSMs. In contrast to Hopf insulators, however, the preimages of two distinct points on S^2 are not necessarily linked in dipolar WSMs. Thus, the linking number or Hopf invariant vanishes. In the ‘DWSM’ phase, the preimages of the “north” (red) and “south” (blue) poles of S^2 – defined by the simultaneous vanishing of n_1 and n_2 with $\hat{n}_3 = \pm 1$, respectively – intersect at $k_z = \pm \cos^{-1}(\Delta - 2 \pm \delta_3)$ along the Γ – Z line, as shown in Fig. 1a. Since $H(\mathbf{k})$ commutes with $\mathcal{S}_4^z$ on the polar-preimages, these intersections are locations of band-inversions with respect to $\mathcal{S}_4^z$. The existence of the intersections is symmetry protected, and remains robust against $\delta_\perp$ as long as $|\delta_\perp| < \delta_3$. Because $\vec{n}(\mathbf{k}) = \vec{0}$ at the intersection of the preimages of the north and south poles, these locations correspond to the Weyl points (see Fig. 2b).

A. Topology of bulk states

By enclosing the band-crossing points by GSs, we determine the net Berry flux emanating from these band singularities to be 2π up to an overall sign, as illustrated in Fig. 1a. Therefore, the band-crossing points are unit-strength monopoles of the Berry curvature, and we identify them as WPs. When a GS encloses both WPs on a fixed side of the k_z -axis, the net Berry flux passing through this surface vanishes. The region within the GS is topologically non-trivial, however, as a non-zero net Berry *dipole-*

flux pierces the surface. The existence of the dipole-flux can be understood by appealing to the topological critical point at $(\delta_3, \delta_\perp) = (0, 0)$: as $\delta_3 \rightarrow 0$, the pair of WPs at $|k_z| = \cos^{-1}(\Delta - 2 \pm \delta_3)$ collide to yield a pair of band singularities at $|k_z| = \cos^{-1}(\Delta - 2)$, which act as sources of the dipole-flux (i.e. Berry dipoles; see Fig. 1b). On an infinitesimal GS enclosing the latter band-crossing points, the Berry flux obtains a staggered form with the northern (southern) hemisphere supporting a 2π (-2π) net flux. Thus, for $\delta_3 > 0$, when a Gaussian surface encloses the monopole-antimonopole pair at $|k_z| = \cos^{-1}(\Delta - 2 \pm \delta_3)$, a net dipole-flux survives. This unusual behavior of the WPs in dipolar WSMs is summarized with further details in section II.C. of the ‘smemental Materials’ (SI)³⁰. The presence of both monopole and dipole fluxes indicates that topologically non-trivial k_z planes in the BZ are not limited to being 2D Chern insulators.

Indeed, the k_z -planes between $k_z = \pm \cos^{-1}(\Delta - 2 + \delta_3)$ (red planes in Fig. 1a) support a dipolar version of the Berry flux which reflects the presence of a skyrmionium texture³¹ for $\vec{n}(\mathbf{k})$. Skyrmioniums are composed of two oppositely charged but non-overlapping skyrmions; consequently, they do not support a finite Chern number. It is possible to define a quantized topological invariant, however, that distinguishes a skyrmionium-carrying plane from an ordinary 2D insulator. In order to emphasize its origin in a staggered distribution of Berry curvature on the $\mathbf{k}_\perp$ plane, we call this topological invariant “staggered Chern number”, and define it as

$$\mathfrak{C}_{\text{stagg}}(k_z) = \frac{1}{2\pi} \int d\mathbf{k}_\perp \mathcal{B}_z(\mathbf{k}_\perp, k_z) f_{\text{stagg}}(\mathbf{k}_\perp), \quad (4)$$

where $\mathcal{B}_z(\mathbf{k}_\perp, k_z)$ is the $\hat{k}_z$ component of the Berry curvature, and $f_{\text{stagg}}(\mathbf{k}_\perp)$ is a weight function that is determined by $H(\mathbf{k})$ such that $\int d\mathbf{k}_\perp f_{\text{stagg}}(\mathbf{k}_\perp) = 0$. Since $\mathfrak{C}_{\text{stagg}}$ is effectively the difference of Chern numbers of the same magnitude but opposite signs, it equals the Chern number when the staggering is absent. If a quantized staggering of flux is present, then $\mathfrak{C}_{\text{stagg}}$ will be an even integer, in analogy to familiar mirror³² or spin³³ Chern numbers. While on mirror- or spin-Chern number carrying planes the Chern number is staggered in an internal sub-space, here, it is staggered in the momentum space. **An alternative formulation of $\mathfrak{C}_{\text{stagg}}$ that does not require an explicit knowledge of f_{stagg} , but utilizes effective Su-Schrieffer-Heeger³⁴ forms of the Hamiltonian on the mirror axes, is provided in the SI³⁰.**

We plot $\mathfrak{C}_{\text{stagg}}$ as a function of k_z in Fig. 2c, and relegate the details of the calculation to the SI³⁰, along with the explicit form of $f_{\text{stagg}}(\mathbf{k}_\perp)$. We obtain $\mathfrak{C}_{\text{stagg}} = 2$ for all planes with $|k_z| < \cos^{-1}(\Delta - 2 + \delta_3)$, indicating intertwining of two regions with opposite Chern numbers. We note that the $k_z = 0$ and π planes of the dipolar WSM are identical to respective planes in the Moore-Ran-Wen class of models of Hopf insulators²⁹. Consequently, *these special planes in Hopf insulators are also characterized by a quantized staggered-Chern number*. The staggering of the Chern number in dipolar WSMs, as well as Hopf insulators, originates from non-trivial 2D winding numbers supported by corresponding $\mathbf{k}_\perp$ planes of the system governed by $h(\mathbf{k}) = \vec{u} \cdot \vec{\Gamma}$. In the present case, $h(\mathbf{k})$ describes a $\mathbb{Z}_2$ -chiral DSM with a pair of band-crossings along the k_z -axis. The k_z -planes lying between the Dirac points are characterized by the relative-Chern number³⁵, which is an $\mathbb{Z}$ -valued 2D bulk invariant and a variant of the spin-Chern number, and it is protected by both bulk- and spin-gaps^{36,37}. The non-trivial $\mathfrak{C}_{\text{stagg}}$ is

thus a manifestation of the non-trivial relative- or spin-Chern number supported by the k_z -planes of chiral DSMs³⁵. Thus, under the Hopf map, the *staggering of the Chern number in a Kramers degenerate subspace maps to its staggering in momentum space*.

The k_z -planes separating WPs on the *same* side of the k_z -axis, i.e. $\cos^{-1}(\Delta - 2 + \delta_3) < |k_z| < \cos^{-1}(\Delta - 2 - \delta_3)$ (green planes in Fig. 1a), carry a Chern number $\mathcal{C} = -1$, as shown in Fig. 2c. By contrast, all planes with $|k_z| > \cos^{-1}(\Delta - 2 - \delta_3)$ are topologically trivial with $\mathcal{C}_{\text{stag}} = 0 = \mathcal{C}$. Thus, the k_z -plane hosting a type-A [-B] WP (see inset in Fig. 2b) may be interpreted as the topological critical point separating a staggered-Chern and a Chern insulator [a Chern and an ordinary insulator]. Here, while the k_z -plane hosting a type-B WP supports a half-integer Chern and staggered-Chern number of equal magnitude, the k_z -plane hosting type-A WP supports a half integer Chern number and a *distinct* half-integer staggered Chern number³⁰.

B. Surface and hinge states

The states on the (100) and (010) surfaces are sensitive to the texture of $\hat{n}(\mathbf{k})$ in the bulk. As a representative example, we portray the topologically protected states on the (100) surface in Fig. 3a. Fermi arcs are found to connect the projections of WPs on the same side of the k_z axis, indicating their origin in the non-trivial Chern insulating planes. The staggered-Chern or dipolar planes have a distinct topological response to surface terminations, which is reminiscent of *sp*-Dirac semimetals^{35,38}. In particular, generic dipolar planes support a pair of gapped edge states. On the $k_z = 0$ (and π) plane $\mathcal{S}_4^z$ reduces to $\mathcal{C}_4^z$, which allows it to support gapless edge states. This leads to the single copy of Dirac cone centered at the $\bar{\Gamma}$ point of the surface-BZ. The pair of degenerate surface states at the $\bar{\Gamma}$ point is protected by a quantized 1D winding number in the bulk along the k_x -axis³⁰. The states on the (001) surface exist as long as the $\mathcal{S}_4^z$ symmetry is preserved. As shown in Fig. 3b, these states form a Dirac cone-like feature about the zone-center of the (001) surface BZ, with the band-crossing point absent. The center of the (001) surface BZ corresponds to the projection of the rotation axis, which accounts for the lack of normalizability at this point³⁵.

While generic k_z -planes in the region $|k_z| < \cos^{-1}(\Delta - 2 + \delta_3)$ support gapped edge states, they also support corner-localized zero modes, which are protected by the anti-unitary mirror symmetries. These corner localized modes stack along the $\hat{z}$ -direction to give rise to hinge localized zero-modes. In Fig. 3c and 3d we identify these hinge-states by exactly diagonalizing the Hamiltonian with periodic boundary condition only along $\hat{z}$. In Section III of the SI³⁰ (also see Ref.³⁹) we detail the presence of a quantized, one-dimensional winding number along the diagonal axes, which protect the corner localized zero-modes in accordance with Ref.⁴⁰. Thus, surface- and hinge-localized Fermi arcs are simultaneously present in dipolar WSMs, which is reminiscent of higher-order topological semimetals^{23,24,35,41}.

C. Vortex-bound states

In the position-space, field-theoretical calculations have proven that the bulk topological invariant in the ground

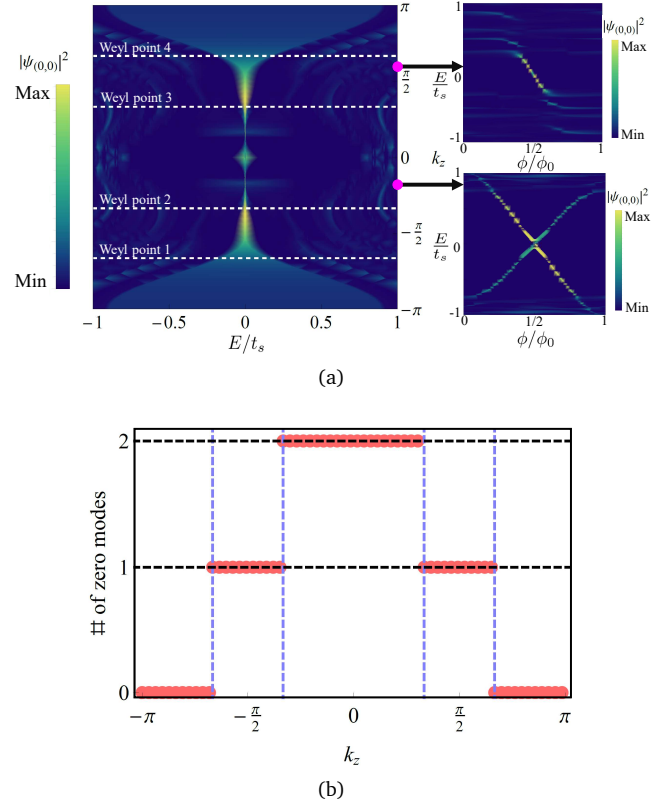


FIG. 4. Vortex-bound states as a diagnostic of bulk topology. (a) Probability density (PD) at the location of vortex for 120 states closest to zero energy. Left: PD for vortex-flux $\phi = \phi_0/2$ as a function of energy and k_z . Here, ϕ_0 is the quantum of flux. Zero modes exist for *all* topologically non-trivial planes. Right: PD as a function of ϕ and energy, fixing k_z . Both Chern (top right) and staggered-Chern planes (bottom right) support charge pumping. (b) Number of vortex-bound zero modes upon insertion of magnetic flux tube with $\phi = \phi_0/2$. The distinct number of zero-modes establishes the different topological character of $\mathbf{k}_\perp$ planes as a function of k_z .

state can be computed through insertion of an electromagnetic π -flux vortex. In particular, the number of states bound to the vortex corresponds to the magnitude of the quantized flux in the ground state of a 2D topological insulator^{35,42–46}. Moreover, as the magnitude of flux carried by the vortex is tuned between zero and the flux quanta, $\phi_0 = h/e$, the vortex-bound modes are pumped across the bulk band-gap.

To unambiguously examine the topological response of the two-dimensional planes stacked along $\hat{k}_z$, we consider them as independent layers of 2D insulators, labeled by k_z . In the position-space for each layer, we insert a vortex tube at the origin, carrying a flux ϕ . We exactly diagonalize the resultant Hamiltonian on a 20×20 lattice, which yields a k_z -dependent energy spectrum, as shown in Fig. 4a. The unit strength Chern planes at $\cos^{-1}(\Delta - 2 + \delta_3) < |k_z| < \cos^{-1}(\Delta - 2 - \delta_3)$ support a single vortex-bound state, which is pumped across the bulk band-gap as a function of ϕ . The dipolar or staggered-Chern planes at $|k_z| < \cos^{-1}(\Delta - 2 + \delta_3)$ support *two* states at the vortex, each corresponding to a Chern sector. Since the Chern sectors carry opposite Chern numbers, these vortex-bound states are pumped in opposite directions as a function of ϕ , reminiscent of spin-Hall insulators^{42,43,47,48}. Importantly, as demonstrated in Fig. 4b, when the strength of flux is held fixed at $\phi = \phi_0/2$, the number of vortex-bound modes can

be used as a quantized diagnostic of the topology of the 2D layers. Since both $\mathcal{C}_{\text{stagg}}$ and the number of vortex-bound zero modes effectively count the number of Chern sectors in each layer, they have an identical response (c.f. Figs. 2c and 4b).

IV. CONCLUSION

We introduced a class of WSMs where both dipole- and monopole-flux carrying planes are present. A topological invariant, the staggered Chern number, is formulated for diagnosing the presence of quantized dipolar-flux. Through flux insertions, the dipolar planes are shown to have a topological response that is analogous to generalized spin-Hall insulators. With the help of a two-band model, we demonstrated that surface and hinge states are reminiscent of higher order topological semimetals. A detailed comparison among dipolar, higher-order, and conventional WSMs is presented in the SI³⁰, where we show that the clearest distinction between a dipolar and a higher-order WSM arises in the Landau level spectra.

The two-band model discussed here follows a similar principle of construction as Hopf-insulators. Therefore, we expect it would be possible to simulate it within the same platforms proposed for realizing the latter^{49–55}. Further, it is possible to construct variants of the same model, with potentially more exotic topological singularities^{56–59}. Both considerations are left to future works.

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